Backward Induction and Subgame Perfection
Economics 302 - Microeconomic Theory II: Strategic Behavior

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(with thanks to Anke Kessler)
Sequential Games of Perfect Information
2 Backward Induction
3 Stackelberg Competition
4 Subgame-Perfect Equilibrium
Most Important Things to Learn

1. Basic concepts: extensive form (game tree), strategies in sequential games, perfect information
2. Know why NE may not work well in sequential games
3. Using backward induction, and knowing when it yields a unique solution
4. Solving for the Stackelberg outcome, and understanding why it differs from the Cournot outcome
5. Definitions: information set, subgame, subgame-perfect equilibrium (SPE)
6. Know how to find SPE
7. Know the difference between SPE and NE
Introduction to Sequential Games

- Up to now: studied games where players move simultaneously.
- But often, people/firms observe what others do before acting. Does it make a difference?
- Example: Battle of the Sexes

<table>
<thead>
<tr>
<th></th>
<th>Ballet</th>
<th>Hockey</th>
</tr>
</thead>
<tbody>
<tr>
<td>Girl</td>
<td>3,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Hockey</td>
<td>0,0</td>
<td>1,4</td>
</tr>
</tbody>
</table>

- If simultaneous move:

- What happens if Girl texts Guy: "I’m going to the ballet, and my phone is dying. See you there!"
- So if Girl has the opportunity to send that text (and, for whatever reason, Guy doesn’t), will she do it?
A good way to represent a sequential game is the **extensive form**, often called a "game tree."

Consider the Battle of the Sexes, with the girl (player 1) first texting the guy "Ballet" (B) or "Hockey" (H), and then becoming out of reach.

Each branch is an action. Let’s call the guy’s (player 2’s) actions B’ and H’ to distinguish them from the girl’s.

Each non-terminal node is a place where the specified player has to make a decision.

Each terminal node is an outcome: a combination of actions, just like before.

The numbers below each terminal node are the payoffs from the outcome corresponding to the node. As usual, the first number is player 1’s payoff, the second is player 2’s, etc.
We will first study games of **perfect information**: one player acts at a time, and each player sees all previous actions.

Simultaneous-move games are NOT games of perfect information (when at least two players have at least two actions each).

Then, we will look at games that do not have perfect information.

Example of the latter: playing a prisoner’s dilemma more than once.

Note: don’t confuse *perfect* information with *complete* information!
A player’s strategy specifies a probability distribution over her actions at each node where she plays, regardless of whether that node is reached.

In other words, a strategy is a player’s full contingency plan.

In our example, the Guy’s strategy must include what he would do if the Girl’s chooses H, even if we don’t expect the Girl to choose H.

Example 1: Guy chooses H’ regardless of what Girl does. (B→H’,H→H’)

Example 2: Guy chooses the same thing as the Girl. (B→B’,H→H’)

Just like before, a strategy profile is a collection of each player’s strategy.
Let’s find the pure-strategy NE in this game. As before, we use the normal form:

\[(B \rightarrow B', H \rightarrow B') \ (B \rightarrow B', H \rightarrow H') \ (B \rightarrow H', H \rightarrow B') \ (B \rightarrow H', H \rightarrow H')\]

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>H</th>
</tr>
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<td><strong>B</strong></td>
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Are these pure-strategy NE all realistic?
Backward Induction

- Idea: should require that players play a best-response (given what they know) **at all nodes, even those that are not reached**.
- Strategy profiles satisfying the above are called **subgame-perfect (Nash) equilibria (SPE or SPNE)** in games of complete information.
- In perfect information games, solving for SPEs is particularly easy: just start at the terminal nodes to infer what players will do at the last step. Given that, figure out what happens at the second-to-last step, and so on.
- This procedure is called **backward induction**.
- When is there a **unique** SPE in perfect information games?
- Is every SPE a **NE**?
- Is every **NE** a SPE?
Consider Rock-Paper-Scissors, but suppose player 2 sees what player 1 does before acting.

Payoff is 1 for a win, -1 for a loss, and 0 for a tie.

Draw this game in extensive form, and find its SPE(s) using backward induction.
In Battle of the Sexes, players gain from committing to a course of action.

As a result, there is a first-mover advantage: the Guy would like to threaten to go to the hockey game after the Girl has gone to the ballet dance, but cannot do so credibly.

As we saw, NE allows for such non-credible threats, while SPE doesn’t.

By contrast, in Rock-Paper-Scissors, flexibility creates a second-mover advantage.

There are also games where neither is the case.
Problems with Backward Induction

- May not be reasonable when game is long and/or complicated: chess, centipede game.
  - This is a similar problem as in ISD: we assume that players can do long chains of reasoning, that they trust others to do so, that they trust others to trust others to do so, and so on...
- Even if you accept this assumption, you need a further assumption when a player has multiple best responses at a node: players correctly anticipate what others will do, even though this cannot be deduced by logic alone.
  - This is an assumption we also made for NE.
- Philosophical aside: unexpected hanging paradox.
Back to oligopolies: suppose there are two firms, and Firm 1 picks quantity before Firm 2.

For example, signs contract with distributors, buys lots of inputs, etc.

Simplest case: both firms have the same constant marginal cost $c$, produce a homogeneous good, and face linear market demand $P = a - bQ$.

We use backward induction to solve for a subgame-perfect equilibrium.
Application: Stackelberg Model (II)

- We know from our analysis of the Cournot model that Firm 2’s best response to $q_1$ is
  $$q_2 = \frac{a - c}{2b} - \frac{q_1}{2}$$

- By backward induction, instead of taking as given a constant $q_2$, Firm 1 will take as given Firm 2’s above best response: Firm 1 knows that $q_2$ now depends on $q_1$.

- Firm 1’s profit function:

  $$q_1(a - b(q_1 + q_2) - c)$$
  $$= q_1(a - b(q_1 + \frac{a - c}{2b} - \frac{q_1}{2}) - c)$$
  $$= \frac{1}{2}((a - c)q_1 - bq_1^2)$$
Taking the first-order condition and rearranging gives:

\[ q_1 = \frac{a - c}{2b} \]

Plugging back into Firm 2’s best response function gives:

\[ q_2 = \frac{a - c}{4b} \]

Compare to Cournot outcome:

\[ q_1 = q_2 = \frac{a - c}{3b} \]
Stackelberg profits are:

\[ \pi_1 = \frac{1}{8} \frac{(a - c)^2}{b}, \quad \pi_2 = \frac{1}{16} \frac{(a - c)^2}{b} \]

Compare to Cournot profits:

\[ \pi_1 = \pi_2 = \frac{1}{9} \frac{(a - c)^2}{b} \]

Who benefits, and why?

Graphical representation (if time permits)
Exercise

Consider the same problem, but Firm 1 has a cost of 2c, while Firm 2 still has a cost of c.

Solve for the SPE.
Recap

- We introduced sequential games of perfect information, and solved them using backward induction.
- Perfect information: one player acts at a time, and each player sees all previous actions.
- We represented these games using the extensive form (game tree).
- Backward induction: start at the bottom of the game tree, figure out the best response(s) at each node, and work our way up the tree.
- We said that the resulting strategy profile(s) is/are SPE(s).
- Now: generalize the concept of SPE to games without perfect information.
Example

- Player 1 first plays Top, Middle or Bottom.
- Player 2 only finds out whether player 1 has played Bottom. If so, he plays Left or Right; if not, he plays In or Out.
- Payoffs are: (1,1) after (Top, In), (0,0) after (Top, Out), (0,0) after (Middle, In), (1,1) after (Middle, Out), (0.6,0.6) after (Bottom, Left), (0,0) after (Bottom, Right).
- How do we draw the game tree of such a game?
- Sometimes, the player taking an action doesn’t know where he/she is because he/she didn’t see how other players played earlier.
An information set is a set of nodes where:

1. the same player is acting at all nodes in the set; and
2. the player that is acting knows that she is at the information set, but cannot tell which node she is at within the information set.

Note: If a player knows that she is at a particular node, then that node is the only element in its information set.

Now, a player’s strategy must specify a course of action for each information set (rather than each node) where he/she acts: you can’t play differently at nodes that you can’t distinguish!
Let’s find pure-strategy profiles that make sense as solutions of this game.

Common sense that player 2 must play Left after Bottom.

Then we solve for the NE of the remaining simplified game.

(We will find the pure ones in class, and you can try to find the others on your own.)

To get the SPEs of the original game, combine the NEs of the simplified game with the fact that player 2 must play Left after Bottom.

Note: The original game as has NEs that are not SPEs. Can you find one?
Subgames

- Idea: some parts of the game tree can stand alone as a game. These are called **subgames**.
- Example: The game we considered, after Player 1 plays Bottom.
- Definition: a node $h$’s **successors** are all the nodes after $h$, all the way to the terminal nodes (end of the game tree).
- Definition: Suppose you have a game $G$. A **subgame of $G$** consists of a single non-terminal node and all its successors with the property that every information set of $G$ is either entirely inside or entirely outside that set of nodes.
- The last part of the definition can be rephrased: no information set of $G$ contains both nodes inside and nodes outside of a subgame.
Way to remember the definition: think of information sets as spider webs. Subgames are parts of the tree (except for terminal nodes) that you can detach by snapping a single branch and without tearing a web.

Note: The whole game is always a subgame.

How many subgames were there in our example?
Once you understand subgames, the definition of subgame-perfect equilibrium is simple:

- **A subgame perfect equilibrium is a strategy profile where a Nash equilibrium is played in each subgame.**

To solve for SPE, do what we have been doing! Start with the small subgames toward the end of the tree, and work backwards.

As you work backwards, you will be solving bigger and bigger subgames.

Backward induction is a special case of this procedure: in games of perfect information, every non-terminal node and its successors are a subgame.
We’ll look at examples of various sequential games to get you some practice.

Then, we’ll focus on repeated games. These are sequential games formed by repeating a simultaneous-move game over and over again.

In keeping with the market power theme, an important application of repeated games is collusion in an oligopoly.
Exercise 1

- Player 1 plays T or B.
  - If T, game ends with payoffs (1,0).
  - If B, player 2 plays L or R, leading to payoffs (0,1) and (3,1) respectively.
- Draw the game tree, identify the subgame(s), find the SPE.
- Is there any NE that is not subgame-perfect?
Exercise 2

- Players 1 and 2 first simultaneously pick Big or Small.
- They observe each other’s choice, and then simultaneously pick High or Low.
- If you were to draw the game tree, how many subgames would there be? How many terminal nodes (i.e. outcomes)?
A large number of firms (say 100) first simultaneously decide whether to enter or exit. Firms that exit get payoff 0.

The firms that enter play a Cournot game with constant marginal cost 3, fixed cost 40, and demand $Q = 45 - P$.

Is this a game of perfect information?

Find all pure-strategy SPEs.

Can you think of another pure-strategy NE?
Exercise 4 (from Fudenberg and Tirole, p441)

Solve for the pure-strategy SPE(s) in these games.

1. Player 1 first chooses Up or Not Up. If Up, game ends. If Not Up, simultaneous-move game where player 1 chooses Middle or Down, and player 2 chooses Left or Right.

2. Player 1 first chooses Up, Middle or Down. If Up, game ends. If Middle or Down, player 2 chooses Left or Right without observing player 1’s action.

Payoffs: (2,2) after Up, (3,1) after (Middle, Left), (1.5,0) after (Middle, Right), (0,0) after (Down, Left), (1,1) after (Down, Right).

This odd property of SPE inspired some other solution concepts way beyond the scope of this course...