Self-Control and Bargaining[*]

Shih En Lu[†]

June 2016

Abstract

This paper examines a bargaining game with alternating proposals where sophisticated quasi-hyperbolic discounters negotiate over an infinite stream of payoffs. In Markov perfect equilibrium, payoffs are almost always unique, and a small advantage in self-control can result in a large advantage in payoff. In subgame-perfect equilibrium, a multiplicity of payoffs and delay can arise, despite the complete information setting. Markov perfect equilibria are the best subgame-perfect equilibria for the agent with more self-control, and the worst for the agent with less self-control. Naïveté can help a player by increasing their reservation value.

Keywords: Self-Control, Bargaining, Time Inconsistency, Quasi-Hyperbolic Discounting

JEL Codes: C78, D90

[*]DOI: dx.doi.org/10.1016/j.jet.2016.05.003.

[†]Department of Economics, Simon Fraser University, Burnaby, BC V5A 1S6, Canada. Email: shihenl@sfu.ca.

1 Introduction

In many bilateral bargaining situations, the surplus to be divided takes the form of a stream, only part of which is immediately realized. As a result, the parties face tradeoffs between deals offering them more immediate surplus and deals offering them more future surplus. For example, a loan's repayment schedule would give the borrower higher short-term payoff and lower utility in later periods if it is backloaded. An employment contract with a higher starting wage but lower wage growth potential similarly shifts the worker's surplus from the future to the present.

In such settings, the party with a higher intertemporal weight on the future has an incentive to propose taking less current surplus in exchange for more future surplus. Such propositions, if accepted, are detrimental to the other party if the source of its high weight on the present is a lack of self-control, which has been extensively documented among individuals.[1] Moreover, the latter agent's bargaining power decreases if players anticipate that her future selves will lack self-control and accept such detrimental offers. This paper shows that, when bargaining over a stream of surplus, limited self-control indeed greatly affects the equilibrium outcome and the participants' welfare. Small changes in levels of self-control can drastically alter equilibrium predictions, and delay - often observed in real-world situations - can arise even under complete information.

Empirical studies of various types of loans, which often result from a bargaining process between the consumer and the lender, point to patterns where self-control issues can play an important role. For example, Attanasio, Goldberg and Kyriazidou (2008) find that demand for car loans in the United States is much more responsive to the maturity date than to the interest rate, and Karlan and Zinman (2008) find the same for microfinance loans in South Africa. The design of the latter study is particularly interesting: all borrowers had access to the same terms, but those that were presented with a longer "suggested maturity" took out more or larger loans. Thus, for some individuals, the decision to borrow may well be partly due to an impulse: the need for funds is weak enough that many do not bother inquiring about other terms[2] if the suggested maturity is short.[3] Therefore, if the lender

[1] For example, many might choose to procrastinate today on work due tomorrow, yet prefer to complete work to be done in either 7 or 8 days at the earlier opportunity. See Frederick, Loewenstein and O'Donoghue (2002) for a summary of early experimental findings.

[2] The letters proposing a suggested maturity explicitly included the mention "Loans available in other sizes and terms."

[3] Similarly, in the US car loan setting, to the extent that longer maturities lead to larger loans (for nicer cars, as opposed to more loans for basic ones), it appears likely that self-control is part of the story alongside other explanations such as liquidity constraint.

were bargaining strategically, as it would in a non-experimental setting, it could entice the customer to borrow more (or to borrow the same amount at a higher interest rate) by offering a loan with a longer term, *i.e.* by reducing its demand for near-term surplus and increasing its demand for future surplus. Other studies argue that agents with worse self-control are more prone to take up offers of high-interest loans: see Bertrand and Morse (2011) and Gathergood (2012) for payday loans, Gathergood (2012) for store card, mail order catalogue and doorstep credit loans, and Shui and Ausubel (2004) for credit card offers with a low teaser rate for a shorter-than-usual period (such that competing offers would lead to lower total interest costs).

Another area where limited self-control potentially plays a role in the bargaining process is employment contracts. An employer can attempt to take advantage of a potential employee's limited self-control by proposing a signing bonus (with a lower regular wage) or by requesting a non-compete covenant (which limits the worker's future outside options).[4] These features may of course serve other purposes: for example, the standard purpose of non-compete covenants is the protection of intellectual property. However, such clauses are also present in contracts within industries where intellectual property is, at best, a minor consideration (Starr, Bishara and Prescott (2015)). Therefore, employers may view non-compete covenants as a way to relieve future upward pressure on the worker's wage[5] that requires little increase in the starting wage due to the worker's present bias.[6] Similarly, it appears plausible that part of the purpose of signing bonuses is to appeal to some workers' desire for instant gratification.[7]

One way to model limited self-control is *quasi-hyperbolic discounting*, which posits that an agent's sequence of discount factors is $1, \beta\delta, \beta\delta^2, \beta\delta^3, ...$ with $\beta, \delta \in (0, 1)$.[8] This paper stud-

[4]Parsons and Van Wesep (2013) use a contracting approach to show that in a setting where the worker's utility of consumption varies over time, firms would optimally offer contracts with higher pay coinciding with periods of high expenditure, *e.g.* holidays.

[5]Garmaise (2011) finds empirical evidence that, as theory predicts, non-compete agreements reduce compensation growth.

[6]Starr, Bishara and Prescott (2015) find that most workers subject to such a clause did not bargain over it, and the most common reason for not doing so, given by over half of these workers, was that the contract's terms seemed reasonable overall. (Of course, "not bargaining" corresponds to a situation where the employer's initial bargaining offer is deemed acceptable.)

[7]Signing bonuses may also serve as a way for firms to signal a good match to new workers (Van Wesep (2010)). However, even in collective bargaining with existing employees, where the signaling motive is mostly absent, signing bonuses are sometimes offered by the employer to entice union members to ratify a contract. For example, the 2015 deals between United Automobile Workers and Fiat Chrysler, GM and Ford all featured signing bonuses, as did the 2014 agreement between the British Columbia government and the British Columbia Teachers' Federation.

[8]This discount function was first proposed by Phelps and Pollak (1968), who used it to model intergen-

ies a two-player alternating-offer bargaining game played between quasi-hyperbolic agents. For the main analysis, quasi-hyperbolic agents are assumed to be *sophisticated*, *i.e.* aware that they will suffer from self-control problems in future periods. Unlike in Ståhl (1972) and Rubinstein (1982), the stream of surpluses to be shared is infinite, with one (perfectly divisible) unit available each period: each offer specifies an allocation of the entire stream of surpluses, and the game ends when an offer is accepted. If there is delay, surplus from the period(s) preceding the agreement vanishes. In many economically relevant situations, such as employment relations and partnerships, the stream of surpluses to be shared occurs over a time horizon that does not have a definite end, and delay results in lost opportunities.

Section 3 studies Markov perfect equilibria[9] in this paper's bargaining game played between two quasi-hyperbolic agents that have the same discount factor δ, but potentially different β. Like in standard Rubinstein-Ståhl bargaining, agreement is immediate, and equilibrium payoffs are unique when $\beta_1 \neq \beta_2$, where β_i denotes player i's β. However, player 1's payoff is discontinuous in β_1: it jumps up as β_1 goes from slightly below β_2 to slightly above β_2. Moreover, for a given value of $\min\{\beta_1, \beta_2\}$, the set of possible equilibrium outcomes[10] is independent of the value of $\max\{\beta_1, \beta_2\}$ as long as $\beta_1 \neq \beta_2$. The intuition for these results is that the player with higher β maximizes her share of future surplus when proposing, while the player with lower β maximizes his share of current surplus. From period $t-1$'s perspective, payoff v from period t is worth $\beta\delta v$ if achieved using surplus from period t, but δv if achieved using surplus from later periods. Therefore, only the β for the agent obtaining current surplus in the game's continuation matters, which implies that the agent with the higher β acts like an exponential discounter.

Section 4 performs the analysis from Section 3, but for subgame-perfect equilibria. Here, equilibrium payoffs are no longer almost always unique. As explained above, a given future payoff may correspond to different current reservation values; this potentially gives rise to multiple equilibrium payoffs. For many parameter values, this potential multiplicity is realized and sustains equilibria where the player with higher β maximizes current surplus, while the player with lower β maximizes future surplus. Unlike in Markov perfect equilibrium,

erational saving, interpreting the factor β as a measure of the current generation's altruism toward future generations. More recently, since Laibson (1997), the application of quasi-hyperbolic discounting to individual intertemporal preferences has received substantial attention. Papers such as Angeletos *et al.* (2001) and Laibson, Repetto and Tobacman (2007) suggest that quasi-hyperbolic discount functions explain empirical data substantially better than exponential ones. Gul and Pesendorfer (2005) and, more specifically, Montiel Olea and Strzalecki (2014) provide foundations for such preferences.

[9] In the sense that players' strategies may depend on time, but not on other payoff-irrelevant history.

[10] Because players share the same δ, they agree on the relative valuation of surplus from future periods. Therefore, there can be multiple ways of achieving the unique equilibrium payoffs.

players can be incentivized to make such offers because continuation values following a rejection of these offers are allowed to be lower than for other offers. Because obtaining current surplus hurts one's reservation value in earlier periods, and the player with lower β cannot commit against doing so in Markov perfect equilibrium, Markov perfect equilibria are the worst subgame-perfect equilibria for the player with lower β, and the best ones for the player with higher β. Furthermore, as a result of the multiplicity of continuation play, delay may occur in equilibrium even though bargaining occurs between only two parties, with complete and perfect information.

Section 5 considers some extensions: time-varying surplus, agents having different δ, non-transferable utility, naïveté, and a small amount of incomplete information about β.

Economists have used quasi-hyperbolic preferences mainly to model individual decision-making.[11] Some have also studied interactions between time-consistent and quasi-hyperbolic agents.[12] This paper contributes to a growing literature instead studying interactions between time-inconsistent agents.[13] Most relatedly, some papers have considered sophisticated non-exponential discounters engaging in Rubinstein-Ståhl bargaining, where, unlike in this paper, the entire surplus is realized upon agreement. Kodritsch (2014) shows that when agents exhibit present bias, agreement is immediate, and subgame-perfect equilibrium is unique. In the case of quasi-hyperbolic discounters with parameters β_i and δ_i, the equilibrium is the same as the equilibrium with exponential agents whose discount factors are $\beta_i\delta_i$.[14] By contrast, multiple equilibria and delay are possible in this paper because offers must specify a division of both present and future surplus.[15] It follows that, with quasi-hyperbolic discounting, even when the parties have the same discount function, the problem of bargaining over a stream of payoffs cannot be reduced to bargaining over the discounted

[11] For example, O'Donoghue and Rabin (1999a, 1999b and 2001) study procrastination.

[12] For example, Della Vigna and Malmendier (2004) study firms facing quasi-hyperbolic consumers, and Bisin, Lizzeri and Yariv (2015) examine government policy with time-inconsistent voters.

In a bargaining setting, Akin (2007 and 2009) studies play between an exponential discounter and a quasi-hyperbolic discounter that has incomplete information about the extent of her own self-control problems.

[13] Chade, Prokopovych and Smith (2008) study repeated games between quasi-hyperbolic discounters with parameters β and δ. They show that such a game's payoff set is contained within the payoff set obtained when the players are replaced by exponential discounters with discount factor Δ, such that $1 + \beta\delta + \beta\delta^2 + ... = 1 + \Delta + \Delta^2 + ...$ By contrast, in this paper's bargaining game, payoff uniqueness holds with exponential discounters, but can fail with quasi-hyperbolic agents.

[14] Ok and Masatlioglu (2007) and Pan, Webb and Zank (2015) axiomatize alternative models of time preference and apply them to Rubinstein-Ståhl bargaining, with similar results.

[15] Rusinowska (2004), Ok and Masatlioglu (2007) and Kodritsch (2014) note that multiple equilibria and delay can occur in Rubinstein-Ståhl bargaining when agents do not have present bias, *e.g.* whose discount factors for payoffs $1, t, t+1$ periods from now, denoted d_1, d_t, d_{t+1}, satisfy $d_1 > \frac{d_{t+1}}{d_t}$ for some t. Because quasi-hyperbolic discounting implies present bias, the source of multiple equilibria and delay in this paper is quite different.

aggregate surplus.

Observe that Kodritsch's (2014) result implies that in equilibrium, present bias is indistinguishable from time-consistent impatience in complete-information Rubinstein-Ståhl bargaining. This is not surprising given the fact that, in equilibrium, players do not exert self-control at any point: the proposer always demands as much of the current surplus as she can, and the receiver always accepts the offer. Therefore, in order to study the impact of self-control problems on bargaining, it is important that the bargaining occurs over both current and future surplus simultaneously, as is the case in this paper.

Sarafidis (2006), Akin (2007) and Haan and Hauck (2014) study Rubinstein-Ståhl bargaining between potentially naïve agents, who mistakenly believe in each period that their own future selves' preferences are consistent with current preferences.[16] All assume that naïve players believe that their opponent shares their beliefs about both players' future preferences. In Sarafidis (2006), naïfs believe that all agents are exponential discounters in the future; in Akin (2007), naïfs are sophisticated about their opponent's preferences; in Haan and Hauck (2014), either may be the case,[17] but unlike in both other papers, sophisticates also believe that their opponent shares their beliefs about players' future preferences. These papers all show that naïfs may benefit from their naïveté, and that delay may arise because naïfs underestimate a sophisticated opponent's reservation value. The analysis in Section 5.4 of this paper differs from existing work not only in that the surplus is a stream, but also in that a naïve player i is aware that their opponent j believes that i's future selves will have self-control problems - that is, the players agree to disagree, with j believing that i is overoptimistic about future self-control. Despite the latter difference, which ensures that players always know each other's current reservation value, there remains scope for delay - now intentional - even in Markov perfect equilibrium: j may find i's reservation value too high due to i's erroneous belief about the future.

2 Model

2.1 Self-Control

This paper uses quasi-hyperbolic discounting (Phelps and Pollak (1968) and Laibson (1997)) to model time-inconsistency and self-control issues. With quasi-hyperbolic discounting, in

[16] Naïveté was suggested by Strotz (1956). Ali (2011) derives conditions under which an agent learns about her own preferences and becomes sophisticated.

[17] Players may even be sophisticated about their own preferences, but naïve about their opponent's.

period t, the agent applies discount factor $\beta\delta^{\tau-t}$ to a payoff obtained in period $\tau > t$, where $\beta \in (0, 1]$ and $\delta \in (0, 1)$. When $\beta < 1$, such preferences are time-inconsistent: for example, if $\delta > 0.8$, an agent prefers a payoff of 4 tomorrow to a payoff of 5 in two days, but if $\beta\delta < 0.8$, the agent's choice reverses tomorrow. Throughout this paper (except in Section 5.4), quasi-hyperbolic agents are *sophisticated*, *i.e.* they are fully aware of their preferences in future periods.

2.2 The Bargaining Game

Time is discrete, utility is transferable, and the surplus available in each period is 1. In period t, player $t(\bmod 2)+1$ proposes a division $(x, \overrightarrow{1} - x)$, where $x = (x^t, x^{t+1}, ...) \in [0, 1]^\infty \equiv X$ is the stream of payoffs kept by player $t(\bmod 2) + 1$, and $\overrightarrow{1} - x$ is the opponent's stream of payoffs. The other player then decides whether to accept or reject the proposal. If the proposal is accepted, the game ends, and players receive the specified payoffs. If the proposal is rejected, the surplus from period t is lost, and play moves on to period $t + 1$.

Formally, at the beginning of period $t > 0$, let the history h^t of the game consist of the proposals in periods $0, ..., t-1$ (trivially, for the game to reach t, all responses must have been rejections). Let H^t be the set of all possible h^t, and let $H^0 = \{h^0\}$ be a singleton containing only the trivial history. Then a pure strategy for player 1 is a pair of functions (f, g) where $f : \cup_{k=0}^{\infty} H^{2k} \to X$ and $g : \cup_{k=0}^{\infty}(H^{2k+1} \times X) \to \{accept, reject\}$, and a pure strategy for player 2 is a pair of functions (f, g) where $f : \cup_{k=0}^{\infty}(H^{2k} \times X) \to \{accept, reject\}$ and $g : \cup_{k=0}^{\infty} H^{2k+1} \to X$.

Two solution concepts are used in this paper: subgame-perfect Nash equilibrium (SPNE) and Markov perfect equilibrium (MPE). SPNE is defined in the usual way, except that an agent's selves from different periods are considered different players, and therefore maximize their utility taking both other agents' and other selves' strategies as given. MPE is defined as follows:

Definition: A strategy profile is a MPE if it is a SPNE where the strategies depend only on t.

This definition of MPE is weaker than requiring stationarity: offers are not allowed to vary across histories in the same period, but they are allowed to vary across periods.

3 Markov-Perfect Equilibria

This section investigates MPE in the bargaining game between quasi-hyperbolic agents. One player may have more self-control than the other (higher β), but for simplicity, players are assumed to share the same discount factor δ (this assumption is relaxed in Section 5.2). Given this assumption, with standard exponential discounters, the bargaining game would essentially collapse into the standard Rubinstein-Ståhl game with surplus $\frac{1}{1-\delta}$: in any SPNE, player 1 obtains surplus with present value $\frac{1}{1+\delta}\frac{1}{1-\delta} = \frac{1}{1-\delta^2}$, while player 2's payoff is $\frac{\delta}{1-\delta^2}$.

Let the sequence of discount factors be $\{1, \beta_1\delta, \beta_1\delta^2, ...\}$ for player 1, and $\{1, \beta_2\delta, \beta_2\delta^2, ...\}$ for player 2.

Proposition 1: A MPE of the bargaining game exists, and in any MPE, player 1's offer in period 0 is accepted. When $\min\{\beta_1, \beta_2\} \geq \frac{1}{\delta(1+\delta)}$,[18] player 1's aggregate payoff v_1 is as follows:

a) If $\beta_1 > \beta_2$, $v_1 = \frac{\beta_1}{\beta_2}(\frac{1}{1-\delta^2})$. Player 2 obtains all of the period-0 surplus.

b) If $\beta_1 < \beta_2$, $v_1 = \frac{1}{1-\delta^2} - \frac{\delta}{1-\delta}(1-\beta_1)$. Player 1 obtains all of the period-0 surplus.

c) If $\beta_1 = \beta_2 = \beta$, $v_1 \in \left[\frac{1}{1-\delta^2} - \frac{\delta}{1-\delta}(1-\beta), \frac{1}{1-\delta^2}\right]$.

Proof: All proofs are provided in the Appendix.

Because players agree on the relative value of payoffs in different future periods, even when total payoffs are unique, the equilibrium itself is not unique. For example, in Case a, v_1 can come from any combination of payoffs from period 1 on. However, when $\beta_1 \neq \beta_2$, the allocation of the current-period surplus is unique: if the proposer has the higher β, she pays her opponent starting with the current surplus, whereas if the proposer has the lower β, her demand will include the entire current surplus. The condition $\min\{\beta_1, \beta_2\} \geq \frac{1}{\delta(1+\delta)}$ guarantees that, in equilibrium, when the proposer has the higher β, she offers her opponent at least the entire current surplus.

As expected, player 1's payoff increases in β_1. It also decreases in β_2 when $\beta_1 > \beta_2$, but, interestingly, is independent of β_2 when $\beta_1 < \beta_2$. Note that when $\beta_1 = \beta_2 = 1$, all of the above values are equal to $\frac{1}{1-\delta^2}$, as in Rubinstein-Ståhl bargaining with total surplus $\frac{1}{1-\delta}$.

The payoffs can be interpreted by imagining each player being endowed with the surpluses from periods where they propose. Therefore, from an exponential discounting standpoint,

[18] This case encompasses the relevant range of parameters for most applications: for example, any parameters satisfying $\beta + \delta \geq \frac{1+\sqrt{5}}{2} \approx 1.618$ satisfy this condition.
The results for the case $\min\{\beta_1, \beta_2\} < \frac{1}{\delta(1+\delta)}$ are given in the Appendix.

player 1's endowment is worth $\frac{1}{1-\delta^2}$. In Case a, player 1, as the more patient player, effectively trades the current surplus against future surplus worth $\frac{1}{\beta_2}$ from the perspective of an exponential discounter, or $\frac{\beta_1}{\beta_2}$ from the perspective of player 1. Therefore, player 1 increases her endowment by a factor of $\frac{\beta_1}{\beta_2}$ by trading away her then-current surplus every time she proposes.

In Case b, player 1, now the less patient player, incurs a loss from trade of $(1-\beta_1)\delta$ in every period. This occurs because, in the MPE of every subgame that starts with a proposal, player 1 obtains the entire current surplus (either because she asks to keep her then-current endowment, or because player 2 trades his then-current endowment with her), which is worth $\delta - (1-\beta_1)\delta$ from the previous period's perspective. This intuition explains why, when $\beta_1 < \beta_2$, v_1 does not depend on β_2: when β_2 varies, as long as it remains above β_1, the same trades occur, resulting in the same losses for player 1. Similarly, when $\beta_1 > \beta_2$, player 1's exponentially discounted (with factor δ) surplus $\frac{1}{\beta_2}(\frac{1}{1-\delta^2})$ does not depend on β_1.

In Case c, a range of surpluses (whose extremes correspond to Cases a and b) is possible because trades can go either way: every period, the proposer is indifferent whether to trade her then current-surplus. In the exponential case ($\beta = 1$), the surplus becomes unique because such trades do not matter: the loss $(1-\beta)\delta$ is nil.

To illustrate the effect of β on a player's bargaining strength[19], Figure 1 plots $1-(1-\delta)\frac{r_2}{\beta_2}$ against β_1, where r_2 is player 2's reservation value in period 0, $\beta_2 = \frac{2}{3}$, and $\delta = 0.95$. Since r_2 is entirely derived from future payoffs, $\frac{r_2}{\beta_2}$ is simply the period-0 exponentially discounted present value of player 2's continuation payoff. This quantity is expressed as a share through multiplication by $1-\delta$, and subtracting the result from 1 yields a measure of player 1's bargaining strength. This quantity represents player 1's best possible exponentially discounted share if she were to keep the period-0 surplus. That is, it does not include player 1's gain (from an exponential perspective) from trading away period-0 surplus when $\beta_1 > \beta_2$, and therefore does not confound this direct gain (which is small from the perspective of player 1's period-0 self when β_1 only slightly exceeds β_2) with player 1's underlying bargaining position, which depends on off-path play.[20]

[19] This exercise is not meant to assess welfare: it is difficult to do so in a clear-cut way for quasi-hyperbolic discounters because what is good for one period's self may not be good for another. Lu (2016) shows that one can interpret a quasi-hyperbolic discounter as a modified Fudenberg and Levine (2006) dual self that does not care about future self-control costs; a natural measure of welfare would then be the exponentially discounted present value of payoffs minus self-control cost.

[20] Instead directly taking the payoff from player 1's period-0 perspective would not be very informative: the value of a given stream of payoffs could vary with β_1. This is problematic even if, for example, the payoff is normalized through division by the value for the entire stream of surplus, $1 + \frac{\beta_1\delta}{1-\delta}$: as $\beta_1 \to 0$, almost the

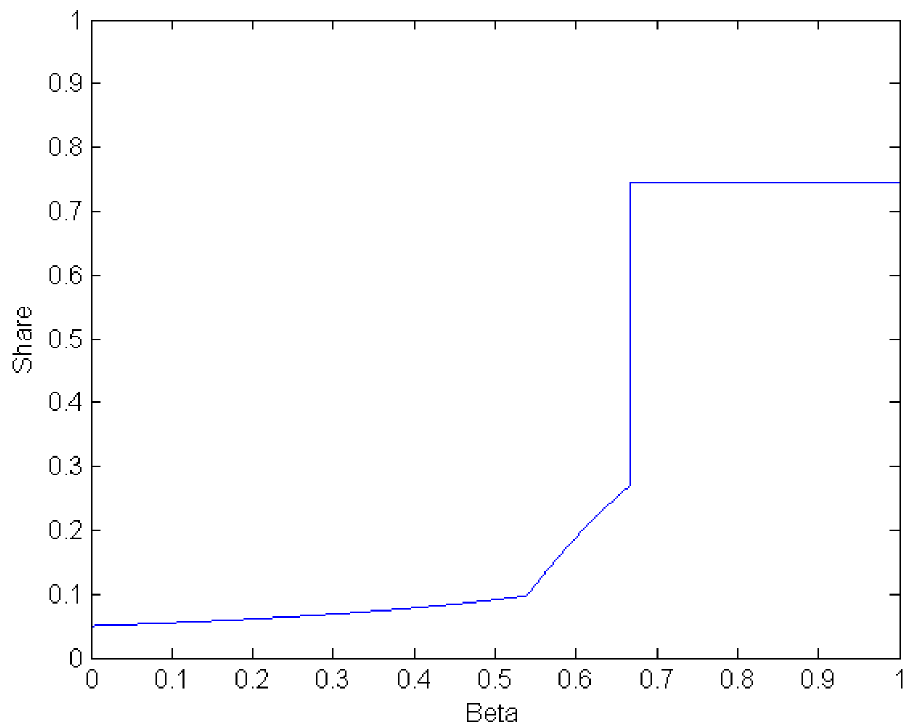

Figure 1: Player 1's MPE bargaining strength vs. β_1, for $\beta_2 = \frac{2}{3}$ and $\delta = 0.95$

The most striking feature of Figure 1 occurs at $\beta_1 = \frac{2}{3}$: player 1's payoff makes a large jump of 0.475, and is then flat.[21] This discontinuity arises because in future off-path play, as explained earlier, only the self-control problem of the agent with lower β is exploited. Therefore, β_1 moving from slightly below to slightly above $\beta_2 = \frac{2}{3}$ has the same effect on equilibrium play as β_1 jumping from slightly below $\frac{2}{3}$ to 1 and β_2 jumping from 1 to $\frac{2}{3}$.

The kink located at $\beta_1 = \frac{1}{\delta(1+\delta)} \approx 0.54$ is the point past which player 1 is no longer satisfied with only obtaining current surplus when player 2 proposes. As a result, further increases in β_1 become more costly to player 2.

4 Subgame-Perfect Equilibria

This section investigates SPNE in the bargaining game between quasi-hyperbolic agents. Like for Section 3, players may have different β, but are assumed to share the same discount factor δ.

Proposition 2: Player 1's supremum and infimum aggregate SPNE payoffs $\overline{v_1}$ and $\underline{v_1}$ satisfy the following properties:

a) If $\beta_1 < \beta_2$, then $\underline{v_1}$ is player 1's MPE payoff; if $\beta_1 > \beta_2$, then $\overline{v_1}$ is player 1's MPE payoff;

b) For any $v \in [\underline{v_1}, \overline{v_1}]$, there is a SPNE where player 1's payoff is v; and

c) When $\min\{\beta_1, \beta_2\} \geq \frac{1}{\delta(1+\delta)}$, then a multiplicity of SPNE payoffs exists if and only if $\frac{\beta_1}{\beta_2} \in [1-(1-\beta_1)\delta, \frac{1}{1-(1-\beta_2)\delta}]$.[22]

The proof of Proposition 2 provides the equations that determine $\overline{v_1}$ and $\underline{v_1}$. They are piecewise linear, and hence straightforward to solve. Part c is restated in the Appendix to explicitly give the values of $\underline{v_1}$ and $\overline{v_1}$ for the case $\min\{\beta_1, \beta_2\} \geq \frac{1}{\delta(1+\delta)}$.[23] These values correspond, at $\beta_1 = \beta_2$, to the minimum and maximum MPE payoffs from Proposition 1.

entire weight would be put on the current period payoff. The ensuing conclusion, that player 1 is very well off when β_1 is very low, is misleading: it is only valid for the period-0 self, who is happy getting the period-0 surplus and little else.

[21] The size of the jump is $\frac{1-\beta}{\beta}\delta$ when $\min\{\beta_1, \beta_2\} \geq \frac{1}{\delta(1+\delta)}$.

[22] The Appendix shows that SPNE payoff multiplicity also arises when $\min\{\beta_1, \beta_2\} < \frac{1}{\delta(1+\delta)}$ and the β's are sufficiently close.

[23] As noted previously, the case $\min\{\beta_1, \beta_2\} \geq \frac{1}{\delta(1+\delta)}$ covers typical parameter values. The number of cases to consider when $\min\{\beta_1, \beta_2\} < \frac{1}{\delta(1+\delta)}$ is large.

This observation and part a imply that when $\min\{\beta_1, \beta_2\} \geq \frac{1}{\delta(1+\delta)}$, $\underline{v_1}$ and $\overline{v_1}$ are continuous at $\beta_1 = \beta_2$ despite the large discontinuity in MPE payoffs there.

The intuition for part a of Proposition 2 is as follows. As explained in Section 3, demanding more of the current surplus is bad from the previous period's perspective. This implies that allocating the then-current surplus to their opponent in each future period (in exchange for more later surplus) is beneficial to a player's bargaining position. For the player with higher β, this is achieved by efficient offers, which must occur in MPE because play in any period is independent from the history of offers. By contrast, the player with lower β would like to commit future selves to trading away the then-current surplus, which is impossible in MPE. Therefore, MPEs correspond to the SPNEs where the player with lower β maximally worsens her bargaining position in earlier periods at every history where she proposes, and where the player with higher β always avoids doing so.

The proof of part b of Proposition 2 constructs a class of strategy profiles that achieves any payoff $v \in [\underline{v_1}, \overline{v_1}]$. Non-MPE payoffs are achieved by incentivizing, at certain histories, the player with lower β to trade away her then-current surplus and the player with higher β to keep it. Essentially, player 1 demands v efficiently in period 0, and player 2 is supposed to accept. Whenever a player deviates, that player is punished in the game's continuation.[24] The player with lower β is punished with a MPE,[25] while the player with higher β is punished with continuation play where, at every history, she obtains her minimum SPNE payoff *as inefficiently (with as much current surplus) as possible* - thus hurting her reservation in the previous periods - subject to her opponent obtaining at least his MPE payoff at every history. This potentially inefficient continuation constitutes a SPNE because both players have an incentive to reject efficient offers and thereby jump to their best SPNE. In particular, suppose $\beta_1 > \beta_2$. In the punishment profile for player 1, which achieves her payoff $\underline{v_1}$, she does not demand $\underline{v_1}$ efficiently, which would leave player 2 with his highest possible continuation value $\overline{w_2}$. Rather, she demands $\underline{v_1}$ inefficiently, which therefore gives player 2 a lower payoff. However, were player 1 to deviate by increasing both players' payoffs with a more efficient offer, player 2 would reject: doing so would entitle him to future payoffs worth $\overline{w_2}$ from the current perspective, while accepting would yield a lower payoff as long as player 1 is getting more than $\underline{v_1}$.

Payoff multiplicity does not arise for all parameter values: sometimes, the SPNE profile

[24] Except, of course, when the proposer is even more generous to the receiver than in the receiver's best SPNE, or when the receiver accepts an offer that she is supposed to reject. Neither of these deviations is profitable.

[25] In a coalitional bargaining setting with deadline, Ambrus and Lu (2015) also provide an example of a non-Markovian SPNE sustained by a MPE.

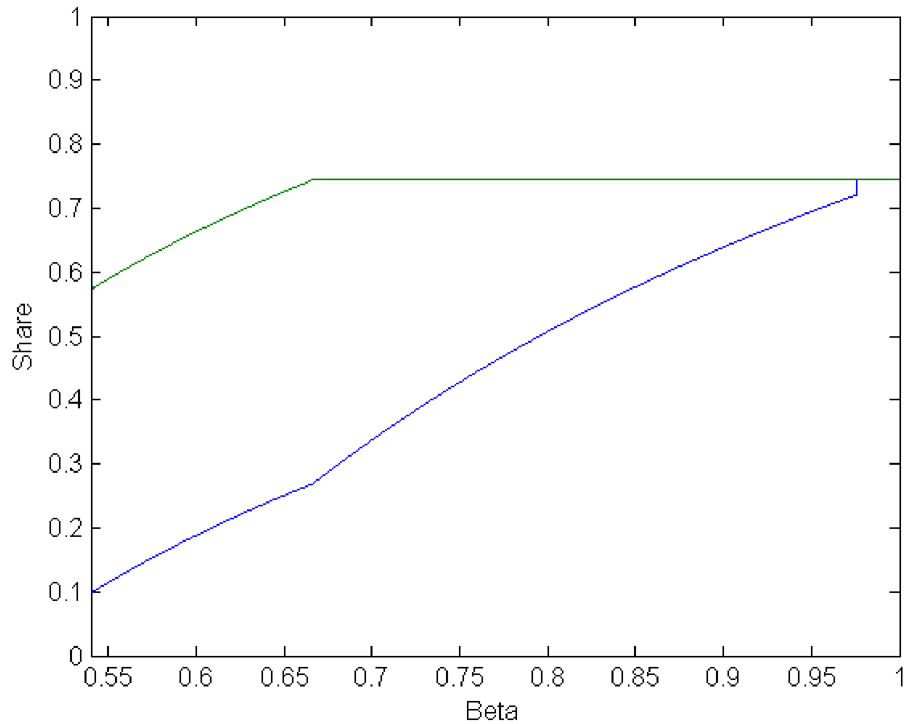

Figure 2: Player 1's SPNE bargaining strength vs. β_1, for $\beta_2 = \frac{2}{3}$ and $\delta = 0.95$

must be efficient, and therefore yield MPE payoffs. Part c of Proposition 2 shows that SPNE payoff multiplicity will arise if β_1 and β_2 are sufficiently close, which is intuitive given part a of Proposition 2 and MPE payoff multiplicity at $\beta_1 = \beta_2$. It also shows that SPNE payoffs are more likely to be unique when δ is small: when this is the case, the payoff impact of future inefficiency is small, so only a small degree of present inefficiency can be sustained. For δ low enough, iterating this reasoning leads to zero inefficiency. For a similar reason, SPNE payoffs are unique when $\max\{\beta_1, \beta_2\}$ is very close to 1: here, the impact of future inefficiency on the reservation value of the player with higher β is small, while the cost of inefficiency becomes relatively large. Once again, this limits the amount of inefficiency that can be sustained, and iterating this reasoning rules out any inefficiency.

Figure 2 is the analog of Figure 1 for SPNE: it plots player 1's maximum and minimum bargaining strengths as a function of β_1, fixing $\beta_2 = \frac{2}{3}$, and $\delta = 0.95$. The range of β_1 is $[0.54, 1]$, which ensures $\beta_1 > \frac{1}{\delta(1+\delta)}$.

As implied by Proposition 2a, the plot in Figure 1 corresponds to the bottom curve in Figure 2 for $\beta_1 < \frac{2}{3}$, and to the top curve for $\beta_1 > \frac{2}{3}$. When viewed in light of Proposition

2b, Figure 2 shows that the multiplicity of equilibrium payoffs is severe for β_1 around $\frac{2}{3}$, and diminishes as β_1 increases. The curves coincide for the high values of β_1 corresponding to the case $\frac{\beta_1}{\beta_2} > \frac{1}{1-(1-\beta_2)\delta}$, where $\underline{v_1} = \overline{v_1}$. Unlike the MPE plot, the curves are continuous at $\beta_1 = \frac{2}{3}$, but the bottom curve is discontinuous at the point where the SPNE payoffs become unique.

An implication of payoff multiplicity is that there exist SPNEs with delay, unlike MPEs and unlike with time-consistent agents. For example, it is straightforward to construct SPNEs where, in periods 0 through $T-1$, the proposer demands the whole surplus and the opponent only accepts offers preferable to their best SPNE continuation payoff, and in period T, a moderate division is proposed and accepted. Deviations from demanding the whole surplus can be punished with jumping to the player's worst SPNE continuation, and this enforces delay as long as getting the moderate payoff later is preferable to getting the worst SPNE payoff today.

For pure-strategy SPNE, the lower bound on payoffs in Proposition 2 implies an upper bound on delay: agreement must occur early enough so that both players receive at least their minimum SPNE payoff from the perspective of period 0. For example, suppose $\beta_1 \geq \frac{1}{\delta(1+\delta)}$ and $\frac{\beta_1}{\beta_2} \in [1-(1-\beta_1)\delta, 1]$, so that in period 0, $\underline{v_1} = \frac{1}{1-\delta^2} - \frac{\delta}{1-\delta}(1-\beta_1)$ (see Appendix), while player 2's reservation value must be at least

$$\begin{aligned}\frac{\beta_2}{\beta_1}(1+\frac{\beta_1\delta}{1-\delta}-\overline{v_1}) &= \frac{\beta_2}{\beta_1}(1+\frac{\beta_1\delta}{1-\delta}-\frac{1}{1-\delta^2}+\frac{\delta}{1-\delta}(1-\frac{\beta_1}{\beta_2})) \\ &= \frac{\beta_2}{\beta_1}\frac{\delta}{1-\delta^2}-\frac{\delta}{1-\delta}(1-\beta_2).\end{aligned}$$

Then, if there is delay, the sum of exponentially discounted aggregate payoffs must be at least

$$\begin{aligned}&\frac{1}{\beta_1}\left[\frac{1}{1-\delta^2}-\frac{\delta}{1-\delta}(1-\beta_1)\right]+\frac{1}{\beta_2}\left[\frac{\beta_2}{\beta_1}\frac{\delta}{1-\delta^2}-\frac{\delta}{1-\delta}(1-\beta_2)\right] \\ = &\frac{1}{1-\delta}\left[\frac{1}{\beta_1}-\delta(\frac{1}{\beta_1}+\frac{1}{\beta_2}-2)\right].\end{aligned}$$

Thus, again letting T denote the length of delay, we must have, in SPNE,

$$\delta^T \geq \frac{1}{\beta_1}-\delta(\frac{1}{\beta_1}+\frac{1}{\beta_2}-2).$$

- As β_1 increases or as β_2 decreases, the right-hand side decreases, which increases the upper bound on T. This is not surprising given Figure 2: as the β's become closer, the difference between the maximum and minimum SPNE payoffs increases, which leads to more scope for delay.

- As δ increases, the left-hand side increases and the right-hand side decreases, which increases the upper bound on T. As players become more patient, on the one hand, delay becomes less costly, and on the other hand, the poor trades in the continuation of a player's worst SPNE have more weight. Both of these increase the amount of acceptable delay.

On the other hand, the length of the delay can be unbounded in SPNEs with mixing. For example, if the β's are close, it is possible to construct SPNEs where, whenever proposing, each player demands somewhat more than half of every period's surplus. Deviations are again punished by jumping to the player's worst SPNE. Whenever responding on path, each player mixes in such a way that players are indifferent between rejecting and accepting at all histories where they respond. Nevertheless, the *expected* length of delay remains bounded for the same reason as above.

5 Extensions

Sections 5.1 to 5.3 examine the implications of some modifications to the bargaining game of Section 2.2. In each of these cases, a multiplicity of SPNE payoffs can be obtained for suitable parameter values in the same way as in Section 4, *i.e.* by defining continuation play where the player with higher β inefficiently obtains more current surplus than in MPE, which causes her to have lower reservation value than in MPE in preceding periods. The analysis below focuses on MPE.

Sections 5.4 and 5.5 discuss the impact on MPE of certain changes to players' information about β: players may be *naïve* about their own self-control problems (mistakenly believing that their future selves have $\beta = 1$), and they may not exactly know the opponent's β.

5.1 Time-Varying Surplus

Suppose that the size of the surplus available in each period is $s_t > 0$, with $\sum_{t=0}^{\infty} \delta^t s_t < \infty$. Let $\beta_i > \beta_j$ in the quasi-hyperbolic model, and let v_k^t be agent k's aggregate payoff when she proposes in period t. The following result is qualitatively similar to Proposition 1, and shows that the payoff expressions remain tractable in many cases.

Proposition 3: Suppose that at every t' where player i proposes,

$$\delta\left[\sum_{k=0}^{\infty} s_{t'+1+2k}\delta^{2k} - (1-\beta_j)\sum_{k=1}^{\infty} s_{t'+1+k}\delta^{k}\right] - (1-\beta_j)\delta s_{t'+1} \geq s_{t'}.$$

Then aggregate MPE payoffs are:

$$\begin{aligned} v_i^t &= \frac{\beta_i}{\beta_j}\sum_{k=0}^{\infty} s_{t+2k}\delta^{2k}, \text{ and} \\ v_j^t &= \sum_{k=0}^{\infty} s_{t+2k}\delta^{2k} - (1-\beta_j)\sum_{k=1}^{\infty} s_{t+k}\delta^{k}. \end{aligned}$$

The condition for Proposition 3 is the analog of $\beta_j \geq \frac{1}{\delta(1+\delta)}$ in Proposition 1. It ensures that whenever player i proposes in a period t', player j's continuation value is above $s_{t'}$, so that the entire current surplus $s_{t'}$ would be offered to player j. Like in Section 3, as $\beta_i - \beta_j \to 0$, the payoffs do not converge to each other.

The payoffs in Proposition 3 can be interpreted using the same thought experiment as those in Proposition 1: both players are endowed with the surplus from the periods where they propose. Whenever player i proposes, she trades her current surplus away for future surplus, and the gains from trade increase her payoff by a factor of $\frac{\beta_i}{\beta_j}$. Because, in every future period, player j would obtain the then-current surplus, her reservation value suffers by $(1-\beta_j)$ times the value of all future surplus.

5.2 Different Discount Factors

Suppose that instead of sharing the same δ, agents have discount factors δ_1 and δ_2. In this case, the equations relating players' payoffs are piecewise linear, with infinitely many segments. Therefore, even in the exponential case, a general explicit solution would involve infinitely many cases. Below is a qualitative analysis of this situation.

In order to offer player l one unit of utility using surplus t periods in the future, player k must give up $\frac{\beta_k}{\beta_l}\left(\frac{\delta_k}{\delta_l}\right)^t$ units of utility. In MPE, when player k proposes, she will first offer surplus from periods where this "exchange rate" is the lowest.

When $(\beta_1 - \beta_2)(\delta_1 - \delta_2) > 0$, the exchange rate is increasing for the player with higher β (henceforth player i), and decreasing for player $j \neq i$. Therefore, like in the equal δ case, player i will offer as little current surplus as possible, while player j will demand all current surplus. It follows that equilibrium play will once again be as if $\beta_i = 1$.

When $\delta_i < \delta_j$, however, player i first offers the farthest-away surplus (t^* or more periods from the proposal for some t^*), then the current surplus, and finally future-but-close surplus (between 1 and $t^* - 1$ periods from the proposal), and player j does the reverse. If β_i is only slightly larger than β_j so that $\beta_i \delta_i < \beta_j \delta_j$, then $t^* = 1$, so the equilibrium would be as if $\beta_j = 1$. As $\frac{\beta_i}{\beta_j}$ increases, t^* increases, which eventually leads to player i offering current surplus when proposing and to player j demanding current surplus when proposing. When this occurs, player j no longer acts as if $\beta_j = 1$, which means that player i's payoff jumps up. Therefore, just like in the equal δ case, v_1 is discontinuous in β_1, though the discontinuity no longer occurs at $\beta_1 = \beta_2$.

5.3 Non-Transferable Utility

Suppose players' instantaneous utility in period t from receiving share $x^{t'}$ is $u(x^{t'})$, where u is a smooth, increasing and strictly concave function, so that player i's utility function in period t is $u(x_i^t) + \beta_i \sum_{t'=t+1}^{\infty} \delta^{t'-t} u(x_i^{t'})$. Let $\beta_i > \beta_j$.

If $\frac{u'(0)}{u'(1)} > \frac{\beta_i}{\beta_j}$, then, when proposing in a MPE, player i will never offer the entire current surplus, and player j will never demand the entire current surplus. Therefore, while, like with transferable utility, player i benefits from her superior self-control, she does not take full advantage of it due to a desire to smooth surplus intertemporally. In particular, v_1 is no longer discontinuous at $\beta_1 = \beta_2$ due to the smoothness and strict concavity of u, but it would still increase very fast in β_1 around $\beta_1 = \beta_2$ if u is close to linear.

5.4 Naïveté

Introducing naïveté in the context of games raises several issues. First, are both players naïve, or is only one of them naïve? Second, are naïve players aware of their opponent's self-control problems? Third, is players' naïveté common knowledge?[26] An exhaustive examination of the possibilities is outside the scope of this paper. The following explores MPE in the case of one naïve player (player 1) that is aware of her opponent's time inconsistency - perhaps she is over-optimistic about her own self-control - and whose naïveté is common knowledge.[27] It is

[26] There is also the conceptual issue of how having incorrect beliefs about one's own future behavior can be consistent with equilibrium analysis.

[27] In particular, this means that player 1 knows that player 2 believes that player 1 is quasi-hyperbolic and naïve.

useful to note that when player i makes an offer, since player j's preferences and information are common knowledge, player j's reservation value is common knowledge as well.[28]

If $\beta_1 > \beta_2$, the equilibrium from Section 3, denoted σ^*, remains valid. First, both naïve and sophisticated selves agree that when player 1 makes an offer, she only demands current surplus if she also demands all future surplus. Second, when player 2 makes an offer, he always demands the entire current surplus. Therefore, when player 1 decides whether to accept an offer, she weighs streams of payoffs that are exclusively in the future. Thus, despite her naïveté, she still believes, in any period, that player 2's future offers in σ^* are all acceptable. As a result, there is no disagreement about future play, and reservation values remain the same as in the sophisticated case.

However, if $\beta_1 < \beta_2$, the analysis qualitatively changes. Here, if player 2 makes an acceptable offer in period t, he will always meet player 1's reservation value while offering her as little future surplus as possible; this is common knowledge. However, at time t, the players disagree about play at time $t+1$: player 1 believes that she will act like an exponential discounter and "trade away" the surplus from time $t+1$, while player 2 believes that player 1 will demand the entire surplus from time $t+1$. This implies that player 1's reservation value is too high from the perspective of player 2. If the time-t surplus is sufficiently large relative to the time-$t+1$ surplus - in particular, if the surplus is constant - player 2 will still meet player 1's reservation value, as losing the time-t surplus is worse.[29] This will increase player 1's payoff relative to the sophisticated case. However, if the time-t surplus is sufficiently low, then player 2 will not make an acceptable offer, and delay must arise in that subgame. Either way, player 1's naïveté hurts player 2, as it forces player 2 to either accommodate player 1 or to effectively give up his time-t endowment by postponing agreement.[30]

[28] As noted below, they may not agree on future play. The point is that they agree on player j's beliefs about future play.

[29] The gain in player 1's period-t reservation value that would arise from player 1 trading away instead of demanding the entire time-$t+1$ surplus (denoted s_{t+1}) at time $t+1$ is $\left(\frac{\beta_1}{\beta_2}\delta - \beta_1\delta\right) s_{t+1} = \frac{\beta_1}{\beta_2}\delta(1-\beta_2)s_{t+1} < s_{t+1}$. Thus, if $s_t = s_{t+1}$, player 2 will meet player 1's inflated reservation value.

[30] What if player 1 were partially naïve, *i.e.* believes that her future selves' β is between β_1 and 1? If $\beta \in (\beta_2, 1)$, then the reasoning from this paragraph applies. If $\beta \in (\beta_1, \beta_2)$, then σ^* remains an MPE. Even in the latter case, player 1 will mistakenly believe (as she would if more naïve) that she will reject future offers by player 2. However, this does not affect play: player 2's reservation value when player 1 proposes is unchanged, and when player 2 proposes, player 1's reservation value, which is based on her offer next period, is also unaffected.

5.5 Incomplete Information about β

Suppose that, while β_1 remains common knowledge, β_2 is player 2's private information, and may take one of two values: β_L and $\beta_H = \beta_L + \varepsilon$, where $\varepsilon > 0$. Let $p \in (0,1)$ be player 1's prior on $\beta_2 = \beta_L$. Then for any fixed p, if ε is sufficiently small, there exists a weak perfect Bayesian equilibrium where the outcome is the same as in the complete information MPE with $\beta_2 = \beta_H$.

The reasoning is simple: the gain that player 1 can realize from type β_L by making a less generous offer that type β_H would reject vanishes as $\varepsilon \to 0$. The positive probability of losing the proposer advantage is then much more costly; player 1 therefore strictly prefers making an offer acceptable to type β_H. This intuition is found in Rubinstein's (1985) study of one-sided incomplete information about δ in the standard Rubinstein-Ståhl model.[31] The result is particularly striking in the case $\beta_L < \beta_1 < \beta_H$: as seen in Section 3, the MPE payoffs of types β_L and β_H can differ greatly. The key is that when type β_L proposes, she has an incentive to mimic type β_H if deviating leads to player 1 putting probability 1 on β_L and to the continuation being the corresponding MPE. In this paper's model, mimicking β_H is costly because β_L has to offer current surplus instead of claiming all of it; however, the cost of doing so is small if ε is small, while the cost of deviating is large. Therefore, in periods where player 1 proposes, type β_L has a reservation value almost as high as type β_H's, and the aforementioned intuition applies.

6 Conclusion

This paper studies the implications of limited self-control in bargaining: for example, borrowers may take a longer-than-necessary amortization period when negotiating loans, while workers may place a disproportionately high weight on the signing bonus when negotiating terms of employment. When quasi-hyperbolic discounters engage in alternating-offer bargaining over a stream of surplus, equilibrium predictions have several noteworthy features. In MPE, even a small difference in the degree of present bias can confer a large advantage to the less present-biased party. In SPNE, there is often a wide range of possible payoffs, and delay is possible despite complete information. Lu (2016) shows that equilibria obtained from an alternative model of limited self-control, Fudenberg and Levine's (2006) dual-self model,

[31] Rubinstein provides an argument for selecting the equilibrium where player 1 gives up on screening when the possible values for player 2's δ are sufficiently close.

are better behaved, but it remains an open question whether the dual-self model describes the behavior of agents with limited self-control better than quasi-hyperbolic discounting.

A natural extension to this paper would be to study the possibility of renegotiation. Indeed, with time-inconsistency, an agreement reached today can fail to be Pareto efficient tomorrow, so agents may agree to modify the division of surplus after a deal is reached.

Acknowledgements

I am grateful to S. Nageeb Ali, Attila Ambrus, David Freeman, John Friedman, Lisa Kahn, Sebastian Kodritsch, David Laibson, Qingmin Liu, Dina Mishra, Marciano Siniscalchi, Andrea Wilson, Leeat Yariv, two anonymous referees, and especially Drew Fudenberg for helpful conversations and suggestions. I would also like to thank seminar participants at Harvard University, and participants at the "Logic, Game Theory and Social Choice 8" and "8th Pan-Pacific Conference on Game Theory" joint conference and the 90th Annual Conference of the Western Economic Association International for their questions and comments. Funding: this work was supported by a Simon Fraser University President's Research Start-Up Grant.

Appendix: Proofs

Proposition 1 (continued): When $\min\{\beta_1, \beta_2\} < \frac{1}{\delta(1+\delta)}$:

a) If $\beta_1 > \beta_2$, $v_1 = 1 + \frac{\beta_1\delta}{1-\delta} - \frac{\beta_2\delta}{1-\beta_2\delta^2}$. Player 2 only obtains period-0 surplus $\frac{\beta_2\delta}{1-\beta_2\delta^2}$.

b) If $\beta_1 < \beta_2$, $v_1 = 1 + \frac{\beta_1^2\delta^2}{1-\beta_1\delta^2}$. Player 1 obtains all of the period-0 surplus.

c) If $\beta_1 = \beta_2 = \beta$, $v_1 \in \left[1 + \frac{\beta^2\delta^2}{1-\beta\delta^2}, 1 + \frac{\beta\delta}{1-\delta} - \frac{\beta\delta}{1-\beta\delta^2}\right]$.

Proof of Proposition 1: At any period t where player i is the proposer, let w_j^t be player j's continuation value based on expected future play if the offer is rejected.

Observation 1: If player i is best-responding at time t, then:

- she makes an offer where player j achieves payoff w_j^t efficiently, *i.e.* maximizing j's share of the current surplus if $\beta_j < \beta_i$, and minimizing it if $\beta_j > \beta_i$; and

- player j accepts player i's offer with probability 1.

Proof of Observation 1: It is obvious that making an offer giving player j utility above w_j^t is suboptimal.

If the offer is accepted but w_j^t is achieved inefficiently, player i would do better by offering $w_j^t + \varepsilon$ efficiently for sufficiently small $\varepsilon > 0$.

It remains to be shown that it cannot be optimal for player i to make an offer rejected with positive probability. Notice that player i's payoff from offering $w_j^t + \varepsilon$ efficiently is at least $1 - \varepsilon$ more than her time-t valuation of the expected stream of payoffs following a rejection: she can demand $1-\varepsilon$ of the time-t surplus as well as the expected future shares she would receive following a rejection. Therefore, for $\varepsilon < 1$, offering $w_j^t + \varepsilon$ efficiently is better than offering less than w_j^t, which would result in a rejection. Moreover, for ε sufficiently small, it is also better than offering w_j^t and being rejected with positive probability. □

By Observation 1, when player i is the proposer, her offer must be accepted in MPE. Therefore, her payoff v_i is

$$v_i = 1 + \frac{\beta_i\delta}{1-\delta} - \beta_i y - (1-\beta_i)y_0, \quad (1)$$

where y_t is the share of the period t surplus being offered to the other player (denote the current period as period 0), and $y = \sum_{t=0}^{\infty} \delta^t y_t$.

The receiver j gets $\beta_j y + (1-\beta_j)y_0$ if he accepts, and $\delta v_j - (1-\beta_j)\delta x_0$ if he rejects and his next-period offer is accepted (which occurs in equilibrium in that subgame), where v_j is his payoff next period (in the relevant subgame) when making the offer, and x_0 is the portion of the then-current pie that he would keep next period (*i.e.* 1 minus the y_0 in j's proposal next period). By Observation 1, player i makes player j indifferent between accepting and rejecting, so she equates the two payoff expressions for player j and sets

$$y = \frac{\delta v_j - (1-\beta_j)(y_0 + \delta x_0)}{\beta_j}. \tag{2}$$

Substituting (2) into (1) gives

$$v_i = 1 + \frac{\beta_i \delta}{1-\delta} - \frac{\beta_i}{\beta_j}\delta v_j + \frac{\beta_i}{\beta_j}(1-\beta_j)\delta x_0 + (\frac{\beta_i}{\beta_j} - 1)y_0. \tag{3}$$

Now suppose $\beta_1 > \beta_2$.

Since $\frac{\beta_2}{\beta_1} - 1 < 0$, player 2 sets $y_0 = 0$ - that is, he keeps all of the current surplus when proposing. Therefore, in the equation for v_1, $x_0 = 1$.

Since $\frac{\beta_1}{\beta_2} - 1 > 0$, player 1 maximizes y_0 when she proposes. Therefore, she sets $y_0 = 1$ when, upon rejecting, player 2's value next period is $v_2 \geq 1 + \frac{1-\beta_2\delta}{\delta}$: in this case, player 2's reservation value is $\beta_2\delta + \delta(v_2 - 1) \geq 1$. If instead $v_2 < 1 + \frac{1-\beta_2\delta}{\delta}$, player 1 sets $y_0 = y = \delta v_2 - \delta + \beta_2\delta$. Thus, in the equation for v_2, either $x_0 = 0$ or $x_0 = 1 - \delta v_2 + \delta - \beta_2\delta$.

Following Shaked and Sutton (1984), let $\overline{v_i}$ and $\underline{v_i}$ be the supremum and infimum, respectively, of player i's MPE payoff when proposing. Because v_i and v_j are negatively related in (3), $\overline{v_1}$ corresponds to a next-period opponent payoff of $\underline{v_2}$ and vice versa. Thus we have:

Case I: $\beta_2\delta + \delta(\underline{v_2} - 1) \geq 1$

$$\begin{aligned}
\overline{v_1} &= 1 + \frac{\beta_1\delta}{1-\delta} - \frac{\beta_1}{\beta_2}\delta\underline{v_2} + \frac{\beta_1}{\beta_2}(1-\beta_2)\delta + (\frac{\beta_1}{\beta_2} - 1) \\
\underline{v_2} &= 1 + \frac{\beta_2\delta}{1-\delta} - \frac{\beta_2}{\beta_1}\delta\overline{v_1}
\end{aligned}$$

Case II: $\beta_2\delta + \delta(\underline{v_2} - 1) < 1$

$$\begin{aligned}
\overline{v_1} &= 1 + \frac{\beta_1\delta}{1-\delta} - \delta\underline{v_2} + \delta - \beta_2\delta \\
\underline{v_2} &= 1 + \frac{\beta_2\delta}{1-\delta} - \frac{\beta_2}{\beta_1}\delta\overline{v_1} + \frac{\beta_2}{\beta_1}(1-\beta_1)\delta(1 - \delta\underline{v_2} + \delta - \beta_2\delta)
\end{aligned}$$

The payoff expressions for the Case a's in the two parts of Proposition 1 are found using straightforward algebraic manipulations. Substituting $\underline{v_2}$ into the conditions for Cases I and II shows that Case I corresponds to the part of Proposition 1 stated in the main text ($\beta_2 \geq \frac{1}{\delta(1+\delta)}$), and Case II corresponds to $\beta_2 < \frac{1}{\delta(1+\delta)}$.

For the same reason, $\underline{v_1}$ corresponds to a next-period opponent payoff of $\overline{v_2}$ and vice versa. Therefore, the same equations determine $\underline{v_1}$ and $\overline{v_2}$, which implies payoff uniqueness.

Checking existence of MPE can be done by construction, using the above payoffs. Just like in Rubinstein-Ståhl bargaining, at every history, the proposer always offers the opponent a stream of surplus worth the latter's reservation value, and the receiver accepts if and only if the offer is worth at least her reservation value. The only additional consideration here is that, as discussed above, the proposer must offer as much of the then-current surplus as possible if she has higher β, and keep all of it if she has lower β.

If instead $\beta_1 < \beta_2$, simply swap the subscripts in the above; this yields the payoffs for the Case b's.

The rest of the proof deals with the remaining case: $\beta_1 = \beta_2 = \beta$. The proposer's payoff function becomes

$$v_i = 1 + \frac{\beta\delta}{1-\delta} - \delta v_j + (1-\beta)\delta x_0.$$

Because y_0 is now absent from the expression, the proposer (in particular next period's) is indifferent between maximizing or minimizing y_0. Thus, x_0 for the (current) proposer can take a range of values. The best-case scenario involves $x_0 = 1$, as was the case for player 1 when $\beta_1 > \beta_2$, and the worst-case scenario involves x_0 being minimized, as was the case for player 2 when $\beta_1 > \beta_2$. Therefore, plugging $\beta_1 = \beta_2 = \beta$ into the payoffs in the Case a's provides upper bounds for player 1's MPE payoff, and plugging $\beta_1 = \beta_2 = \beta$ into the payoffs in the Case b's yields lower bounds. It is straightforward to build MPEs achieving these bounds: simply take a MPE with $\beta_1 > \beta_2$ or $\beta_1 < \beta_2$, and substitute $\beta_1 = \beta_2 = \beta$. ■

Proposition 2 (continued):
c) When $\min\{\beta_1, \beta_2\} \geq \frac{1}{\delta(1+\delta)}$:
- if $\frac{\beta_1}{\beta_2} \in [1, \frac{1}{1-(1-\beta_2)\delta}]$, $\underline{v_1} = \frac{\beta_1}{\beta_2}(\frac{1}{1-\delta^2}) - (1-\beta_1)\frac{\delta}{1-\delta} < \overline{v_1}$,
- if $\frac{\beta_1}{\beta_2} \in [1-(1-\beta_1)\delta, 1]$, $\overline{v_1} = \frac{1}{1-\delta^2} - \frac{\delta}{1-\delta}(1-\frac{\beta_1}{\beta_2}) > \underline{v_1}$,
- otherwise, $\underline{v_1} = \overline{v_1}$; and

d) When $\min\{\beta_1, \beta_2\} < \frac{1}{\delta(1+\delta)}$, then $\underline{v_1} < \overline{v_1}$ if $(\beta_1 - \beta_2)^2 \leq \delta^2\beta_1\beta_2(1-\min\{\beta_1,\beta_2\})^2$.

Proof of Proposition 2: Suppose $\beta_i > \beta_j$, and define x_0, y_t and y as in the proof of Proposition 1.

Proof of Statement a: Note that $\overline{v_i}$ cannot correspond to player i making a rejected offer: her reservation value next period cannot exceed $\delta\overline{v_i}$, so player j cannot offer player i more than $\delta\overline{v_i}$ in equilibrium. That offer is worth at most $\delta^2\overline{v_i}$ from the current perspective, so we'd have $\overline{v_i} \leq \delta^2\overline{v_i}$, or $\overline{v_i} \leq 0$, which is impossible.

In order for player i's offer to be accepted, player j must get at least his reservation value. Therefore, $\overline{v_i}$ at most corresponds to meeting player j's reservation value efficiently (*i.e.* player j must either get the entire current surplus or get no future surplus), when such reservation value corresponds to player j obtaining $\underline{v_j}$ efficiently next period (which maximizes (3) from the proof of Proposition 1).

When player j proposes, he can guarantee acceptance by giving player i slightly more than her reservation value. Therefore, $\underline{v_j}$ at least corresponds to meeting player i's reservation value efficiently, when such reservation value corresponds to player i obtaining $\overline{v_i}$ efficiently next period (which minimizes (3) from the proof of Proposition 1).

It follows that the same equations as in the MPE case provide an upper bound for $\overline{v_i}$ and a lower bound for $\underline{v_j}$. These bounds are achieved since any MPE is a SPNE. □

Before proving the remaining statements, an upper bound for $\overline{v_j}$ and a lower bound for $\underline{v_i}$ are defined below. The proof of Statement b will show that these bounds are tight.

Let $\underline{w_k}$ and $\overline{w_k}$ be the infimum and supremum, respectively, of the reservation value for player $k \in \{1, 2\}$ when play following rejection is a SPNE. Note that player k's reservation value satisfies

$$w_k = \delta v_k - (1 - \beta_k)\delta x_0, \tag{4}$$

where v_k is player k's expected payoff in the subgame following a rejection, and x_0 is the expected share of next period's surplus that player k would obtain following a rejection. Therefore, $\underline{w_j}$ corresponds to player j obtaining $\underline{v_j}$ efficiently (*i.e.* while maximizing x_0) next period, which is what occurs in MPE. Thus, from MPE payoffs, we know that

$$\underline{w_j} = \begin{cases} \frac{\beta_j\delta}{1-\delta} - \frac{\delta^2}{1-\delta^2} & \text{if } \beta_j \geq \frac{1}{\delta(1+\delta)} \\ \frac{\beta_j\delta}{1-\beta_j\delta^2} & \text{if } \beta_j < \frac{1}{\delta(1+\delta)} \end{cases} . \tag{5}$$

Upper bound for $\overline{v_j}$

$\overline{v_j}$ cannot correspond to player j making a rejected offer for the same reason as for player i, in the proof of Statement a. Therefore, at best, player j offers player i a value of $\underline{w_i}$ efficiently, which implies $y_0 = 0$. Player i's reservation value satisfies $w_i = \delta v_i - (1 - \beta_i)\delta x_0$, so $\underline{w_i}$ corresponds at least to player i getting $\underline{v_i}$ and x_0 (here the expected share of next period's pie that player i keeps in next period's offer) being maximized subject to player j getting at least $\underline{w_j}$ next period. Thus, from (3), we have

$$\overline{v_j} \leq 1 + \frac{\beta_j \delta}{1-\delta} - \frac{\beta_j}{\beta_i}\delta \underline{v_i} + \frac{\beta_j}{\beta_i}(1-\beta_i)\delta x_0.$$

The following derives an upper bound for x_0. From the perspective of player i next period, the value of player i's then-future payoffs is at least $\underline{v_i} - x_0$, while the value of player j's then-future payoffs (still from the perspective of i) is $\frac{\beta_i}{\beta_j}(\underline{w_j} - 1 + x_0)$. Therefore, we have the resource constraint $\underline{v_i} - x_0 + \frac{\beta_i}{\beta_j}(\underline{w_j} - 1 + x_0) \leq \frac{\beta_i \delta}{1-\delta}$, or $x_0 \leq \frac{\frac{\beta_i \beta_j \delta}{1-\delta} - \beta_i(\underline{w_j}-1) - \beta_j \underline{v_i}}{\beta_i - \beta_j}$. Replacing x_0 by this bound gives

$$\overline{v_j} \leq 1 + \frac{\beta_j \delta}{1-\delta} - \frac{\beta_j}{\beta_i}\delta \underline{v_i} + \frac{\beta_j}{\beta_i}(1-\beta_i)\delta \min\left\{1, \frac{\frac{\beta_i \beta_j \delta}{1-\delta} - \beta_i(\underline{w_j}-1) - \beta_j \underline{v_i}}{\beta_i - \beta_j}\right\}. \tag{6}$$

Lower bound for $\underline{v_i}$

Player i can ensure acceptance by offering player j a payoff of $\overline{w_j}$ efficiently, *i.e.* by setting $y_0 = \min\{1, \overline{w_j}\}$. By (4), to maximize w_j, next period, v_j needs to be maximized and x_0 (here the expected share of next period's pie that player j keeps in next period's offer) minimized. However, these are at odds because the former requires an efficient offer, while the latter implies inefficiency. Specifically, the tradeoff is $v_j \leq \overline{v_j} - (1-x_0)(1-\frac{\beta_j}{\beta_i})$: $1 - x_0$ is the amount of current surplus inefficiently traded from j to i, and $1 - \frac{\beta_j}{\beta_i}$ is the per-unit cost of such trade. Plugging this into (4) yields

$$w_j \leq \delta\left[\overline{v_j} - (1-x_0)(1-\frac{\beta_j}{\beta_i})\right] - (1-\beta_j)\delta x_0. \tag{7}$$

The coefficient on x_0 is $-\beta_j \delta(\frac{1}{\beta_i} - 1)$, so to find an upper bound on w_j, x_0 needs to be minimized (which implies that $\overline{w_j}$ does not correspond to $\overline{v_j}$, but rather to a continuation where j gives i her infimum reservation value $\underline{w_i}$ inefficiently by maximizing the share of the then-current pie given to i). Therefore, substituting $v_j = \overline{v_j} - (1-x_0)(1-\frac{\beta_j}{\beta_i})$ and $y_0 = \min\{1, \overline{w_j}\}$ into (3), we have

$$\underline{v_i} \geq 1 + \frac{\beta_i \delta}{1-\delta} + \delta(\frac{\beta_i}{\beta_j} - 1) - \frac{\beta_i}{\beta_j}\delta \overline{v_j} + (1-\beta_i)\delta x_0 + (\frac{\beta_i}{\beta_j} - 1)\min\{1, \overline{w_j}\}, \tag{8}$$

where:

(i) $x_0 = \max\{0, 1-\underline{w_i}, \frac{\beta_j(\underline{w_i}-1)+\beta_i \underline{v_j} - \frac{\beta_i\beta_j\delta}{1-\delta}}{\beta_i-\beta_j}\}$: x_0 is minimized subject to not giving player i more than $\underline{w_i}$ (so the share kept x_0 must be at least $1-\underline{w_i}$), and subject to players i and j getting at least $\underline{w_i}$ and $\underline{v_j}$ respectively (which implies the resource constraint $\underline{v_j} - x_0 + \frac{\beta_j}{\beta_i}(\underline{w_i}-1+x_0) \leq \frac{\beta_j\delta}{1-\delta}$, or $x_0 \geq \frac{\beta_j(\underline{w_i}-1)+\beta_i \underline{v_j} - \frac{\beta_i\beta_j\delta}{1-\delta}}{\beta_i-\beta_j}$);

(ii) $\underline{w_i} = \frac{\beta_i}{\beta_j}\left(1+\frac{\beta_j\delta}{1-\delta} - \overline{v_j}\right)$: as discussed in the derivation of the upper bound for $\overline{v_j}$, $\underline{w_i}$ corresponds to player i getting payoff valued at $1+\frac{\beta_j\delta}{1-\delta} - \overline{v_j}$ from player j's perspective; since all of this payoff is from future periods, it is worth $\frac{\beta_i}{\beta_j}\left(1+\frac{\beta_j\delta}{1-\delta} - \overline{v_j}\right)$ to player i; and

(iii) $\overline{w_j} = \delta\left[\overline{v_j} - (1-x_0)(1-\frac{\beta_j}{\beta_i})\right] - (1-\beta_j)\delta x_0$: from (7).

Substituting (ii) into (i) and simplifying (iii) yields

$$x_0 = \max\{0, \frac{\beta_i}{\beta_j}\overline{v_j} - \left(\frac{\beta_i}{\beta_j} + \frac{\beta_i\delta}{1-\delta} - 1\right), 1 - \frac{\beta_i(\overline{v_j}-\underline{v_j})}{\beta_i-\beta_j}\} \quad (9)$$

$$\overline{w_j} = \delta\left[\overline{v_j} - (1-\frac{\beta_j}{\beta_i}) - \beta_j(\frac{1}{\beta_i}-1)x_0\right] \quad (10)$$

Equations (6) and (8) constitute a system of two piecewise linear inequalities in $\overline{v_j}$ and $\underline{v_i}$: $\underline{w_j}$ is defined in (5), $\overline{w_j}$ is defined in (10) in terms of $\overline{v_j}$ and x_0, and x_0 is defined in (9) in terms of $\overline{v_j}$ and $\underline{v_j}$ (which is player j's MPE payoff). They define a region in $\mathbb{R}^2$ of possible values for $\underline{v_i}$ and $\overline{v_j}$, which is always non-empty: it can be verified that the MPE payoffs always satisfy both conditions with equality. The desired bounds correspond to the lowest $\underline{v_i}$ and highest $\overline{v_j}$ within the region. In fact, there must be a single point $(\underline{v_i}^*, \overline{v_j}^*)$ in the region that corresponds to both values, *i.e.* a single solution to the system of inequalities that simultaneously minimizes $\underline{v_i}$ and maximizes $\overline{v_j}$. The reason is that the upper bound for $\overline{v_j}$ depends negatively on $\underline{v_i}$ (as seen by inspection of (6)), and the lower bound on $\underline{v_i}$ also depends negatively on $\overline{v_j}$ (as can be shown by algebraic manipulations of (8)).

Moreover, both (6) and (8) must hold with equality at $(\underline{v_i}^*, \overline{v_j}^*)$. To see this, note that given the negative slopes of both constraints, the only other option is $\underline{v_i}^* = 0$, with (6) giving a higher upper bound for $\overline{v_j}$ at $\underline{v_i} = 0$ than the lower bound for $\overline{v_j}$ implied by (8). The former is at most

$$1 + \frac{\beta_j\delta}{1-\delta} + \frac{\beta_j}{\beta_i}(1-\beta_i)\delta.$$

For the latter, note that if $\underline{v_i} = 0$, then $\underline{w_i} = 0$, so we are in the portion of (8) where

$x_0 = 1 - \underline{w_i} = \frac{\beta_i}{\beta_j}\overline{v_j} - \left(\frac{\beta_i}{\beta_j} + \frac{\beta_i\delta}{1-\delta} - 1\right)$. Thus

$$\begin{aligned} 0 &\geq 1 + \frac{\beta_i\delta}{1-\delta} + \delta(\frac{\beta_i}{\beta_j} - 1) - \frac{\beta_i}{\beta_j}\delta\overline{v_j} + (1-\beta_i)\delta(\frac{\beta_i}{\beta_j}\overline{v_j} - \left(\frac{\beta_i}{\beta_j} + \frac{\beta_i\delta}{1-\delta} - 1\right)) \\ \frac{\beta_i}{\beta_j}\delta\overline{v_j} - (1-\beta_i)\delta\frac{\beta_i}{\beta_j}\overline{v_j} &\geq 1 + \frac{\beta_i\delta}{1-\delta} + \delta(\frac{\beta_i}{\beta_j} - 1) + (1-\beta_i)\delta(1 - \frac{\beta_i}{\beta_j} - \frac{\beta_i\delta}{1-\delta}) \\ \frac{\beta_i^2}{\beta_j}\delta\overline{v_j} &\geq 1 + \frac{\beta_i^2\delta}{\beta_j} + \frac{\beta_i^2\delta^2}{1-\delta} \\ \overline{v_j} &\geq \frac{\beta_j}{\beta_i^2\delta} + 1 + \frac{\beta_j\delta}{1-\delta} \end{aligned}$$

Therefore, we would need

$$\begin{aligned} 1 + \frac{\beta_j\delta}{1-\delta} + \frac{\beta_j}{\beta_i}(1-\beta_i)\delta &> \frac{\beta_j}{\beta_i^2\delta} + 1 + \frac{\beta_j\delta}{1-\delta} \\ \beta_i(1-\beta_i)\delta^2 &> 1, \end{aligned}$$

which is impossible.

To summarize, there is a point $(\underline{v_i}^*, \overline{v_j}^*)$ where (6) and (8) both hold with equality that simultaneously minimizes $\underline{v_i}$ and maximizes $\overline{v_j}$. This implies that continuation strategy profiles where, at every history where player i is to be punished, players inefficiently maximize i's share of the then-current surplus subject to resource constraints (as is assumed in the derivation of (6) and (8)) generate reservation values leading to $\underline{v_i}^*$ or $\overline{v_j}^*$ (depending on who is player 1) in period 0. The proof of Statement b verifies that such continuation profiles can be made optimal at every history, and shows that they can support the entire range of payoffs.

Proof of Statement b: Consider the following classes of SPNE, which achieve by construction any $v \in [\underline{v_1}, \overline{v_1}]$.

For $\beta_1 > \beta_2$

1. In period 0:

a) Player 1 demands payoff v efficiently.

b) Player 2 accepts player 1's equilibrium offer.

c) Otherwise, player 2 accepts player 1's offer iff he gets strictly more than $\overline{w_2}$.

2. In period $2k+1$, $k \in \mathbb{Z}_+$: If player 2 deviated in period $2k$ by rejecting an offer that he was supposed to accept, play a MPE. If player 2 rejected an offer that he was supposed

to reject:

a) Player 2 offers player 1 $\underline{w_1}$. This is done as inefficiently as possible (maximizing player 1's share of the current surplus), subject to the constraint that player 2 must obtain at least $\underline{v_2}$.

b) Player 1 accepts this offer.

c) Otherwise, player 1 accepts player 2's offer iff she gets strictly more than her MPE reservation value.

3. In period $2k$, $k \in \mathbb{Z}_{++}$: If player 1 rejected in period $2k$ an offer that she was supposed to reject, play a MPE. If player 1 deviated by rejecting an offer that she was supposed to accept:

a) Player 1 demands $\underline{v_1}$. This is done as inefficiently as possible (maximizing player 1's share of the current surplus), subject to the constraint that player 2 must obtain at least $\underline{w_2}$.

b) Player 2 accepts this offer.

c) Otherwise, player 2 accepts player 1's offer iff he gets strictly more than $\overline{w_2}$.

Sequential rationality is checked below.

1a) Since $\underline{v_1}$ corresponds to player 1 offering player 2 a payoff of $\overline{w_2}$ efficiently, any other acceptable demand would give player 1 less than $\underline{v_1} \leq v$. Making a rejected offer is clearly suboptimal, as it would lead to payoff $\underline{w_1}$ next period.

1b) If player 1 were to demand $\overline{v_1}$ efficiently, player 2's payoff would be equal to his MPE reservation value $\underline{w_2}$. Thus, player 1's offer gives player 2 at least $\underline{w_2}$. Since rejecting this offer would lead to a MPE, it is optimal for player 2 to accept.

1c) By construction, the continuation after a rejection yields a reservation value of $\overline{w_2}$.

2a) Any other acceptable offer would give player 2 payoff below his MPE payoff $\underline{v_2}$. Making a rejected offer would give player 2 $\underline{w_2}$, which is even worse.

2b) By construction, the continuation after a rejection yields a reservation value of $\underline{w_1}$.

2c) Since the continuation after a rejection is a MPE, player 1's reservation value is her MPE reservation value.

3a) See 1a.

3b) Since the continuation after a rejection is a MPE, player 2's reservation value is his MPE reservation value $\underline{w_2}$.

3c) See 1c.

For $\beta_1 < \beta_2$ (this strategy profile satisfies sequential rationality for the same reasons as above)

1) In period 0:

- Player 1 demands payoff v efficiently.

- Player 2 accepts player 1's equilibrium offer.

- Otherwise, player 2 accepts player 1's offer iff he gets strictly more than his MPE reservation value.

2) In period $2k+1$, $k \in \mathbb{Z}_+$: If, in period $2k$, player 2 rejected an offer that he was supposed to reject, play a MPE. If player 2 deviated by rejecting an offer that he was supposed to accept:

- Player 2 demands $\underline{v_2}$. This is done as inefficiently as possible (maximizing player 1's share of the current surplus), subject to the constraint that player 1 must obtain at least $\underline{w_1}$.

- Player 1 accepts this offer.

- Otherwise, player 1 accepts player 2's offer iff she gets strictly more than $\overline{w_1}$.

3) In period $2k$, $k \in \mathbb{Z}_{++}$: If, in period $2k$, player 1 deviated by rejecting an offer that she was supposed to accept, play a MPE. If player 1 rejected an offer that she was supposed to reject:

- Player 1 offers player 2 $\underline{w_2}$. This is done as inefficiently as possible (maximizing player 1's share of the current surplus), subject to the constraint that player 1 must obtain at least $\underline{v_1}$.

- Player 2 accepts this offer.

- Otherwise, player 2 accepts player 1's offer iff he gets strictly more than his MPE reservation value. □

Proof of Statement c: In the $(\underline{v_i}, \overline{v_j})$ plane, (6) has two segments with slope $-\frac{\beta_j}{\beta_i}\delta$ when $\underline{v_i}$ is small, and $-\frac{\beta_j}{\beta_i}\delta - \frac{\beta_j}{\beta_i}(1-\beta_i)\delta\frac{\beta_j}{\beta_i-\beta_j} = -\frac{\beta_j(1-\beta_j)}{\beta_i-\beta_j}\delta$ when $\underline{v_i}$ is large enough for the resource constraint to bind. Therefore, the curve is concave, and the set of solutions to (6) is convex.

If $\beta_j \geq \frac{1}{\delta(1+\delta)}$, we know that in MPE, $w_j \geq 1$. Since $\overline{w_j}$ depends positively on $\overline{v_j}$ by (10) and (9), $\overline{w_j} \geq 1$ at all points above the MPE payoff. Therefore, in the $(\underline{v_i}, \overline{v_j})$ plane, (8) has three segments where the inverse of the slope is $-\frac{\beta_i}{\beta_j}\delta + (1-\beta_i)\delta\frac{\beta_i}{\beta_j} = -\frac{\beta_i^2}{\beta_j}\delta$ when $x_0 = 1 - \underline{w_i}$ (*i.e.* when $\overline{v_j}$ is large), $-\frac{\beta_i}{\beta_j}\delta$ when $x_0 = 0$, and $-\frac{\beta_i}{\beta_j}\delta - (1-\beta_i)\delta\frac{\beta_i}{\beta_i-\beta_j} = -\frac{\beta_i(\beta_i-\beta_i\beta_j)}{\beta_j(\beta_i-\beta_j)}\delta$ when the resource constraint binds (*i.e.* when $\overline{v_j}$ is small). Therefore, the curve is convex, and the set of solutions to (8) is convex.

Suppose (6) and (8) intersect on the portion of (6) where the slope is $-\frac{\beta_j}{\beta_i}\delta$ (the flatter portion on the left) and on the portion of (8) where $x_0 = 0$, *i.e.* where the slope is $-1/(\frac{\beta_i}{\beta_j}\delta)$ (the portion in the middle). Here, (8) is steeper than (6), and the convexity of the solution sets implies that they cannot intersect to the left of this point. It follows that such an intersection would correspond to the desired bounds. Consider the corresponding system of equations:

$$\overline{v_j} = 1 + \frac{\beta_j \delta}{1-\delta} - \frac{\beta_j}{\beta_i}\delta \underline{v_i} + \frac{\beta_j}{\beta_i}(1-\beta_i)\delta \quad (11)$$

$$\underline{v_i} = 1 + \frac{\beta_i \delta}{1-\delta} + \delta(\frac{\beta_i}{\beta_j} - 1) - \frac{\beta_i}{\beta_j}\delta \overline{v_j} + (\frac{\beta_i}{\beta_j} - 1) \quad (12)$$

Solving the system of equations (11) and (12) yields the payoff values in statement c. The conditions that need to be satisfied are:

i. $\frac{\frac{\beta_i\beta_j\delta}{1-\delta} - \beta_i(\frac{\beta_j\delta}{1-\delta} - \frac{\delta^2}{1-\delta^2} - 1) - \beta_j \underline{v_i}}{\beta_i - \beta_j} \geq 1$, so that we are not on the right segment of (6)

$\Longleftrightarrow \beta_i(\frac{\delta^2}{1-\delta^2} + 1) - \beta_j \left[\frac{\beta_i}{\beta_j}(\frac{1}{1-\delta^2}) - (1-\beta_i)\frac{\delta}{1-\delta}\right] \geq \beta_i - \beta_j$

$\Longleftrightarrow \beta_j(1-\beta_i)\frac{\delta}{1-\delta} \geq \beta_i - \beta_j$

$\Longleftrightarrow \beta_j(1-\beta_i)\delta \geq (\beta_i - \beta_j)(1-\delta)$

$\Longleftrightarrow -\beta_i\beta_j\delta \geq \beta_i - \beta_j - \beta_i\delta$

$\Longleftrightarrow \beta_j \geq \beta_i(1 - \delta + \beta_j\delta)$

This is equivalent to the condition stated in Proposition 2.

ii. $\frac{\beta_i}{\beta_j}\overline{v_j} - \left(\frac{\beta_i}{\beta_j} + \frac{\beta_i\delta}{1-\delta} - 1\right) \leq 0$, so that we are not on the left segment of (8)

$\Longleftrightarrow \frac{\beta_i}{\beta_j}\frac{1}{1-\delta^2} - \frac{\beta_i}{\beta_j}\frac{\delta}{1-\delta}(1 - \frac{\beta_j}{\beta_i}) \leq \frac{\beta_i}{\beta_j} + \frac{\beta_i\delta}{1-\delta} - 1$

$\Longleftrightarrow \frac{1}{1-\delta^2} - \frac{\delta}{1-\delta}(1 - \frac{\beta_j}{\beta_i}) \leq 1 + \frac{\beta_j\delta}{1-\delta} - \frac{\beta_j}{\beta_i}$

$\Longleftrightarrow \frac{1}{1-\delta}\frac{\beta_j}{\beta_i} \leq \frac{1}{1-\delta} - \frac{1}{1-\delta^2} + \frac{\beta_j\delta}{1-\delta}$

$\Longleftrightarrow \frac{\beta_j}{\beta_i} \leq 1 - \frac{1}{1+\delta} + \beta_j\delta$

Since $\frac{\beta_j}{\beta_i} \leq 1$, it is sufficient to have $-\frac{1}{1+\delta} + \beta_j\delta \geq 0$. This is guaranteed since $\beta_j \geq \frac{1}{\delta(1+\delta)}$.

iii. $1 - \frac{\beta_i(\overline{v_j} - \underline{v_j})}{\beta_i - \beta_j} \leq 0$, so that we are not on the right segment of (8)

$\Longleftrightarrow \beta_i(\frac{1}{1-\delta^2} - \frac{\delta}{1-\delta}(1 - \frac{\beta_j}{\beta_i}) - \left[\frac{1}{1-\delta^2} - \frac{\delta}{1-\delta}(1-\beta_j)\right]) \geq \beta_i - \beta_j$

$\Longleftrightarrow \beta_i\frac{\delta}{1-\delta}(\frac{\beta_j}{\beta_i} - \beta_j) \geq \beta_i - \beta_j$

$\Longleftrightarrow \beta_j(1-\beta_i)\frac{\delta}{1-\delta} \geq \beta_i - \beta_j$

This is equivalent to condition i.

iv. $\overline{w_j} \geq 1$: This must hold, as explained above.

Therefore, either all constraints are satisfied, in which case (11) and (12) indeed define $\overline{v_j}$ and $\underline{v_i}$, or conditions i and iii simultaneously fail, in which case both resource constraints bind, *i.e.* (11) and (12) intersect on the right segment of both (6) and (8). In the latter case, due to the shape of (6) and (8), they cannot cross at any other point; the intersection must then be the MPE, so SPNE payoffs are unique. □

Proof of Statement d: Since the curves touch at the MPE payoff $(\overline{v_i}, \underline{v_j})$, it follows that the solution set includes payoffs with $\underline{v_i} < \overline{v_i}$ and $\overline{v_j} > \underline{v_j}$ if (6) is no flatter than (8) at that point.

- Since the resource constraints bind at $(\overline{v_i}, \underline{v_j})$, the slope of (6) is $-\frac{\beta_j(1-\beta_j)}{\beta_i-\beta_j}\delta$, as derived at the beginning of the proof of Statement c.

- For (8), we know from (9) that when the resource constraint binds, $x_0 = 1 - \frac{\beta_i(\overline{v_j}-\underline{v_j})}{\beta_i-\beta_j}$. Thus, by (10), $\overline{w_j}$ depends positively on $\overline{v_j}$, so that assuming $\overline{w_j} < 1$ provides an upper bound on the inverse slope of (8). Substituting $x_0 = 1 - \frac{\beta_i(\overline{v_j}-\underline{v_j})}{\beta_i-\beta_j}$ into (10) and (10) into (8) yields an inverse slope of $\frac{-\beta_i(1-\beta_j)}{\beta_i-\beta_j}\delta$.

Therefore, our sufficient condition becomes

$$\frac{\beta_j(1-\beta_j)}{\beta_i-\beta_j}\delta \geq \frac{\beta_i-\beta_j}{\beta_i(1-\beta_j)}\frac{1}{\delta}.$$

This is equivalent to $\beta_i\beta_j(1-\beta_j)^2\delta^2 \geq (\beta_i-\beta_j)^2$, as desired. ■

Proof of Proposition 3: Observation 1 in the proof of Proposition 1 still holds, so the reasoning used to derive (3) remains valid. Once again, player j will demand the entire current surplus whenever he proposes. Therefore, if his payoff as proposer in period $t+1$ is v_j^{t+1}, then his reservation value in the period t is $\delta v_j^{t+1} - \delta(1-\beta_j)s_{t+1}$. Player i maximizes the share of current surplus offered to player j when she proposes, so she will offer the entire current surplus in period t whenever

$$s_t \leq \delta v_j^{t+1} - \delta(1-\beta_j)s_{t+1}. \tag{13}$$

In that case, we have

$$v_j^t = s_t + \beta_j \sum_{k=1}^{\infty} s_{t+k}\delta^k - \frac{\beta_j}{\beta_i}\delta v_i^{t+1}, \text{ and} \tag{14}$$

$$v_i^t = \frac{\beta_i}{\beta_j}s_t + \beta_i \sum_{k=1}^{\infty} s_{t+k}\delta^k - \frac{\beta_i}{\beta_j}\delta v_j^{t+1} + \frac{\beta_i}{\beta_j}(1-\beta_j)\delta s_{t+1}. \tag{15}$$

The payoffs from in Proposition 3 solve this system of equations, and the stated condition corresponds to (13).

To see that the stated payoffs must be unique, suppose that instead, there exists another sequence of MPE payoffs $\{V_i^{t'}\}$ for player i, so that at some t, $V_i^t = v_i^t + \varepsilon$. By (15), if (13) is satisfied at t, there must also exist a sequence of MPE payoffs $\{V_j^{t'}\}$ for player j, with $V_j^{t+1} = v_j^{t+1} - \frac{\beta_j}{\beta_i \delta}\varepsilon$. By (14), if (13) is satisfied at $t+2$, we must then have $V_i^{t+2} = v_i^{t+2} + \frac{1}{\delta^2}\varepsilon$. Iterating this argument, we obtain $V_i^{t+2k} = v_i^{t+2k} + \frac{1}{\delta^{2k}}\varepsilon$ for all $k \geq 0$.

Let $S_t = \sum_{n=0}^{\infty} s_{t+n}\delta^n$ be the total discounted surplus available in period t. Note that $\frac{\delta^{2k} S_{t+2k}}{S_t} = 1 - \frac{\sum_{n=0}^{2k-1} s_{t+n}\delta^n}{S_t} \rightarrow 0$ as $k \rightarrow \infty$ since $\sum_{t=0}^{\infty} \delta^t s_t < \infty$. It follows that $\varepsilon = 0$.

Finally, if (13) fails at any period where i proposes, it can be shown that the coefficient on ε would be even greater in magnitude, which strengthens the argument. ■

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