Branching Program size lower bounds via Projective Dimension

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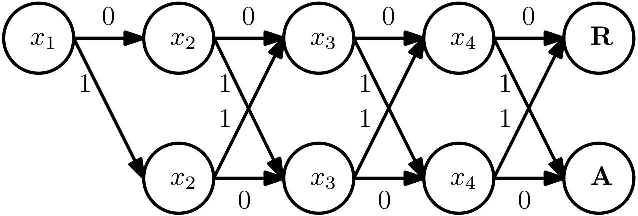
Theory Lunch, Technion

Outline

1. Branching Programs - model and motivation
2. Projective Dimension and BP size lower bounds
3. Gap Between Projective Dimension and BP Size
4. Bridging the Gap : Bitwise Projective Dimension
5. A lower bound for Bitwise Projective Dimension that matches state of the art Branching program lower bound
6. Discussions and Future Work

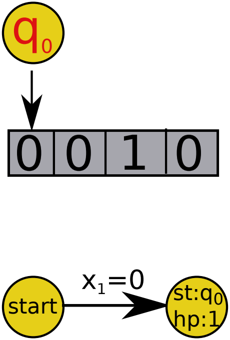
Branching Programs

- Directed Acyclic Graphs with designated start, accept and reject nodes
- Each node queries a variable
- Edges emanating out of a node are labeled by the bit value of the variable queried by the variable



$\mathsf{PARITY}_4 = x_1 \oplus x_2 \oplus x_3 \oplus x_4$

Connection to space bounded Turing Machines



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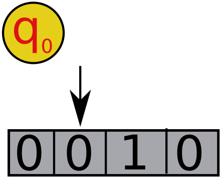

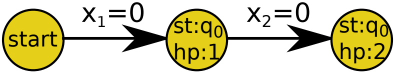

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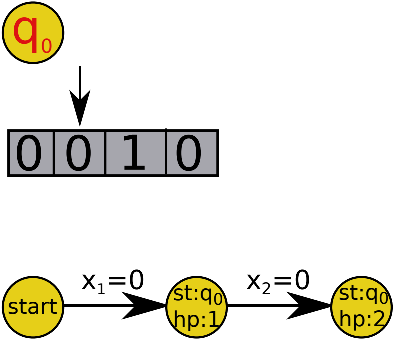

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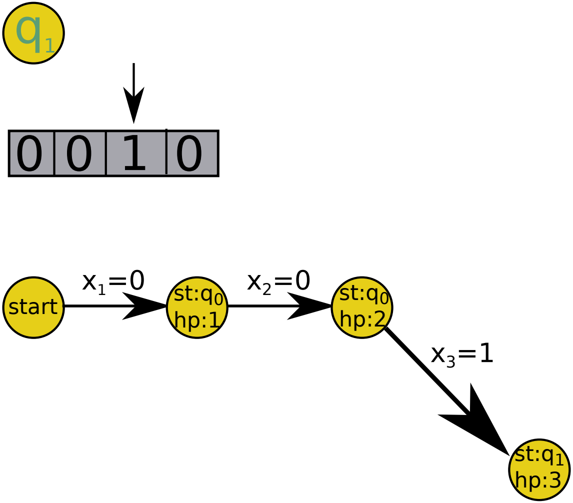

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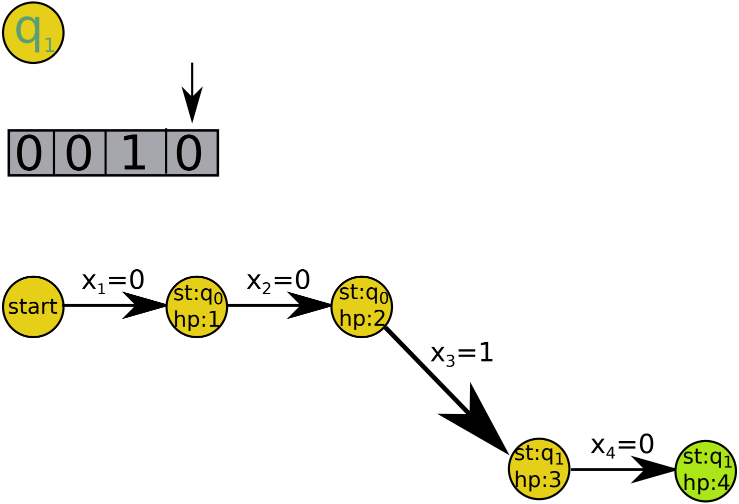

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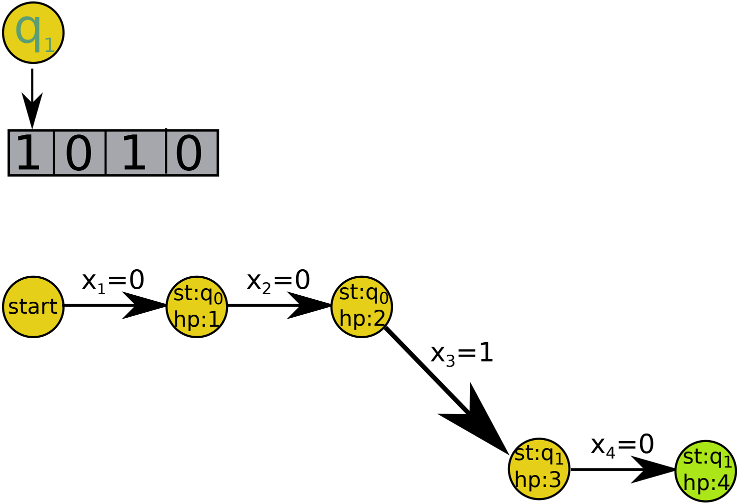

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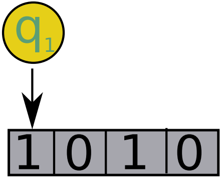

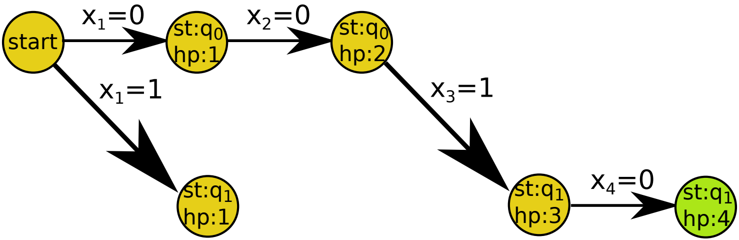

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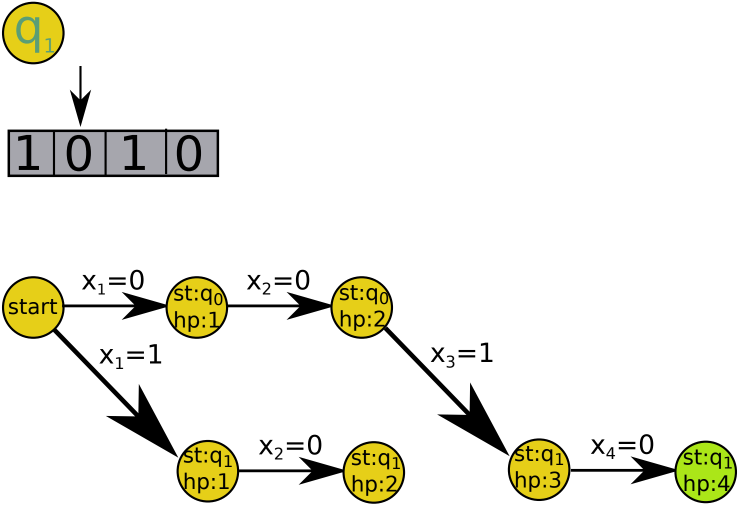

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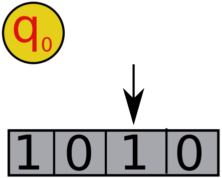

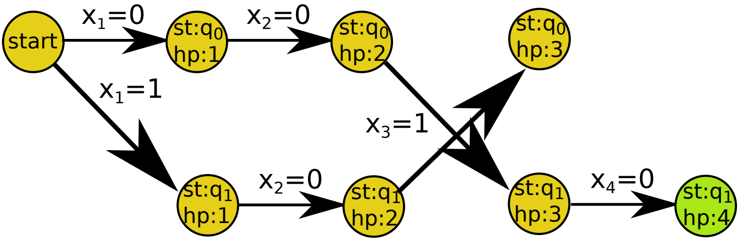

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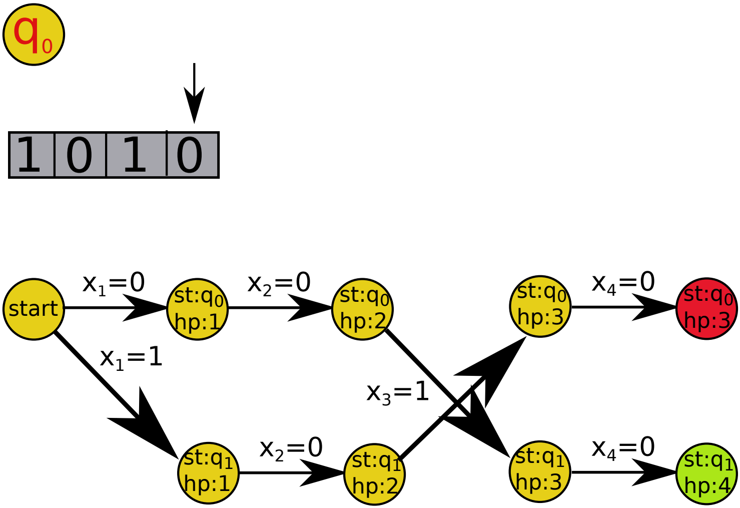

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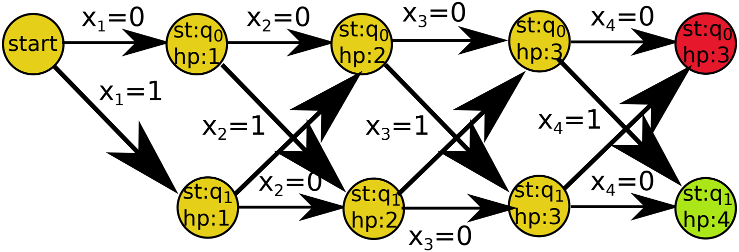

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- For every TM with space bound S there is a Deterministic Branching Program with size $2^{O(S)}$
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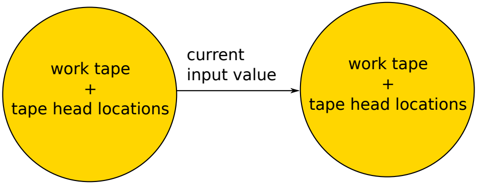

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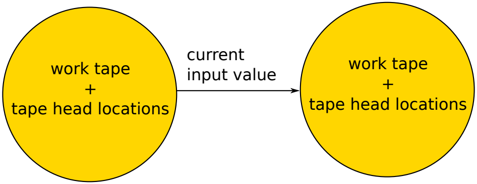

State of the art of BP size lower bounds

- For deterministic branching programs it is $n^2/\log^2 n$ by Nechiporuk **from 60's**
- Nechiporuk's method applies for many functions. We consider the **Element Distinctness** function
  - $\mathrm{ED}_m : \{0,1\}^{n=m2\log m} \to \{0,1\}$
  - m inputs $x_1,\ldots,x_m$ each representing a number in $[m^2]$
  - $f(x_1,\ldots,x_m) = 1$ iff no two x_i, x_j are equal
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Projective Dimension

- Measure on bipartite graphs introduced by Pudlak and Rodl
- Graph $G(U, V, E)$. Assign subspaces from $\mathbb{F}^d$ to vertices so that

$$(x, y) \in E \iff \phi(x) \cap \phi(y) \neq \{0\}$$

- Smallest such d : $\mathrm{pd}_{\mathbb{F}}(G)$.

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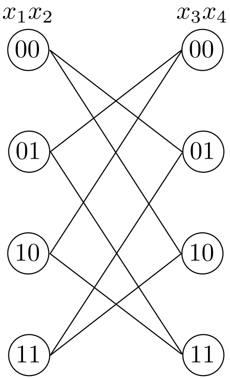

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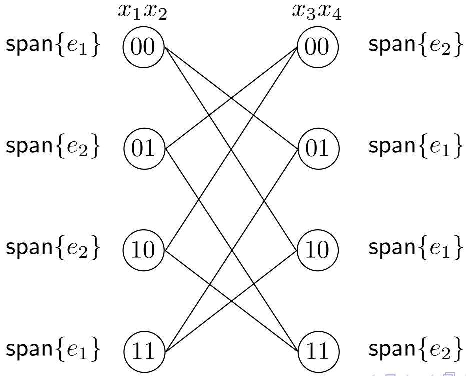

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Theorem, (Pudlak and Rodl (1992))

Over any $\mathbb{F}$, $\text{bpsize}(f) \geq \text{pd}_{\mathbb{F}}(G_f)$.

- To define the bipartite graph G_f associated with a function f on $2n$ variables, take some natural partition of the variable set into two equal parts

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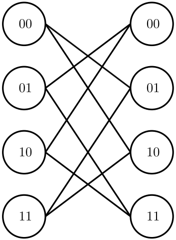
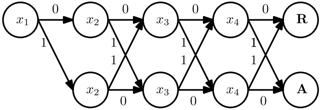


G_f

$f(x_1, x_2, x_3, x_4) = x_1 \oplus x_2 \oplus x_3 \oplus x_4$



Branching program computing $f = \mathsf{PARITY}_4$

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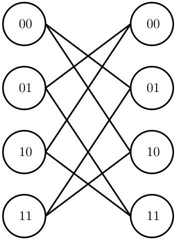

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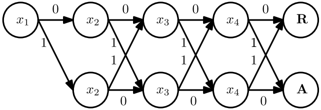

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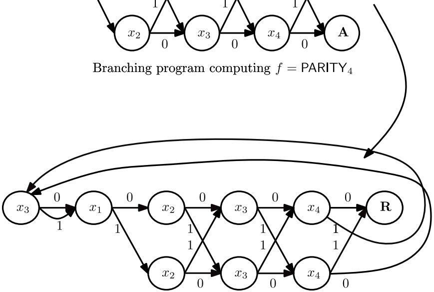

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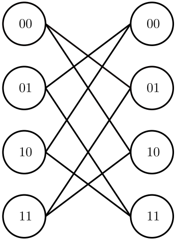

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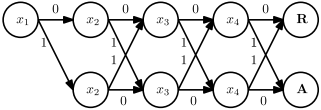

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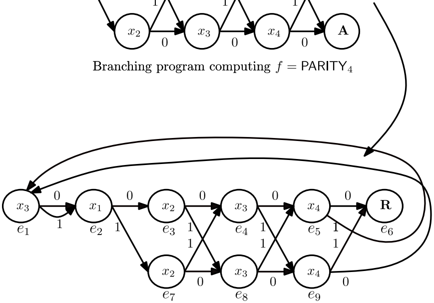

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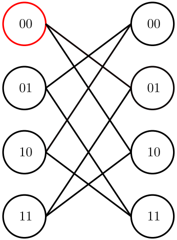

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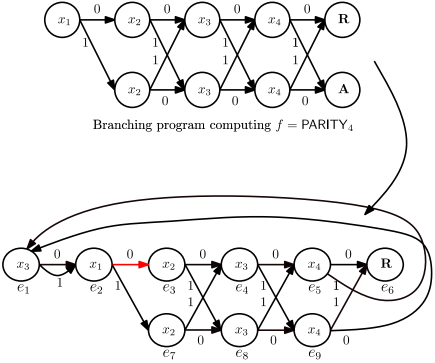

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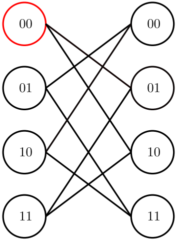
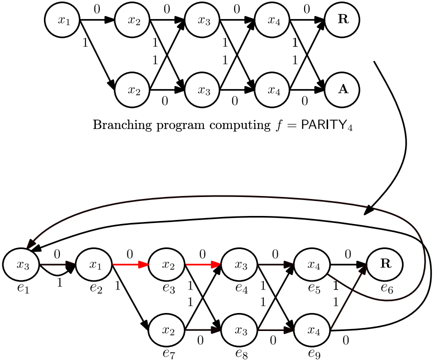


G_f

$f(x_1, x_2, x_3, x_4) = x_1 \oplus x_2 \oplus x_3 \oplus x_4$



Branching program computing $f = \mathsf{PARITY}_4$

Modified graph giving subspace assignment for G_f

Proof of the Pudalk Rodl theorem

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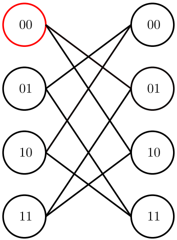
$e_3 - e_4, e_7 - e_8\}$



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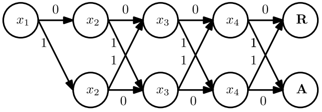

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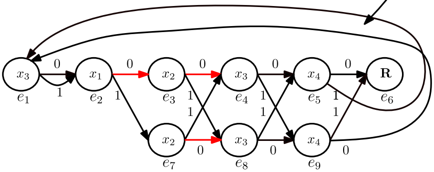

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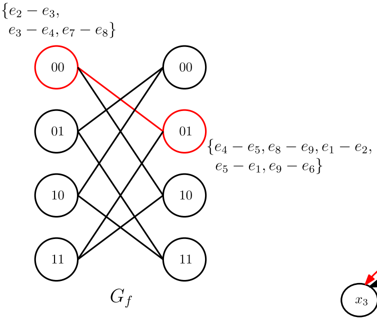

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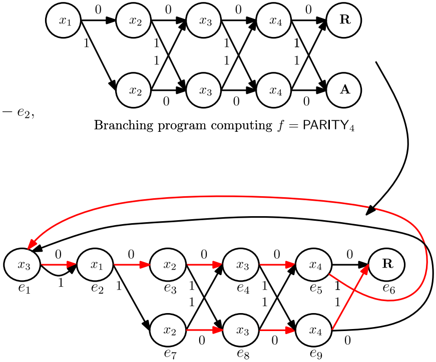

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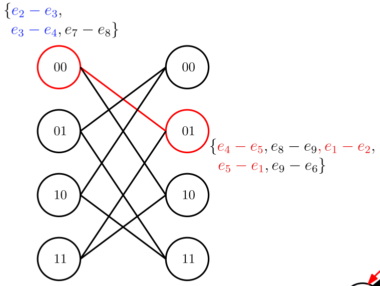

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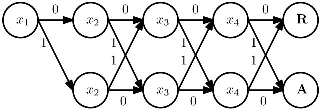

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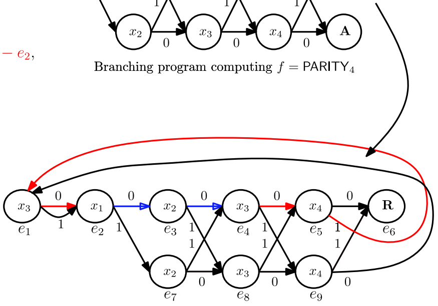

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Proof contd.

- Let (x, y) be an input. And H_x be the edge-subgraph of the branching program whose edges query variables in x. Similarly define H_y.
- After the transformation $f(x, y) = 1$ iff $H_x \cup H_y$ contains a cycle
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Known Bounds on $\text{pd}_{\mathbb{F}}$

- (Existential) N vertex bipartite G such that

$\text{pd}_{\mathbb{F}}(G)$	Field	Result
$\Omega\left(\sqrt{\frac{N}{\log N}}\right)$	Infinite	Babai et.al, 2002
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- (Explicit) $G =$ Complement of N perfect matchings. $\text{pd}_{\mathbb{R}}(G) = \Omega(\log N)$
- (Upper bounds) Bipartite G, $\text{pd}_{\mathbb{R}}(G) = O\left(\frac{N}{\log N}\right)$
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Our Result, a similar result known for Formulas and Graph Complexity by Jukna

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Bridging the gap

- The gap example gave an assignment which is of low projective dimension, but it may not be easy (read poly in n) to describe
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Bitwise Decomposable Projective Dimension

Definition

For $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, $bpdim(f) \leq d$ if there exists $\mathscr{C} = \{U_i^a \mid a \in \{0,1\}, i \in [n]\}$, $\mathscr{D} = \{V_i^a \mid a \in \{0,1\}, i \in [n]\}$, such that

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Main Result

$$\text{bitpdim}(f) = \Omega(\text{bpsize}(f)^{1/6})$$

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- This can be done by blowing up the size of the given branching program by a factor of at most 4.

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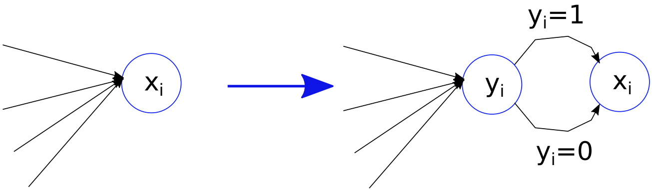

Branching programs from bitpdim

Theorem

$\text{bitpdim}(f) \leq d(n) \implies \text{bpsize}(f) \leq (d(n))^6$

Proof.

(Sketch)

- We describe a space bounded algorithm which given the bitpdim assignment as an advice, and two inputs (x, y) computes whether $f(x, y) = 1$.
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- $f(x, y) = 1$ iff there is a cycle in G^*
- check for a cycle in G^*. Can be done in space $5 \log |G^*|$
- $|G^*| = \text{bitpdim}(f)$.

□

Branching programs from bitpdim

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$\text{bitpdim}(f) \leq d(n) \implies \text{bpsize}(f) \leq (d(n))^6$

Proof.

(Sketch) Given (x, y)

- implicit G, vertices – standard basis vectors in ϕ, $(u, v) \in E(G^*)$ iff $e_u - e_v \in U_i^{x_i}$ or $V_j^{y_j}$.
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- The best known pd lower bound is linear
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- Recall the function ED.
  - $\mathrm{ED}_m : \{0,1\}^{n=m2\log m} \to \{0,1\}$
  - m inputs $x_1, \ldots, x_m$ each representing a number in $[m^2]$
  - $f(x_1, \ldots, x_m) = 1$ iff no two x_i, x_j are equal
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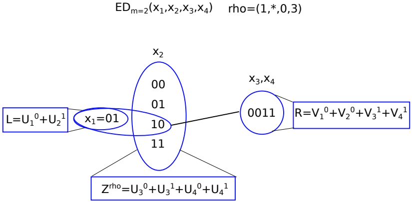

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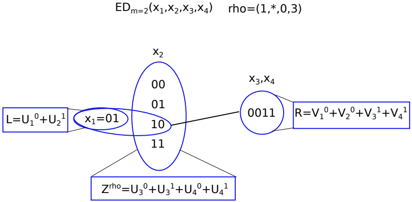

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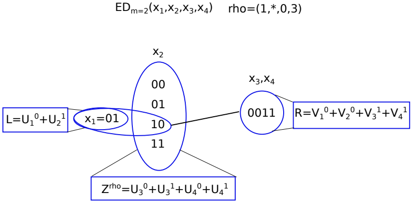

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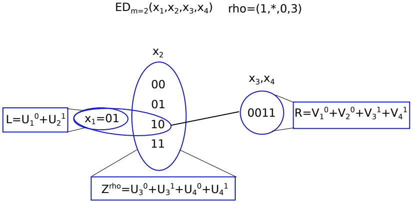

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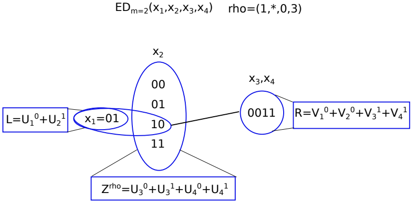

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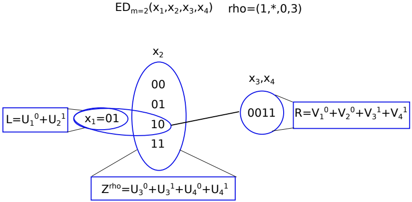

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