A Continuous-Time Model of Multilateral Bargaining*

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Abstract

We propose a finite-horizon continuous-time framework for coalitional bargaining, in which players can make offers at random discrete times. In our model: (i) expected payoffs in Markov perfect equilibrium (MPE) are unique, generating sharp predictions and facilitating comparative statics; (ii) MPE are the only subgame perfect Nash equilibria (SPNE) that can be approximated by SPNE of nearby discrete-time bargaining models. We investigate the limit MPE payoffs as the time horizon goes to infinity and players get infinitely patient. In convex games, we establish that the set of these limit payoffs achievable by varying recognition rates is exactly the core of the characteristic function.

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1 Introduction

Coalitional bargaining investigates situations in which different groups of players can reach an agreement and divide the surplus generated by the coalition being formed. In these games, a proposer has to choose both a coalition to approach and a division of the surplus that the coalition generates. A relatively large literature investigates both general coalitional bargaining situations and important subclasses thereof, such as the legislative bargaining model of Baron and Ferejohn (1989) and its many applications. However, for general coalitional values, the applicability of the proposed models is limited by (typically severe) multiplicity of equilibria and analytical intractability.[1]

This paper proposes a coalitional bargaining framework, set in continuous time[2], where players get random opportunities to approach others and make proposals according to independent Poisson processes.[3] The model can be regarded as a limit of discrete-time models in which the time duration between consecutive periods goes to zero, but the probability that some player is recognized at a given time period also goes to zero. The possibility of no one being recognized at a given period distinguishes discrete-time approximations of our model from discrete-time random-recognition coalitional bargaining games typically considered in the literature (see Okada (1996)). Another key feature of our model is that we assume a fixed deadline for negotiations. Our motivation for this is twofold. First, in many real-world bargaining situations, there are natural deadlines for negotiations.[4] Second, we use the resulting model to select among subgame perfect Nash equilibria (SPNE) of the infinite-horizon model by studying equilibrium payoffs in the limit as the deadline gets infinitely far away.[5]

[1] The models in this literature are multilateral extensions of the bilateral dynamic bargaining games of Stahl (1972) and Rubinstein (1982). For an incomplete list of papers investigating coalitional bargaining with general coalitional values, see Gul (1989), Chatterjee *et al.* (1993), Perry and Reny (1994), Moldovanu and Winter (1995), Bloch (1996), Okada (1996), Ray and Vohra (1997) and (1999), Evans (1997), Konishi and Ray (2003), and Gomes (2005). For a relatively recent synthesis of the literature, see Ray (2007).

[2] For other continuous-time bargaining games, see Perry and Reny (1993), Sákovics (1993), and Perry and Reny (1994). These papers are set in infinite horizon. The first two focus on two-player bargaining, while Perry and Reny (1994) studies stationary subgame-perfect equilibria (SSPE) in totally balanced games. An important difference between our setting and the above models is that in the latter, players can make a proposal at any point in time, after a fixed amount of delay following their previous proposal.

[3] Players with higher recognition rates can propose more frequently in expectation. This might be either a consequence of institutional features, like certain members of a legislature (party leaders or other elected officials within the legislature) enjoying preferential treatment in initiating proposals, or of how much attention and resources a player can devote to the bargaining procedure at hand. For models in which the right to make an offer is endogenous, see Board and Zwiebel (2012) and Yildirim (2007).

[4] If a professional sports league and its players' association do not reach an agreement by a certain date, then the season needs to be canceled, as happened to the 2004-2005 National Hockey League season. For reaching an out-of-court settlement, the announcement of the verdict poses a final deadline.

[5] Our paper is not the first to examine deadline effects in bargaining. Fershtman and Seidmann (1993) examine bilateral bargaining with a particular commitment; Ma and Manove (1993) study bilateral bargaining with imperfect control over the timing of offers; Norman (2002) investigates legislative bargaining with

We also assume that, once an offer is made, the approached parties react immediately, and all of them have to accept the proposal in order for an agreement to be reached.[6] Once an agreement is reached (by some coalition), the game ends. If a proposal is rejected by any of the approached players, the game continues, and players wait for the next recognition. Two highlighted special cases that fit into this framework are n-player group bargaining, where only the grand coalition can generate positive surplus, and legislative bargaining, where any large enough coalition of players (in the case of simple majority, voting coalitions involving more than half of the players) can end the game by reaching an agreement. Another example is a patent race in which several different coalitions have the opportunity to pool their insights to develop the same technology, but the race ends after some coalition successfully obtains a patent.

Our framework has some convenient features, for any specification of coalitional values and recognition rates. In particular, a Markov perfect equilibrium (MPE), which is an SPNE in which strategies only depend on the payoff-relevant part of the game history, always exists, and expected MPE payoffs are uniquely determined. This facilitates comparative statics with respect to the parameters of the model (the time horizon for negotiations, recognition rates for proposals, and the characteristic function indicating the values of different coalitions).[7] Furthermore, we show that the MPE are the only SPNE of the model that can be approximated by SPNE of nearby discrete-time bargaining models satisfying a regularity condition, which holds generically for discrete-time approximations of the continuous-time model. This provides a microfoundation for focusing on MPE in the continuous-time model, if one regards the latter as a limit of discrete-time environments.

Our main results provide a characterization of the unique MPE payoffs in the limit as the time horizon tends to infinity in games with convex characteristic functions, for patient players. We show that, for any vector of recognition rates, the vector of limit MPE payoffs converges, as the discount rate goes to zero, to a point in the core of the underlying characteristic function. Conversely, for any point in the core, there is a vector of recognition rates such that, as the discount rate goes to zero, the limit MPE payoff vector converge to the given point. Hence, by varying the recognition rates, we can establish an exact

a deadline; finally, Yildiz (2003) and Ali (2006) consider long finite horizon games in which players disagree over their bargaining powers.

[6] The assumption that time only lapses between proposals, but not between a proposal and players' responses, goes back to the original dynamic bargaining models of Stahl (1972) and Rubinstein (1982), and it naturally holds in various settings. For example, in the legislative bargaining context of Baron and Ferejohn (1989), it takes time to prepare a new bill and bring it to a vote, but once voting starts, results are known essentially instantaneously.

[7] In a companion paper (Ambrus and Lu (2010)), we apply our model to legislative bargaining with a long finite time-horizon, and conduct these comparative statics.

equivalence between points of the core of the characteristic function and limit MPE payoffs of the continuous-time bargaining game. We also show that limit MPE payoffs correspond to stationary equilibrium payoffs of the infinite-horizon game.

Our findings complement existing results on noncooperative foundations of the core in coalitional bargaining games, as in Chatterjee *et al.* (1993), Perry and Reny (1994) and Yan (2003). However, our results are novel in that for any vector of recognition rates, the limit MPE payoffs are unique, but varying the recognition rates establishes an exact equivalence between limit MPE payoffs and the core. In some papers, such as Perry and Reny (1994), for a given specification of the model, there can be a severe multiplicity of equilibrium payoffs (including all points of the core), making comparative statics more difficult than in our model. In other models, only one direction of the equivalence relationship holds: either all equilibrium payoffs in a class of games correspond to points of the core (as in Chatterjee *et al.* (1993)), or each point of the core can be supported as an equilibrium payoff vector, for some specification of recognition probabilities (as in Yan (2003)).

In the case of symmetric recognition rates, with a convex characteristic function and patient players, the limit MPE payoffs of our noncooperative game correspond to the point in the core that preserves the highest amount of symmetry in the division of the surplus, subject to the sum of payoffs of members of any coalition being at least the coalition's value. In particular, if equal division is a core allocation, then it is selected as the equilibrium payoff vector.

By providing microfoundations for stationary SPNE in the infinite-horizon version of our game, we contribute to the literature on selecting Markovian equilibria in games with asynchronous moves (see Bhaskar and Vega-Redondo (2002) and Bhaskar, Mailath and Morris (2013)). The literature on coalitional bargaining, and in particular the literature on legislative bargaining, primarily focuses on analyzing stationary SPNE because of the severe multiplicity and relative complexity of SPNE. However, despite the large number of papers using the solution concept in multilateral bargaining games, there is little work on formally justifying this practice.[8] In fact, Norman (2002) provides negative results in this direction: he shows that in legislative bargaining, there can be many non-Markovian SPNE, even if the game is finite. Moreover, even when one restricts attention to specifications that have unique SPNE in finite horizons, expected equilibrium payoffs in general do not converge, as the horizon goes to infinity, to stationary SPNE payoffs of the infinite-horizon version of the

[8] Baron and Kalai (1993) show that the stationary equilibrium is the unique simplest equilibrium in the Baron and Ferejohn legislative bargaining game. Chatterjee and Sabourian (2000) show that noisy Nash equilibrium with complexity costs leads to the unique stationary equilibrium in n-person group bargaining games (that is, when unanimity is required for an agreement). See also Baron and Ferejohn (1989) for informal arguments for selecting the stationary equilibrium in their game.

game.

Lastly, we note that our uniqueness result for MPE differs in important ways from existing uniqueness results in the bargaining literature. In finite-horizon coalitional bargaining games, there generically is a unique SPNE, as shown in Norman (2002) in the context of legislative bargaining. This holds because for generic vectors of recognition probabilities, no player is ever indifferent between approaching any two coalitions of players, so strategies and continuation payoffs can be simply computed by backward induction.[9] This is not the case in our continuous-time framework: in MPE, indifferences are generated endogenously, for open sets of recognition rates, for nondegenerate intervals of time during the game. Therefore, the arguments needed to show uniqueness of MPE in our game are unrelated to those establishing generic uniqueness of SPNE in finite-horizon discrete games. Moreover, our uniqueness result also differs from uniqueness results for stationary SPNE in special classes of infinite-horizon coalitional bargaining games, such as the main result of Eraslan (2002) in the Baron and Ferejohn (1989) legislative bargaining context, or the generalization of this result in Eraslan and McLennan (2012). It is known that those uniqueness results do not extend to all random-proposer coalitional bargaining games (for an explicit example, see Yan (2001)). By contrast, the uniqueness of MPE payoffs in our finite-horizon games holds for coalitional bargaining games with general characteristic functions.

2 The Model

The Cooperative Game

Consider a bargaining situation with set of players $N = \{1, 2, ..., n\}$ and characteristic function $V : 2^N \to \mathbb{R}_+$, where $V(C)$ for $C \subset N$ denotes the surplus that players in C can generate by themselves (without players in $N \backslash C$). We refer to elements of 2^N as *coalitions*. We assume that if $C_1 \subset C_2$, then $V(C_1) \leq V(C_2)$. The core of V is defined as: $\mathcal{C}(V) = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : \sum_{i \in C} x_i \geq V(C) \; \forall \; C \subset N \text{ and } \sum_{i \in N} x_i = V(N)\}$.

The Dynamic Noncooperative Game

The dynamic bargaining game we investigate is defined as follows. The game is set in continuous time, starting at $-T < 0$.[10] There is a Poisson process associated with each player i, with rate parameter $\lambda_i > 0$. The processes are independent from each other. For

[9] The flipside of this simplicity is that strategies (including which coalitions to approach) and continuation values typically do not converge as the time horizon goes to infinity, and instead "jump around."

[10] We use the nonstandard notation of negative time because fixing the deadline at zero facilitates keeping track of reservation values at time t, independently of the length of the game. This is because, in MPE, the latter only depends on the time remaining before the deadline, not on when the game started. This notation allows us to have increasing t as time progresses.

future reference, we define $\lambda \equiv \sum_{i=1}^{n} \lambda_i$. (In an abuse of notation, we will also refer to λ as the vector of rates.) Whenever the process realizes for a player i, she can make an offer $x = (x_1, x_2, ..., x_n)$ to a coalition $C \subseteq N$ satisfying $i \in C$. The offer x must have the following characteristics:

1. $x_j \geq 0$ for all $1 \leq j \leq n$;
2. $\sum_{j=1}^{n} x_j \leq V(C)$.

Players in $C\backslash\{i\}$ immediately and sequentially accept or reject the offer (the order in which they do so turns out to be unimportant). If everyone accepts, the game ends, and all players in N are paid their shares according to x. If an offer is rejected by at least one of the respondents, it is taken off the table, and the game continues with the same Poisson parameters. If no offer has been accepted at time 0, the game ends, and all players receive payoff 0.

We assume that players discount future payoffs using a constant discount rate $r \in (0, \infty)$.[11]

For a formal definition of strategies in the above game, see Appendix A.

3 Properties of Markov Perfect Equilibrium

In this section, we establish the existence of MPE and the uniqueness of MPE expected payoffs. That is, while strategies in our model might not be uniquely determined in MPE, they can only vary in a payoff-irrelevant way.[12] We also show that for a generic sequence of discrete-time games approximating a continuous-time game in our framework, the corresponding expected SPNE payoffs converge to the unique MPE payoffs of the limit game. The latter result provides a microfoundation for focusing on MPE in our model, when we think about the continuous-time game as a limit of discrete-time environments. We then provide an example demonstrating that, for an open set of parameter specifications in our model, there are multiple SPNE with different payoffs. This stands in contrast with the uniqueness of MPE payoffs, and distinguishes our model from finite-horizon discrete-time bargaining games, in which there is generically a unique SPNE that can be obtained by backward induction.

[11] The presence of a deadline, together with the possibility of no recognition occurring over any given time horizon, implies that most of our conclusions also apply to a model with no discounting ($r = 0$). See an earlier circulated version of this paper in which we focused on the case of no discounting.

[12] For example, if there are three players with equal recognition rates and reservation values, then it is payoff-irrelevant whether all players approach the other two players with probability 1/2 each, or whether player 1 always approaches player 2, player 2 always approaches player 3, and player 3 always approaches player 1.

To formally state this section's results, we define discrete-time coalitional bargaining games and the convergence of such games to a continuous-time game.

Let $k \in \mathbb{Z}_{++}$, and fix a set of players N with $|N| = n$, a characteristic function V, a discount rate r, and a vector of recognition rates $\lambda \in \mathbb{R}^n_{++}$. As usual, we abuse notation and denote $\lambda = \lambda_1 + \lambda_2 + ... + \lambda_n$.

Definition: A k-period *discrete random recognition coalitional bargaining game* with time horizon T, denoted $G^k(N, V, \lambda, T)$, is a k-period random-recognition discrete game in which $\frac{T}{k}$ units of time lapse between consecutive periods, and in each period, player i is recognized with probability $\frac{\lambda_i}{\lambda}(1 - e^{-\lambda \frac{T}{k}})$, while with probability $e^{-\lambda \frac{T}{k}}$, no one is recognized. The periods are denoted $\{1, 2, ..., k\}$.

Definition: A sequence of discrete coalitional bargaining games $\{G_j^{k(j)}(N, V, \lambda^j, T)\}_{j=1}^{\infty}$ converges to continuous-time bargaining game $G(N, V, \lambda, T)$ if $k(j) \to \infty$ and $\lambda^j \to \lambda$ as $j \to \infty$.

If $\{G_j^{k(j)}(N, V, \lambda^j, T)\}_{j=1}^{\infty} \to G(N, V, \lambda, T)$, then the recognition process indeed converges to the Poisson process defined for the continuous game: see Billingsley (1995), Theorem 23.2 (p 302). For notational simplicity, we henceforth omit the superscript $k(j)$ for discrete games indexed by j.

Definition: $G^k(N, V, \lambda, T)$ is *regular* if it has a unique SPNE payoff vector.

If the game is regular, then expected continuation payoffs in SPNE are Markovian: they depend only on the time remaining before the deadline. In Claim 1 (all Claims are formally stated and proven in Appendix C), we show that regularity is a generic property among k-period discrete random recognition bargaining games, in the sense that such games are regular for an open and dense set of recognition vectors λ.[13]

We can now state this section's main result. For the remainder of this paper, we let $w_i(t)$ denote player i's MPE continuation value at time t, and call the function w_i player i's continuation value function.

Theorem 1: For every game G in our framework, the following hold:

(i) An MPE exists.

(ii) All MPE have the same continuation value functions w_i.

[13] Norman (2002) established an analogous result in the context of discrete-time legislative bargaining games.

(iii) Suppose the sequence of regular discrete coalitional bargaining games $\{G^k\}$ converges to G. Let $w_i^{G^k}$ be the step function derived from player i's unique SPNE continuation values in G^k.[14] Then $\{w_i^{G^k}\}$ converges uniformly to w_i for all i.

The formal proofs for this paper's Theorems are in Appendix D. The arguments establishing Theorem 1 are sketched below.

To show (i), we prove that if a sequence of regular discrete coalitional bargaining games converges to a continuous-time bargaining game, then there is a subsequence such that the associated (unique) SPNE collections of continuation payoff functions converge uniformly to an MPE collection of continuation payoff functions of the limit game.

We start by constructing strategy profiles in continuous time such that the associated continuation value functions approximate the SPNE continuation payoff functions arbitrarily well as $k \to \infty$. These generated functions are Lipschitzcontinuous, with a uniform Lipschitz constant given by the discount rate, the recognition rates, and $V(N)$. Hence, by the Ascoli-Arzela theorem, there is a subsequence of the games such that the associated continuation payoffs uniformly converge to a limit function (which is Lipschitzcontinuous with the same constant) for each player. To establish that these limit functions constitute the continuation payoff functions of an MPE of the limit game, we first prove a mathematical theorem: at points t where both the continuation payoff functions along the sequence and the limit functions are differentiable (which holds for almost all points of time), the derivatives of the limit functions are in the convex hull of points that can be achieved as limit points of derivatives at points $t_1, t_2, ...$ along the sequence, where $t_k \to t$ as $k \to \infty$. Each of these limit points correspond to (proposer) strategies that are played arbitrarily close to t, and arbitrarily far along in the sequence. It follows that these corresponding strategies are optimal in the limit game, assuming that the limit functions are indeed the continuation payoff functions. We can use this fact to define strategies that are optimal if the continuation payoff functions are given by the limit functions, and at the same time generate the limit functions as the continuation payoff functions of the game.

Claim 2, used in the proof of (ii) as well as in the next section, establishes that in an MPE, a player recognized at time t only approaches coalitions C that maximize $V(C) - \sum_{i \in C} w_i(t)$, *i.e.* the coalitions that are the cheapest to buy relative to the value they generate.

We now describe the proof of (ii). Suppose that there are two MPE, A and B, with different continuation payoff functions. Suppose that t is the earliest time such that continuation payoffs in the two equilibria are equal for all times on the interval $[t, 0]$ (note that such time exists, as equilibrium continuation functions are continuous, and at time 0, all

[14] $w_i^{G^k}$ is formally defined in Appendix B.

players' continuation payoffs are 0 in all equilibria). Let $f_j(\tau) = w_j^A(\tau) - w_j^B(\tau)$ be the difference between player j's payoffs in equilibrium A and in equilibrium B, at time τ. Thus, when $f_j(\tau) > 0$, player j is "more expensive" in equilibrium A than in equilibrium B, at time τ. Similarly, let $g_j(\tau) = \sum_{i \neq j} \lambda_i p_{ij}^A(\tau) - \sum_{i \neq j} \lambda_i p_{ij}^B(\tau)$, where $p_{ij}^X(\tau)$ is the probability that i approaches a coalition that includes j, conditional on i making an offer at time τ in equilibrium X, be the difference in the density of j being approached by another player in equilibrium A relative to equilibrium B.

The proof of (ii) is by contradiction. First, Lemma 3 shows that for all τ, $\sum_{j \in N} f_j(\tau) g_j(\tau) \leq 0$. This derives from the fact that if $f_j(\tau) > 0$, it should become less attractive to approach a coalition that includes j. However, Lemma 4 implies that in order for $f_j(\tau) > 0$ in the first place, $g_j(s)$ must be generally positive for $s \in (\tau, t)$, at least when $f_j(\tau)$ is not too small relative to $f_i(s)$ for all $i \in N$ and $s \in (\tau, t)$. This occurs because j's payoffs are determined by how often j is approached and by the surplus j receives when she is recognized. Near t, the former dominates the latter, which must be similar in A and B. Thus, contrary to Lemma 3, Lemma 4 suggests that f and g should be positively correlated during some time period. The remainder of the proof, Lemma 5, formally pinpoints the contradiction between Lemmata 3 and 4.

Taken together, the above results imply that if a sequence of regular discrete-time coalitional bargaining games converges to a continuous-time coalitional bargaining game, any convergent subsequence of the SPNE continuation payoff functions converges to the unique MPE payoff function of the limit game. This implies that the original sequence of SPNE continuation payoff functions has to be convergent, with the same limit, which establishes (iii).

The following example shows that SPNE that are not MPE may yield different payoffs than those characterized in Theorem 1.

Example 1: $N = \{1, 2, 3\}$, $V(N) = 1$, $V(\{1, 2\}) = V(\{1, 3\}) = V(\{2, 3\}) = \frac{3}{4}$, $V(\{1\}) = V(\{2\}) = V(\{3\}) = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$ and $r = 0$.

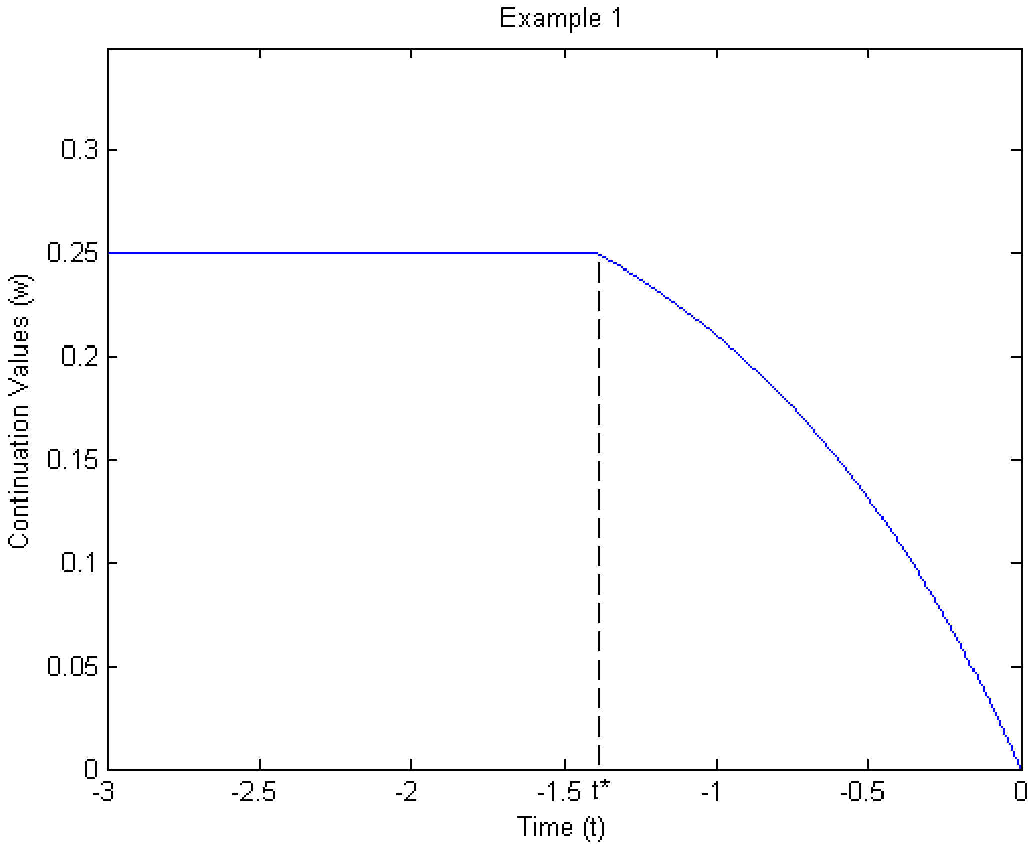

Figure 1

In Example 1, each player's marginal contribution to the grand coalition is $\frac{1}{4}$. In any MPE of the game, for $t > t^* = ln\frac{1}{4}$, each player, when recognized, approaches the grand coalition, and the continuation payoffs of players (that the recognized player has to offer) is $\frac{1}{3}(1 - e^t)$, as shown in Figure 1, which depicts the players' MPE continuation value function. However, when these values reach $\frac{1}{4}$ (at $t = t^*$), in any MPE, players have to switch to proposing to two-player coalitions with probabilities that keep everyone's continuation payoff constant at this level. A player's continuation value cannot decrease below $\frac{1}{4}$ when going back in time, since then every proposer would include the player in the proposed coalition, which would imply that the player's continuation value should increase when going back in time, instead of decreasing. Similarly, a player's continuation value cannot increase above $\frac{1}{4}$, since then all other recognized players would exclude the player from the proposed coalition, which would imply that the player's continuation value should decrease when going back in time, instead of increasing.

Therefore, in Example 1, when the time horizon is long enough, the expected payoffs in MPE are $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. However, it is possible to create a non-Markovian SPNE with higher expected payoffs for all players. After t^*, play follows an MPE. Before t^*, if no offer was rejected so far, any recognized player approaches the grand coalition and offers 0.25 to each of the other players (keeping 0.5 for herself). In this phase, any approached player is supposed to accept an offer if and only if she is offered at least 0.25. However, once an offer is rejected, players switch to an MPE. Note that in the above profile, players' continuation values, provided that no rejection occurred so far, increase strictly above 0.25 before t^*. Nevertheless, they are willing to accept an offer of 0.25 because rejecting an offer moves play to a different phase, in which players' continuation payoffs are exactly 0.25.

The qualitative conclusions from Example 1 carry through to an ε-neighborhood of λ around $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. In particular, for small enough $\varepsilon > 0$, there exists $\delta \geq 0$ such that before time $t^* - \delta$, the MPE continuation payoff of all three players is 0.25. Nevertheless, using exactly the same construction as above (with $t^* - \delta$ instead of t^* as the switching point between the history-dependent and the history-independent phases of the game), one can create an SPNE in which all players' expected payoffs converge to a value near $\frac{1}{3}$ as $t \to -\infty$. This shows that there is an open set of recognition rates, for the given characteristic function, for which there are multiple SPNE with distinct expected payoff vectors.[15]

We conclude this section by discussing a special class of games in our framework where SPNE expected payoffs, and in fact strategies, are unique: $V(C) = 0 \ \forall \ C \neq N$, and $V(N) > 0$; we shall normalize $V(N)$ to 1. This type of specification is often referred to as group bargaining. Only the grand coalition has positive value, so every player's acceptance is needed for an agreement.

In Claim 3, we show that in any SPNE, the n-player group bargaining game ends at the first realization of the Poisson process. SPNE payoff functions are unique, with player i receiving $\frac{\lambda_i+r}{\lambda+r} + \frac{\lambda-\lambda_i}{\lambda+r}e^{(\lambda+r)t}$ when she makes the offer at time t, and continuation payoff $\frac{\lambda_i}{\lambda+r}(1 - e^{(\lambda+r)t})$ when she is not the proposer. This implies that player i's expected payoff converges to $\frac{\lambda_i}{\lambda+r}$ as $T \to \infty$. Moreover, a player's expected payoff, both unconditionally and conditionally on being recognized, is increasing in her recognition rate, at all times.[16] Figure 2 depicts continuation payoffs for $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{3}, \lambda_3 = \frac{1}{6}$ and $r = 0$. The proof uses a similar argument as in Shaked and Sutton (1984).

[15] Slight changes in the characteristic function, or a small positive r, do not alter these conclusions either.

[16] This contrasts with predictions in Perry and Reny (1993) and Sákovics (1993), where being able to make offers more frequently can be disadvantageous for a player.

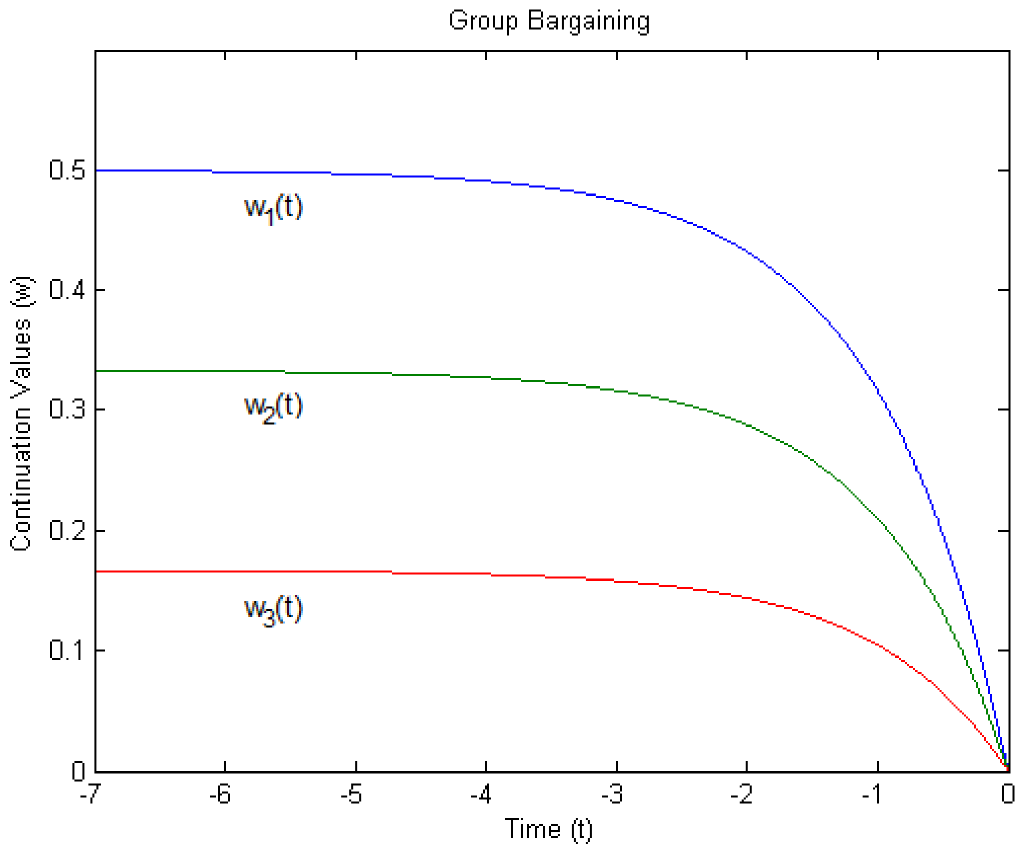

Figure 2

As is well-known in the literature, if the number of players is at least 3, then in an alternating-offer bargaining game with infinite horizon, any division of the surplus can be supported in SPNE if players are patient enough. The same conclusion holds in our framework, again for infinite horizon.[17] On the other hand, as claimed above, in the game with deadline, there is a unique SPNE for any vector of recognition rates.

For the rest of the paper, we restrict attention to MPE.

4 Limit of MPE Payoffs for Long Time Horizon

In this section, we investigate MPE payoffs as the time horizon of the game goes to infinity. To gain some intuition on how limit MPE payoffs depend on recognition rates and the characteristic function, we start with the following example.

Example 2: $N = \{1, 2, 3\}$, $V(N) = 1$, $V(\{1, 2\}) = 0.8$, $V(\{1\}) = \frac{1}{2}$, $V(C) = 0$ for all other coalitions C, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$ and $r = 0$.

[17] In particular, the type of construction in p 63 of Osborne and Rubinstein (1990) supports even the most extreme allocation in which one player gets all of the surplus.

The MPE continuation payoffs are depicted in Figure 3. Going back in time from the deadline, all players' payoffs initially increase at the same rate, as determined by the common Poisson recognition rate. In this region, since continuation values are low and therefore coalitional partners are cheap to buy, players always approach the grand coalition when recognized. However, when player 3's continuation payoff reaches 0.2, her marginal contribution to the grand coalition, the other two players stop approaching her with probability 1, in a way that keeps player 3's continuation payoff constant at 0.2. The other two players' continuation payoffs keep increasing until player 2's payoff reaches 0.3, which is her marginal contribution to the value of coalition $\{1,2\}$. At this point, player 1 starts proposing with positive probability to the singleton coalition involving only herself (that is, excluding player 2), and player 2's continuation payoff is kept constant at 0.3. Finally, player 1's payoff converges to $\frac{1}{2}$, the value she can generate by herself. As $t \to -\infty$, the probability that the grand coalition is approached goes to 1, since as the proposer's surplus shrinks, players 2 and 3 need to be excluded with lower probability for their expected continuation payoffs to be held constant.

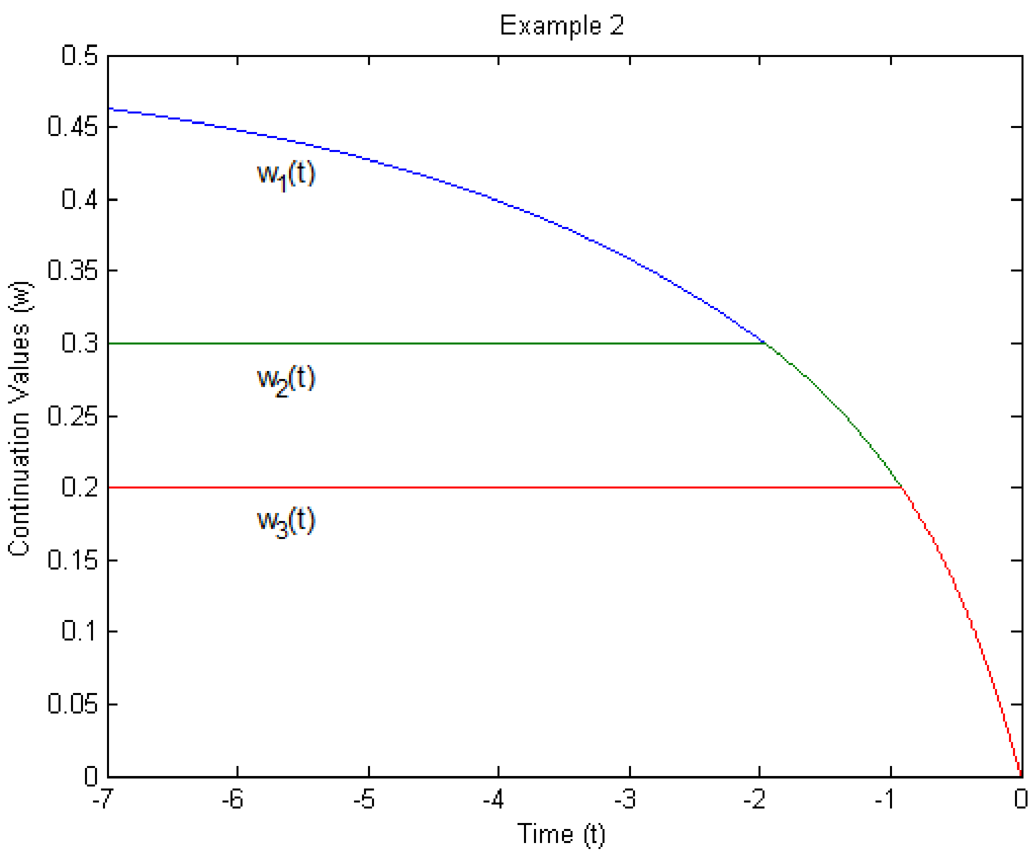

Figure 3

Example 2 demonstrates that the value of a coalition can act as a lower bound on how much players of that coalition can expect in MPE, if players are patient and the time horizon is long. In the case of group bargaining, featured at the end of the Section 3, relative expected payoffs are purely determined by relative likelihoods of being the proposer. In games with more complicated characteristic functions, such as in Example 2, both the recognition rates and the values of coalitions play a role in shaping expected MPE payoffs.

One feature of Example 2 is that, as the time horizon goes to infinity, the probability that a recognized player approaches the grand coalition is 1 (limit efficiency). Moreover, the limit expected payoffs belong to the core of the characteristic form game: for every coalition, the sum of members' limit expected payoffs is at least as much as the coalition's value. We show below that these features generalize to all convex games (games in which a member's marginal contribution to a coalition's value increases in the coalition), and that, in fact, there is a one-to-one mapping between the core and the set of limit payoffs achievable by varying recognition rates when there is no discounting.

Definition: A bargaining game is convex if $V(C \cup A) - V(C) \geq V(C' \cup A) - V(C')$, whenever $C \supset C'$ and $C \cap A = C' \cap A = \varnothing$.

Let $\mathcal{S}(r, V)$ be the set of limit MPE payoffs (as $t \to -\infty$) obtained by varying $\lambda \in \mathbb{R}^n_{++}$, and let $\mathcal{S}(V) = \lim_{r \to 0} \mathcal{S}(r, V)$. Recall that $\mathcal{C}(V)$ denotes the core of V.

Theorem 2: If V is convex, then $\mathcal{S}(V) = \mathcal{C}(V)$.

One direction of Theorem 2, $\mathcal{C}(V) \subseteq \mathcal{S}(V)$, can be established for all games with a nonempty core, even nonconvex games. The idea is to take relative recognition rates proportional to payoffs in a core allocation. Then, if every player always approaches the grand coalition N, expected payoffs as $T \to \infty$ converge to the core allocation at hand, from below. Because the sum of continuation payoffs of any coalition C's members at no time exceeds v_C, approaching N is optimal. This part of Theorem 2 is similar to the main result in Yan (2003) on *ex ante* expected payoffs in stationary SPNE of infinite-horizon random-proposer discrete-time games, with the caveat that in our model, the discount rate has to converge to 0 to achieve the core convergence result because in our model, the expected time before the first proposal is positive. Therefore, the limit case of our model with infinitely patient players can be compared to the model in Yan (2003).[18]

To establish $\mathcal{S}(V) \subseteq \mathcal{C}(V)$ in convex games, we show that in the limit as $r \to 0$ and $t \to -\infty$, the sum of MPE payoffs of members of any coalition C is bounded below by $V(C)$. We first show that for any player i, $w_i(t) < V(\{i\})$ implies that i is included in any

[18] Yan shows that if players are sufficiently patient and the vector of recognition probabilities is outside the core, the resulting stationary SPNE allocation is inefficient, but does not examine whether the inefficiency vanishes as players become patient.

coalition approached by any recognized player, from which it follows that $w_i(t)$ is bounded below by a function converging to $V(\{i\})$. We then iteratively establish the result for all larger coalitions.

We can say more about limit payoffs and how coalitions form far from the deadline in convex games. Let $S(t) = \arg\max_{C \subseteq N}\{V(C) - w_C(t)\}$ be the set of optimal coalitions to approach at time t, $\lambda_C = \sum_{i \in C} \lambda_i$, and $w_C(t) = \sum_{i \in C} w_i(t)$.

Theorem 3: For a convex game V and small enough r, there exist a partition P^* of the set of players N, a coalition $C^* \in P^*$ and a time $\widehat{t} < 0$ such that, in MPE:

i) for all $t < \widehat{t}$, $S(t) \equiv S^*$ is constant, $\{C^*, N\} \subseteq S^*$, and every element of S^* is the union of C^* and elements of P^*;

ii) $\lim_{t \to -\infty} w_i(t) = \frac{\lambda_i}{r + \lambda_C} V(C^*)$ for all $i \in C^*$; and

iii) for any $D \in P^*$ with $D \neq C^*$, there exists $E \in S^*$ such that $w_D(t) = V(E) - V(E \backslash D) \equiv w_D$ for all $t < \widehat{t}$, and $\lim_{t \to -\infty} w_i(t) = \frac{\lambda_i}{\lambda_D} w_D$ for all $i \in D$.

Theorem 3(i) states that for patient enough players, when the deadline is far, the set of optimal coalitions to approach does not change and must include N. However, only a subset of players C^* is part of any optimal coalition. We will refer to to this set of players as essential. Non-essential players, who are approached by others with probability strictly less than 1 far from the deadline, are partitioned into cells of players with the property that either all or none of them are in any coalition forming far from the deadline. Parts (ii) and (iii) of Theorem 3 reveal that the limit payoffs of essential versus non-essential players are determined differently. Essential players split the value that they can create by themselves, while the total limit payoff of players in any non-essential cell is equal to that cell's marginal contribution. In both cases, within each cell, limit payoffs are proportional to recognition rates.

In the focal case of symmetric recognition rates, Theorem 3 implies that the limit payoff of player i, as $r \to 0$ and $t \to -\infty$, is $\frac{1}{|C^*|} V(C^*)$ if $i \in C^*$, and $\frac{1}{|D|} w_D$ for all $i \in D \in P^* \backslash \{C^*\}$. For example, when $\frac{V(N)}{|N|}(1, 1, ..., 1) \in \mathcal{C}(V)$, it is the limit MPE payoff as $r \to 0$ and $t \to -\infty$: in that case, as argued when proving Theorem 2, we have $C^* = N$. In general, coalitions' marginal contributions put upper bounds on the limit payoffs of different players, which breaks the equality of the allocation in the limit payoff. However, within each partition cell of P^*, all players receive the same limit payoffs. Therefore, in convex games, our model's limit MPE payoffs with symmetric recognition rates motivate a solution concept that selects the most symmetric allocation in the core compatible with coalitional constraints.

We now sketch the proof of Theorem 3. First, we show that for r and t small enough, the extra surplus from being the proposer relative to being approached is small. We use this fact to show that $w_C(t) \leq V(N) - V(N \backslash C)$: otherwise, proposals by players outside C would not include all of C, and without a sufficiently large proposer surplus to compensate for exclusion, $w_C(t)$ would drop as we move away from the deadline. This implies that when players are patient enough, far away from the deadline, N is always an optimal coalition to approach.

Next, we establish the partitional structure described in Theorem 3(i) by showing that both the intersection (which must be nonempty) and the union of two sets from $S(t)$ must also be elements of $S(t)$. We then show that, far away from the deadline, the proposer surplus $V(N) - w_N(t)$ is weakly monotonic in t, and that therefore, before a certain time $\widehat{t}$, $S(t)$ is constant, which implies Theorem 3(i). Taking P^* to be the coarsest partition satisfying Theorem 3(i) and solving for $w_i(t)$ for $t \leq \widehat{t}$ yields the rest of Theorem 3.

Because $\arg\max_{C \subseteq N}\{V(C) - w_C\}$ is upper-hemicontinuous in w_C, the set of optimal coalitions to approach remains S^* if the continuation values correspond to the limits in Theorem 3. Claim 4 uses this fact to establish that the limit payoffs from Theorem 3 constitute stationary equilibrium payoffs in the infinite-horizon version of the game, when r is low.

We conclude the section by showing that in non-convex games, even when the core is nonempty, the limit equilibrium payoff vector might be outside the core, and that there can be inefficiency in the limit, in that inefficient coalitions form with strictly positive probability.

Example 3: $N = \{1,2,3,4\}$, $V(N) = 1$, $V(\{1,2,3\}) = \frac{1}{2}$, $V(\{1,2\}) = V(\{2,3\}) = V(\{3,1\}) = \frac{3}{8}$, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{15}$, $\lambda_4 = \frac{4}{5}$, and $r = 0$. This game has a nonempty core; for example, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \in \mathcal{C}(V)$. It can be shown that in the limit as the deadline gets infinitely far away, expected payoffs converge to the inefficient allocation $(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2})$, outside the core. In particular, far away from the deadline, only player 4 approaches the grand coalition, and all other players, when recognized, form inefficient two-player coalitions.[19]

5 Discussion: Extensions

Our model can be extended in many directions. Some extensions, like incorporating asymmetric information, are beyond the scope of this paper. Others are relatively straightforward; we discuss two of these below.

[19]This game is not totally balanced, because the restriction of the game with only players $\{1,2,3\}$ has an empty core (even though the whole game has a nonempty core). Claim 5 provides a totally balanced - but nonconvex - game where limit MPE payoffs are also outside the core.

5.1 Infinite Horizon

Without a deadline, our model yields very similar results to a discrete-time model in which a proposer is selected randomly in every period (with perhaps a positive probability of no one being selected). For example, in the group bargaining case, in both models, any given allocation of the surplus can be supported in stationary SPNE for r low enough, and there is only one stationary SPNE. In our framework, payoffs are characterized by:

$$w_i = \int_0^\infty [\lambda_i e^{-(\lambda+r)\tau}(1 - \sum_{j\in N\setminus\{i\}} w_j) + \sum_{j\in N\setminus\{i\}} \lambda_j e^{-(\lambda+r)\tau} w_i] d\tau.$$

The solution of this system is $w_i = \frac{\lambda_i}{r+\lambda}$, the same as the limit payoffs in the finite-horizon model as the horizon goes to infinity.[20]

5.2 Gradually Disappearing Pies

Our model assumes that the surplus generated by any coalition stays constant until a certain point of time (the deadline) and then discontinuously drops to zero. Although there are many situations in which there is such a highlighted point of time that makes subsequent agreements infeasible, in other cases, it is more realistic to assume that the surpluses start decreasing at some point, but only go to zero gradually. For example, agreeing upon broadcasting the games of a sports season yields diminishing payoffs once the season started, but if there are games remaining in the season, a fraction of the original surplus can still be attained.

Some of our results can be extended to this framework. For example, the case of group bargaining remains tractable when $V(N)$ is time-dependent, even without assuming specific functional forms. Indeed, if $V(N)(t)$ is continuous and nonincreasing, and there is some time t^* at which $V(N)$ becomes zero, our argument for the uniqueness of SPNE payoffs applies with minor modifications. Continuation payoff functions are then $w_i(t) = \lambda_i \int_t^\infty e^{-(\lambda+r)(\tau-t)} V(N)(\tau) d\tau$, so payoffs remain proportional to recognition rates at all times, and since the grand coalition always forms, the sum of expected payoffs across all players is simply the expected size of the pie at the next recognition (0 after t^*). Even if we do not assume that there is a time t^* as above, but instead only that $V(N)(t)$ is nonincreasing and $\lim_{t\to\infty} V(N)(t) = 0$, it is possible to show uniqueness of MPE payoffs. It is an open question whether this uniqueness result for gradually disappearing pies extends to general coalitional bargaining.

[20] For small r, this is a special case of Theorem 4, since group bargaining games are convex.

6 Conclusion

In this paper, we propose a tractable noncooperative framework for coalitional bargaining, which can be used to derive sharp predictions with respect to the division of the surplus. In subsequent research, we plan to extend the framework to settings with asymmetric information, as well as situations in which a successful agreement by a proper subcoalition does not end the game, and the remaining players can continue bargaining with each other.

In a companion paper (Ambrus and Lu (2010)), we apply our model to legislative bargaining, where there is a natural upper bound for negotiations: the end of the legislature's mandate. We characterize limit payoffs when the time horizon for negotiations goes to infinity, and show that there is a discontinuity between long finite-horizon legislative bargaining and infinite-horizon legislative bargaining. In particular, even in the limit, the model with deadline puts restrictions on the distributions of surplus that can be achieved by varying recognition probabilities of different players, leading to a lower-dimensional subset of all feasible distributions.[21] In future work, we also plan to extend our finite-horizon continuous-time random recognition framework to spatial bargaining situations like in Baron (1991) and Banks and Duggan (2000).

[21] In contrast, Kalandrakis (2006) shows that in the infinite-horizon Baron and Ferejohn legislative bargaining model, any division of the surplus can be achieved as an expected stationary SPNE payoff if recognition probabilities can be freely specified.

7 References

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8 Appendix A: Formal Definition of Strategies

First, we define the set of possible histories of the game formally. We need to consider two types of histories.

For any $t \in [-T, 0]$, a time-t *proposer-history* consists of:

(i) recognition times $-T \leq t_1, ..., t_k < t$ for $k \in \mathbb{Z}_+$;

(ii) proposers assigned to the above recognition times $i_{t_1}, ..., i_{t_k} \in N$;

(iii) feasible proposals (C_{t_m}, x_{t_m}) by i_{t_m}, where $C_{t_m} \subseteq N$ and $x_{t_m} \in \mathbb{R}^n_+$, for every $m \in \{1, ..., k\}$; and

(iv) acceptance-rejection responses $(y^{t_m}_{j_{m,1}}, ..., y^{t_m}_{j_{m,n_m}})$ such that, for every $m \in \{1, ..., k\}$: $n_m = |C_{t_m}| - 1$, $j_{m,1}, ..., j_{m,n_m} \in C_{t_m}$, $j_{m,k'} \neq j_{m,k''}$ if $k' \neq k''$, $y^{t_m}_{j_{m,k'}} \in \{accept, reject\}$ for every $k' \in \{1, ..., n_m\}$, and $y^{t_m}_{j_{m,k'}} = reject$ for some $k' \in \{1, ..., n_m\}$.

Let $\mathcal{H}^p_t$ denote the set of all time-t proposer-histories, and let $\mathcal{H}^p = \underset{t\in[-T,0]}{\cup} \mathcal{H}^p_t$.

For any $t \in [-T, 0]$, a time-t *responder-history* of $i \in N$ consists of:

(i) a time-t proposer history $h^p_t \in \mathcal{H}^p_t$;

(ii) a time-t proposer $j \in N\backslash\{i\}$;

(iii) a feasible proposal at time $t : (C_j, x_j)$ by j, where $C_j \subseteq N$ and $x_j \in \mathbb{R}^n_+$, such that $i \in C_j$;

(iv) previous acceptance-rejection decisions at time $t : (y_{j_1}, ..., y_{j_m})$ such that $j_1, ..., j_m \in C_j\backslash\{i, j\}$, $j_{k'} \neq j_{k''}$ if $k' \neq k''$, and $y_{j_{k'}} \in \{accept, reject\}$ for every $k' \in \{1, ..., m\}$.

Let $\mathcal{H}^{r_i}_t$ denote all time-t responder-histories of i, and let $\mathcal{H}^{r_i} = \underset{t\in[-T,0]}{\cup} \mathcal{H}^{r_i}_t$.

Next we construct metrics on the spaces of different types of histories, which we will use to impose a measurability condition on strategies.

Define a metric d^p on $\mathcal{H}^p$ such that $d^p(h^p_t, \widetilde{h}^p_{t'}) < \varepsilon$ for $\varepsilon > 0$ iff (i) $|t - t'| < \varepsilon$; (ii) h^p_t and $\widetilde{h}^p_{t'}$ have the same number of recognition times $k \in \mathbb{Z}_+$; (iii) denoting the recognition times of h^p_t and $\widetilde{h}^p_{t'}$ by $t_1, ..., t_k$ and $t'_1, ..., t'_k$, $|t_l - t'_l| < \varepsilon \; \forall \; l \in \{1, ..., k\}$; (iv) $i_{t_l} = i'_{t'_l} \; \forall \; l \in \{1, ..., k\}$, where i_{t_l} is the proposer assigned at t_l by h^p_t and $i'_{t'_l}$ is the proposer assigned at t'_l by $\widetilde{h}^p_t$; (v) $C_{t_l} = C'_{t'_l} \; \forall \; l \in \{1, ..., k\}$, where C_{t_l} is the approached coalition at t_l in h^p_t and $C'_{t'_l}$ is the approached coalition at t'_l in $\widetilde{h}^p_t$; (vi) $||x_{t_l} - x'_{t_l}|| < \varepsilon$, where $|| \cdot ||$ stands for the Euclidean norm in $\mathbb{R}^n$, and x_{t_l} and $x'_{t'_l}$ are the proposed allocations at t_l in h^p_t and at t'_l in $\widetilde{h}^p_t$; (vii) $y^{t_l} = \widetilde{y}^{t'_l} \; \forall \; l \in \{1, ..., k\}$, where y^{t_l} and $\widetilde{y}^{t'_l}$ are the vector of acceptance-rejection responses at t_l in h^p_t and at t'_l in $\widetilde{h}^p_t$.

Define a metric d^{r_i} on $\mathcal{H}^{r_i}$ such that $d^{r_i}(h^{r_i}_t, \widetilde{h}^{r_i}_{t'}) < \varepsilon$ for $\varepsilon > 0$ iff (i) $|t - t'| < \varepsilon$; (ii) $d^p(h^p_t, \widetilde{h}^p_{t'}) < \varepsilon$, where h^p_t and $\widetilde{h}^p_{t'}$ are the proposer-histories belonging to $h^{r_i}_t$ and $\widetilde{h}^{r_i}_{t'}$; (iii) the time-t proposer in $h^{r_i}_t$ and the time-t' proposer in $\widetilde{h}^{r_i}_{t'}$ is the same player $j \in N\backslash\{i\}$; (iv) $C_j = C'_j$ and $||x_j - x'_j|| < \varepsilon$, where (C_j, x_j) is the time-t proposal in $h^{r_i}_t$ and (C'_j, x'_j) is the time-t' proposal in $\widetilde{h}^{r_i}_{t'}$; (v) $(y_{j_1},...,y_{j_m}) = (y'_{j_1},...,y'_{j_m})$, where $(y_{j_1},...,y_{j_m})$ is the previous acceptance-rejection responses in $h^{r_i}_t$ and $(y'_{j_1},...,y'_{j_m})$ is the previous acceptance-rejection responses in $\widetilde{h}^{r_i}_{t'}$.

The set of proposer action choices of player i at any $t \in [-T, 0]$, denoted by A^p_i, is defined as $\{(C_i, x_i)|i \in C_i \subseteq N, \sum_{j\in N} x^j_i \leq V(C_i),\ x_i \geq 0\}$. Define a metric $d^{a_{p,i}}$ on A^p_i such that $d^{a_{p,i}}((C_i, x_i), (C'_i, x'_i)) < \varepsilon$ for $\varepsilon > 0$ iff $C_i = C'_i$ and $||x_i - x'_i|| < \varepsilon$.

Definition: A pure strategy of player i in game G is a pair of functions: a proposal function $\mathcal{H}^p \to A^p_i$ that is measurable with respect to the σ-algebras generated by d^p and $d^{a_{p,i}}$, and a responder function $\mathcal{H}^{r_i} \to \{accept, reject\}$ that is measurable with respect to the σ-algebra generated by d^{r_i} and the σ-algebra belonging to the discrete topology on $\{accept, reject\}$.

The measurability requirement on pure strategies is imposed to ensure that the expected payoffs of players are well-defined after any history.[22]

9 Appendix B: Embedding Continuation Payoff Functions of Discrete Games in Continuous Time

For regular discrete coalitional bargaining game $G^k(N, V, \lambda, T)$, let $w^k_i(m)$ be player i's SPNE continuation value before the realization of the recognition process in period m, for $m \in \{1, 2, ..., k\}$ (thus, $w^k_i(k) = \frac{1-e^{-\lambda\frac{T}{k}}}{\lambda}\lambda_i$). Let $w^k_i(k+1) = 0$. We extend these regular SPNE continuation payoff functions to continuous time.

Definition: For all $t \in [-T, 0]$, let $w^{G^k}_i(t) = e^{-r\Delta} w^k_i(\lfloor \frac{T+t}{T} k \rfloor + 1)$, where Δ satisfies $e^{-r\Delta}(1 - e^{-\lambda\frac{T}{k}}) = \frac{\lambda}{r+\lambda}(1 - e^{-(r+\lambda)\frac{T}{k}})$.

Note that for k high enough, $\Delta \in (0, \frac{T}{k})$. Also note that as $k \to \infty$, $\Delta \to 0$.

The definition is consistent with the following setup: place the mth period of the discrete game at time $-\frac{k-(m-1)}{k}T + \Delta$. For $t \in [-\frac{k-(m-1)}{k}T, -\frac{k-m}{k}T)$ (which corresponds to the mth of the k $\frac{T}{k}$-sized intervals in $[-T, 0]$), $w^{G^k}_i(t)$ is simply $w^k_i(m)$ discounted from the perspective of time $-\frac{k-(m-1)}{k}T$. At time $-\frac{k-(m-1)}{k}T$, a player receiving an expected payoff x with density $\lambda e^{-\lambda(\tau + \frac{k-(m-1)}{k}T)}$ throughout $[-\frac{k-(m-1)}{k}T, -\frac{k-m}{k}T)$ has value $x \int_0^{\frac{T}{k}} \lambda e^{-(\lambda+r)\tau} d\tau =$

[22] We do not need the measurability assumption to make sure that strategies lead to well-defined outcomes for any realization of the Poisson processes. In contrast with differential games, the conceptual problems pointed out in Alós-Ferrer and Ritzberger (2008) do not arise in our context.

$x\frac{\lambda(1-e^{-(\lambda+r)\frac{T}{k}})}{r+\lambda}$, while a player receiving the same expected payoff x at time $-\frac{k-(m-1)}{k}T+\Delta$ with probability $\int_0^{\frac{T}{k}} \lambda e^{-\lambda\tau}d\tau = 1-e^{-\lambda\frac{T}{k}}$ has value $xe^{-r\Delta}(1-e^{-\lambda\frac{T}{k}})$. Thus, our definition of Δ implies that any player will be indifferent between the continuous and the discrete recognition process specified above.

10 Appendix C: Additional Formal Results

Claim 1: $U = \{\lambda \in \mathbb{R}^n_{++} | G^k(N,V,\lambda,T) \text{ is regular}\}$ is open and dense.

Proof of Claim 1: Let $S = \{v \in \mathbb{R}^n_+ | \exists C_1, C_2 \in 2^N \text{ s.t. } C_1, C_2 \in \arg\max_{C \ni i}(V(C) - \sum_{j\in C} v_j) \text{ for some } i \in N\}$. This is the set of reservation payoff vectors for which at least one player has at least two different optimal coalitions to approach.

Let $v^k(m)$ denote an SPNE reservation value vector in period $m \in \{1,...,k\}$ in $G^k(N,V,\lambda,T)$. Since k is fixed in the following proof, we abbreviate by writing $v(m)$. Note that in any $G^k(N,V,\lambda,T)$, $v(k-1) = e^{-r\frac{T}{k}}V(N)\left[\frac{1-e^{-\lambda\frac{T}{k}}}{\lambda}(\lambda_1,\lambda_2,...,\lambda_n)\right]$. When we vary λ, we will write $v^\lambda(m)$.

Suppose that the reservation value is arbitrarily given to be v in period m, and all players play optimally in that period. Then, we denote the set of reservation value vectors attainable in period $m-1$ as $F(v,\lambda)$, where F is a correspondence. Note that $v \notin S \Leftrightarrow F(v,\lambda)$ is single-valued, in which case we denote its unique element as $f(v,\lambda)$. Since S is a finite collection of $(n-1)$-dimensional hyperplanes, the set on which F is single-valued is open and dense (call this set W) within $\mathbb{R}^n_+ \times \mathbb{R}^n_{++}$;[23] within W f is clearly continuous.

Openness: Suppose $\lambda \in U$. By definition, for all $m \in \{1,2,...,k-1\}$, $(v^\lambda(m),\lambda) \in W$, with $v^\lambda(k-1) = e^{-r\frac{T}{k}}V(N)\left[\frac{1-e^{-\lambda\frac{T}{k}}}{\lambda}(\lambda_1,\lambda_2,...,\lambda_n)\right]$ and $v^\lambda(m) = f(v^\lambda(m+1),\lambda)$. Now note that because f is continuous and W is open, for any λ' close enough to λ, we have that for all $m \in \{1,2,...,k-1\}$, $v^{\lambda'}(m)$ is close to $v^\lambda(m)$, where $v^{\lambda'}(k-1) = e^{-r\frac{T}{k}}V(N)\left[\frac{1-e^{-\lambda'\frac{T}{k}}}{\lambda'}(\lambda'_1,\lambda'_2,...,\lambda'_n)\right]$ and $v^{\lambda'}(m) = f(v(m+1),\lambda')$. Again due to the openness of W, this implies that $\lambda' \in U$.

Density: We show that a payoff $v^\lambda(t)$ can be changed in "any direction" in $\mathbb{R}^n$ by perturbing λ. To do so, we argue that in the linear approximation of changes in $v^\lambda(t)$ with respect to changes in λ, the transformation has full rank. This will allow us to break any indifferences at t using infinitesimal changes in λ.

[23] Obviously, if $(v,\lambda) \in W$, then $(v,\lambda') \in W$ as well.

When $v(m) \notin S$, we can write:

$$v_i(m-1) = \left[\frac{\lambda_i}{\lambda}(1-e^{-\lambda\frac{T}{k}})\max_{C\ni i}(V(C) - \sum_{j\in C} v_j(m)) + \left[1-(1-p_i(m))(1-e^{-\lambda\frac{T}{k}})\right] v_i(m)\right] e^{-r\frac{T}{k}}$$

where $p_i(m)$ is the probability that i is included in period m's proposal given that there is one. Note that in a neighborhood of $v(m) \notin S$, $\arg\max_{C\ni i}(V(C) - \sum_{j\in C} v_j(m))$ is single-valued and constant. Fixing λ and in such a neighborhood, $v_i(m-1)$ is linear in each $v_j(m)$, so we can write $f(v(m)+\delta,\lambda) - f(v(m),\lambda) = A_m\delta$, where A_m is an $n\times n$ matrix. Note that the ith column of A_m must have a strictly positive ith element and have all other elements weakly negative. Similarly, fixing $v(m) \notin S$, we note that each $v_i(m-1)$ is infinitely differentiable in each component of λ, so we have the linear approximation $f(v(m),\lambda+\gamma) - f(v(m),\lambda) \approx B_m\gamma$, where B_m is an $n\times n$ matrix. Just like A_m, the ith column of B_m must have a strictly positive ith element and have all other elements weakly negative. We have $f(v(m)+\delta,\lambda+\gamma) - f(v(m),\lambda) \approx A_m\delta + B_m\gamma$. Define $B_k = D_\lambda[e^{-r\frac{T}{k}}V(N)\frac{1-e^{-\lambda\frac{T}{k}}}{\lambda}(\lambda_1,\lambda_2,...,\lambda_n)]$.

Fix $\varepsilon > 0$, and suppose $\lambda \notin U$. Then $\exists\tau$ such that $v(\tau) \in S$. Let t be the largest such τ. We still have that all $m \in \{t+1, t+2, ..., k-1\}$, $F(.,.)$ is single-valued in a neighborhood of $(v(m),\lambda)$, with $v(k-1) = e^{-r\frac{T}{k}}V(N)\left[\frac{1-e^{-\lambda\frac{T}{k}}}{\lambda}(\lambda_1,\lambda_2,...,\lambda_n)\right]$ and $v(m) = f(v(m+1),\lambda)$. Let $\lambda^1 = \lambda+\gamma$ be in a neighborhood of λ. Then we have the linear approximation $v^{\lambda'}(t) \approx v^\lambda(t) + (A_{t+1}A_{t+2}...A_{k-1}B_k + A_{t+1}A_{t+2}...A_{k-2}B_{k-1} + ... + A_{t+1}B_{t+2} + B_{t+1})\gamma \equiv v^\lambda(t) + M\gamma$. Since the set of matrices with strictly positive diagonal entries and weakly negative entries elsewhere is closed under addition and multiplication, M must retain that property. Thus, M has full rank. Therefore, $\exists\lambda^1$ within distance $\frac{\varepsilon}{2}$ of λ such that $v^{\lambda^1}(\tau) \notin S$ for all $\tau \geq t$.

Now with λ^1, go back in time until the next indifference point, and iterate the argument with $\frac{\varepsilon}{4}, \frac{\varepsilon}{8}$, etc. Since there is a finite number of periods $k+1$, there is a finite number, say q, of indifferences to be broken. So λ^q, which is by construction within ε of λ, ensures that $G^k(N,V,\lambda^q,T)$ is regular. ■

Claim 2: In any MPE, at any $t \leq 0$ where $i \in N$ is recognized, she approaches a coalition $C \in \arg\max_{D\ni i} V(D) - \sum_{j\in D\setminus\{i\}} w_j(t)$ and offers exactly $w_j(t)$ to every $j \in C\setminus\{i\}$. Furthermore, the offer is accepted with probability 1.

Proof of Claim 2: Note that $\sum_{j\in N} w_j(t) \leq V(N) - e^{\lambda t}$, where $e^{\lambda t} > 0$ is the probability that no one has the chance to make an offer during $[t,0]$. Furthermore, in any MPE, if $C \subset N$ is approached by i at t, and every $j \in N\setminus\{i\}$ is offered strictly more than $w_j(t)$, then the offer has to be accepted by everyone with probability 1. Therefore, player i can guarantee a payoff strictly larger than $w_i(t)$ by approaching N and offering $w_j(t)+\varepsilon$ to every $j \in N \backslash \{i\}$ for small enough $\varepsilon > 0$. On the other hand, a rejected offer results in continuation payoff $w_i(t)$ for i. Next, note that approaching a coalition C and offering strictly less than $w_j(t)$ to some $j \in C$ results in rejection of the offer with probability 1, and is therefore not optimal. Approaching a coalition C and offering $w_j(t)+\varepsilon$ for $\varepsilon > 0$ to some $j \in C$ is also suboptimal, because offering instead $w_j(t)+\varepsilon/n$ to every $j \in C\backslash\{i\}$ results in acceptance of the offer with probability 1 and strictly higher payoff. Therefore, whatever coalition C is approached, player i has to offer exactly $w_j(t)$ to every $j \in C\backslash\{i\}$. It cannot be that this offer is accepted with probability less than 1, since then player i could strictly improve her payoff by offering slightly more than $w_j(t)$ to every $j \in C\backslash\{i\}$, and that offer would be accepted with probability 1. Finally, it cannot be that $C \notin \arg\max_{D \subset N} V(D) - \sum_{j \in D\backslash\{i\}} w_j$, since then approaching some $C' \in \arg\max_{D \subset N} V(D) - \sum_{j \in D\backslash\{i\}} w_j$ instead, and offering slightly more than $w_j(t)$ to every $j \in C'\backslash\{i\}$ would result in a strictly higher payoff. ■

Claim 3: In any SPNE, the n-player group bargaining game ends at the first realization of the Poisson process for any player as follows: an offer is made to N and all players accept. SPNE payoff functions are unique, with player i receiving $\frac{\lambda_i+r}{\lambda+r} + \frac{\lambda-\lambda_i}{\lambda+r} e^{(\lambda+r)t}$ when she makes the offer at time t, and $\frac{\lambda_i}{\lambda+r}(1-e^{(\lambda+r)t})$ when she is not the proposer.

Proof of Claim 3: Let $\overline{v_i}(t)$ and $\underline{v_i}(t)$ be the supremum and the infimum, respectively, over all SPNE and all histories preceding t, of player i's share when she makes an offer at time t. Let $\overline{w_i}(t)$ and $\underline{w_i}(t)$ be the supremum and the infimum, respectively of player i's share when no player is making an offer, over all SPNE, histories and $j \neq i$.

Note that the density of i being the next recognized player, at x time units from the current time, is $\lambda_i e^{-\lambda x}$, and payoffs received at that point are discounted by a factor e^{-rx}.

First, note that $\overline{v_i}(t) + \sum_{j\neq i} \underline{w_j}(t) = 1$, since this will be true in an SPNE where i offers everyone $\underline{w_j}(t)$ and takes the rest, and where, if any such offer by i is rejected, we move to a SPNE giving a continuation value of $\underline{w_j}(t)$ to the first rejector.[24]

Consider the following profile:

1. When any player $k \neq i$ makes an offer, the offer to player i must be $\overline{w_i}(t)$, and the offer to all $j \neq i, k$ is $\underline{w_j}(t)$. If k offers less to any player, the offer is rejected by that player;

[24]Strictly speaking, at this point in the argument, it is possible that $\underline{w_j}(t)$ is not attained in any SPNE. However, since values arbitrarily close to it are attained in some SPNE, $\overline{v_i}(t)$ can be arbitrarily close to $1 - \sum_{j\neq i} \underline{w_j}(t)$, which implies that $\overline{v_i}(t) + \sum_{j\neq i} \underline{w_j}(t) = 1$. To simplify the exposition, we proceed in the proof as if all suprema and infima are attained, keeping in mind that we are referring to arguments analogous to the one presented in this footnote.

if player $j \neq i, k$ is the rejector, we move to an SPNE giving player k an expected payoff of $\underline{w_k}(t)$, and if player i is the rejector, we move to an SPNE giving player i an expected payoff of $\overline{w_i}(t)$. If k makes the correct offer and player j is the first rejecting the offer, then we move to an equilibrium giving $\underline{w_j}(t)$ to j.

2. When i makes an offer, she gives herself $\overline{v_i}(t)$ and gives $\underline{w_j}(t)$ to all $j \neq i$, as specified above.

To show that the profile is an SPNE, we need to verify that it indeed exists, i.e. that offers are feasible. Note that player k's offer is feasible if $\overline{w_i}(t) + \sum_{j\neq i} \underline{w_j}(t) \leq 1$. But this must be true since the sum of all continuation values in any SPNE must be less than 1, and the SPNE where $\overline{w_i}(t)$ is attained has a sum of continuation values at t of at least $\overline{w_i}(t) + \sum_{j\neq i} \underline{w_j}(t)$. As established above, player i's offer is feasible. We also need to check that players' actions are optimal. The only case where this is not trivial is that when k makes an offer, she may prefer to make one that is rejected by i. However, this will not be the case in an interval close to 0 where the probability of any future recognition $\leq \frac{1}{n}$, since then $\overline{w_k}(t) \leq \frac{1}{n} = 1 - \frac{n-1}{n} \leq 1 - \sum_{i\neq k} \overline{w_i}(t)$, so k will want the offer to be accepted. Denote this interval $[s, 0]$ (so $s = \frac{1}{\lambda}\ln(\frac{n-1}{n})$).

The above profile is of course the best possible one for i, so on $[s, 0]$ we have:

$$\begin{aligned}
\overline{w_i}(t) &= \int_t^0 \left[\lambda_i e^{-(\lambda+r)(\tau-t)}\overline{v_i}(\tau) + \sum_{j\neq i} \lambda_j e^{-(\lambda+r)(\tau-t)}\overline{w_i}(\tau)\right] d\tau \\
&= \int_t^0 \left[\lambda_i e^{-(\lambda+r)(\tau-t)}(1 - \sum_{j\neq i} \underline{w_j}(\tau)) + \sum_{j\neq i} \lambda_j e^{-(\lambda+r)(\tau-t)}\overline{w_i}(\tau)\right] d\tau
\end{aligned}$$

Since $\overline{w_i}(t)$ is the integral of a continuous function, its derivative exists, so:

$$\begin{aligned}
\overline{w_i}'(t) &= -\lambda_i(1 - \sum_{j\neq i} \underline{w_j}(t)) - \sum_{j\neq i} \lambda_j \overline{w_i}(t) \\
&\quad +(\lambda + r)\int_t^0 \left[\lambda_i e^{-(\lambda+r)(\tau-t)}(1 - \sum_{j\neq i} \underline{w_j}(\tau)) + \sum_{j\neq i} \lambda_j e^{-(\lambda+r)(\tau-t)}\overline{w_i}(\tau)\right] d\tau \\
&= -\lambda_i(1 - \sum_{j\neq i} \underline{w_j}(t)) - \sum_{j\neq i} \lambda_j \overline{w_i}(t) + (\lambda + r)\overline{w_i}(t) \\
&= (\lambda_i + r)\overline{w_i}(t) - \lambda_i(1 - \sum_{j\neq i} \underline{w_j}(t)).
\end{aligned}$$

Similarly, we note that $\underline{v_i}(t) + \sum_{j\neq i} \overline{w_j}(t) = 1$ on $[s, 0]$, since this occurs when i offers everyone $\overline{w_j}(t)$ and takes the rest, and where, if i gives any less than $\overline{w_j}(t)$ to a player, we move to a SPNE giving a continuation value of $\overline{w_j}(t)$ to the first rejector. On $[s,0]$, $\overline{w_j}(t)$ and the probability of a future recognition are close to 0, so it will be optimal for i to make such an offer. By a similar argument as above, we can show that:

$$\underline{w_i}'(t) = (\lambda_i + r)\underline{w_i}(t) - \lambda_i(1 - \sum_{j \neq i} \overline{w_j}(t))$$

Thus on a nontrivial interval $[s,0]$, we have a system of $2n$ differential equations continuous in t, and Lipschitz continuous in $2n$ unknown functions with initial values $\overline{w_i}(0) = \underline{w_i}(0) = 0$. By the Picard-Lindelof theorem, this initial value problem has a unique solution. It is easy to check that the following functions constitute the solution:

$$\overline{w_i}(t) = \underline{w_i}(t) = \frac{\lambda_i}{\lambda + r}(1 - e^{(\lambda+r)t}) \equiv w_i(t)$$

The above argument can be iterated for $[2s,s]$ since the game ending at s with payoffs $w_i(s)$ is simply a scaled version of the original game, and so on. ■

Claim 4: In convex games, for small enough r, MPE limit payoffs $(\lim_{\tau\to-\infty} w_i(\tau))$ constitute stationary equilibrium payoffs.

Proof of Claim 4: Take P^* from the game with deadline, as defined in Theorem 3. (Below, we refer to P^* and C^* simply as P and C.) Suppose that limit stationary payoffs are as follows: $w_i = \frac{\lambda_i}{\lambda_D} v_D$ for all $i \in D \in P \backslash \{C\}$, and $w_i = \frac{\lambda_i}{r+\lambda_C} V(C)$ for all $i \in C$. Let q_D be the stationary probability that $D \in P$ is approached conditional on a player being recognized. Note that $\int_0^\infty e^{-(\lambda+r)\tau} d\tau = \frac{1}{\lambda+r}$. We need to verify that the following condition is satisfied: $w_i = \frac{1}{\lambda+r}(\lambda_i \frac{r}{r+\lambda_C} V(C) + \lambda q_D w_i)$, where D is the cell of P containing i, with $q_C = 1$ and $q_D \in [0,1]$ for all $D \in P \backslash \{C\}$.

It is easy to check that our condition is satisfied for $i \in C$. For $i \notin C$, the condition becomes: $q_D = \frac{\lambda+r}{\lambda} - \frac{\lambda_D}{\lambda v_D} \frac{r}{r+\lambda_C} V(C)$. This expression is clearly continuous in r and approaches 1 as $r \to 0$. We need only check that it is decreasing in r in a right neighborhood of $0 \Longleftrightarrow 1 < \frac{\lambda_D}{v_D} V(C) \frac{1}{r+\lambda_C}$ for small enough $r \Longleftrightarrow \frac{V(C)}{\lambda_C} > \frac{v_D}{\lambda_D}$.

In the game with deadline, we have the following upper bound for $w_i(t)$, corresponding to the case where i is always approached prior to $\widehat{t}$:

$$
\begin{aligned}
w_i(t) &< e^{-(\lambda+r)(\widehat{t}-t)}w_i(\widehat{t}) + \int_t^{\widehat{t}} \left[\lambda_i e^{-(\lambda+r)(\tau-t)}(V(C)-w_C(\tau)) + \lambda e^{-(\lambda+r)(\tau-t)}w_i(\tau)\right] d\tau \\
&= [w_i(\widehat{t}) - \frac{\lambda_i}{\lambda_C}w_C(\widehat{t})]e^{r(t-\widehat{t})} + \frac{\lambda_i}{\lambda_C}[w_C(\widehat{t}) - \frac{\lambda_C}{r+\lambda_C}V(C)]e^{(r+\lambda_C)(t-\widehat{t})} + \frac{\lambda_i}{r+\lambda_C}V(C)
\end{aligned}
$$

This bound converges to $\frac{\lambda_i}{r+\lambda_C}V(C)$. Thus, $\lim_{\tau\to-\infty} w_i(\tau) = \frac{\lambda_i}{\lambda_D}v_D \leq \frac{\lambda_i}{r+\lambda_C}V(C) < \frac{\lambda_i}{\lambda_C}V(C)$, which implies $\frac{V(C)}{\lambda_C} > \frac{v_D}{\lambda_D}$, as desired. ■

Claim 5: Limit payoffs as $r \to 0$ and $t \to -\infty$ can be outside the core in non-convex but totally balanced games.

Proof of Claim 5: Consider the following example: $N = \{1,2,3,4,5,6\}$, $V(\{1,2,3\}) = V(\{3,4,5\}) = V(\{5,6,1\}) = 2$, $V(C) = 2$ for all $C \neq N$ that includes at least two odd players and an even player, $V(N) = 3$, $V(C) = 0$ for all other coalitions C, $\lambda_1 = \lambda_3 = \lambda_5 = \frac{1}{9}$, and $\lambda_2 = \lambda_4 = \lambda_6 = \frac{2}{9}$.

Note that V has nonempty core $\{(1,0,1,0,1,0)\}$. The core of the subgame with players 1 through 5 is also nonempty: $\{(0,0,2,0,0)\}$, and the same holds for all subgames with three odd and two even players. Finally, any other subgame where surplus can be generated includes only one surplus-generating three-player coalition, so any allocation where these three players split the total surplus of 2 is in the core. Therefore, V is totally balanced.

We now verify that with $r = 0$, payoffs converge to $\frac{1}{3}$ for all players, far from the core allocation. The reasoning below can be extended to small positive r.

Going back in time from the deadline, all players will approach the grand coalition N until the time t^* where payoffs reach $(.2, .4, .2, .4, .2, .4)$. Prior to t^*, all proposals shift to three-player coalitions generating surplus 2. This is because for approaching N to be optimal at a time t, the sum of continuation values for two even players and one odd player cannot exceed 1, which implies $\sum_i w_i(t) + \sum_{i \text{ is even}} w_i(t) \leq 3$. Since all surplus-generating coalitions other than N generate a surplus of 2, we must have $\sum_i w_i(t) \geq 2$ for all $t < t^*$, with strict inequality if N is approached with positive probability between t and t^*. Therefore, N can only be approached with positive probability between t and t^* if $\sum_{i \text{ is even}} w_i(t) < 1$.

Standard computations then show that if all proposals prior to t^* are made to three-player coalitions generating surplus 2, all players' payoffs converge to $\frac{1}{3}$, so that, in fact, N is not approached with positive probability before t^*. ■

11 Appendix D: Proofs of Theorems

11.1 Proof of Theorem 1

STATEMENT (i): There exists an MPE.

Proof of Statement (i): We prove the following result, which implies existence:

Suppose that the sequence of regular discrete coalitional bargaining games $\{G_j(N, V, \lambda^j, T)\}_{j=1}^{\infty}$ ($G_1, G_2, ...$ for short) converges to continuous-time bargaining game $G(N, V, \lambda, T)$. Then the sequence has a subsequence $\{G_{j_h}\}_{h=1}^{\infty}$ such that $\{w^{G_{j_h}}(.)\}_{h=1}^{\infty}$, the sequence of SPNE payoff functions, converges uniformly. Moreover, for any such subsequence, the limit of $\{w^{G_{j_h}}(.)\}_{h=1}^{\infty}$ corresponds to the continuation payoff functions of an MPE of $G(N, V, \lambda, T)$.

The following lemma is used in proving the result:

Lemma 1: Suppose $f^1 \equiv (f_1^1, ..., f_n^1), f^2 \equiv (f_1^2, ..., f_n^2), ...$ is a sequence of collections of functions, where $f_j^k : [0, T] \to \mathbb{R}$ are Lipschitzcontinuous with Lipschitzconstant L, for every $k \in \mathbb{Z}_{++}$ and $j \in \{1, ..., n\}$. Moreover, suppose that the sequence converges uniformly to $f \equiv (f_1, ..., f_n)$, where each f_j is also Lipschitzcontinuous with Lipschitz constant L. Let Ξ be the set of all subsequences of $f^1, f^2, ...$ For any $t \in [0, T]$, let $D(t) = \{x \in \mathbb{R}^n | \exists (f^{j_1}, f^{j_2}, ...) \in \Xi$ and $t_1, t_2, ... \to t$ s.t. $\nabla f^{j_i}(t_i) \to x$ as $i \to \infty\}$. Then f differentiable at t implies $\nabla f(t) \in co(D(t))$, where co stands for the convex hull operator.

Proof of Lemma 1: First we show that $co(D(t))$ is closed. Consider a sequence of points in $D(t)$, $x_1, x_2, ...$, converging to $x \in \mathbb{R}^n$. This means there are $(f^{j_1^m}, f^{j_2^m}, ...) \in \Xi$ and $t_1^m, t_2^m, ... \to t$ s.t. $\nabla f^{j_i^m}(t_i^m) \to x_m$ as $i \to \infty$, for every $m \in \mathbb{Z}_{++}$. Let $k(.)$ be such that $|\nabla f^{j_{k(m)}^m}(t_{k(m)}^m) - x_m| < \frac{\varepsilon}{m}$ and $|t_{k(m+1)}^{m+1} - t| < |t_{k(m)}^m - t|$. Then the sequence $\nabla f^{j_{k(1)}^1}(t_{k(1)}^1), \nabla f^{j_{k(2)}^2}(t_{k(2)}^2), ...$ converges to x, and $t_{k(1)}^1, t_{k(2)}^2, ... \to t$, hence $x \in D(t)$. This implies that $D(t)$ is closed. Since $-L \le D_i(t) \le L$ for every $i \in N$, $D(t)$ is compact. Hence, $co(D(t))$ is compact.

For every $\delta \ge 0$, let $co^\delta(D(t)) = \{x \in \mathbb{R}^n | d(x, co(D(t))) \le \delta\}$, where $d(x, co(D(t)))$ is the Hausdorff-distance between point x and set $co(D(t))$. Suppose the statement does not hold. Then, since $co(D(t))$ is closed, there is $\delta > 0$ such that $\nabla f(t) \notin co^\delta(D(t))$. By definition of $D(t)$, there exist $n_{\varepsilon(\delta)}(t)$, a relative $\varepsilon(\delta)$-neighborhood of t in $[0, T]$, and $k \in \mathbb{Z}_+$ such that for any $k' \ge k$ and for any $t' \in n_\varepsilon(t)$ at which $f^{k'}$ is differentiable, $\nabla f^{k'}(t') \in co^\delta(D(t))$. Then for any $t' \in n_\varepsilon(t)$ and any $k' \ge k$, $f^{k'}(t') - f^{k'}(t) \in (t' - t) co^\delta(D(t))$. However, $\nabla f(t) \notin co^\delta(D(t))$ implies that there is $t' \in n_\varepsilon(t)$ such that $f(t') - f(t) \notin (t' - t) co^\delta(D(t))$. This contradicts that $f^1, f^2, ...$ converges uniformly to f. $\square$

We first consider an arbitrary k-period regular discrete game of the form $G^k(N, V, \lambda^k, T)$.

Notation: Let s^k denote a pure strategy SPNE strategy profile in $G^k(N,V,\lambda^k,T)$, for every $k \in \mathbb{Z}_{++}$. Let $C_i^k(m)$ denote the coalition that player i approaches in s^k in period m, for $m \in \{1,...,k\}$.

Based on s^k, for every $i \in N$, define as follows strategy $\widehat{s}_i^k$ of i in the continuous-time game $G(N,V,\lambda^k,T)$:

Divide $[-T,0]$ into m equal intervals. If i is recognized in the m^{th} interval (*i.e.* at $t=-T+(m-\alpha)\frac{T}{k}$ for $\alpha \in [0,1)$ and $m \in \{1,...,k\}$), she approaches $C_i^k(m)$ and offers $e^{-r\frac{T}{k}}w_j^k(m+1)$ to every $j \in C_i^k(m)\backslash\{i\}$. If player i is approached in the m^{th} interval by any player, then she accepts the offer if and only if it is at least $e^{-r\frac{T}{k}}w_j^k(m+1)$.

Let $\widehat{w}_i^k(t)$ be player i's continuation value in the continuous-time game generated by the profile $\widehat{s}^k=(\widehat{s}_1^k,\widehat{s}_2^k,...,\widehat{s}_n^k)$, and let $\widehat{w}^k(t)=(\widehat{w}_1^k(t),\widehat{w}_2^k(t),...,\widehat{w}_n^k(t))$.

Fact 1: For any $\varepsilon>0$, there is a $k^\varepsilon \in \mathbb{Z}_{++}$ such that for any $k>k^\varepsilon$, $\widehat{s}^k=(\widehat{s}_1^k,\widehat{s}_2^k,...,\widehat{s}_n^k)$ is an ε-perfect equilibrium of G^k.

Note that by construction, whenever $t=-T+m\frac{T}{k}$, we have $\widehat{w}^k(t)=w^{G^k}(t)$, for every $m \in \{1,...,k\}$. (Recall that $w_i^{G^k}(t)$ is a step-function derived from the discrete game payoffs.)

Second, note that given $\widehat{s}_{-i}^k$, strategy $\widehat{s}_i^k$ specifies an optimal action for i if she is recognized, at every $t \in [-T,0]$.

Next, we bound the suboptimality of $\widehat{s}_i^k$ when i considers an offer. Observe that as we approach the end of the m^{th} interval (*i.e.* for $t=-T+(m-\alpha)\frac{T}{k}$, as $\alpha \searrow 0$), $\widehat{w}_i^k(t) \to w_i^{G^k}(-T+m\frac{T}{k})=e^{-r\Delta}w_i^k(m+1)$. Given that $\widehat{s}^k$ is Markovian, the optimal action for i in G^k when she is approached by any other player at $t=-T+(m-\alpha)\frac{T}{k}$ for $\alpha \in [0,1)$ and $m \in \{1,...,k\}$ is, independently of payoff-irrelevant history, to accept the offer if it is at least $\widehat{w}_i^k(t)$, and reject it otherwise. Instead, strategy $\widehat{s}_i^k$ specifies that i accepts the offer if and only if it is at least $e^{-r\frac{T}{k}}w_i^k(m+1)$; hence, after some histories, $\widehat{s}_i^k$ specifies a suboptimal action for i. However, since $\widehat{w}_i^k(t)$ is between $e^{-r\Delta}w_i^k(m)$ and $e^{-r\Delta}w_i^k(m+1)$, the difference between the expected payoff resulting from following $\widehat{s}_i^k$ versus choosing the optimal action at t is bounded by $\left|w_i^k(m)-w_i^k(m+1)\right|+w_i^k(m+1)(e^{-r\Delta}-e^{-r\frac{T}{k}})$. Given that the probability of any recognition between $t=-T+(m-1)\frac{T}{k}$ and $t=-T+m\frac{T}{k}$ is $1-e^{-\lambda^k\frac{T}{k}}$, $\left|w_i^k(m)-w_i^k(m+1)\right| \leq V(N)(1-e^{-(\lambda^k+r)\frac{T}{k}})$. Thus, as $k\to\infty$, since $\Delta, \frac{T}{k}\to 0$, we have $\left|w_i^k(m)-w_i^k(m+1)\right|+w_i^k(m+1)(e^{-r\Delta}-e^{-r\frac{T}{k}})\to 0$. This means that for any $\varepsilon>0$, there is a $k^\varepsilon \in \mathbb{Z}_{++}$ such that for any $k>k^\varepsilon$, $\widehat{s}^k$ specifies an ε-perfect equilibrium of G^k (which is also Markovian, by construction).

We now return to our original sequence $\{G_j^{k(j)}(N,V,\lambda^j,T)\}_{j=1}^\infty \equiv G_1, G_2, ...$

Fact 2: Uniform convergence of $\widehat{w}^k(.)$ along a subsequence of $G_1, G_2, ...$

Define $\widehat{t}^k(\tau) = \lceil (T+\tau)\frac{k}{T} \rceil$. By construction,

$$\widehat{w}_i^k(t) = \int_t^0 e^{-(r+\lambda^k)(\tau-t)}[\lambda_i^k \left(V[C_i^k(\widehat{t}^k(\tau))] - \sum_{j \in C_i^k(\widehat{t}^k(\tau)) \setminus \{i\}} e^{-r\frac{T}{k}} w_j^k(\widehat{t}^k(\tau)+1) \right) + \sum_{j \neq i: i \in C_j^k(\widehat{t}^k(\tau))} \lambda_j^k e^{-r\frac{T}{k}} w_i^k(\widehat{t}^k(\tau)+1)] d\tau.$$

It is easy to see that for every $i \in N$ and $k \in \mathbb{Z}_+$, $\widehat{w}_i^k(.)$ is Lipschitzcontinuous with Lipschitz constant $(r+\lambda^k)V(N)$. Moreover, all $\widehat{w}_i^k(.)$ are uniformly bounded by 0 below and $V(N)$ above. Therefore, returning to our sequence $G_1, G_2, ...$ (and, for simplicity, now indexing our continuation value functions by the index of the corresponding game rather than the number of periods), by the Ascoli-Arzela theorem (see Royden (1988), p169), the sequence of functions $\{\widehat{w}^j(.)\}_{j=1}^\infty$ has a subsequence $\{\widehat{w}^{j_h}(.)\}_{h=1}^\infty$ that converges uniformly to functions $\widehat{w}^*(.) = (\widehat{w}_1^*(.), ..., \widehat{w}_n^*(.))$, as $h \to \infty$. Moreover, because $\lambda^{j_h} \to \lambda$ as $h \to \infty$, each $\widehat{w}_i^*(.)$ is also Lipschitzcontinuous with constant $(r+\lambda)V(N)$. Without loss of generality, assume that the original sequence $G_1, G_2, ...$ is convergent.

Facts 1 and 2 taken together establish that if strategies are history-independent and continuation payoff functions are given by $\widehat{w}^*(.)$, then when approached at t, an optimal strategy for j' is accepting the offer iff it gives her at least $\widehat{w}_{j'}^*(t)$. Below, we complete the proof by constructing optimal strategies for proposers that generate these payoff functions.

Let $\mathcal{T}$ stand for the set of points in $[-T, 0]$ where $\widehat{w}_i^*(.)$ and $\widehat{w}_i^j(.)$ are differentiable, for every $i \in N$ and $j \in \mathbb{Z}_{++}$. Since the above functions are all Lipschitzcontinuous, $[-T, 0] \setminus \mathcal{T}$ is a null set.

By Lemma 1, for any $t \in \mathcal{T}$, $\nabla \widehat{w}^*(t) \in co(D(t))$. By Caratheodory's theorem, there exist points $x_1, ..., x_{n+1} \in co(D(t))$ such that $\nabla \widehat{w}(t) = \alpha_1 x_1 + ... + \alpha_{n+1} x_{n+1}$ for $\alpha_1, ..., \alpha_{n+1} \geq 0$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$. For every $m \in \{1, ..., n+1\}$, let $G_{m_1}, G_{m_2}, ...$ be a subsequence of $G_1, G_2, ...$ and $t^{m_1}, t^{m_2}, ...$ be a sequence of points in $[-T, 0]$ converging to t such that $\nabla \widehat{w}^{m_h}(t^{m_h}) \to x_m$. Because there are only a finite number of coalitions, $G_{m_1}, G_{m_2}, ...$ has a subsequence $G_{\widehat{m}_1}, G_{\widehat{m}_2}, ...$ such that for every $i \in N$ and $\widehat{m}_h \in \mathbb{Z}_{++}$, $C_i^{\widehat{m}_h}(t^{\widehat{m}_h}) = C_i^{m*}$ for some $C_i^{m*} \in 2^N$. If approaching C_i^{m*} and offering $\widehat{w}_p^{\widehat{m}_h}(t^{\widehat{m}_h})$ to every player $p \in C_i^{m*}$ is an optimal strategy for i in $G_{\widehat{m}_h}$ at $t^{\widehat{m}_h}$ given $\widehat{s}^{\widehat{m}_h}$, then by upper hemicontinuity of the best-response correspondence, approaching C_i^{m*} and offering $\widehat{w}_p^*(t)$ to every $p \in C_i^{m*}$ is an optimal strategy for i in $G(N, V, \lambda, T)$ at t, provided that any approached player p' at any point of time t' accepts an offer iff the offer to her is at least $\widehat{w}_{p'}^*(t')$.

Since the above holds for all $m \in \{1, ..., n+1\}$, the strategy of approaching C_i^{m*} with probability α_m (and offering $\widehat{w}_p^*$ to to every $p \in C_i^{m*}$) is an optimal strategy for i in $G(N, V, \lambda, T)$ at t, provided that any approached player p' at any point of time t' accepts an offer iff the offer to her is at least $\widehat{w}_{p'}^*(t')$.

Consider now the following Markovian strategy profile s^* in $G(N, V, \lambda, T)$: (i) For any $i \in N$ and any $t \in [-T, 0]$, if i is approached at t, she accepts the offer iff it gives her at least $\widehat{w}_i^*(t)$; (ii) For any $i \in N$ and any $t \in \mathcal{T}$, if i is recognized at t, she approaches C_i^{m*} with probability α_m and offers $\widehat{w}_p^*(t)$ to every $p \in C_i^{m*} \backslash \{i\}$, for every $m \in \{1, ..., n+1\}$; (iii) For any $i \in N$ and any $t \in [-T, 0] \backslash \mathcal{T}$, if i is recognized at t, she approaches some coalition $C \in \underset{C' \in 2^N : i \in C'}{\arg\max} \, (V(C') - \sum_{p \in C' \backslash \{i\}} \widehat{w}_p^*(t))$ and offers $\widehat{w}_p^*(t)$ to every $p \in C \backslash \{i\}$.

By construction, if all players follow the above Markovian strategies, then the gradient of the continuation payoff function at t is exactly $\nabla \widehat{w}^*(t)$ at every $t \in \mathcal{T}$. Note also that $\widehat{w}^*(0) = 0$ and continuation payoffs at 0 are also equal to 0. Given that both the continuation payoff functions given the above strategies, and $\widehat{w}_i^*(.)$ for all $i \in N$ are Lipschitzcontinuous, this implies that the continuation payoff functions generated by the above strategies are exactly $\widehat{w}^*(.)$. Since we established above the optimality of these strategies given that continuation payoffs are $\widehat{w}^*(.)$, we constructed an MPE of $G(N, V, \lambda, T)$ such that the continuation payoffs defined by the MPE are given by $\widehat{w}^*(.)$.

Finally, note that $\sup_{t \in [-T, 0]} |w_i^{G_j}(t) - \widehat{w}_i^j(t)| \leq (r + \lambda^j) V(N) \frac{T}{k(j)}$, where the right-hand sides goes to 0 as $j \to \infty$. Hence, the sequence of SPNE continuation payoff functions $\{w^{G_{j_h}^{k(j_h)}}(.)\}_{h=1}^{\infty}$ converges to the same limit as any convergent subsequence $\widehat{w}^{j_1}(.), \widehat{w}^{j_2}(.), ...$, as Statement (i) claims. ■

STATEMENT (ii): All MPE have the same continuation value functions w_i.

Proof of Statement (ii): We start with the following mathematical result.

Lemma 2: Suppose $g(.)$ is an integrable function, $w(.)$ is a Lipschitz continuous function, and $w(t) > 0$. Let $G(x) = \int_t^x g(\tau) d\tau$ and $H(x) = \max_{s \in [t,x]} \left| \int_t^s g(\tau) w(\tau) d\tau \right|$.

There exist $c, \varepsilon > 0$ such that whenever $\delta \in [t, t + \varepsilon]$, we have $c \, |G(\delta)| \leq H(\delta)$.

Proof of Lemma 2:

$$\begin{aligned}
|G(\delta)| &= \left|\int_t^\delta g(\tau)d\tau\right| \\
&= \frac{1}{w(t)}\left|\int_t^\delta g(\tau)w(\tau)d\tau + \int_t^\delta g(\tau)\left[w(t)-w(\tau)\right]d\tau\right| \\
&\leq \frac{1}{w(t)}\left(H(\delta) + \left|\int_t^\delta g(\tau)\left[w(t)-w(\tau)\right]d\tau\right|\right) \\
&= \frac{1}{w(t)}\left(H(\delta) + \left|\int_t^\delta g(\tau)w(\tau)\left[\frac{w(t)}{w(\tau)}-1\right]d\tau\right|\right)
\end{aligned}$$

Let $f(\tau) = \frac{w(t)}{w(\tau)}$. Because w is Lipschitz continuous and $w(t) \neq 0$, f is Lipschitz continuous in a non-empty interval $(t, t+\varepsilon)$; let L be f's Lipschitz bound within $(t, t+\varepsilon)$. A Lipschitz continuous function is differentiable almost everywhere, so we can write:

$$\begin{aligned}
\left|\int_t^\delta g(\tau)w(\tau)\left[\frac{w(t)}{w(\tau)}-1\right]d\tau\right| &= \left|\int_t^\delta g(\tau)w(\tau)\left[f(\tau)-f(t)\right]d\tau\right| \\
&= \left|\int_t^\delta g(\tau)w(\tau)\int_t^\tau f'(s)ds d\tau\right| \\
&= \left|\int_t^\delta f'(s)\int_s^\delta g(\tau)w(\tau)d\tau ds\right| \\
&\leq \int_t^\delta |f'(s)|\left|\int_s^\delta g(\tau)w(\tau)d\tau\right| ds \\
&\leq 2LH(\delta)\delta
\end{aligned}$$

The last step follows because $\left|\int_s^\delta g(\tau)w(\tau)d\tau\right| \leq \left|\int_t^\delta g(\tau)w(\tau)d\tau\right| + \left|\int_t^s g(\tau)w(\tau)d\tau\right| \leq 2H(\delta)$.

Thus,

$$|G(\delta)| \leq \frac{1}{w(t)}(1+L\delta)H(\delta)$$

Picking $c = \frac{w(t)}{1+L\varepsilon}$ completes the proof of Lemma 2. □

We proceed by contradiction. Suppose two MPE, A and B, of the same bargaining game with characteristic function V and recognition rates $(\lambda_1, ..., \lambda_n)$ do not have the same continuation value functions.

Let $p_{ij}(t)$ be the probability of j receiving an offer at time t given that i is recognized at that time.

Define $f_j(\tau) = w_j^A(\tau) - w_j^B(\tau)$ and $g_j(\tau) = \sum_{i\neq j} \lambda_i p_{ij}^A(\tau) - \sum_{i\neq j} \lambda_i p_{ij}^B(\tau)$.

Lemma 3 is used in the proof of Lemma 5, which contradicts Lemma 4.

Lemma 3: For all τ, $\sum_{j\in N} f_j(\tau)g_j(\tau) \leq 0$.

Proof of Lemma 3: Let $f_{C,C'}(\tau) \equiv f_C(\tau) - f_{C'}(\tau) \equiv \sum_{i\in C} f_i(\tau) - \sum_{i\in C'} f_i(\tau)$.

For $k = A, B$, let $p^k_{iC}(t)$ be the probability of C (and only C) receiving an offer at time t given that i is recognized at that time, in equilibrium k.

Let $g_{iC}(\tau) = p^A_{iC}(\tau) - p^B_{iC}(\tau)$, $\Delta_i(\tau) = \{C|g_{iC}(\tau) > 0\}$, and $\nabla_i(\tau) = \{C|g_{iC}(\tau) < 0\}$. Define:[25]

$$g_{iC,C'}(\tau) = \left\{ \begin{array}{c} \frac{-g_{iC}(\tau)g_{iC'}(\tau)}{\sum_{D\in\Delta_i(\tau)} g_{iD}(\tau)} \text{ if } C \in \Delta_i(\tau) \text{ and } C' \in \nabla_i(\tau) \\ -g_{iC',C}(\tau) \text{ if } C' \in \Delta_i(\tau) \text{ and } C \in \nabla_i(\tau) \\ 0 \text{ if } g_{iC}(\tau) = 0,\ g_{iC'}(\tau) = 0, \text{ or } g_{iC}(\tau)g_{iC'}(\tau) > 0 \end{array} \right\}$$

Finally, let:

$$g_{C,C'}(\tau) = \sum_{i\in N} \lambda_i g_{iC,C'}(\tau)$$

Observe that $g_{C,C'}(\tau)$ has a simple interpretation: it measures the frequency of proposals gained by coalition C from C' in equilibrium A relative to equilibrium B. It is easy to verify that $\sum_{C'\in 2^N} g_{C,C'}(\tau) = -\sum_{C'\in 2^N} g_{C',C}(\tau) \equiv g_C(\tau)$, and $g_i(\tau) = \sum_{C\ni i} g_C(\tau)$.

By optimality, we must have $f_{C,C'}(\tau)g_{iC,C'}(\tau) \leq 0$ for all i. For example, if $f_{C,C'}(\tau) > 0$, coalition C has become more expensive relative to C'. It is therefore not optimal for any i to approach C more often while simultaneously approaching C' less often; thus, in this case, we cannot have $g_{iC,C'}(\tau) > 0$. It follows that, for all $C, C' \in 2^N$ and $\tau < 0$,

$$f_{C,C'}(\tau)g_{C,C'}(\tau) = \sum_{i\in N} f_{C,C'}(\tau)\lambda_i g_{iC,C'}(\tau) \leq 0$$

[25] Since $\sum_{D\in\Delta_i(\tau)} g_{iD}(\tau) = \sum_{D\in\nabla_i(\tau)} g_{iD}(\tau)$, this definition is symmetric.

Now note that:

$$\begin{aligned}
\sum_{(C,C')\in 2^N\times 2^N} f_{C,C'}(\tau)g_{C,C'}(\tau) = & \begin{array}{l}\sum_{(C,C')\in 2^N\times 2^N} f_C(\tau)g_{C,C'}(\tau)\\ -\sum_{(C,C')\in 2^N\times 2^N} f_{C'}(\tau)g_{C,C'}(\tau)\end{array}\\
&= \sum_{C\in 2^N} f_C(\tau)g_C(\tau) + \sum_{C'\in 2^N} f_{C'}(\tau)g_{C'}(\tau)\\
&= 2\sum_{C\in 2^N}\left(\sum_{i\in C} f_i(\tau)\right) g_C(\tau)\\
&= 2\sum_{j\in N} f_j(\tau)\left(\sum_{C\ni j} g_C(\tau)\right)\\
&= 2\sum_{j\in N} f_j(\tau)g_j(\tau)
\end{aligned}$$

Thus, $\sum_{j\in N} f_j(\tau)g_j(\tau) \leq 0$, as Lemma 3 states. $\square$

Observe that if $\lambda_j = 0$, the only possible MPE continuation value for j is 0 at all times. The game is then equivalent to an alternative game with players $N\backslash\{j\}$ and characteristic function $V'(C) = V(C\cup\{j\}), \forall C \subseteq N\backslash\{j\}$. So we assume without loss of generality that $\lambda_j > 0, \forall j \in N$.

The rest of the proof requires the introduction of some extra notation.

Let $v_j(\tau) = \max_{C\ni i}\left\{v(C) - \sum_{i\in C\backslash\{j\}} w_i(\tau)\right\}$. Then we can write:

$$w_j(t') = \int_{t'}^{0} e^{-(\lambda+r)(\tau-t')}\left[\lambda_j v_j(\tau) + w_j(\tau)\left(\sum_{i\neq j}\lambda_i p_{ij}(\tau)\right)\right] d\tau \qquad (1)$$

Let $t = \min\{\tau | f_i(t') = 0, \forall t' \in [\tau, 0], \forall i \in N\}$.

Since $w_j(0) = 0$, there must be some nontrivial interval just before time 0 where proposing to a coalition of value $V(N)$ is strictly optimal for everyone. When the only such coalition is N, MPE payoffs are clearly unique within this interval; the same can be shown if multiple coalitions have value $V(N)$.[26] It follows that $t < 0$. Since $\lambda_j > 0$ for all j, we have $w_j(t) > 0$.

[26]Note that at $t = 0$, the left derivative of continuation value functions must exist since $w_i(0) = 0$ (so the probability of being approached when someone is recognized, which may be discontinuous, does not affect the rate of change of w_i). In fact, we have $w_i'(0) = -\lambda_i V(N)$. So when there are coalitions $C_1, ..., C_m \neq N$ with $i \in C_j$ and $V(C_j) = V(N)$ for $j = 1, ..., m$, it must be true that in a neighborhood of 0, i only proposes to C_k with positive probability if $\sum_{l\in C_k}\lambda_l \leq \sum_{l\in C_j}\lambda_l$ for $j = 1, ..., m$. When there are multiple such coalitions, if feasible, they will be approached such that their continuation values are equalized; if equalization cannot be achieved, those having the choice between many such coalitions will propose to the cheapest one.

Let $F_j(\tau) = \max_{x\in[\tau,t]} |f_j(x)|$ and $F(\tau) = \max_{i\in N} F_i(\tau)$. We have $F_j(t) = F(t) = 0$, and these functions are non-increasing.

Since $w_j(\tau)$ exists, and all terms other than $\sum\limits_{i\neq j} \lambda_i p_{ij}(\tau)$ in the integrand in (1) are continuous in τ, $\sum\limits_{i\neq j} \lambda_i p_{ij}(\tau)$ must be integrable. Therefore, $g_j(\tau)$ must be integrable as well. Thus we can define:

$$h_j(t') = \int_{t'}^{t} e^{-(\lambda+r)(\tau-t')} w_j^B(\tau) g_j(\tau) d\tau$$

Let $H_j(\tau) = \max_{x\in[\tau,t]} |h_j(x)|$ and $H(\tau) = \max_{i\in N} H_i(\tau)$. Note that $H_j(t) = H(t) = 0$, and these functions are non-increasing.

Finally, let $R(t') = \sum_{j\in N} \frac{f_j(t')h_j(t')}{w_j^B(t')}$.

Lemma 4: For every $\varepsilon > 0$, there exists $t' \in [t-\varepsilon, t)$ such that $R(t')$ is positive and at least on the order of magnitude of $H(t')^2$, and $F(t')$ and $H(t')$ share the same order of magnitude.

Proof of Lemma 4: By (1), for any $t' < t$, we can write:

$$\begin{aligned} f_j(t') &= \int_{t'}^{t} e^{-(\lambda+r)(\tau-t')} \left[\begin{array}{c} \lambda_j(v_j^A(\tau) - v_j^B(\tau)) + (w_j^A(\tau) - w_j^B(\tau)) \left(\sum\limits_{i\neq j} \lambda_i p_{ij}^A(\tau) \right) \\ + w_j^B(\tau) \left(\sum\limits_{i\neq j} \lambda_i p_{ij}^A(\tau) - \sum\limits_{i\neq j} \lambda_i p_{ij}^B(\tau) \right) \end{array} \right] d\tau \\ &= \int_{t'}^{t} e^{-(\lambda+r)(\tau-t')} \left[\lambda_j \left(v_j^A(\tau) - v_j^B(\tau) \right) + f_j(\tau) \left(\sum\limits_{i\neq j} \lambda_i p_{ij}^A(\tau) \right) \right] d\tau + h_j(t') \qquad (2) \end{aligned}$$

Using the triangle inequality and the facts that on $\tau \in [t', t]$, $e^{-(\lambda+r)(\tau-t')} < 1$ and

$\sum_{i\neq j} \lambda_i p_{ij}^A(\tau) < \lambda$, we have the following bounds:

$$\begin{aligned}
|f_j(t')| - |h_j(t')| &\leq |f_j(t') - h_j(t')| \\
|f_j(t') - h_j(t')| &\leq \left|\lambda_j \int_{t'}^{t} e^{-(\lambda+r)(\tau-t')}\left[v_j^A(\tau) - v_j^B(\tau)\right] d\tau\right| \\
&\quad + \left|\int_{t'}^{t} e^{-(\lambda+r)(\tau-t')} f_j(\tau)\left(\sum_{i\neq j}\lambda_i p_{ij}^A(\tau)\right) d\tau\right| \\
&\leq \lambda_j(t-t') \sum_{i\in N\setminus\{j\}} F_i(t') + \lambda(t-t')F_j(t') \\
&< \lambda(t-t')(n-1)F(t') + \lambda(t-t')F(t') \\
&= \lambda(t-t')nF(t') \qquad (3)
\end{aligned}$$

Observation: If $t - t' < \frac{1}{\lambda n}$, we have:

$$F(t') \leq \frac{H(t')}{1-\lambda(t-t')n}$$

Proof of Observation: There must exist $i \in N$ and $\tau \in [t', t)$ such that:

$$\begin{aligned}
F(t') &= |f_i(\tau)| \\
&< |h_i(\tau)| + \lambda(t-\tau)nF(\tau) \\
&\leq H(t') + \lambda(t-t')nF(t')
\end{aligned}$$

The observation is obtained by rearranging this inequality. □

By (3):

$$\begin{aligned}
f_j(t')h_j(t') &= h_j(t')^2 - h_j(t')\left[h_j(t') - f_j(t')\right] \\
&> h_j(t')^2 - |h_j(t')|\,\lambda(t-t')nF(t') \qquad (4)
\end{aligned}$$

Note that (4) is quadratic in $|h_j(t')|$ and thus minimized if $|h_j(t')| = \frac{\lambda(t-t')nF(t')}{2}$. So if $t - t' < \frac{1}{\lambda n}$, we have:

$$\begin{aligned}
\frac{f_j(t')h_j(t')}{w_j^B(t')} &> -\frac{\lambda^2 n^2}{4w_j^B(t')}(t-t')^2 F(t')^2 \\
&\geq -\frac{\lambda^2 n^2}{4w_j^B(t')\left[1-\lambda(t-t')n\right]^2}(t-t')^2 H(t')^2 \qquad (5)
\end{aligned}$$

The last step follows from the Observation.

Let $S = \{\tau < t | H(\tau) = \max_{i \in N} |h_i(\tau)|\}$, and for all $\tau \in S$, let $i(\tau) \in N$ be such that $H(\tau) = \left|h_{i(\tau)}(\tau)\right|$. Because H is continuous and $H(t) = 0$, we have $S \cap [\tau, t) \neq \varnothing$ for all $\tau < t$.

By (4), for any $t' \in S \cap (t - \frac{1}{\lambda n}, t]$, we have:

$$\begin{aligned} \frac{f_{i(t')}(t') h_{i(t')}(t')}{w^B_{i(t')}(t')} &> \frac{1}{w^B_{i(t')}(t')} \left[H(t')^2 - \lambda(t-t')nH(t')F(t')\right] \\ &\geq \frac{1}{w^B_{i(t')}(t')} \left[1 - \frac{\lambda(t-t')n}{1-\lambda(t-t')n}\right] H(t')^2 \end{aligned} \tag{6}$$

Again, the last step follows from the Observation.

Since w^B_j is Lipschitz continuous, $w^B_j(t')$ must be on the order of $w^B_j(t) \neq 0$. Thus, taking $(t-t')$ small and combining (5) and (6) imply that $R(t') = \sum_{j \in N} \frac{f_j(t') h_j(t')}{w^B_j(t')}$ is at least on the order of $H(t')^2$, as desired.

Furthermore, by (3), for $t' \in S$, we have:

$$F(t') \geq \left|f_{i(t')}(t')\right| > \left|h_{i(t')}(t')\right| - \lambda(t-t')nF(t') = H(t') - \lambda(t-t')nF(t')$$

Thus:

$$F(t') \geq \frac{H(t')}{1 + \lambda(t-t')n}$$

Combining the above inequality with the observation yields $F(t') \in \left[\frac{H(t')}{1+\lambda(t-t')n}, \frac{H(t')}{1-\lambda(t-t')n}\right]$ for $t' \in S \cap (t - \frac{1}{\lambda n}, t)$. Therefore, as $t' \to t$, $F(t')$ and $H(t')$ have the same order of magnitude when $t' \in S$, as Lemma 4 states. □

Lemma 5: Suppose $t - t'$ is small. Then if $R(t')$ is positive, its order of magnitude is at most $F(t')H(t')(t-t')$.

Note: Lemma 5 contradicts Lemma 4 and concludes the proof of Statement (ii).

Proof of Lemma 5: We have:

$$\frac{d}{ds} f_j(s) h_j(s) \mid_{s=\tau} = f'_j(\tau) h_j(\tau) - f_j(\tau) w^B_j(\tau) g_j(\tau) + (\lambda + r) f_j(\tau) h_j(\tau)$$

Since f_j and w_j^B are differentiable, $R(\tau) = \sum_{j \in N} \frac{f_j(\tau) h_j(\tau)}{w_j^B(\tau)}$ is differentiable as well. Thus:

$$\begin{aligned} R'(\tau) &= \sum_{j \in N} \left(\frac{f_j'(\tau) h_j(\tau) - f_j(\tau) w_j^B(\tau) g_j(\tau) + (\lambda + r) f_j(\tau) h_j(\tau)}{w_j^B(\tau)} - \frac{f_j(\tau) h_j(\tau) w_j^{B\prime}(\tau)}{w_j^B(\tau)^2} \right) \\ &= -\sum_{j \in N} f_j(\tau) g_j(\tau) + \sum_{j \in N} \frac{f_j'(\tau) h_j(\tau)}{w_j^B(\tau)} + \sum_{j \in N} \frac{f_j(\tau) h_j(\tau)}{w_j^B(\tau)} \left(\lambda + r - \frac{w_j^{B\prime}(\tau)}{w_j^B(\tau)} \right) \\ &\geq \sum_{j \in N} \frac{f_j'(\tau) h_j(\tau)}{w_j^B(\tau)} + \sum_{j \in N} \frac{f_j(\tau) h_j(\tau)}{w_j^B(\tau)} \left(\lambda + r - \frac{w_j^{B\prime}(\tau)}{w_j^B(\tau)} \right) \end{aligned}$$

The last step follows from Lemma 3.

We now derive a lower bound on $R'(\tau)$ near t. (Equivalently, we seek an upper bound on how negative $R'(\tau)$ can be.) We examine the two terms in turn.

1. First, we focus on $\sum_{j \in N} \frac{f_j'(\tau) h_j(\tau)}{w_j^B(\tau)}$.

Differentiating (2) yields:

$$f_j'(\tau) = -\lambda_j \left[v_j^A(\tau) - v_j^B(\tau) \right] - f_j(\tau) \left(\sum_{i \neq j} \lambda_i p_{ij}^A(\tau) \right) - w_j^B(\tau) g_j(\tau) + (\lambda + r) f_j(\tau)$$

Thus:

$$\frac{f_j'(\tau)}{w_j^B(\tau)} = -g_j(\tau) + \frac{1}{w_j^B(\tau)} \left[-\lambda_j \left[v_j^A(\tau) - v_j^B(\tau) \right] - f_j(\tau) \left(\sum_{i \neq j} \lambda_i p_{ij}^A(\tau) \right) + (\lambda + r) f_j(\tau) \right]$$

Also from (2):

$$h_j(\tau) = f_j(\tau) - \int_\tau^t e^{-(\lambda + r)(s - \tau)} \left[\lambda_j \left(v_j^A(s) - v_j^B(s) \right) + f_j(s) \left(\sum_{i \neq j} \lambda_i p_{ij}^A(s) \right) \right] ds$$

Therefore,

$$\begin{aligned} &\sum_{j \in N} \frac{f_j'(\tau) h_j(\tau)}{w_j^B(\tau)} \\ = \; &\sum_{j \in N} \left[\begin{array}{c} -f_j(\tau) g_j(\tau) + g_j(\tau) \int_\tau^t e^{-(\lambda + r)(s - \tau)} \left[\lambda_j \left(v_j^A(s) - v_j^B(s) \right) + f_j(s) \left(\sum_{i \neq j} \lambda_i p_{ij}^A(s) \right) \right] ds \\ + \frac{h_j(\tau)}{w_j^B(\tau)} \left[-\lambda_j \left[v_j^A(\tau) - v_j^B(\tau) \right] - f_j(\tau) \left(\sum_{i \neq j} \lambda_i p_{ij}^A(\tau) \right) + (\lambda + r) f_j(\tau) \right] \end{array} \right] \end{aligned}$$

The first term $\sum_{j\in N} -f_j(\tau)g_j(\tau)$ is nonnegative by Lemma 3, and can thus be ignored. The second term (occupying the rest of the first line) will be examined later. The last term (occupying the second line) has maximum order of magnitude $H(\tau)F(\tau)$.

2. We now turn our attention to $\sum_{j\in N} \frac{f_j(\tau)h_j(\tau)}{w_j^B(\tau)}\left(\lambda + r - \frac{w_j^{B\prime}(\tau)}{w_j^B(\tau)}\right)$. Again, since we are only interested in orders of magnitude, and since w_j^B is Lipschitz continuous, the relevant quantity is $f_j(\tau)h_j(\tau)$. This term has order of magnitude at most $F(\tau)H(\tau)$.

Denote a positive term on the order of x or less by $\mathcal{O}(x)$. Combining our examination of the two terms, we see that there is a lower bound for $R'(\tau)$ equal to:

$$\begin{aligned}&\sum_{j\in N} g_j(\tau)\int_\tau^t e^{-(\lambda+r)(s-\tau)}\left[\lambda_j\left(v_j^A(s)-v_j^B(s)\right)+f_j(s)\left(\sum_{i\neq j}\lambda_i p_{ij}^A(s)\right)\right]ds\\&-\mathcal{O}(F(\tau)H(\tau))\end{aligned}$$

Therefore, there is an upper bound for $R(t') = R(t) - \int_{t'}^t R'(\tau)d\tau = -\int_{t'}^t R'(\tau)d\tau$ equal to:

$$\begin{aligned}&-\sum_{j\in N}\int_{t'}^t g_j(\tau)\left(\int_\tau^t e^{-(\lambda+r)(s-\tau)}\left[\lambda_j\left(v_j^A(s)-v_j^B(s)\right)+f_j(s)\left(\sum_{i\neq j}\lambda_i p_{ij}^A(s)\right)\right]ds\right)d\tau\\&+\mathcal{O}(F(t')H(t')(t-t'))\\\equiv\ &\overline{R}(t')+\mathcal{O}(F(t')H(t')(t-t'))\end{aligned}$$

To complete the proof, it suffices to show that $\overline{R}(t')$ has order of magnitude at most $F(t')H(t')(t-t')$.

Changing the order of integration for $\overline{R}(t')$ yields:

$$\begin{aligned}\overline{R}(t') &= -\sum_{j\in N}\int_{t'}^t\left(\int_{t'}^s e^{-(\lambda+r)(s-\tau)}g_j(\tau)d\tau\right)\left(\lambda_j\left[v_j^A(s)-v_j^B(s)\right]+f_j(s)\left(\sum_{i\neq j}\lambda_i p_{ij}^A(s)\right)\right)ds\\&< \lambda n F(t')\sum_{j\in N}\int_{t'}^t\left|\int_{t'}^t e^{-(\lambda+r)(s-\tau)}g_j(\tau)d\tau-\int_s^t e^{-(\lambda+r)(s-\tau)}g_j(\tau)d\tau\right|ds\end{aligned}$$

We now apply Lemma 2, with $e^{-(\lambda+r)(s-\tau)}g_j(\tau)$ playing the role of $g(\tau)$ and $w_j^B(\tau)e^{-(\lambda+r)(2\tau-s-t')}$ playing the role of $w(\tau)$. We have that $\left|\int_{t'}^t e^{-(\lambda+r)(s-\tau)}g_j(\tau)d\tau\right|$ is of order at most $H_j(t') \leq H(t')$. Similarly, $\left|\int_s^t e^{-(\lambda+r)(s-\tau)}g_j(\tau)d\tau\right|$ is of order at most $H(s) \leq H(t')$. Therefore, the summation has order at most $(t-t')H(t')$, and thus $\overline{R}(t')$ has maximum order $F(t')H(t')(t-t')$. This concludes the proof of Statement (ii). ■

STATEMENT (iii): Suppose the sequence of regular discrete coalitional bargaining games $\{G^k\}$ converges to G. Then $\{w_i^{G^k}\}$ converges uniformly to w_i (player i's MPE continuation value function in G) for all i.

Proof of Statement (iii): Statement (ii) and the proof of statement (i) together imply that all convergent subsequences of $\{\widehat{w}^k(.)\}$ (as defined in the proof of statement (i)) converge to w_i, with respect to the uniform topology. The set of uniformly bounded Lipschitz continuous functions on $[-T, 0]$ with Lipschitz constant $(r + \lambda + \varepsilon)V(N)$ is compact with respect to the uniform topology for any $\varepsilon > 0$, from which it follows that the sequence $\{\widehat{w}^k(.)\}$ itself converges uniformly to the vector $(w_1, w_2, ..., w_n)$. Again, note that $\sup\limits_{t\in[-T,0]} |w_i^{G_j}(t) - \widehat{w}_i^j(t)| \leq (r + \lambda^j)V(N)\frac{T}{k(j)}$, where the right-hand sides goes to 0 as $j \to \infty$. Hence, $\{w_i^{G^k}\}$ converges to the same limit as the sequence $\{\widehat{w}_i^k(.)\}$, namely w_i. ■

11.2 Proof of Theorem 2

Theorem 2: If V is convex, then $\mathcal{S}(V) = \mathcal{C}(V)$.

Theorem 2 is the combination of Lemma 6 (which implies $\mathcal{C}(V) \subseteq \mathcal{S}(V)$) and Lemma 7 (which implies $\mathcal{S}(V) \subseteq \mathcal{C}(V)$).

Lemma 6: For every $x \in \mathcal{C}(V)$, there exist recognition rates $\{\lambda_i\}_{i\in N}$ such that the expected MPE payoffs converge to $\frac{x}{1+r}$ as $T \to \infty$.

Proof of Lemma 6: The statement holds vacuously if $\mathcal{C}(V) = \varnothing$, so we assume $\mathcal{C}(V) \neq \varnothing$. Normalize payoffs with $V(N) = 1$. Let $x \in \mathcal{C}(V)$, set $\lambda_i = x_i$ (so $\sum_i \lambda_i = V(N) = 1$), and specify strategy profile σ as follows:

For every $i \in N$, if player i is recognized at $t \in [-T, 0]$, she approaches N and offers exactly $w_j(t)$ to every $j \in N\backslash\{i\}$. If player i is approached at t, then independently of who approached her and what coalition was approached, she accepts the offer if and only if she is offered at least $w_i(t)$.

By Claim 3, if play proceeds according to σ, expected payoffs are given by $\frac{x_i}{1+r}(1 - e^{(1+r)t})$ for every $i \in N$. Note that $w_j(t) < x_j\ \forall\ t \in [-T, 0]$ and $j \in N$. Since $x \in \mathcal{C}(V)$, this implies that $\sum\limits_{j\in C} w_j(t) < \sum\limits_{j\in C} x_j = V(N) - \sum\limits_{j\in N\backslash C} x_j \leq V(N) - V(N\backslash C)$ for any $C \subset N$. Therefore, when recognized, the optimal coalition to approach is always N.

Combining the above observation with Claim 2, we conclude that σ is an MPE. Thus, the MPE expected payoffs are given by $\frac{x_j}{1+r}(1 - e^{-(1+r)T})$ for every $i \in N$. As $T \to \infty$, the expected MPE payoff of player i converges to $\frac{x_i}{1+r}$. □

Lemma 7: Let $\lambda_M = \min_{i\in N} \lambda_i$. If V is convex, then for any $\varepsilon > 0$, there exists

T^* such that in any MPE of a game with $T > T^*$, continuation values satisfy $\sum\limits_{i \in C} w_i(t) \geq \frac{\lambda_M}{\lambda_M + r} V(C) - \varepsilon$, $\forall\ C \subseteq N$ and $t \leq -T^*$.

Proof of Lemma 7: We proceed by induction. First, note that for any $i \in N$ and any $t \leq 0$, $w_i(t) < V(\{i\})$ implies that $V(C \cup \{i\}) - w_i(t) > V(C)$ for any $i \notin C$. This and Claim 2 imply that at any time where $w_i(t) < V(\{i\})$ in a MPE, any recognized player $j \neq i$ will include player i in the approached coalition and offer her exactly $w_i(t)$. Furthermore, note that if player i has the chance to make an offer at t, then she can guarantee a payoff of at least $V(\{i\})$ by approaching herself. This implies that $w_i(t)$ is bounded below by $\int\limits_t^0 [\lambda_i e^{-(\lambda+r)(\tau-t)} V(\{i\}) + \sum\limits_{j \neq i} \lambda_j e^{-(\lambda+r)(\tau-t)} w_i(\tau)] d\tau$. It is easy to check that this implies $w_i(t) \geq \frac{\lambda_i}{\lambda_i + r} V(\{i\})(1 - e^{(\lambda_i + r)t})$ in every MPE. Therefore, if $T_1(\varepsilon) = \min\limits_{i \in N} \frac{1}{\lambda_i + r} ln \frac{\varepsilon(\lambda_i + r)}{\lambda_i V(\{i\})}$, then for any $t \leq T_1(\varepsilon)$ and $i \in N$, $w_i(t) \geq \frac{\lambda_i}{\lambda_i + r} V(\{i\}) - \varepsilon$, for every $\varepsilon > 0$.

Assume now that for some $K \in \{1, ..., n-1\}$, there exists a finite $T_K(\varepsilon)$ for any $\varepsilon > 0$ such that for every $C \subset N$ with $|C| \leq K$, it holds that $\sum\limits_{i \in C} w_i(t) \geq \frac{\lambda_M}{\lambda_M + r} V(C) - \varepsilon$, $\forall\ t \leq T_K(\varepsilon)$. We show below that this implies that for any $\varepsilon > 0$, there exists a finite $T_{K+1}(\varepsilon)$ such that for every $C \subset N$ with $|C| \leq K + 1$, it holds that $\sum\limits_{i \in C} w_i(t) \geq \frac{\lambda_M}{\lambda_M + r} V(C) - \varepsilon, \forall\ t \leq T_{K+1}(\varepsilon)$.

Fix any $\varepsilon > 0$ and any C with $|C| = K + 1$. From the induction assumption, $\sum\limits_{i \in C'} w_i(t) \geq \frac{\lambda_M}{\lambda_M + r} V(C') - \varepsilon$, $\forall\ t \leq T_K(\varepsilon)$ and $C' \subsetneq C$. Consider now any such t, and assume that $\sum\limits_{i \in C} w_i(t) < \frac{\lambda_M}{\lambda_M + r} V(C) - \varepsilon$. Suppose that there is $i \in N$ such that i does not approach everyone in C with probability 1 at t. Let D be such that there is a positive probability that D is approached at t by i, and $C \nsubseteq D$. Since $t \leq T_K(\varepsilon)$, $\sum\limits_{i \in C \cap D} w_i(t) \geq \frac{\lambda_M}{\lambda_M + r} V(C \cap D) - \varepsilon$. Then $\sum\limits_{i \in C} w_i(t) < \frac{\lambda_M}{\lambda_M + r} V(C) - \varepsilon$ implies $\sum\limits_{i \in C \setminus D} w_i(t) < \frac{\lambda_M}{\lambda_M + r} V(C) - \frac{\lambda_M}{\lambda_M + r} V(C \cap D)$. Convexity of V then implies $\sum\limits_{i \in C \setminus D} w_i(t) < \frac{\lambda_M}{\lambda_M + r}[V(D \cup C) - V(D)]$. By Claim 2, i could strictly improve her payoff by approaching $D \cup C$ instead of D, a contradiction. Therefore, for any $C \subset N$ for which $|C| \leq K + 1$, $\sum\limits_{i \in C} w_i(t) < \frac{\lambda_M}{\lambda_M + r} V(C) - \varepsilon$ and $t \leq T_K(\varepsilon)$ imply that everyone in C is approached by every player at t with probability 1. By the same argument as in the first step, there exists $T_{K+1}^C(\varepsilon)$ such that $\sum\limits_{i \in C} w_i(t) \geq \frac{\lambda_M}{\lambda_M + r} V(C) - \varepsilon$, $\forall\ t \leq T_{K+1}^C(\varepsilon)$. Then for $T_{K+1}(\varepsilon) = \min \left\{ \min\limits_{C : |C| = K+1} T_{K+1}^C(\varepsilon), T_K(\varepsilon) \right\}$, for every $C \subset N$ with $|C| \leq K + 1$, it holds that $\sum\limits_{i \in C} w_i(t) \geq \frac{\lambda_M}{\lambda_M + r} V(C) - \varepsilon$, $\forall\ t \leq T_{K+1}(\varepsilon)$. $\square$

Proof of Theorem 2: Taking $r \to 0$ in the statement of Lemma 6 yields $\mathcal{C}(V) \subseteq \mathcal{S}(V)$. Taking $r \to 0$ in the statement of Lemma 7 implies that $\lim_{r \to 0} \lim_{t \to -\infty} \sum\limits_{i \in C} w_i(t)$, if defined, cannot be below $V(C)$. This in turn implies $\mathcal{S}(V) \subseteq \mathcal{C}(V)$ if $\mathcal{S}(V)$ is well-defined, which is guaranteed by Theorem 3. ■

11.3 Proof of Theorem 3

STATEMENT (i): For a convex game V and small enough r, there exist a partition P^* of the set of players N, a coalition $C^* \in P^*$, and a time $\widehat{t} < 0$ such that, in MPE, for all $t < \widehat{t}$, $S(t) \equiv S^*$ is constant, $\{C^*, N\} \subseteq S^*$, and every element of S^* is the union of C^* and elements of P^*.

Let $v_C = V(N) - V(N \backslash C)$, the marginal contribution of coalition C to the value of the grand coalition, for any $C \in 2^N$.

Lemma 8: If V is convex, there exists $\widehat{r}$ such that whenever $r \in (0, \widehat{r})$, $\exists\ t$ such that $\forall t' \leq t$, $N \in S(t')$.

Proof of Lemma 8: It is sufficient to show that $\exists\ t$ such that $w_C(t') \leq v_C$, for every $t' < t$ and $C \subset N$. We proceed by induction.

First, suppose $|C| = 1$. Note that conditional on a player being recognized t', player i's expected payoff is $\frac{\lambda_i}{\lambda} \max_{D \ni i}\{V(D) - w_D(t')\} + p_i(t')w_i(t')$, where $p_i(t')$ is the probability that i is included in the proposal (which may or may not be her own) at t'. If $w_i(t') > v_{\{i\}}$, then no one in $N \backslash \{i\}$ would include i in a proposal at t', and player i's expected payoff becomes $\frac{\lambda_i}{\lambda} \max_{D \ni i}\{V(D) - w_D(t')\} + \frac{\lambda_i}{\lambda} w_i(t')$. By Lemma 7, for r low enough, $\exists \delta < 1$ and t_i such that at all $t' < t_i$ where $w_i(t') > v_{\{i\}}$ (and hence no one in $N \backslash \{i\}$ includes i in a proposal), $\max_{D \ni i}\{V(D) - w_D(t')\}$ is small enough that player i's expected payoff is less than $\delta w_i(t')$. This implies that going back in time, w_i will eventually reach $v_{\{i\}}$, and can then never increase from that value. Thus, with low enough r, $\exists\ t^1$ s.t. our claim holds for all $|C| = 1$.

Now let t^m be such that our claim holds for all coalitions of size at most m. Let $|C| = m + 1$, and $t' < t^m$. Note that if $w_C(t') > v_C$, we have $w_C(t') > V(D) - V(D \backslash C)$ for all $D \supset C$. Thus C cannot be entirely included in the proposal of anyone outside C. Moreover, for any $E \subset C$, $w_{C \backslash E}(t') \leq v_{C \backslash E}$ by the induction hypothesis, so $w_E(t') > v_C - v_{C \backslash E} = V(N \backslash (C \backslash E)) - V(N \backslash C) \geq V(D \cup E) - V(D)$ for all $D \subseteq N \backslash C$. Thus no part of C can be included in the proposal of anyone outside C. Due to Lemma 7, by an argument similar to the $|C| = 1$ case, it follows that $\exists\ t^{m+1}$ such that our claim holds for $|C| = m + 1$. □

Lemma 9: Let $D, E \in S(t)$. If V is convex, for all $t \leq 0$:

a) $D \cap E \in S(t)$, and

b) $D \cup E \in S(t)$.

Proof of Lemma 9: Suppose that $D \cap E \notin S(t)$. Then $V(D \cap E) - w_{D\cap E}(t) < V(E) - w_E(t)$, so $w_{E\setminus(D\cap E)}(t) < V(E) - V(D \cap E)$. By convexity, $V(E) - V(D \cap E) \leq V(D \cup E) - V(D)$. Therefore, we have $w_{E\setminus(D\cap E)}(t) < V(D \cup E) - V(D)$, which implies that it is strictly better to approach $D \cup E$ than D, contradicting $D \in S(t)$. Note that this implies $D \cap E \neq \varnothing$ and $\cap_{S\in S(t)} S \in S(t)$.

Similarly, if $D \cup E \notin S(t)$, then $V(D \cup E) - w_{D\cup E}(t) < V(E) - w_E(t)$, so $w_{D\setminus(D\cap E)}(t) > V(D \cup E) - V(E)$. By convexity, $V(D \cup E) - V(E) \geq V(D) - V(D \cap E)$. Therefore, we have $w_{D\setminus(D\cap E)}(t) > V(D) - V(D \cap E)$, which implies that it is strictly better to approach $D \cap E$ than D, contradicting $D \in S(t)$. □

Proof of Statement (i): By Lemma 8, we only consider $r < \widehat{r}$ and t small enough such that $N \in S(t)$. Let $C(t) = \cap_{C\in S(t)} C$, and let $P(t)$ be the coarsest partition of N such that each element of $S(t)$ can be expressed as a union of cells defined by $P(t)$. By Lemma 9a, $C(t) \in P(t)$. Therefore, to prove Theorem 3(i), it suffices to show that $S(t)$ is constant for small enough t. We need two more lemmata for our argument.

Lemma 10: For small enough t, for all $D \in P(t)\setminus\{C(t)\}$, we have $w_D(t) = V(D \cup E) - V(E)$, where $E = \cup_{C\in S(t):D\cap C=\varnothing} C$.

Proof of Lemma 10: By Lemma 9, $E \in S(t)$. It suffices to show that $D \cup E \in S(t)$: once that is done, we have $V(D \cup E) - w_{D\cup E}(t) = V(E) - w_E(t) \Longrightarrow w_D(t) = V(D \cup E) - V(E)$.

By Lemma 8, we take t small enough so that $N \in S(t)$. We proceed by contradiction: suppose $D \cup E \notin S(t)$, so that $D \cup E \neq N$. By the definition of $P(t)$, for any $F \in P(t)$ with $F \subseteq N\setminus(D \cup E)$, there must exist an optimal coalition $C(D, F) \in S(t)$ such that either:

- $D \subset C(D, F)$ and $F \nsubseteq C(D, F)$; or
- $F \subset C(D, F)$ and $D \nsubseteq C(D, F)$.

However, if the latter were the case, we would have $F \subseteq E$. Thus for any $F \in P(t)$ with $F \subseteq N\setminus(D\cup E)$, there exists $C(D, F) \in S(t)$ such that $D \subset C(D, F)$ and $F \nsubseteq C(D, F)$. This implies that $\cap_{F\subseteq N\setminus(D\cup E)} C(D, F) \equiv C(D)$ contains D and no one outside $D \cup E$. Therefore, $C(D) \cup E = D \cup E$, and by Lemma 9, it is an element of $S(t)$. □

Lemma 11: There exists a time to the left of which $V(N) - w_N(\cdot)$ is weakly monotonic.

Proof of Lemma 11: Note that within a time interval $[t_1, t_2]$ where $C(\cdot) = C$, we have:

$$w_C(\tau) = e^{-(\lambda+r)(t_2-\tau)} w_C(t_2) + \int_{\tau}^{t_2} \left[\lambda_C e^{-(\lambda+r)(\tau'-\tau)}(V(C) - w_C(\tau')) + \lambda e^{-(\lambda+r)(\tau'-\tau)} w_C(\tau')\right] d\tau'.$$

Thus, $w'_C(\tau) = r w_C(\tau) - \lambda_C(V(C) - w_C(\tau))$, which implies that $w_C(\cdot)$ is monotonic (increasing if and only if $w_C(\cdot) > \frac{\lambda_C}{r+\lambda_C} V(C)$), so the proposer surplus $V(C) - w_C(\cdot) = V(N) - w_N(\cdot)$

is monotonic. It follows that any non-monotonicity in $V(N) - w_N(\cdot)$ can only occur when $C((\cdot)$ changes.

Suppose $C(t) = C$, so $\max_{D: C(t) \not\subseteq D}\{V(D) - w_D(t)\} < V(C) - w_C(t)$. Because continuation value functions are Lipschitz continuous, $\exists \delta > 0$ such that $C \subseteq C(t')$ for all $t' \in (t-\delta, t+\delta)$. There are therefore two (not mutually exclusive) ways in which $C(\cdot)$ can change at t:

a) for all $\varepsilon > 0$, there exist $t' \in (t, t+\varepsilon)$ and $D \subseteq N \backslash C$ (with $D \neq \varnothing$) such that $C(t') = C \cup D$; and

b) for all $\varepsilon > 0$, there exist $t' \in (t-\varepsilon, t)$ and $D \subseteq N \backslash C$ (with $D \neq \varnothing$) such that $C(t') = C \cup D$.

By continuity of $w_D(.)$, at t, it must be equally attractive to approach C and $C \cup D$. Therefore, we must have $w_D(t) = V(C \cup D) - V(C)$.

Let $f_E(\tau)$ denote the hypothetical derivative of $w_E(\cdot)$ at τ, assuming that at τ, E is always approached and obtains $V(E)$ when one of its members is recognized. We have:

$$
\begin{aligned}
f_{C \cup D}(t) &= (r + \lambda_C + \lambda_D)(w_C(t) + w_D(t)) - (\lambda_C + \lambda_D)V(C \cup D) \\
&= [rw_C(t) - \lambda_C(V(C) - w_C(t))] + [rw_D(t) - \lambda_D(v(C) - w_C(t))] \\
&\quad +(\lambda_C + \lambda_D)(w_D(t) - [V(C \cup D) - V(C)]) \\
&= f_C(t) + [rw_D(t) - \lambda_D(v(C) - w_C(t))].
\end{aligned}
$$

Since $C(t') = C \cup D$, we have $w_D(t') < V(C \cup D) - V(C) = w_D(t)$.

- If we are in scenario a), this implies $w'_D(\tau) = rw_D(\tau) - \lambda_D(v(N) - w_N(\tau)) < 0$ for some $\tau \in (t, t')$. Because $v(N) - w_N(t) = v(C) - w_C(t)$ and by continuity, we must have $rw_D(t) - \lambda_D(v(C) - w_C(t)) \leq 0$. Thus $f_{C \cup D}(t) = f_{C(t')}(t) \leq f_{C(t)}(t)$, where $t' > t$.

- If we are in scenario b), by a similar argument, we have $f_{C(t')}(t) \geq f_{C(t)}(t)$ where $t' < t$.

Going back in time (t' to t in case a, t to t' in case b), we see that the derivative $f_{C(.)}(t)$ cannot decrease from changing $C(\cdot)$. Therefore, $w'_{C(\cdot)}(\cdot)$ cannot jump downward at t when we go back in time. Because $w'_{C(\cdot)}(\cdot)$ maintains its sign when $C(\cdot)$ does not change, it cannot go from positive to negative, going back in time - thus it will eventually maintain its sign all the way to $-\infty$. Because $V(C(\cdot)) - w_{C(\cdot)}(\cdot)$ is always continuous (even when $C(\cdot)$ changes), we conclude that $V(C(\cdot)) - w_{C(\cdot)}(\cdot)$, and hence $V(N) - w_N(\cdot)$, must be monotonic before a certain time. $\square$

Now we conclude the proof of Theorem 3(i) by showing that $S(t)$ is constant before a certain time. As mentioned earlier, throughout our argument, we only consider early enough t such that $N \in S(t)$.

Each $S(t)$ corresponds to a partition $P(t)$ where, by Lemma 10, the continuation value of every cell except $C(t)$ is uniquely determined. Therefore, going back in time, for $S(t)$ to last, it must be that $w'_D(.) = 0$ for all $D \in P(t) \backslash \{C(t)\}$. $w'_D(.)$ depends on $w_D(.)$, which is fixed, the proposer surplus $V(N) - w_N(.)$, and the frequency with which D is approached, which we denote q_D. The set of feasible q_D is limited by finitely many weak linear inequalities corresponding to the structure of $S(t)$ and the fact that when a player is recognized, the sum of her proposal probabilities must equal 1. Therefore the range of $V(N) - w_N(.)$ for which $S(t)$ can be maintained is the union of finitely many closed intervals - denote it $R(S(t))$.

By Lemma 11, $V(N) - w_N(.)$ is monotonic for early enough times - the rest of the argument only considers these times. Since $V(N) - w_N(.)$ is also bounded, it must converge to some limit v^*.

If $V(N) - w_N(.) \to v^*$ from above (which we will assume for the rest of the argument; the other case is symmetric) and $[v^*, V(N) - w_N(t)] \subseteq R(S(t))$ for some t, then $S(t)$ will be maintained forever, and we are done. If not, then at some time before t, $S(.)$ will change. Since $S(.)$ must exist at all times, $\cup_{t' \leq t} R(S(t'))$ must include $[v^*, V(N) - w_N(t)]$. Because N is finite, the set of possible $S(.)$ is finite as well. Therefore, there must exist $\widehat{t} < t$ such that $[v^*, V(N) - w_N(\widehat{t})] \subseteq R(S(\widehat{t}))$, which implies that $S(\widehat{t})$ will be maintained forever. ■

STATEMENTS (ii) AND (iii): For a convex game V, small enough r, and P^*, C^* defined as in Theorem 3(i):

ii) $\lim_{t \to -\infty} w_i(t) = \frac{\lambda_i}{r+\lambda_C} V(C^*)$ for all $i \in C^*$; and

iii) for any $D \in P^*$ with $D \neq C^*$, there exists $E \in S^*$ such that $w_D(t) = V(E) - V(E \backslash D) \equiv w_D$ for all $t < \widehat{t}$, and $\lim_{t \to -\infty} w_i(t) = \frac{\lambda_i}{\lambda_D} w_D$ for all $i \in D$.

Proof of Statement (ii): For each $i \in C^*$ and $t < \widehat{t}$, we have:

$$w_i(t) = e^{-(\lambda+r)(\widehat{t}-t)} w_i(\widehat{t}) + \int_t^{\widehat{t}} \left[\lambda_i e^{-(\lambda+r)(\tau-t)} (V(C^*) - w_{C^*}(\tau)) + \lambda e^{-(\lambda+r)(\tau-t)} w_i(\tau)\right] d\tau.$$

Thus $w'_i(t) = r w_i(t) - \lambda_i (V(C^*) - w_{C^*}(t))$ and $w'_{C^*}(t) = r w_{C^*}(t) - \lambda_{C^*}(V(C^*) - w_{C^*}(t))$. Therefore, $w_{C^*}(t) = (w_{C^*}(\widehat{t}) - \frac{\lambda_{C^*}}{r+\lambda_{C^*}} V(C^*)) e^{(r+\lambda_C)(t-\widehat{t})} + \frac{\lambda_{C^*}}{r+\lambda_{C^*}} V(C^*)$. Algebraic manipulations yield:

$$w_i(t) = [w_i(\widehat{t}) - \frac{\lambda_i}{\lambda_{C^*}} w_{C^*}(\widehat{t})] e^{r(t-\widehat{t})} + \frac{\lambda_i}{\lambda_{C^*}} [w_{C^*}(\widehat{t}) - \frac{\lambda_{C^*}}{r+\lambda_{C^*}} V(C^*)] e^{(r+\lambda_{C^*})(t-\widehat{t})} + \frac{\lambda_i}{r+\lambda_{C^*}} V(C^*). ■$$

Proof of Statement (iii): We make two observations. First, by Lemma 10 and Theorem 3(i), at all times $t < \widehat{t}$ and for all $D \in P^* \backslash \{C^*\}$, we must have $w_D(t) = V(E) - V(E \backslash D) \equiv w_D$

for some $E \in S^*$. Thus, by similar computations as in the proof of statement (ii),

$$w'_D(t) = -\lambda_D(V(C^*) - w_{C^*}(t)) - \lambda p_D(t) w_D + (\lambda + r) w_D = 0,$$

where $p_D(t)$ is the probability that a proposal at time t includes D. Second, for all $i \in D$, we have:

$$w'_i(t) = -\lambda_i(V(C^*) - w_{C^*}(t)) - \lambda p_D(t) w_i(t) + (\lambda + r) w_i(t).$$

Combining these two observations yields $w'_i(t) = (V(C^*) - w_{C^*}(t))(\frac{\lambda_D}{w_D} w_i(t) - \lambda_i)$. Since $V(C^*) - w_{C^*}(t) = V(N) - w_N(t) > 0$, $w_i(.)$ is monotonic as $t \to -\infty$, so it converges. Since $\lim_{t\to-\infty}(V(C^*) - w_{C^*}(t)) > 0$, we must have $\lim_{t\to-\infty} \frac{\lambda_D}{w_D} w_i(t) - \lambda_i = 0$, as desired. ■