Almost Fully Revealing Cheap Talk with Imperfectly Informed Senders*

Attila Ambrus[†] and Shih En Lu[‡]

This Version: October 2014

Abstract

We show that in multi-sender communication games where senders imperfectly observe the state, if the state space is large enough, then there can exist equilibria arbitrarily close to full revelation of the state as the noise in the senders' observations gets small. In the case of replacement noise, where the senders observe the true state with high probability, we show this under mild assumptions, for both unbounded and large bounded state spaces. In the case of continuous noise, where senders observe a signal distributed continuously over a small interval around the true state, we establish this for unbounded state spaces. The results imply that when there are multiple experts from whom to solicit information, if the state space is large, then even when the state is observed imperfectly, there are communication equilibria that are strictly better for the principal than delegating the decision right to one of the experts.

JEL Codes: C72, D82, D83

Keywords: Imperfect Information, Cheap Talk, Multi-senders, Communication

---

*DOI: dx.doi.org/10.1016/j.geb.2014.09.001. © 2014. This manuscript version is made available under the CC-BY-NC-ND 4.0 license http://creativecommons.org/licenses/by-nc-nd/4.0/. Previous versions of the paper were circulated under the title "Robust Almost Fully Revealing Equilibria in Multi-Sender Cheap Talk".

[†]Department of Economics, Duke University, Durham, NC 27708, USA; email: attila.ambrus@duke.edu

[‡]Department of Economics, Simon Fraser University, Burnaby, BC V5A 1S6, Canada; email: shihenl@sfu.ca

1 Introduction

In sharp contrast to the predictions of cheap talk models with a single sender (Crawford and Sobel (1982), Green and Stokey (2007)), if a policymaker has the chance to consult multiple experts who each observe the state, there exist equilibria in which the policymaker always learns the true state. This observation was first made by Krishna and Morgan (2001a), while Battaglini (2002) gives necessary and sufficient conditions for the existence of such fully revealing equilibria. An important implication of these results is that with a large enough state space, under the best equilibrium for the sender, retaining the decision right and consulting multiple experts is superior to delegating the decision power to one of the informed agents. In contrast, as Dessein (2002) shows, the best outcome from communicating with one expert can be strictly worse than delegating the decision to the expert. In particular, if the expert's bias is small enough, delegation is optimal.[1]

This paper revisits the above questions by departing from the assumption that each sender observes the state exactly, and instead investigating a more realistic situation in which senders observe the state with a small amount of noise.[2] There are reasons to think that in such an environment, the qualitative conclusions from multi-sender cheap talk games would significantly change. In particular, some of the equilibrium constructions provided in the literature require the senders to exactly reveal the true state, and punish individual deviations by rendering an action that is bad for both senders in the case of nonmatching reports. The latter is possible because after an out-of-equilibrium profile of messages by the experts, nothing restricts the beliefs of the receiver. Such constructions obviously break down if the experts observe the state with noise, however small, since nonmatching reports then occur along the equilibrium path. Indeed, Battaglini (2002) shows that for a particular type of noise structure, in a 1-dimensional state space, there cannot exist any fully revealing equilibrium with two senders biased in opposite directions. However, it is not investigated how close information revelation can get to full revelation.

In this paper, we show that if the state space is large enough (relative to the experts' biases), then for types of noise structures commonly used in the literature, there exist equilibria arbitrarily close to full revelation of the state as the noise in the senders' observations vanishes. We consider two common types of noise. The first one is *replacement noise*, where each sender observes the true state with high probability, but observes the realization of a random variable independent of the state with low probability. In such a context, we

[1] In the political science literature, Gilligan and Krehbiel (1987) establishes a similar point in the context of legislative decision-making.

[2] Relatedly, Krehbiel (2001) addresses the issue of empirical plausibility of equilibria in a multi-sender cheap talk context.

show that with one round of simultaneous cheap talk, as the probabilities of observing the true state approach 1, there exist equilibria arbitrarily close to full revelation under weak conditions on the noise structure. Then, we investigate the case of *continuous noise*, where the senders' signals, conditional on the state, follow a distribution with bounded support. Constructing equilibria with fine information revelation is more involved in this environment, as it is a 0-probability event that the senders' observations coincide. Here, we show that if the state space is unbounded, then under mild assumptions, there exists a fully revealing equilibrium. For large bounded state spaces, we provide a more limited result, showing that if there are two rounds of communication, then under certain restrictions on the noise structure, there exists an equilibrium in which one of the senders' signal is fully revealed.

The starting point for our basic construction is an equilibrium in the noiseless limit game where, for any message m_1 sent by sender 1 and any message m_2 sent by sender 2 along the equilibrium path, there is a set of states *with positive measure* where the prescribed message profile is (m_1, m_2). Put simply, any combination of messages that are used in equilibrium are on the equilibrium path, as in the multi-dimensional construction of Battaglini (2002), and the receiver's action following any of these message profiles can be determined by Bayes' rule. A new feature of our construction is that any pair of equilibrium messages is sent with strictly positive probability.

The above type of construction constitutes an equilibrium whenever the set of states where (m_1', m_2) or (m_1, m_2') is prescribed (for $m_1' \neq m_1$ and $m_2' \neq m_2$) is far away, relative to the senders' biases, from the set of states where (m_1, m_2) is prescribed. We show that even when the set of states corresponding to each message pair is small, the above condition can be satisfied when the state space is sufficiently large. Thus, the action taken by the receiver can be made arbitrarily close to the true state with *ex ante* probability arbitrarily close to 1, if the state space is large enough. In such equilibria, the receiver's expected utility is arbitrarily close to her expected utility in the case of full revelation of the state.

When the state space is a large bounded interval of the real line, we propose the following construction. The state space is partitioned into n^2 intervals ("cells") with equal size, for some large n corresponding to the number of each sender's equilibrium messages. The $n \times n$ different combinations of the equilibrium messages are then assigned such that any two cells in which a sender sends the same message are far from each other. Essentially, cells are labeled with 2-digit numbers in a base-n number system in a particular way, and one sender is supposed to report the first digit of the interval from which she received a signal, while the other sender is supposed to report the second digit of the interval from which she received a signal. The construction relies on the fact that if the state space is large, then n can be taken to be such that $\frac{1}{n^2}$ times the length of the state space (the size of the cells) is small, but $\frac{1}{n}$ times the length of the state space (which is roughly the distance between cells in which a sender sends the same message) is large.

In the proposed equilibrium, the receiver solicits different pieces of information from the senders. This is done by transforming the 1-dimensional state space to a 2-dimensional one, through discretization and rearrangement, such that each message is of very limited use, but their combination reveals the state with high precision.[3] For this reason, we view our contribution as more normatively relevant, for situations where the policymaker can propose a mechanism, but cannot commit to action choices (so that the latter have to be sequentially rational). We note that for a fixed bounded state space, our constructions do not necessarily yield the best equilibrium for the receiver. However, if the state space is large enough relative to the biases, they provide a recipe to construct equilibria close to full revelation of information, irrespective of the fine details of the game (distribution of the noise, preferences of the senders), for a remarkably large class of games. In particular, the constructions allow for state-dependent biases for the senders. Moreover, we do not require either the single-crossing condition on the senders' preferences that is usually assumed in the literature, or the assumption that the *sign* of a sender's bias remains constant over the state space. Thus, our results hold for games outside the Crawford and Sobel framework.

In the case of continuous noise, the above construction breaks down, as even for very small noise, in states near a boundary between two cells, senders can be highly uncertain about the cell where the other sender's signal lies. Senders getting signals right around cell boundary points may therefore deviate from prescribed play, which can cause the prescribed strategy profile to unravel far from the boundary points.

Nevertheless, when the supports of the senders' signals are bounded conditional on a state realization, and the state space is unbounded, it is possible to construct a fully revealing equilibrium. The key condition for the result is that the support of a sender's signal conditional on the state strictly increases (that is, both boundaries of the interval support increase), at a rate bounded away from zero. This is a mild requirement, and in particular, it is satisfied when each sender's signal is the sum of the true state and an independent noise term with interval support, a standard specification in the global games literature (starting with Carlsson and van Damme (1993) and Frankel *et al.* (2003)). If this condition holds, then provided that the other sender truthfully reports her signal, any misreporting by a sender results with positive probability in a pair of signal reports that are not compatible (cannot realize in any state). An unbounded state space makes it possible to specify out-

[3] Ivanov (2012) proposes an optimal mechanism with similar features in a communication game with one sender, in which the receiver can endogenously determine the type of information the sender can learn, and there can be multiple rounds of learning and communication.

of-equilibrium beliefs for the receiver that lead to actions very far away from the reported signals if they are incompatible. In fact, we provide a construction that yields an expected payoff of $-\infty$ to a sender after any misreporting. This construction crucially relies on both the boundedness of the signal distributions conditional on a state, and on the unboundedness of the state space. It is similar to the equilibrium provided in Mylovanov and Zapechelnyuk (2013) in that deviations from truthtelling induce the harshest possible punishment, but our construction does not require commitment power on the side of the receiver (the action choices are sequentially rational), and it is in an environment with noisy state observations and unbounded state space.

For bounded state spaces, we provide a construction that requires two rounds of public communication by the senders and, under certain restrictions on the noise structure, constitutes an equilibrium that fully reveals the signal of one of the senders. In the second round, this equilibrium prescribes a strategy profile very similar to the construction described above for replacement noise, involving combinations of messages from the senders identifying cells in a partition of the state space. The main complications are that the sizes of the partition cells vary in a specific way that depends on the noise structure, and that instead of one fixed partition, there is a continuum of partitions that can be played along the equilibrium path. The partition used in the second round is announced in the first round by sender 1. In particular, for every signal s_1 that player 1 can receive, there is exactly one equilibrium partition with a cell exactly consisting of the support of sender 2's signal s_2 conditional on s_1. We show that for large state spaces, we can take the loss from a coordination failure high enough so that sender 1 chooses to announce this partition in order to ensure successful coordination.

An interesting feature of this construction is that even though, initially, no small subset of the state space is common p-belief between the senders for positive p, after the first-round communication, it becomes a common 1-belief between the senders that both of their signals and the true state are in a small interval. This aspect of our construction is potentially relevant in games outside the sender-receiver framework. In particular, the infection arguments used in global games rely on the nonexistence of nontrivial events that are common p-belief among players, for p close enough to 1, as described for example in Morris *et al.* (1995).[4] We show that strategic communication can create such events, albeit in a different type of game: in global games with cheap talk, the communication stage is followed by actions from the senders of messages, while in our game they are followed by an action from a third party. As far as we know, existing work on global games with pre-play cheap talk (Baliga and Morris (2002), Baliga and Sjostrom (2004), Acharya and Ramsay (2013); see also p71 in

[4] The concept of common p-belief for $p < 1$ was introduced by Monderer and Samet (1989).

the survey paper Morris and Shin (2003)) only consider one round of communication. Our analysis suggests the potential importance of considering the possibility of multiple rounds of communication (as well as mediated communication), in related games.[5]

The closest papers to ours in the literature investigating cheap talk communication between multiple senders and a receiver are Battaglini (2002), Battaglini (2004), Ambrus and Takahashi (2008), Eső and Fong (2008) and Lu (2014).[6] Battaglini (2002) shows the nonexistence of fully revealing equilibria for replacement noise in one-dimensional state spaces, but does not address the question of how close equilibria can get to full revelation. We show that for large state spaces, there are such equilibria arbitrarily close to full revelation. To reconcile these results, note that a finer and finer interval partition of the real line segment, such as the one in our construction, does not have a well-defined limit as the sizes of the intervals in the partition go to zero. Correspondingly, even in the noiseless limit game, we only consider equilibria close to full revelation (in which the receiver only learns that the state is from a small interval around the true state), and show that there exist such equilibria robust to small amounts of replacement noise. Battaglini (2004) shows the existence of a fully revealing equilibrium in a specific model with continuous noise, if the state space is a multi-dimensional Euclidean space and the prior distribution is diffuse. This result requires restrictive assumptions, and in particular does not extend to situations where the prior distribution is proper. Ambrus and Takahashi (2008) focus on the case of perfectly informed senders, but also show the nonexistence of fully revealing equilibria that satisfy a robustness criterion (diagonal continuity), indirectly motivated by noisy state observations, for compact state spaces. The equilibria in our paper do not satisfy diagonal continuity, with the exception of the construction for bounded continuous noise and unbounded state space. However, diagonal continuity is not a natural requirement for either games with replacement noise, where senders receive the same signals with high probability, or in games with multiple rounds of communication, where the first round of communication can establish common p-belief (for p close to 1) between the senders that the state is in a small interval of the state

[5] In cheap talk games without noise, it has been shown that adding rounds of communication can improve information transmission; see for example Krishna and Morgan (2001b) and Krishna and Morgan (2004). Golosov *et al.* (2014) considers a sender-receiver game in which there are multiple rounds of action choices, besides communication. For mediation between one sender and one receiver, see Goltsman *et al.* (2009), Ivanov (2010) and Ambrus *et al.* (2013). There is also an earlier literature investigating what communication equilibria between multiple senders and a receiver can be replicated without a mediator; see for example Forges (1988).

[6] For multi-sender cheap talk games in which the senders observe the state with substantial noise, see Austen-Smith (1990a, 1990b, 1993), Wolinsky (2002) and Ottaviani and Sørensen (2006). McGee and Yang (2013) present a model of multi-sender cheap talk in which different senders observe different dimensions of the state. Lai *et al.* (2014) and Vespa and Wilson (2012) are recent experimental contributions on multi-sender cheap talk games.

space - as in our equilibrium construction. Eső and Fong (2008) analyze a continuous-time dynamic multi-sender game with discounting and construct a fully revealing equilibrium that is robust to replacement noise under certain assumptions, including that the receiver is more patient than the senders.

Lu (2014) takes the approach that the *ex ante* distributions of senders' observations may not be common knowledge even while there is common knowledge that the senders' observations are near the state. He shows that, generically, the only equilibria of the noiseless game that are robust to such situations (in the sense that there exists a "nearby" strategy profile where all players play approximate best responses as evaluated under their own belief about the noise) are multi-sender analogs of the one-sender Crawford and Sobel (1982) equilibria - and therefore do not approach full revelation. By contrast, this paper examines equilibria in the noisy game, where noise is commonly known. Therefore, taken together, these two papers imply that common knowledge of the noise structure can significantly enhance information transmission.

The rest of the paper is organized as follows. In Section 2, we introduce the basic model and some terminology. In Section 3, we establish our main results for one-dimensional state spaces in games with replacement noise. In Section 4, we examine the case of continuous noise. Finally, in Section 5, we discuss extensions of the model. Specifically, we describe how some of our results extend to multidimensional state spaces, to discrete state spaces, to models in which noise is also introduced at other points in the game, and to situations in which the receiver has commitment power.

2 Model

The model features two senders, labeled 1 and 2, and one receiver. The game starts with sender 1 observing signal s_1 and sender 2 observing signal s_2 of a random variable $\theta \in \Theta$, which we call the *state*. We refer to Θ as the *state space*, and assume that it is a closed and connected subset (not necessarily proper) of $\mathbb{R}$.[7] The prior distribution of θ is given by F, which we assume exhibits a density function f that is strictly positive and continuous on Θ.[8]

We will consider both games in which the senders observe the state perfectly (*noiseless limit games*, where $s_1 = s_2 = \theta$), and games in which senders observe the state with small noise, for two types of noise structures: *replacement noise* (Section 3), where each sender

[7] Closedness is assumed for notational convenience only. None of the results depend on this assumption. For an extension of the model to multi-dimensional state spaces, see Section 5.

[8] Although we assume a proper prior distribution throughout, our results from Section 3 readily extend to the case where the state space is an unrestricted Euclidean space and the prior is diffuse, as in Battaglini (2004).

observes the true state with high probability, and *bounded continuous noise* (Section 4), where each sender observes a signal that follows a continuous distribution around the state.

After observing their signals, the senders simultaneously send public messages $m_1 \in M_1$ and $m_2 \in M_2$. We assume that M_1 and M_2 are Borel sets having the cardinality of the continuum. In the baseline game, after observing the above messages, the receiver chooses an action $y \in \mathbb{R}^d$, and the game ends. In Section 4, we consider an extended version of this game, with an additional round of cheap talk. Formally, after the senders send public messages m_1 and m_2, they send another pair of messages $m_1' \in M_1$ and $m_2' \in M_2$. After observing the sequence of message pairs (m_1, m_2), (m_1', m_2'), the receiver chooses an action and the game ends.

We assume that the receiver's utility function $v(\theta, y)$ is continuous, strictly concave in y, and that $v(\theta, \cdot)$ attains its maximum value of 0 at $y = \theta$. We also assume that sender i's utility function $u_i(\theta, y)$ is continuous, that it is strictly concave in y, and that $u_i(\theta, \cdot)$ attains its maximum value of 0 at $y = \theta + b_i(\theta)$. We refer to $\theta + b_i(\theta)$ as sender i's ideal point at state θ, and to $b_i(\theta)$ as sender i's bias at state θ. Note that neither the signals or the messages directly enter the players' utility functions.

We also maintain the following two assumptions throughout the paper.

A1: For every $y \in \mathbb{R}$, $\int_{\mathbb{R}} f(\theta) v(\theta, y) d\theta$ is finite.

A2: For any $\delta \geq 0$, there exists $K(\delta) > 0$ such that, for any $\theta \in \Theta$, $u_i(\theta, a') < u_i(\theta, a)$ whenever $|a - \theta| \leq \delta$ and $|a' - \theta| \geq K(\delta)$, $\forall\ i = 1, 2$.[9]

A1 requires that the expected utility of the receiver from choosing any action is well-defined under the prior. A2 posits that neither sender becomes infinitely more sensitive to the chosen action being in some directions from the true state than in other directions. In the case of symmetric loss functions around ideal points, which is assumed in most of the literature, A2 is equivalent to requiring that there is a universal bound on the magnitude of senders' biases - without it, the biases could tend to infinity as $\theta \to \pm\infty$. The assumption automatically holds in the case of state-independent biases assumed, for example, in Battaglini (2002, 2004).

The solution concept we use is weak perfect Bayesian equilibrium, defined in the context of our model as follows.

For the baseline game with one communication round, let $H(\theta, s_1, s_2)$ be the c.d.f. of the joint probability distribution of the state and the sender's signals, and for $i \in \{1, 2\}$, let $H_i^{s_i}$

[9] A2 is not implied by u_i being strictly concave in the action. For example, take $\delta = 0$. While a constant K such that $u_i(\theta, a') < u_i(\theta, \theta)$ whenever $|a' - \theta| \geq K$ automatically exists for *each* θ, without A2, there would be no guarantee that we can require the same finite K to apply for *all* θ.

be the marginal distribution of (θ, s_{-i}) conditional on s_i. An *action rule* of the receiver is a measurable function $y : M_1 \times M_2 \to \mathbb{R}$, and a *belief rule* of the receiver is a measurable function $\mu : M_1 \times M_2 \to \Delta(\Theta)$. For every $i \in \{1, 2\}$, sender i's *signaling strategy* is a measurable function $m_i : \Theta \to M_i$.[10]

Definition: Action rule $\widehat{y}$, belief rule $\widehat{\mu}$, and signaling strategies $\widehat{m}_i$ $(i \in \{1, 2\})$ constitute a pure strategy weak perfect Bayesian Nash equilibrium if:

(1) $\forall\, i \in \{1, 2\}$ and $s_i \in \Theta$, $\widehat{m}_i(s_i)$ solves $\max\limits_{m_i \in M_i} \int\limits_{(\theta, s_{-i}) \in \Theta^2} u_i(\theta, \widehat{y}(m_i, \widehat{m}_{-i}(s_{-i})))dH_i^{s_i}$,

(2) $\forall\, (m_1, m_2) \in M_1 \times M_2$, $\widehat{y}(m_1, m_2)$ solves $\max\limits_{y \in \mathbb{R}} \int\limits_{\theta \in \Theta} v(\theta, y)d\widehat{\mu}(m_1, m_2)$,

(3) $\widehat{\mu}(m_1, m_2)$ is obtained from $\widehat{m}_1(\cdot)$ and $\widehat{m}_2(\cdot)$ by Bayes' rule, whenever possible.

We use this weak notion of perfect Bayesian Nash equilibrium mainly because there is no universally accepted definition of perfect Bayesian Nash equilibrium with continuous action spaces. We note that in the equilibrium constructed for Section 3, there are no out-of-equilibrium message pairs, and Bayes' rule pins down the receiver's beliefs after any possible message pair. Such equilibria satisfy the requirements of any reasonable definition of perfect Bayesian equilibrium.

We henceforth refer to weak perfect Bayesian Nash equilibrium simply as equilibrium.

3 Replacement Noise

Throughout this section, we assume that the senders observe the state with replacement noise, defined as follows.

Definition: In a *game with replacement noise*, there are random variables $\tau_1, \tau_2 \in \Theta$, independent of each other and of θ, and distributed according to c.d.f. G with a continuous density function g strictly positive on Θ. Then, conditional on any $\theta \in \Theta$, $s_i = \begin{cases} \theta \text{ with probability } p \\ \tau_i \text{ with probability } 1-p \end{cases}$ for $i \in \{1, 2\}$, for some $p \in (0, 1)$.[11]

3.1 Large Bounded State Space

First, we consider the case where $\Theta = [-T, T]$ for some $T \in \mathbb{R}_{++}$, and show that for every $\varepsilon, \delta > 0$, if T is large enough and the noise parameter is low enough, then there exists an

[10] Since we only construct pure strategy equilibria, we do not formally introduce mixed strategies here.

[11] We only assume that τ_1 and τ_2 are independent of each other for expositional simplicity. The results can be extended for arbitrary correlation between the two variables.

equilibrium of the cheap talk game in which, at every state, the probability that the distance between the induced action and the state is smaller than δ is at least $1-\varepsilon$.

To establish this result, we consider the following signaling profile for the senders. For any $T \geq K(\delta)$, let $n_{\delta,T}$ be the largest integer such that $\frac{T}{n_{\delta,T}} \geq K(\delta)$. Partition Θ to $n_{\delta,T}$ equal intervals, to which we will refer as *blocks*. Note that the size of each block is $\frac{2T}{n_{\delta,T}}$, which is by construction between $2K(\delta)$ and $4K(\delta)$. Next, further partition each block into $n_{\delta,T}$ equal subintervals, to which we will refer as *cells*. We will use $I_{j,k(i,j)}$ to denote the jth cell in the ith block, where $k(i,j) = \left\{ \begin{smallmatrix} i+j-1 \text{ if } i+j-1 \leq n \\ i+j-1-n \text{ if } i+j-1>n \end{smallmatrix} \right.$. Thus, block i is partitioned into the following $n_{\delta,T}$ cells: $\{(1,i),(2,i+1),...,(n_{\delta,T}-i+1,n_{\delta,T}),(n_{\delta,T}-i+2,1),...,(n_{\delta,T},i-1)\}$, and there is a total of $n_{\delta,T}^2$ cells. For completeness, assume that the cells in the partition are closed on the left and open on the right, with the exception of cell $I_{n_{\delta,T},n_{\delta,T}}$, which is closed at both ends. Define signaling profile $(m_1^{\delta,T}, m_2^{\delta,T})$ such that for every $j,k \in \{1,...,n_T\}$, after receiving signal $s_1 \in I_{j,k}$, sender 1 sends message m_1^j, and after receiving signal $s_2 \in I_{j,k}$, sender 2 sends message m_2^k. Figure 1 illustrates this signaling profile.

Let $y^{\delta,T}$ be an action rule that maximizes the receiver's expected payoff given $(m_1^{\delta,T}, m_2^{\delta,T})$. Note that $y(m_1^j, m_2^k)$ is uniquely defined for $j,k \in \{1,...,n_{\delta,T}\}$ for any noise structure we consider, since the conditional beliefs of the receiver after receiving such message pairs are given by Bayes' rule, and the receiver's utility function is strictly concave. As for out-of-equilibrium messages $m_i \neq m_i^j$ for all $j \in \{1,...,n_{\delta,T}\}$, assume that the receiver interprets each as having the same meaning as some message sent in equilibrium. No sender will then have an incentive to deviate to an out-of-equilibrium message.

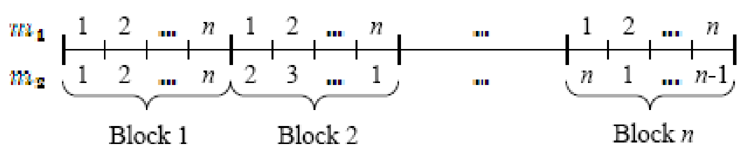

Figure 1: Signaling profile for large bounded intervals

Proposition 1: For every $\delta > 0$, there exists $T(\delta) > 0$ such that if $T > T(\delta)$, then strategy profile $(m_1^{\delta,T}, m_2^{\delta,T}, y^{\delta,T})$ constitutes an equilibrium in the noiseless limit game, and for every $\theta \in \Theta$, we have $|y^{\delta,T}(m_1^{\delta,T}(\theta), m_2^{\delta,T}(\theta)) - \theta| < \delta$.

Proof: By construction, the receiver plays a best response in the proposed profile, so we only need to check the optimality of the senders' strategies.

Note that $n_{\delta,T} \to \infty$ as $T \to \infty$. Since, by construction, $\frac{2T}{n_{\delta,T}} \leq 4K(\delta)$ for any $T \geq K(\delta)$, the above implies that the cell size, $\frac{2T}{n_{\delta,T}^2}$, goes to 0 as $T \to \infty$. Note that for every $j,k \in$

$\{1, ..., n_{\delta,T}\}$ and $\theta \in I_{j,k}$, the assumptions on v imply that if both senders play according to the prescribed profile, then the action induced at θ lies within $I_{j,k}$.

Also by construction, if the other sender plays the prescribed strategy, all other actions that a sender could induce by sending a different message than prescribed are more than $\frac{n_{\delta,T}-2}{n_{\delta,T}}$ times the block size away. The latter is by construction at least $2K(\delta)$, so if $n_{\delta,T} > 4$, those actions are more than $K(\delta)$ away.

The above imply that there exists $T(\delta) > 0$ such that $|y^{\delta,T}(m_1^{\delta,T}(\theta), m_2^{\delta,T}(\theta)) - \theta| < \delta$ if $T > T(\delta)$, and any deviation by a sender, given strategy profile $(m_1^{\delta,T}, m_2^{\delta,T}, y^{\delta,T})$, would induce an action y by the receiver such that $|u - \theta| > K(\delta)$. By the definition of $K(\delta)$, this implies that there is no profitable deviation by either sender. ■

Intuitively, the proposed construction is an equilibrium because the cell associated with message pair (m_1^j, m_2^k) for any $j, k \in \{1, ..., n_{\delta,T}\}$ is far away from any cell in which the prescribed message pair is either (m_1, m_2^k) with $m_1 \neq m_1^j$, or (m_1^j, m_2) with $m_2 \neq m_2^k$. This holds for large T even given a small cell size, which ensures that the distance between states and induced actions is small.

Next, we show that if noise parameter $1-p$ is small enough, then profile $(m_1^{\delta,T}, m_2^{\delta,T}, y^{\delta,T})$ remains an equilibrium in a game with replacement noise.

Proposition 2: Suppose $\delta > 0$ and $T > T(\delta)$. Then for any noise distribution G, there exists $\underline{p}(G) < 1$ such that $p > \underline{p}(G)$ implies that in a game with replacement noise structure (G, p), strategy profile $(m_1^{\delta,T}, m_2^{\delta,T}, y^{\delta,T})$ constitutes an equilibrium.

Proof: Note that since both f and g are continuous and strictly positive on the compact Θ, as $p \to 1$, given signaling strategies $(m_1^{\delta,T}, m_2^{\delta,T})$, the conditional distribution of θ given message pair (m_1^j, m_2^k) in the game with replacement noise, denoted $\mu(\theta, m_1^j, m_2^k)$, converges weakly to the conditional distribution of θ given message pair (m_1^j, m_2^k) in the noiseless limit game, for every $j, k \in \{1, ..., n_{\delta,T}\}$. To see this, note that letting $f(m_1^j, m_2^k) = \int_{I_{m_1^j, m_2^k}} f(\theta) d\theta$, we have

$$
\begin{aligned}
\mu(\theta, m_1^j, m_2^k) &= \frac{\int_{I_{m_1^j,m_2^k}} h(\theta, s_1, s_2) ds_1 ds_2}{\int_\Theta \int_{I_{m_1^j,m_2^k}} h(\theta, s_1, s_2) ds_1 ds_2 d\theta} \\
&= \begin{cases} \frac{(p^2+2p(1-p)f(m_1^j,m_2^k)+(1-p)^2 f(m_1^j,m_2^k)^2) f(\theta)}{D(m_1^j,m_2^k,p)} & \text{if } \theta \in I_{m_1^j,m_2^k} \\ \frac{(p(1-p)f(m_1^j,m_2^k)+(1-p)^2 f(m_1^j,m_2^k)^2) f(\theta)}{D(m_1^j,m_2^k,p)} & \begin{array}{c}\text{if } \theta \in I_{m_1^j,m_2} \text{ for some } m_2 \neq m_2^k, \\ \text{or } \theta \in I_{m_1,m_2^k} \text{ for some } m_1 \neq m_1^j\end{array} \\ \frac{(1-p)^2 f(m_1^j,m_2^k)^2 f(\theta)}{D(m_1^j,m_2^k,p)} & \text{otherwise} \end{cases},
\end{aligned}
$$

where $D(m_1^j, m_2^k, p)$ is the integral over Θ of the numerators. As $p \to 1$, the numerator in each of the latter two lines $\to 0$, while the numerator in the first line $\to f(\theta)$.

Then since the expected payoff of the receiver resulting from choosing some action y after message pair (m_1^j, m_2^k) is continuous with respect to the weak topology in the conditional distribution of θ given (m_1^j, m_2^k), the theorem of the maximum implies that $y^{\delta,T}$ is continuous in p, even at $p = 1$. This implies that the expected payoff of sender i resulting from sending message m_i^l after receiving signal s_i is continuous in p, for every $i \in \{1, 2\}$, $l \in \{1, ..., n_{\delta,T}\}$ and $s_i \in \Theta$, even at $p = 1$. Moreover, in the noiseless limit game, after signals $s_1, s_2 \in I_{j,k}$, sending message m_1^j yields a strictly higher expected payoff for sender 1 than m_1^l for $l \neq j$, and sending message m_2^k yields a strictly higher expected payoff for sender 2 than m_2^l for $l \neq k$. Thus, the same holds for p close enough to 1. This establishes the claim. ■

The intuition behind Proposition 2 is that the receiver's optimal action rule given $(m_1^{\delta,T}, m_2^{\delta,T})$ is continuous in p, even at $p = 1$. Therefore, the expected payoff of a sender when sending different messages after a certain signal changes continuously in p as well. Since in the noiseless limit game, a sender strictly prefers to send the prescribed message to sending any other equilibrium message, the same holds for noisy games with p high enough.

Note that the above propositions imply that for any $\delta > 0$, if the state space is large enough, then there is an equilibrium of the noiseless limit game, where the action induced at any state is at most δ away from the state, that is robust to replacement noise in a strong sense: it can be obtained as a limit of equilibria of games with vanishing replacement noise, for any noise distribution G.

The propositions also imply the following result.

Corollary 1: Fix payoff functions $v(.,.)$ and $u_i(.,.)$, $i = 1, 2$, defined over $\mathbb{R}^2$ satisfying A1. Take any sequence of games with bounded interval state spaces $[-T_1, T_1]$,$[-T_2, T_2]$,... , state distributions $F_1, F_2, ..$, noise distributions G_1,G_2,... and payoff functions (v^1, u_1^1, u_2^1), (v^2, u_1^2, u_2^2), ... such that v^j, u_1^j and u_2^j are restrictions of v, u_1 and u_2 to $[-T_j, T_j] \times \mathbb{R}$. If $T_i \to \infty$ as $i \to \infty$, then there exists a sequence of noise levels $p_1, p_2, ...$ with $p_i < 1$ for every $i \in \mathbb{Z}_{++}$ and $p_i \to 1$ as $i \to \infty$, such that there is a sequence of equilibria of the above games with equilibrium outcomes converging to full revelation in $\mathbb{R}$.

Corollary 1 contrasts with Proposition 2 in Battaglini (2002), which establishes that if the senders' biases are above some threshold, then there does not exist a fully revealing equilibrium robust to replacement noise in a one-dimensional state space, no matter how large the state space.[12] To reconcile these results, it is useful to observe that although the

[12] Battaglini's result is stated in an environment where each player's loss depends only on $(a - b_i(\theta))^2$, and

sequence of outcomes induced by the sequence of equilibria from Corollary 1 converges to full revelation of the state in $\mathbb{R}$, such sequences of equilibrium strategy profiles do not have a well-defined limit in the noiseless limit game with state space $\mathbb{R}$. This is because the limit of a sequence of interval partitions where the sizes of the intervals converge to zero is not well-defined.

3.2 Unbounded State Space

In this subsection, we analyze the case where $\Theta = \mathbb{R}$. We show that the equilibrium construction introduced in the previous subsection can be extended to this case when the prior distribution of states has thin enough tails.

The state space is still partitioned into n^2 cells, and combinations of the n equilibrium messages are allocated to different cells in the same order as before. The difference is that in the case of an unrestricted state space, only the middle $n^2 - 2$ cells can be taken to be small; the extreme cells are infinitely large. Hence, in the equilibria we construct, even with no noise, the implemented action will be far away from the state with nontrivial probability in states in the extreme cells. But if the profile is constructed such that the middle $n^2 - 2$ cells cover interval $[-T, T]$ for large enough T, then for small noise, the *ex ante* probability that the induced action is within a small neighborhood of the realized state can be made close to 1.

The extra assumption needed for this construction guarantees that for large enough block size, even in the extreme cells, the senders prefer inducing the action corresponding to the cell instead of deviating and inducing an action in a different block.

Let $y_{\theta,d,L}$ (respectively, $y_{\theta,d,R}$) be the optimal action for the receiver when her belief about the true state follows density d truncated to $(-\infty, \theta]$ (respectively, $[\theta, \infty)$).

A3: There exist $C, Z > 0$ such that $|y_{\theta,f,L} - \theta| < Z$ for all $\theta < -C$, and $|y_{\theta,f,R} - \theta| < Z$ for all $\theta > C$.

A3 requires the tail of the prior distribution to be thin enough: it is satisfied if f converges quickly enough to 0 at $-\infty$ and ∞, relative to how fast the loss functions at moderate states diverge to infinity as the action goes to $-\infty$ or ∞. For example, if v exhibits quadratic loss invariant in θ, a sufficient condition for A3 is that $\lim_{x\to-\infty} \frac{F(x)}{f(x)}$ and $\lim_{x\to\infty} \frac{1-F(x)}{f(x)}$ exist. This is clearly true if f converges to 0 exponentially fast, so for v quadratic, A3 holds

does so in the same way in every state. However, its proof extends to our more general setting if for every θ, each sender's bias is large enough (in the sense that if $u_i(\theta, \theta) = u_i(\theta, a)$ and $a \neq \theta$, then $|a - \theta|$ must be large), and if the extent to which the receiver's payoff's sensibility to $|a - \theta|$ varies across states is bounded.

for exponential distributions or for any distribution that converges to 0 faster, such as the normal.

Proposition 3: Suppose $\frac{f(\theta)}{g(\theta)} \geq b > 0$, for every $\theta \in \Theta$. If $\Theta = \mathbb{R}$ and A3 holds, then for every $\delta, \eta > 0$, there exists $\underline{p} < 1$ such that, in a noisy game with $p > \underline{p}$, $|y - \theta| < \delta$ with *ex ante* probability at least $1 - \eta$.

Proof: Consider the following strategy profile in the noiseless limit game.

Let T be such that $F(T) - F(-T) = 1 - \frac{\eta}{2}$. As in Section 3.1, partition $\mathbb{R}$ into n blocks. Blocks 2 through $n-1$ are equally sized and large enough so that each is bigger than $K(\delta) + 2\delta$, and they together cover $[-T, T] \cup [-C, C]$, where C is the corresponding constant in A3.

For each $k \in \{1, ..., n\}$, we will further partition block k into n cells, labeled as in Section 3.1. Block 1 minus the leftmost cell and block n minus the rightmost cell are each bigger than $\max\{K(Z), K(\delta) + \delta\}$, where Z is the corresponding constant in A3. We choose n large enough so that each of the middle $n^2 - 2$ cells, which are of equal size, is smaller than δ. For the sake of completeness, let each of the middle $n^2 - 2$ cells be closed on the left and open on the right.

Label the cells as in Section 3.1, and consider the following strategy profile:

- when s_1 falls in cell (j, k), sender 1 sends message m_1^j;

- when s_2 falls in cell (j, k), sender 2 sends message m_2^k;

- $y(m_1^j, m_2^k)$ is an optimal response to m_1^j, m_2^k given the above strategies, for every $j, k \in \{1, ..., n\}$;

- the receiver associates any out-of-equilibrium message to a message sent by that player in equilibrium, and after any other message pair, the receiver chooses the corresponding $y(m_1^j, m_2^k)$ for some $j, k \in \{1, ..., n\}$.

This profile constitutes an equilibrium in the noiseless limit game, which has the property that $|y - \theta| < \delta$ with *ex ante* probability at least $1 - \frac{\eta}{2}$. This is because message pairs are allocated to cells in a way that at any state, any action that a sender could induce other than the prescribed one is strictly worse for the sender than the prescribed action.

Analogous arguments as the ones used in the proof of Proposition 2 establish that for large enough p, the above profile still constitutes an equilibrium. The assumption that $\frac{f(\theta)}{g(\theta)} \geq b > 0$ guarantees that for p large, senders believe with high probability that they have observed the correct state. Moreover, it is easy to see that for large enough p, conditional on the state being in the middle $n^2 - 2$ cells, the probability that $|y - \theta| < \delta$ is at least $1 - \frac{\eta}{2}$. Then, the *ex ante* probability of $|y - \theta| < \delta$ is at least $1 - \eta$, concluding the proof. ■

Proposition 3 implies that, if A3 holds, then in a game where the state space is the real line, for any $\delta > 0$, there exists an equilibrium robust to small replacement noise in which the distance between any state and the action induced in that state is less than δ with high *ex ante* probability. The thinness of the tail of the distribution then also implies that the *ex ante* expected payoff of the receiver in equilibrium can closely approximate the maximum possible payoff value 0, obtained in a truthful equilibrium.

4 Bounded Continuous Noise

Our construction from Section 3 requires that, regardless of their signal s_i, each sender believes that the other sender's signal s_j lies in the same cell as s_i with high probability. This occurs with replacement noise due to the high probability that both senders observe the state exactly. However, if signals follow a continuous distribution around the state and are not perfectly correlated, then the probability that they coincide is 0. As a result, when s_i is sufficiently near the boundary between two cells, the probability that s_j lies on the other side of the boundary is non-negligible. In that case, the senders may have an incentive to second-guess their signal in order to reduce the probability of a coordination failure (which we call *miscoordination*) that would result in the action being in a different block, or in order to make miscoordination less costly by changing the action that follows it. But doing so in states near the boundary can trigger a departure from the originally prescribed strategy profile in other states and lead to an unraveling of the equilibrium construction from Section 3.[13]

In this section, we allow for the distribution of s_i to exhibit continuous density, and focus on the case where, conditional on θ, the supports of the signals s_i are small relative to the state space ("bounded"). Our main result is that if the state space is unbounded, then under mild conditions on the noise structure, there exists an equilibrium in which the receiver perfectly learns both senders' signals. For large bounded state spaces, we show a more limited result: given additional assumptions on the noise structure, there exists an equilibrium with two rounds of messages by the senders that fully reveals one of the senders' signal.

[13] For example, suppose θ follows a uniform distribution, and both senders independently observe s_i distributed $U[\theta-\varepsilon, \theta+\varepsilon]$ (whenever $[\theta-\varepsilon, \theta+\varepsilon] \subseteq \Theta$). Suppose θ is the boundary between $I_{1,2}$ and $I_{2,3}$ (which are in block 2) in the construction from Section 3. Suppose also that between the miscoordinations resulting an action in $I_{1,3}$ (in block 3) and $I_{2,2}$ (in block 1), sender 1 prefers the former, while sender 2 has symmetric preferences.

If sender 2 follows the prescribed strategy, then when sender 1 sees s_1 slightly to the right of θ, she will prefer sending message 1 instead of 2: although she thinks that sender 2 is slightly more likely to send 3 than 2, she minds $I_{1,3}$ less than $I_{2,2}$. However, sender 2 will respond to that by sending 2 instead of 3 when s_2 is slightly to the right of θ. As a result, sender 1 will shift her "boundary" between messages 1 and 2 further right, and so on. The equilibrium profile unravels.

4.1 Unbounded State Space: Fully Revealing Equilibrium

It is well known that in noiseless games with large enough state space, full revelation can occur in equilibrium, given out-of-equilibrium beliefs by the receiver that make any deviation sufficiently costly. Proposition 4 shows that for many noise structures where the senders' signals are never too far apart, a similar reasoning holds when $\Theta = \mathbb{R}$. The main difference with the noiseless case is that if m_i is only slightly different from s_i, then an out-of-equilibrium message pair only occurs with small probability. The construction that we propose ensures that the receiver's strategy is sufficiently extreme off the equilibrium path that even small deviations by the senders are deterred.

The following notation is useful to state our assumptions about the noise structure. Let $g(s_1, s_2, \theta)$ be the joint density of (s_1, s_2, θ). Let $S_i(\theta)$ be the support of s_i conditional on θ, and let $\Theta_i(s)$ be the support of θ conditional on $s_i = s$. Finally. let $\underline{s_i}(\theta)$ and $\overline{s_i}(\theta)$ be the infimum and supremum of $S_i(\theta)$, and let $\underline{\theta_i}(s)$ and $\overline{\theta_i}(s)$ be the infimum and supremum of $\Theta_i(s)$.

Assumption 4 ensures that whenever sender j does not truthfully report her signal ($m_j \neq s_j$), but sender i does ($m_i = s_i$), then there is a positive probability that (m_1, m_2) does not correspond to a pair of signals that can arise with truthful reporting, *i.e.* that $g(m_1, m_2, \theta) = 0$ for all θ.

A4: For all θ and i, $S_i(\theta)$ are nontrivial intervals, and $\underline{s_i}(.)$ and $\overline{s_i}(.)$ are both continuous and increasing at a rate of at least C, for some $C > 0$.[14]

Next, we also need the probability that $g(m_1, m_2, \theta) = 0$ for all θ (*i.e.* that m_1 and m_2 are incompatible signals) to increase sufficiently fast from 0 as m_j moves away from s_j, assuming $m_i = s_i$. This is guaranteed by A5.

A5: For any $j = 1, 2$ and $s_j \in \mathbb{R}$, there exists $\nu_j(s_j) > 0$ such that $g(s_1, s_2, \theta) > \nu_j(s_j)$ whenever $\theta \in \Theta_j(s_j)$ and $s_i \in S_i(\theta)$.

Proposition 4: Suppose A4 and A5 hold and $\Theta = \mathbb{R}$. Then there exists an equilibrium where both s_1 and s_2 are revealed to the receiver.[15]

[14] This is equivalent to assuming that for all s and i, $\Theta_i(s)$ are nontrivial intervals, and $\underline{\theta_i}(.)$ and $\overline{\theta_i}(.)$ are both strictly increasing and Lipschitz continuous, with Lipschitz constant $1/C$. One can see this by noting that the joint support of (θ, s_i) is the joint support of (s_i, θ) reflected about the 45° line.

[15] In an earlier version of this paper, we provided a fully revealing equilibrium without assuming A5, but with the presence of an impartial mediator.

The idea for the proof of Proposition 4 is to make the senders' expected utility from any deviation $-\infty$. Suppose, without loss of generality, that m_2 is slightly below s_2, while $m_1 = s_1$. Then, by A4, $x(m_1, m_2) \equiv \underline{\theta_1}(m_1) - \overline{\theta_2}(m_2)$ is positive (and small) with positive probability. In this case, we let the receiver's belief be $\theta = 1/x(m_1, m_2)^2$, which means that her optimal action is $a = 1/x(m_1, m_2)^2$. A5 guarantees that the density of $x(m_1, m_2)$ for small $x(m_1, m_2)$ is on the order of $x(m_1, m_2)$. This, combined with $a = 1/x(m_1, m_2)^2$ and the concavity of u_i in the action, ensure that the expected utility after any deviation is indeed $-\infty$. The full proof in the Appendix shows how this can be done for deviations that are not necessarily small.

4.2 Bounded State Spaces: Two Rounds of Communication

The result in the previous subsection applies to a broad class of bounded continuous noise structures, and provides a fully revealing equilibrium for arbitrary levels of noise. However, the construction requires the state space to be unbounded. For bounded state spaces, we propose an alternative construction, which can be implemented through two rounds of public cheap talk. The drawbacks of this construction are that it only reveals s_1, is only valid for a more restricted class of noise structures, and is significantly more involved than the equilibrium used for Proposition 4.[16]

Formally, we consider a multi-stage game in which θ is realized and each sender i observes s_i in stage 0. In stage 1, the senders send messages m_1 and m_2 that are public (*i.e.* observed by all players before the next stage). After observing these messages, in stage 2, the senders simultaneously send public messages m_1' and m_2'. Lastly, in stage 3, the receiver chooses an action.

Strategies are defined as follows. The action rule of the receiver becomes a measurable function $y : (M_1 \times M_2)^2 \to \mathbb{R}$, and the belief rule is now a measurable function $\mu : (M_1 \times M_2)^2 \to \Delta(\Theta)$. Sender i's signaling strategy is a pair of measurable functions $m_i : S_i \to \Delta(M_i)$ and $m_i' : S_i \times M_1 \times M_2 \to \Delta(M_i)$. Sender i's belief in stage 1 is determined by Bayes' rule, while her belief rule in stage 2 is $\mu_i : S_i \times M_1 \times M_2 \to \Delta(\Theta \times S_j)$. We use weak perfect Bayesian Nash equilibrium, defined analogously as in Section 2, as our solution concept.

We impose the following assumptions A6-A8 regarding the distribution of signals and preferences:

A6: There exists $\varepsilon > 0$ such that, conditional on the state θ, we have $s_i \in [\theta - \varepsilon, \theta +$

[16] The construction we propose here can also be used for unbounded state space, in which case it has the advantage of not relying on out-of-equilibrium beliefs as any combination of messages can occur in equilibrium.

$\varepsilon]$. Furthermore, there exist $\underline{\nu}, \delta, r > 0$ such that for every s_1, (i) the distribution of s_2 conditional on s_1 exhibits a density $f(s_2|s_1) > \underline{\nu}$ everywhere on $S_2(s_1) \equiv [\underline{s_2}(s_1), \overline{s_2}(s_1)]$, (ii) $\overline{s_2}(s_1) - \underline{s_2}(s_1) > \delta$, and (iii) $\underline{s_2}(.)$ and $\overline{s_2}(.)$ are continuous and strictly increasing at a rate of at least r.

An example of a class of noise that satisfies the above restrictions is the following: $s_1 = \theta + \omega_1$ and $s_2 = \theta + \omega_1 + \omega_2$, where $\omega_i \sim F_i$ independently, and F_1 and F_2 are distributions with finite interval supports and densities bounded away from 0. In particular, this holds when s_2 is a mean-preserving spread of s_1 satisfying the above technical conditions.[17]

Let $a(s_1) = \arg\max_a \int v(\theta, a) f(\theta|s_1) d\theta$ be the receiver's optimal action conditional on s_1. Note that $a(s_1)$ is unique because $v(\theta, a)$ is strictly concave in a.

A7: $a(s_1)$ is strictly increasing and Lipschitz continuous in s_1, with Lipschitz constant Λ.

For example, A7 is satisfied when for some $\varepsilon > 0$, the distribution of θ conditional on s_1, denoted $F(\theta|s_1)$, first-order stochastically dominates $F(\theta - \varepsilon(s_1 - s_1')|s_1')$ whenever $s_1 > s_1'$.

A8: For any $\varepsilon \geq 0$, there exists $L(\varepsilon) > 0$ such that for any $a, \theta \in \Theta$, and $a', \theta' \in [\theta - \varepsilon, \theta + \varepsilon]$, we have $u_i(\theta, a') - u_i(\theta, a) = u_i(\theta', a'') - u_i(\theta', a)$ for some $a'' \in [a - L(\varepsilon), a + L(\varepsilon)]$, $\forall$ $i = 1, 2$.

A8 bounds the extent to which senders care more about the outcome in one state of the world than the outcome in another state. Like A2, A8 is automatically satisfied if preferences depend only on $a - \theta$, as assumed in most of the literature on cheap talk and delegation.

When Θ is a large bounded interval of $\mathbb{R}$, we define the *underlying* game (distribution of θ and s_i, preferences) over all of $\mathbb{R}$.

Proposition 5: Suppose A1, A2, A6, A7 and A8 are satisfied in the underlying game and when Θ is truncated to a bounded state space. Then there exists T^* such that whenever Θ is truncated to $[-T, T]$, where $T > T^*$, in a game with two rounds of simultaneous public messages, there exists an equilibrium where s_1 is exactly revealed.

[17] In this example, one sender is better informed than the other, but Propositions 4 and 5 below do not require the noise to be asymmetric, and the lack of atoms assumed in A6 can be relaxed. For example, our results also hold for the following class of noise: $s_1 = \theta + \omega_1$ and $s_2 = \theta + \omega_2$, where $\omega_i \sim F_i$ independently, F_1 and F_2 are distributions with finite interval supports and densities bounded away from 0, and F_1, F_2, or both F_1 and F_2 has/have point masses at the endpoints (the point masses can be arbitrarily small). The presence of atoms (or, in the main text's example, asymmetry) ensures that $f(s_2|s_1)$ is bounded away from 0, as A6 requires.

Below is a summary of the equilibrium construction in the proof of Proposition 5. Because the construction is simpler when $\Theta = \mathbb{R}$, we start by describing this case.

In stage 1, sender 1 sends a message that selects a partition among a continuum of possible partitions, while sender 2 babbles. Each of these partitions has infinitely many cells, each designated by a message pair (m_1', m_2'), where $m_1' \in \{1, 2, ..., n\}$ and $m_2' \in \mathbb{Z}$. Like in the construction in Section 3, a cell designated by message pair (m_1', m_2') is located far from any cell of the form (x, m_2') for $x \in \{1, 2, ..., n\} \backslash \{m_1'\}$ or of the form (m_1', x) for $x \in \mathbb{Z} \backslash \{m_2'\}$, so that if the cell where signals are located is common knowledge among senders, no deviation is profitable. The continuum of possible partitions is such that for any s_1, exactly one partition has a cell that contains $S_2(s_1)$. For this partition, sender 1 knows for sure which cell s_2 lies in. In stage 2, the senders play a continuation strategy profile analogous to the one from Section 3. The combination of stage 1 and stage 2 messages exactly reveals s_1 to the receiver, and does not reveal any additional information. Hence, the receiver plays the optimal action conditional on sender 1's signal being s_1.

We build the collection of partitions for the first stage as follows. First, note that A6 ensures that for any $s_1 \neq s_1' \in S_1$, $S_2(s_1) \nsubseteq S_2(s_1')$. This implies that one can build a set $\mathcal{P}$ of partitions such that, for every s_1, there is a unique partition within $\mathcal{P}$ where sender 1 puts probability 1 on s_2 lying in the same cell as s_1. Furthermore, for every s_1, there exist $s_1' < s_1$ and $s_1'' > s_1$ such that $\underline{s_2}(s_1) = \overline{s_2}(s_1')$ and $\overline{s_2}(s_1) = \underline{s_2}(s_1'')$. Therefore, it is possible to construct $\mathcal{P}$ such that, in every partition in $\mathcal{P}$, every cell can occur on the equilibrium path. Since, by definition, every partition covers $\mathbb{R}$, this implies that for every $s_2 \in \mathbb{R}$, every partition in $\mathcal{P}$ is on the equilibrium path: regardless of the chosen partition, sender 2 cannot know that sender 1 has deviated.

Assumptions A6 (through $\underline{\nu}$ and r) and A7, together with the cost of miscoordination in our construction, ensure that sender 1 chooses the partition where the probability of miscoordination is 0. To see this, note that since both $\underline{s_2}(s_1)$ and $\overline{s_2}(s_1)$ are increasing at a rate bounded away from 0, and the density of $S_2(s_1)$ is bounded away from 0 on its support, any small deviation from the prescribed partition increases the probability of miscoordination by a rate bounded away from 0. Finally, the receiver's action is $a(s_1)$: she receives no information about s_2 other than the fact that it lies in $S_2(s_1)$.

To extend the construction to bounded Θ, we give each partition n^2 cells, like in our basic construction from Section 3. As in Section 3, T needs to be large relative to the senders' biases such that the partition's blocks can be made large enough to discourage deviations. There are two main difficulties for the case of bounded Θ: i) since the size of the cells is determined by the noise structure, to make the number of cells in each partition square, we need to modify the construction; ii) the cells at the ends of the partitions may now be out-of-equilibrium. We discuss these issues and provide the full proof of Proposition 5 in the Appendix.

5 Extensions and Discussion

5.1 Multidimensional State Spaces

Our construction from Section 3 can be readily extended to multidimensional state spaces for replacement noise if the state space is the whole Euclidean space $\mathbb{R}^d$, for $d \geq 2$. In particular, for any $\delta > 0$, instead of partitioning a high-probability portion of the state space into $n^2 - 2$ intervals of size δ, we partition in into $(n^2-2)^d$ d-dimensional hypercubes with edges of size δ. Now take n^d messages for each sender, and index them by $\{1, 2, ..., n\}^d$. Hence, a typical message for sender i is labeled as $m^i_{j_1,...,j_d}$ ($j_1, ..., j_d \in \{1, 2, ..., n\}$). The lth component j_l of each sender's message is determined by the l-coordinate of the cell where that sender's signal is located, in the same way as in Section 3.2. Proceeding like this for all d dimensions results in a strategy profile of the senders such that each pair of possible messages is identified with a unique cell in the above partition. Moreover, the profile is constructed such that at every state, sending a different message than the one corresponding to the cell containing the state results in a message pair identified with a cell far away from the original state, whether the sender deviates in one or more dimensions from the prescribed message. Proving that this profile constitutes an equilibrium for small enough replacement noise is analogous to the proof of Proposition 3.

The equilibrium construction from Proposition 4 can be extended to multidimensional Euclidean spaces with straightforward extensions of assumptions A4 and A5 (for example, it is enough if A4 is required to hold for each dimension, while A5 is unchanged).

For large d-dimensional orthotopes, the construction from Section 3 can be extended in a straightforward manner. For different types of bounded state spaces in $\mathbb{R}^d$, our basic construction cannot be applied directly. However, the same qualitative insight still holds. Suppose the state space can be partitioned into n^2 cells with diameter at most δ, and that there is a bijection from $M \equiv \{m_1^1, ..., m_1^n\} \times \{m_2^1, ..., m_2^n\}$ to the cells in the partition such that for any $(m_1, m_2) \in M$ and for any $(m_1', m_2') \in M$ with either (i) $m_1 = m_1'$ and $m_2 \neq m_2'$, or (ii) $m_1 \neq m_1'$ and $m_2 = m_2'$, it holds that the distance between the partition cells associated with (m_1, m_2) and with (m_1', m_2') are at least $K(\delta)$ away from each other. Then there is an equilibrium of the noiseless limit game that is robust to a small amount of replacement noise.

5.2 Introducing Noise at Different Stages of the Game

We have been investigating equilibria robust to perturbations of a multi-sender cheap talk game in the observations of the senders. This is the type of perturbation most discussed in the literature. However, similar perturbations can be introduced at various other stages of the communication game: in the communication phase (the actual message received by the receiver is not always exactly the intended message by a sender) and in the action choice phase (the policy chosen by the receiver is not exactly the same as the intended policy choice).[18]

The equilibria we propose in this paper are robust with respect to the above perturbations as well. To see this for the case of small perturbations in the receiver's action choice, note that in the equilibria we construct, senders strictly prefer sending the prescribed message to any other equilibrium message (while out-of-equilibrium messages lead to the same intended actions as equilibrium messages). Hence for replacement noise, bounded continuous noise with unbounded state space, and in stage 2 of the two-round construction, if the noise in the action choice is small enough, senders still strictly prefer sending the prescribed message to sending any other message along the path of play. To ensure that there is no deviation in stage 1 of the two-round construction, one would need regularity conditions on the change in the distribution of the realized action when the intended action changes.

For noise in the communication phase, it is more standard to introduce replacement noise, as in Blume *et al.* (2007), since there typically is no natural metric defined on the message space (messages obtain their meanings endogenously, through the senders' strategies). A small modification of our equilibrium constructions from Sections 3 and 4.2[19] makes the profile robust with respect to such noise: partition each sender's message space into subsets, one for each equilibrium message, and associate each subset with a distinct equilibrium message. In the senders' new strategies, after any signal, they select an action randomly (according to a uniform distribution) from the subset of messages associated with the original equilibrium message. The resulting profile remains an equilibrium and induces exactly the same outcome. Moreover, this equilibrium is robust to a small amount of replacement noise, subject to regularity conditions guaranteeing that, when the above strategy profile is played, any message is much more likely to be intended than to be the result of noise.[20]

[18] For an analysis of noisy communication in one-sender cheap talk, see Blume *et al.* (2007). See also Chen *et al.* (2008) for a one-sender cheap talk game in which both the sender and the receiver are certain behavioral types with small probability, a model resembling one in which there is a small replacement noise in both the communication and the action choice stages.

[19] Extending our construction from Section 4.1 leads to the following problem: if there is always a positive probability of incompatible messages, the senders' utilities are $-\infty$ regardless of their strategy.

[20] For a bounded state space, this is the case if the distribution of the replacement noise has a bounded density function, and the probability of replacement noise is small enough.

5.3 Commitment Power

If the receiver can credibly commit to an action scheme as a function of messages received, then there exist constructions simpler than the ones we proposed that are robust to small amount of noise and achieve exact truthful revelation of the state. Here, we only discuss the case of replacement noise and one-dimensional state spaces. Mylovanov and Zapechelnyuk (2013) shows that a necessary and sufficient condition for the existence of a fully revealing equilibrium in a noiseless two-sender cheap talk game with commitment power and bounded interval state space $[-T, T]$ is the existence of a lottery with support $\{-T, T\}$ with the property that at every $\theta \in [-T, T]$, both senders prefer action θ to the above lottery. The sufficiency of this condition is easy to see: the receiver can commit to an action scheme that triggers the above lottery in case of differing messages from the senders.

We observe that given the above action scheme, truthtelling by the senders remains an equilibrium for small enough replacement noise. This is because if the other sender follows a truthtelling strategy, then after receiving signal θ, sending any other message than θ induces the threat lottery with probability 1, while sending message θ induces θ with high probability. The latter outcome is, by construction, preferred by the sender if the state is likely to be θ. The above implies that in case of commitment power, there exists a fully revealing equilibrium robust to replacement noise, even if the state space is relatively small. For example, if senders have symmetric and convex loss functions and are biased in opposite directions, then there exists an equilibrium construction like the one above whenever biases are less than T in absolute value.

5.4 Discrete State Spaces

The constructions in 3.1 and 3.2 extend in a straightforward manner to large discrete state spaces. Consider first the case when the state space is a coarse finite grid of a large bounded interval: $\Theta = \{\theta \in [-T, T] | \theta = -T + k \cdot \varepsilon\}$, where $T \in \mathbb{R}_+$ is large and $\varepsilon \in \mathbb{R}_+$ is small. Define $n_{2\varepsilon,T}$ as in Section 3.1, and partition $[-T, T]$ to $n^2_{2\varepsilon,T}$ equal-sized subintervals. By construction, each partition contains at least one state from Θ. Then it is easy to see that the strategy profile presented in 3.1 gives an equilibrium robust to a small amount of noise, in which the supremum of the absolute distance between any possible state and the action induced in that state is at most 2ε. This construction readily extends to other finite one-dimensional state spaces and implies almost full revelation of the state whenever the distance between the two extreme states is large enough, and the maximum distance between two neighboring states is small enough.

6 Appendix

A Proof of Proposition 4

Consider the following strategy profile:

- Sender i sends $m_i = s_i$.

- If $g(m_1, m_2, \theta) > 0$ for some θ, the receiver's beliefs are derived from Bayes' rule, and the receiver takes the optimal action.

- Otherwise, let $x(m_1, m_2) = \max\{\underline{\theta_1}(m_1) - \overline{\theta_2}(m_2), \underline{\theta_2}(m_2) - \overline{\theta_1}(m_1)\} \geq 0$ be the distance between the set of states compatible with $s_1 = m_1$ and the set of states compatible with $s_2 = m_2$. Also let:

$$k(m_1, m_2) = \inf_{i=1,2; j \neq i; z_j \in S_j(\Theta_i(m_i))} \{\overline{\theta_i}(m_i) - \underline{\theta_i}(m_i), \overline{\theta_j}(z_j) - \underline{\theta_j}(z_j)\}.$$

Note that A4 implies $k(m_1, m_2) > 0$. Suppressing the (m_1, m_2) argument after x and k for readability, define:

$$y(m_1, m_2) = \begin{cases} x(\operatorname{mod} k) & \text{if } x \text{ is not a multiple of } k \\ k & \text{if } x \text{ is a multiple of } k \end{cases}.$$

Then the receiver believes $\theta = 1/y(m_1, m_2)^2$ with probability 1 and plays $a = 1/y(m_1, m_2)^2$.

It is obvious that the receiver's beliefs are consistent with Bayes' rule when possible, and that her action rule is optimal given these beliefs.

It remains to be shown that truthtelling is optimal for the senders. We do so by showing that the sender's expected utility after any deviation is $-\infty$.

Note that by A4, $\underline{\theta_i}(\overline{s_i}(\theta)) = \theta$. Taking $\theta = \overline{\theta_j}(s_j)$, we see that after observing s_j, sender j believes that $\underline{\theta_i}(s_i)$ can be as high as $\overline{\theta_j}(s_j)$. By the same token, sender j also believes that $\overline{\theta_i}(s_i)$ can be as low as $\underline{\theta_j}(s_j)$. This implies that:

- As soon as $\Theta_j(m_j) \neq [\underline{\theta_j}(s_j), \overline{\theta_j}(s_j)]$ (which, by A4, occurs whenever $m_j \neq s_j$), then m_j and s_i are incompatible with positive probability, and assuming truthtelling by i, $x > 0$ with positive probability.

- From sender j's perspective, the interval of possible $\underline{\theta_i}(s_i)$ contains $[\underline{\theta_j}(s_j), \overline{\theta_j}(s_j)]$. Thus, if i reports the truth and $m_j < s_j$, then the support of $x(m_1, m_2)$ either has left endpoint 0 (if $\overline{\theta_j}(m_j) > \underline{\theta_j}(s_j)$), or has size at least $\overline{\theta_j}(s_j) - \underline{\theta_j}(s_j) \geq k(m_1, m_2)$. The same goes for the interval of possible $\overline{\theta_i}(s_i)$ and $m_j > s_j$.

Combining the above observations, we see that if $m_i = s_i$, then player j believes that if she sends any $m_j \neq s_j$, then the support of $y(m_1, m_2)$ has left endpoint 0.

As senders' utilities are concave, in order to conclude that any deviation yields expected utility $-\infty$, it is sufficient to show that $\int_0^\infty (1/y^2)\varphi_j(y|s_j,m_j)dy=\infty$ for any s_j and $m_j\neq s_j$, where $\varphi_j(.|s_j,m_j)$ is the density of y conditional on s_j, $m_j\neq s_j$ and truthtelling by i.

Now let $f_j(s_i|s_j)$ be the density of s_i conditional on s_j. That is,

$$f_j(s_i|s_j)=\frac{\int_{\Theta_j(s_j)} g(s_1,s_2,\theta)d\theta}{\int_{S_i(\Theta_j(s_j))}\int_{\Theta_j(s_j)} g(s_1,s_2,\theta)d\theta ds_i}.$$

Lemma 1: For any s_j, $\exists\varepsilon(s_j)>0$ such that $f_j(s_i|s_j)>\varepsilon(s_j)|\Theta_i(s_i)\cap\Theta_j(s_j)|$.

Proof of Lemma 1: By A5, $g(s_1,s_2,\theta)>\nu_j(s_j)$ for all $\theta\in\Theta_j(s_j)$ and $s_i\in S_i(\theta)$, *i.e.* whenever $\theta\in\Theta_i(s_i)\cap\Theta_j(s_j)$. Letting $\varepsilon(s_j)=\nu_j(s_j)/D(s_j)$, where $D(s_j)$ is the denominator of $f_j(s_i|s_j)$, yields the desired result. □

Without loss of generality, we consider deviations of the form $m_j<s_j$. By Lemma 1 and A4, the density of $\underline{\theta_i}(s_i)$ conditional on s_j is bounded below by $C\varepsilon(s_j)|\Theta_i(s_i)\cap\Theta_j(s_j)|$. This implies that, conditional on s_j, $m_j\neq s_j$ and truthtelling by i, the density of $x=\underline{\theta_i}(s_i)-\overline{\theta_j}(m_j)$ only reaches 0 at the endpoints of its support, from where, as s_i changes, it increases at a rate of at least $C\varepsilon(s_j)$ until $\Theta_i(s_i)\cap\Theta_j(s_j)$ starts losing some elements.[21] Therefore, $\varphi_j(y|s_j,m_j)>yC\varepsilon(s_j)$ for sufficiently small y and all $m_j<s_j$. Since $\int_0^c(1/y^2)yC\varepsilon(s_j)dy=\infty$ for any $c>0$, it follows that $\int_0^\infty(1/y^2)\varphi_j(y|s_j,m_j)dy=\infty$, which concludes the proof. ■

B Proposition 5

The main idea of the construction is the same as described in the main text for unbounded state spaces: build a set of partitions $\mathcal{P}$ so that, in the first round of communication, sender 1 announces a partition with a cell that is exactly $[\underline{s_2}(s_1),\overline{s_2}(s_1)]$. However, in order to use the construction from Section 3 in the second round, we need to make the number of cells in each partition square, which may conflict with partitions covering all of S_2. To address this problem, two disjoint subsets of partitions, $\mathcal{L}$ and $\mathcal{R}$, are constructed, and $\mathcal{P}=\mathcal{L}\cup\mathcal{R}$. Each of these partitions has fine cells in at least half of S_2 so that, for each s_1, at least one of $\mathcal{L}$ and $\mathcal{R}$ has a partition where a cell is exactly $[\underline{s_2}(s_1),\overline{s_2}(s_1)]$; such a partition is unique within $\mathcal{L}$ and/or $\mathcal{R}$. If there are two partitions that sender 1 can announce (one in $\mathcal{L}$ and one in $\mathcal{R}$), he randomizes 50/50 between them. The ensuing play on the equilibrium path is as described in the main text: in the second round, senders send the message corresponding

[21] That is, moving right to left, until $\overline{\theta_i}(s_i)$ drops below $\overline{\theta_j}(s_j)$, and, moving left to right, until $\underline{\theta_i}(s_i)$ increases above $\underline{\theta_j}(s_j)$.

to the cell associated with their signal (as in Section 3), after which the receiver plays the optimal action conditional on sender 1's signal being s_1.

Section B.1 describes the construction of $\mathcal{P}$ in detail and discusses some of the off-equilibrium-path issues that arise, while Section B.2 presents the proof.

B.1 Extending the Construction Described in the Main Text to Bounded State Spaces

Each partition in $\mathcal{L}$ is constructed starting with a cell at the left end of $S_2 \subseteq [-T-\varepsilon, T+\varepsilon]$. This first cell is called a *small extreme* cell if its right endpoint is less than $\overline{s_2}(\min_{s_1 \in S_1} s_1)$, which implies that there is no s_1 for which this cell contains $S_2(s_1)$; otherwise, it is a *regular* cell. Each subsequent cell, except for the rightmost one, is $(\underline{s_2}(z), \overline{s_2}(z)]$ for some $z \in S_1$, where $\underline{s_2}(z)$ is the right boundary of the previous cell; these are all *regular* cells. Once regular cells and, if applicable, the small extreme cell together cover $\frac{3}{4}$ of S_2 and number $n^2 - 1$ for some integer n, we cease creating new cells, and the last cell, called the *large extreme* cell, covers the rest of S_2. By construction, there are n^2 cells in each partition. Partitions in $\mathcal{R}$ are constructed in a similar fashion, but starting at the right end of S_2. Notice that for every element of $\mathcal{L}$, there is a corresponding element of $\mathcal{R}$ that is identical in the area where regular cells of $\mathcal{L}$-partitions and regular cells of $\mathcal{R}$-partitions overlap, and vice versa. A typical partition is illustrated in Figure B.1. The top of the picture displays the partition in $\mathcal{L}$ prescribed for signal s_1, while the bottom of the picture displays the corresponding partition in $\mathcal{R}$. Note that in both partitions, the cell that s_1 belongs to is exactly $S_2(s_1)$.

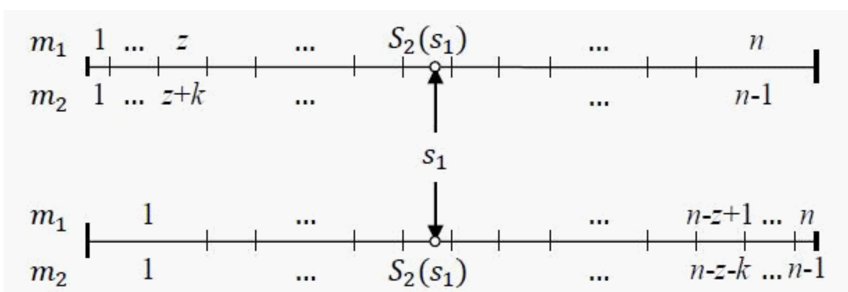

Figure B.1: The $\mathcal{L}$ and $\mathcal{R}$ partitions prescribed for a typical signal of sender 1

In the proposed equilibrium, sender 1 chooses a partition where one of the cells is exactly $[\underline{s_2}(s_1), \overline{s_2}(s_1)]$. By construction, there is at least one such partition in $\mathcal{P}$, and at most one such partition in each of $\mathcal{L}$ and $\mathcal{R}$. When a partition from $\mathcal{L}$ and a partition from $\mathcal{R}$ both satisfy the criterion, which occurs when s_1 is a regular cell of both partitions, sender 1 randomizes 50/50 between them; this is consistent with equilibrium since both partitions lead to the same action, $a(s_1)$.

We also need to deal with extreme cells being out of equilibrium: on the equilibrium path, the senders' round 2 messages (m'_1, m'_2) never point to an extreme cell. This is especially problematic when (m'_1, m'_2) corresponds to the large extreme cell due to its size. We specify the receiver's beliefs after such message pairs such that no profitable deviation is created.[22] Specifically, if $P \in \mathcal{L}$, then the large extreme cell is $(n, n-1)$. In our equilibrium, the receiver's belief after seeing $P \in \mathcal{L}$ and $(n, n-1)$ is the same as if (m'_1, m'_2) had instead been $(n, n-2)$, which corresponds to a cell a block away. As a result:

- When $m'_1 = n$, the rightmost action that sender 2 can induce is in $(n, n-2)$ (by sending $m'_2 = n-1$ or $n-2$). So when s_2 is in cell $(n, n-1)$, sending $m'_2 = n-1$ is still optimal for sender 2.

- When $m'_2 = n-1$, the rightmost action that sender 1 can induce is in $(n, n-2)$ (by sending $m'_1 = n$). This is because, apart from $(n, n-1)$, which cannot be induced, the closest cell to $(n, n-2)$ where $m'_2 = n-1$ is $(1, n-1)$, which is almost a block to the left of $(n, n-2)$.[23] Therefore, when s_1 is in cell $(n, n-1)$, sending $m'_1 = n$ is still optimal for sender 1.

The case where $P \in \mathcal{R}$ is analogous.

B.2 Proof of Proposition 5

1. Constructing partitions for stage 1 messages by sender 1.

Note that by assumption A6, $\overline{s_2}(s_1) - \underline{s_2}(s_1) \geq \delta$ for all s_1.

Let $S_2 = [-\underline{B}, \overline{B}] \subseteq [-T-\varepsilon, T+\varepsilon]$, and let $X = \max_{\{s_1 : \underline{s_2}(s_1) = -T\}} \overline{s_2}(s_1)$.

Define partition $L(x)$, which consists of n^2 cells labeled as in Section 3, as follows for all $x \in (-\underline{B}, X]$. The leftmost cell $(1,1)$ is $[-\underline{B}, x]$. Each subsequent cell, except for the rightmost one $(n, n-1)$, is $(\underline{s_2}(z), \overline{s_2}(z)]$, where $\underline{s_2}(z)$ is the boundary of the previous cell; note that this is well-defined because $\underline{s_2}(.)$ and $\overline{s_2}(.)$ are continuous and strictly increasing. Let n_L be large enough so that if $n = n_L$, the left boundary of the rightmost cell $(n, n-1)$ is greater than $\frac{T}{2}$ for all $L(x)$. This is possible since, by assumption, the size of cells is bounded below by δ and above by 2ε.[24] We will refer to the rightmost cell as the *large extreme* cell, and in partitions where $x < \overline{s_2}(\min_{s_1 \in S_1} s_1)$, we refer to the leftmost cell as the

---

[22] Moreover, in round 2, sender 2 is aware that sender 1 has deviated in round 1 if s_2 is in an extreme cell of the announced partition. At such histories, in our equilibrium, sender 2's beliefs are such that m'_1 is consistent with the cell where s_2 is located.

[23] By construction, $m'_2 = n-1$ is skipped at the boundary between the last two blocks.

[24] To see that for T large enough, it is possible for the size of the first n^2-1 cells to lie between $\underline{B} + \frac{T}{2}$ and $\underline{B} + \overline{B}$, we need to show that as n increases, the size of the first n^2-1 cells will not "skip over" this range. A partition's n^2-2 middle cells will cover at least $(n^2-2)\delta$. If we increase n by 1, we are adding $2n+1$ cells, which cover at most $(2n+1)\varepsilon$. Note that $\frac{(n^2-2)\delta+(2n+1)\varepsilon}{(n^2-2)\delta} \overset{n\to\infty}{\to} 1$, while $\frac{\underline{B}+\overline{B}}{\underline{B}+\frac{T}{2}} \geq \frac{(T+\varepsilon)+(T-\varepsilon)}{(T+\varepsilon)+\frac{T}{2}}$, which increases in T and converges to $\frac{4}{3} > 1$.

small extreme cell. Moreover, call all others cells *regular*. Note that since $\underline{s_2}(s_1)$ and $\overline{s_2}(s_1)$ are strictly increasing, there exists a unique $x \in (-\underline{B}, X]$ such that sender 1 puts probability 1 on s_1 and s_2 being in the same regular cell.

Let n_L (the number of cells per block) be large enough so that the following hold for all s, feasible s_2 given $s_1 = s$, and actions a located $n_L - 3$ cells away from $a(s)$:

(i) $\min_{\theta \in [s-\varepsilon, s+\varepsilon]} \{u_1(\theta, a(s)) - u_1(\theta, a)\} > \frac{\Lambda}{\underline{\nu} r} \max_{\theta \in [s-\varepsilon, s+\varepsilon], a' \in [a(s-4\varepsilon), a(s+4\varepsilon)]} |\frac{\partial}{\partial \alpha} u_1(\theta, a')| \equiv \frac{\Lambda}{\underline{\nu} r} M(s)$, where $\frac{\partial}{\partial \alpha}$ denotes the partial with respect to the receiver's action; and

(ii) $E[u_2(\theta, a(s))|s_1 = s, s_2] - E[u_2(\theta, a)|s_1 = s, s_2] > 0$.

Condition (ii) simply ensures that in stage 2, after sender 2 has learned $s_1 = s$, sender 2 has no incentive to deviate since inducing a cell almost a block away (as happens when she unilaterally deviates given a partition) is not profitable. Note that $\theta \in [s_1 - \varepsilon, s_1 + \varepsilon]$, and that furthermore, as a result, $a(s_1) \in [s_1 - \varepsilon, s_1 + \varepsilon]$. Therefore, as in Section 3, due to A2, it is possible to make the blocks large enough (on the order of $K(\varepsilon)$, by taking n large enough) so that (ii) is satisfied.

Condition (i) does the same for sender 1 and - as shown in the last part of the proof - provides incentives for sender 1 to announce a partition where no miscoordination is possible. Without the min and max functions in condition (i), the existence of n_L would be guaranteed by A2 and the concavity of u_1 with respect to the action. A8 links sender 1's preferences at any $\theta \in [s - \varepsilon, s + \varepsilon]$ to her preferences at the state where $|\frac{\partial}{\partial \alpha} u_1(\theta, a')|$ is maximized, which allows us to take the min and max functions in condition (i).

Similarly, let $Y = \min_{\{s_1 : \underline{s_2}(s_1) = T\}} \underline{s_2}(s_1)$, and define the partition $R'(y)$ for all $y \in [Y, T)$, starting from the right, so that its rightmost $n^2 - 1$ cells cover at least $[-\frac{T}{2}, \overline{B}]$. Define n_R analogously, and let $n = \max\{n_L, n_R\}$.

Note that, for any x, y, the regular cells from $L(.)$ and $R'(.)$ will overlap over at least $[-\frac{T}{2}, \frac{T}{2}]$. Because $\underline{s_2}(s_1)$ and $\overline{s_2}(s_1)$ are strictly increasing over the relevant range, for every $y \in (0, Y]$, there exists $\varphi(y) \in (0, X]$ such that in the range of the overlap, the cells of $L(\varphi(y))$ have the same boundaries as the cells from $R'(y)$. Define $R(x) = R'(\varphi^{-1}(x))$, so that in the area of the overlap, the cells of $L(x)$ and $R(x)$ have the same boundaries.

As a result, for every s_1, there exists a unique x such that sender 1 puts probability 1 on s_1 and s_2 being in the same regular cell for at least one of $L(x)$ and $R(x)$. Denote this quantity $x(s_1)$, and let the set of partitions be $\mathcal{P} = \{L(x), R(x)\}_{x \in (0,X]}$.

2. The strategy profile

- Stage 1: Sender 1 announces a partition from $\mathcal{P}$ such that sender 1 knows which regular cell s_2 lies in - the partition must be either $L(x(s_1))$ or $R(x(s_1))$. If both of these partitions work, sender 1 randomizes 50/50 between them. For the remainder of this proof, we will let P be the announced partition. Sender 2 babbles.

- Stage 2: We distinguish three cases:

a) On the equilibrium path (for sender 1, if no deviation in stage 1; for sender 2, if s_2 lies in a regular cell of P): The senders send m_i' corresponding to the cell of P where s_2 lies, which, by construction, is known to sender 1. Beliefs μ_i are determined according to Bayes' rule. Note that (m_1', m_2') must correspond to a regular cell of P and reveals s_1 when combined with sender 1's announcement in stage 1.

b) Off the equilibrium path for sender 1 (following deviation in stage 1): μ_1 is unchanged from stage 1, and sender 1 plays a best response.

c) Off the equilibrium path for sender 2 (if s_2 lies in an extreme cell of P): Sender 2 sends m_2' corresponding to the cell of P where s_2 lies. Her belief μ_2 remains unchanged with respect to θ, while with respect to s_1, it is such that $m_1'(s_1, m_1, m_2) = 1$ (if s_2 lies in the leftmost cell of P) or $m_1'(s_1, m_1, m_2) = n$ (if s_2 lies in the rightmost cell of P). Note that this is always possible because, on the equilibrium path, $m_1 = P'$ and $m_1' = k$ can occur for all $P' \in \mathcal{P}$ and $k \in \{1, ..., n\}$.

- Stage 3:

a) If (m_1', m_2') corresponds to a regular cell of P, the receiver chooses $a(s_1)$, where s_1 is inferred from P, m_1' and m_2'. Note that this is always the case on the equilibrium path.

b) If, instead, (m_1', m_2') points to the small extreme cell of P, the receiver believes that the state is the endpoint ($-T$ or T) within that extreme cell, and chooses that action.

c) If, finally, (m_1', m_2') points to the large extreme cell of P, the receiver believes that the state is $a(s_1)$ and chooses that action, for s_1 determined as follows:

- if $P = L(x)$ for some x, then s_1 is inferred as if sender 2 had sent $m_2' = n - 2$, so that the cell is the regular cell $(n, n-2)$ instead of the large extreme cell $(n, n-1)$;

- if $P = R(x)$ for some x, then s_1 is inferred as if sender 2 had sent $m_2' = 2$, so that the cell is the regular cell $(1, 2)$ instead of the large extreme cell $(1, 1)$.

We now verify the optimality of this strategy profile for each player. For sender 2, we will assume that $P = L(x)$ for some x. A symmetric argument applies if $P = R(x)$ instead.

3a. Optimality for the receiver

On the equilibrium path, the receiver has learned s_1 exactly, but no information on s_2 other than the fact that it lies in $[\underline{s_2}(s_1), \overline{s_2}(s_1)]$. It is therefore optimal for the receiver to choose $a(s_1)$.

Off the equilibrium path, sequential rationality for the receiver's action choice directly follows from the specified beliefs.

3b. Optimality for sender 2 on the equilibrium path

If s_2 lies in a regular cell of P, then sender 2 believes that m_1' will refer to the cell where s_2 lies. For the same reason as in Section 3's basic construction, sender 2 has no profitable deviation to another regular cell or to the small extreme cell. Deviating to the large extreme cell $(n, n-1)$ is also not profitable: message pair $(n, n-1)$ now leads to the same action as for message pair $(n, n-2)$. Obviously, sender 2 has no incentive to deviate is she is supposed to send $n-2$. When sender 1 is supposed to send n and sender 2 is supposed to send any message other than $n-1$ and $n-2$, then s_2 is at least almost a block away from cell $(n, n-2)$, so once again, there is no incentive to deviate.

3c. Optimality for sender 2 off the equilibrium path

Given μ_2, sender 2 believes, as on the equilibrium path, that coordination will be successful if m_2' corresponds to the cell of P where s_2 lies. If this cell is the small extreme cell, this is optimal as deviating would lead to an action almost a block away. If s_2 is in the large extreme cell $(n, n-1)$, then sender 2 expects $m_1' = n$. The best that sender 2 can send is to send $n-1$ or $n-2$, which results in an action about a block to the left of cell $(n, n-1)$. If sender 2 sends anything else, she would expect the action to be even further left, which is undesirable.

3d. Optimality for sender 1

For the same reasons as for sender 2, there is no profitable deviation to a regular cell or the small extreme cell in stage 2 if sender 1 followed the equilibrium prescription in stage 1. Deviating to the large extreme cell $(n, n-1)$ is also not profitable: this is only possible when $m_2' = n-1$, which implies that s_2 is located in cell $(1, n-1)$ or further to the left. Therefore, inducing an action in cell $(n, n-2)$, which is almost a block to the right of $(1, n-1)$, is suboptimal.

Moreover, it cannot be profitable for sender 1 to announce $L(x)$ instead of $R(x)$ or vice versa - it would either lead to the same action, or to a far away action (if sender 1 also deviates in stage 2, or if the announcement causes s_2 to be in the large extreme cell). For the same reason, if announcing $L(y)$ instead of $L(x)$ is not a profitable deviation, then announcing $R(y)$ instead of $L(x)$ is also not profitable (and vice versa).

It therefore remains to be shown that if sender 1 is supposed to announce $L(x)$, then announcing $L(y)$ instead is not a profitable deviation (and similarly with $R(x)$ and $R(y)$).

Suppose sender 1's signal is s_1 and that she is supposed to announce $L(x)$. Let $c(s_1)$ be the cell of $L(x)$ that would be communicated in stage 2 on the equilibrium path. Then announcing $L(y)$ instead would make sender 1 uncertain about sender 2's message in stage 2. Specifically, the revealed s_1 will be either:

(a) slightly different, if coordination is successful (*i.e.* one of the two cells of $L(y)$ that overlaps with $c(s_1)$ is communicated in stage 2); or

(b) almost one block away or further, in the event of miscoordination.

Without loss of generality, let $L(y)$ be the partition that should be announced if sender 1 had signal $s_1 + \Delta$, where $\underline{s_2}(s_1 + \Delta) \in (\underline{s_2}(s_1), \overline{s_2}(s_1))$. The probability of miscoordination is at least $\underline{\nu} r \Delta$, so the expected loss from miscoordination (point b) is at least $\underline{\nu} r \Delta \frac{\Lambda}{\underline{\nu} r} M(s_1) = \Lambda \Delta M(s_1)$ by the definition of n in step 1. We will show below that the gain from point (a) cannot exceed this amount.

We assume that in stage 2, the cell communicated by sender 1 is the rightmost of the two cells of $L(y)$ that overlaps with $c(s_1)$. This is done without loss of generality as the left case is symmetric, and any other announcement causes miscoordination for sure, which cannot be profitable.

Note that $\Delta < 4\varepsilon$, which allows us to use the bound $M(s_1)$. The expected gain from point (a) is thus:

$$\begin{aligned} \int [u_1(\theta, a(s_1 + \Delta)) - u_1(\theta, a(s_1))] f(\theta | s_1, s_2 &\in [\underline{s_2}(s_1 + \Delta), \overline{s_2}(s_1)]) d\theta \\ &\leq \Lambda \Delta M(s_1) \end{aligned}$$

It is therefore not profitable for sender 1 to deviate. ■

C Acknowledgments

We would like to thank Eduardo Azevedo, Georgy Egorov, Alex Frankel, Drew Fudenberg, Johannes Hörner, Maxim Ivanov, Yuichiro Kamada, Tymofiy Mylovanov, Martin Osborne, Larry Samuelson, anonymous referees and especially Joel Sobel and Satoru Takahashi for useful comments.

7 References

Acharya, A. and K. Ramsay (2013): "The calculus of the security dilemma," *Quarterly Journal of Political Science*, 8, 183-203.

Ambrus, A., E. Azevedo and Y. Kamada (2013): "Hierarchical cheap talk," *Theoretical Economics*, 8, 223-261.

Ambrus, A. and S. Takahashi (2008): "Multi-sender cheap talk with restricted state spaces," *Theoretical Economics*, 3, 1-27.

Austen-Smith, D. (1990a): "Information transmission in debate," *American Journal of Political Science*, 34, 124-152.

Austen-Smith, D. (1990b): "Credible debate equilibria," *Social Choice and Welfare*, 7, 75-93.

Austen-Smith, D. (1993): "Interested Experts and Policy Advice: Multiple Referrals under Open Rule," *Games and Economic Behavior*, 5, 3-43.

Baliga, S. and S. Morris (2002): "Co-ordination, spillovers, and cheap talk," *Journal of Economic Theory*, 105, 450-468.

Baliga, S. and T. Sjostrom (2004): "Arms races and negotiations," *Review of Economic Studies*, 71, 351-369.

Battaglini, M. (2002): "Multiple referrals and multidimensional cheap talk," *Econometrica*, 70, 1379—1401.

Battaglini, M. (2004): "Policy advice with imperfectly informed experts," *Advances in Theoretical Economics*, 4, Article 1.

Blume, A., O. Board, and K. Kawamura (2007): "Noisy Talk", *Theoretical Economics*, 2, 395–440.

Carlsson, H. and E. van Damme (1993): "Global games and equilibrium selection," *Econometrica*, 61, 989–1018.

Chen, Y., N. Kartik, and J. Sobel (2008): "Selecting cheap talk equilibria," *Econometrica*, 76, 117-136.

Crawford, V. and J. Sobel (1982): "Strategic information transmission," *Econometrica*, 50, 1431-1452.

Dessein, W. (2002): "Authority and communication in organizations," *Review of Economic Studies*, 69, 811-838.

Eső, P. and Y. Fong (2008): "Wait and see," *mimeo Northwestern University.*

Forges, F. (1988): "Can sunspots replace a mediator?," *Journal of Mathematical Economics*, 17, 347-368.

Frankel, D., S. Morris and A. Pauzner (2003): "Equilibrium selection in global games with strategic complementarities," *Journal of Economic Theory*, 108, 1-44.

Gillighan, T. and K. Krehbiel (1987): "Collective decisionmaking and standing committees: An informational rationale for restrictive amendment procedures," *Journal of Law, Economics, and Organization*, 3, 287-335.

Golosov, M., V. Skreta, A. Tsyvinski and A. Wilson (2014): "Dynamic strategic information transmission," *Journal of Economic Theory*, 151, 304-341.

Goltsman, M., J. Hörner, G. Pavlov and F. Squintani (2009): "Mediation, arbitration and negotiation," *Journal of Economic Theory*, 144, 1397-1420.

Green, J. and N. Stokey (2007): "A two-person game of information transmission," *Journal of Economic Theory*, 135, 90-104.

Ivanov, M. (2010): "Communication via a strategic mediator," *Journal of Economic Theory*, 145, 869-884.

Ivanov, M. (2012): "Dynamic learning and strategic communication," *mimeo McMaster University.*

Krehbiel, K. (2001): "Plausibility of signals by a heterogeneous committee," *American Political Science Review*, 95, 453-457.

Krishna, V. and J. Morgan (2001a): "Asymmetric information and legislative rules: Some amendments," *American Political Science Review*, 95, 435-452.

Krishna, V. and J. Morgan (2001b): "A model of expertise," *Quarterly Journal of Economics*, 116, 747-475.

Krishna, V. and J. Morgan (2004): "The art of conversation: eliciting information from experts through multi-stage communication," *Journal of Economic Theory*, 117, 147-179.

Lai, E., W. Lim and J. Wang (2014): "Experimental implementations and robustness of fully revealing equilibria in multidimensional cheap talk," *mimeo National Taiwan University.*

Lu, S. (2014): "Coordination-free equilibria in cheap talk games," *mimeo Simon Fraser University.*

McGee, A. and H. Yang (2013): "Cheap talk with two senders and complementary information," *Games and Economic Behavior*, 79, 181-191.

Monderer, D. and D. Samet (1989): "Approximating common knowledge with common beliefs," *Games and Economic Behavior*, 1, 170-190.

Morris, S., R. Rob and H. Shin (1995): "p-dominance and belief potential," *Econometrica*, 145-157.

Morris, S. and H. Shin (2003): "Global games: Theory and applications," in M. Dewatripont, L. Hansen, and S. Turnovsky, eds.: *Advances in Economics and Econometrics, Eighth World Congress, Volume I* (Cambridge University Press, Cambridge), 56-114.

Mylovanov, T. and A. Zapechelnyuk (2013): "Decision rules revealing commonly known events," *Economics Letters*, 119, 8-10.

Ottaviani, M. and P. Sørensen (2006): "The strategy of professional forecasting," *Journal of Financial Economics*, 81, 441-446.

Vespa, E. and A. Wilson (2012): "Communication with multiple senders and multiple dimensions: An experiment," *mimeo University of Pittsburgh.*

Wolinsky, A. (2002): "Eliciting information from multiple experts," *Games and Economic Behavior*, 41, 141-160.