Econometrica, Vol. 86, No. 5 (September, 2018), 1569–1588

ROBUST MECHANISMS UNDER COMMON VALUATION

SONGZI DU
Department of Economics, Simon Fraser University

I construct an informationally robust auction to sell a common-value good. I examine the revenue guarantee of an auction over all information structures of bidders and all equilibria. As the number of bidders gets large, the revenue guarantee of my auction converges to the full surplus, regardless of how information changes as more bidders are added. My auction also maximizes the revenue guarantee when there is a single bidder.

KEYWORDS: Robust mechanism, common-value auction, full surplus extraction, large markets.

1. INTRODUCTION

IN THIS PAPER, I CONSTRUCT an auction mechanism whose revenue guarantee over all information structures and all equilibria converges to the full surplus as the number of bidders gets large, regardless of how information changes as more bidders are added. I assume all bidders have a common value for the auctioned good, so the full surplus is simply the expected common value.

Standard auction formats, such as first or second price auction, in general do not have such an asymptotic guarantee of full surplus extraction. Standard auctions are able to extract the full surplus in large markets when bidders have one-dimensional signals that are symmetrically and smoothly distributed, as shown by Wilson (1977), Milgrom (1979), Pesendorfer and Swinkels (1997), Bali and Jackson (2002), among others. However, when one bidder has proprietary information and is strictly more informed about the value than all other bidders (Engelbrecht-Wiggans, Milgrom, and Weber (1983)), or when there is a resale market that prices the value at the maximum of everyone's signals (Bergemann, Brooks, and Morris (2017a, 2017c)), standard auctions typically fail to extract the full surplus even as the number of bidders goes to infinity. From the perspective of an auctioneer whose platform must accommodate diverse groups of bidders, it is useful to commit to an auction that can be used in a variety of situations, which saves the costs of having to customize an auction design to each specific situation. Such commitment also makes the auction design more familiar and more understandable to the bidders, potentially making them more likely to participate.

The auction that I construct is simple to describe. Suppose there is a single unit of good to sell and $N \geq 1$ bidders with quasi-linear utility and a common value (but potentially different information about the value). Let the message space for each bidder i be the interval $[0, 1]$. I interpret a message $m_i \in [0, 1]$ as a *demand* for a fraction m_i of the good. Buyer i gets $q_i(m_i, m_{-i})$ quantity of allocation (q_i could be the literal quantity if the good is divisible, or the probability of allocation if the good is indivisible) and pays $t_i(m_i)$

Songzi Du: songzid@sfu.ca
I thank Ben Brooks, Gabriel Carroll, Vitor Farinha Luz, Ben Golub, Wei Li, Michael Ostrovsky, Arthur Robson, Joel Sobel, and four anonymous referees for helpful comments and discussions.

 https://doi.org/10.3982/ECTA14993

independent of others' messages. If $m_1 \geq m_2 \geq \cdots \geq m_N$, then ($m_{N+1} \equiv 0$):

$$q_i(m_i, m_{-i}) = \sum_{j=i}^{N} \frac{m_j - m_{j+1}}{j}, \qquad t_i(m_i) = X\bigl(\exp(m_i/A) - 1\bigr),$$

and analogously for any other ordering of $(m_1, m_2, \ldots, m_N)$. That is, the lowest bidder gets $1/N$ of his demand, the second lowest bidder gets that plus $1/(N-1)$ of the difference between his and the lowest demand, and so on.[1] Thus, the total allocation is equal to the highest demand.[2] Moreover, each buyer's payment depends only on his demand and is independent of his final allocation. I call bidder i's payment $t_i(m_i)$ his *bid* in the auction; in this sense, this is an all pay auction. Finally, the constants $A > 0$ and $X > 0$ in the payment rule are parameters that can be optimized for a specific distribution of values (if the seller knows the distribution); intuitively, A and X are choices of units for the demand and the payment, respectively. I call this mechanism the *exponential price auction*.

My main result is that for specific choices of parameters A and X that depend only on the number N of bidders and the upper bound on value, the exponential price auction guarantees an expected revenue (over all information structures and all equilibria) which converges to the full surplus as $N \to \infty$. This equilibrium revenue is guaranteed by the following two features of the auction. First, the exponential payment rule satisfies the linear relationship $t_i(m_i) = A \cdot t_i'(m_i) - X$. In any equilibrium, the marginal payment $t_i'(m_i)$, and hence the payment $t_i(m_i)$ as well, is determined by the marginal surplus $\mathbb{E}[v \cdot \frac{\partial q_i}{\partial m_i} \mid m_i]$ through the first-order condition (a bidder has no incentive to deviate locally from m_i), where $\mathbb{E}$ is based on the joint equilibrium distribution over v and m. Therefore, the expected payment is determined by the expected marginal surplus $\mathbb{E}[v \cdot \frac{\partial q_i}{\partial m_i}]$. Second, the allocation rule satisfies $\frac{\partial q_i}{\partial m_i} = 1/\operatorname{rank}(m_i; m)$, where $\operatorname{rank}(m_i; m)$ is the rank of m_i among $m = (m_1, m_2, \ldots, m_N)$. Therefore, the total expected marginal surplus satisfies $\mathbb{E}[v \cdot \sum_{i=1}^N \frac{\partial q_i}{\partial m_i}] = \mathbb{E}[v] \cdot \sum_{i=1}^N 1/i$, as $\sum_{i=1}^N 1/\operatorname{rank}(m_i; m) = \sum_{i=1}^N 1/i$. Intuitively, for an individual bidder i, the expected marginal surplus $\mathbb{E}[v \cdot \frac{\partial q_i}{\partial m_i}]$ could be heavily distorted from the expected value $\mathbb{E}[v]$, as v is weighted by the factor $\frac{\partial q_i}{\partial m_i}$. For example, there is the winner's curse: a low v is weighted by a high $\frac{\partial q_i}{\partial m_i} = 1/\operatorname{rank}(m_i; m)$ if the other bidders submit low demands m_{-i}. However, for the expected revenue, we sum the expected marginal surplus across all bidders, and in the summation these distortions cancel each other out.

When there is a single bidder ($N = 1$), the exponential price auction is an extension of the random posted price mechanism in Carrasco, Farinha Luz, Kos, Messner, Monteiro, and Moreira (2018). I prove that the exponential price auction gives the optimal revenue guarantee when $N = 1$, which generalizes a result in Carrasco et al. (2018) from binary value distribution to arbitrary distribution. Roesler and Szentes (2017) characterized the optimal information structure for a buyer when the seller best responds to this information structure with a posted price. I show that when $N = 1$, the revenue guarantee of the

[1] The allocation rule is similar to the serial-cost sharing payment rule of Moulin (1994) in a public good provision setting, and to the side-payment rule in the knockout auctions used by a bidding cartel of stamp dealers in the 1990s (Asker (2010)).

[2] The allocation is inefficient, since a positive fraction of the good (which always has a nonnegative value) can remain unallocated. It is easy to make the auction efficient when N is large: just run the auction for the first $N - 1$ bidders and give whatever that is unallocated to the last bidder for free. See Section 6 for another tweak on the exponential price auction that yields efficient allocation.

exponential price auction is exactly the seller's optimal revenue at the Roesler–Szentes information structure, thus proving the optimality of the revenue guarantee.[3] The Roesler–Szentes information structure and the exponential price auction have proved to be useful in the study of a robust dynamic pricing problem by Libgober and Mu (2017).

The setup and methodology in my paper come from Bergemann, Brooks, and Morris (2017a), who analyzed the set of revenue and welfare outcomes in the first price auction as one varies the information structure. Subsequent to this paper and working with the same model, Bergemann, Brooks, and Morris (2017b) characterized an optimal mechanism that achieves the best possible revenue guarantee when there are two bidders and binary common values. The optimal mechanism in Bergemann, Brooks, and Morris (2017b) shares some similar features with the exponential price auction: the bidder's message space is $M_i = [0, 1]$ and can be interpreted as demands for fractions of the good; the allocation rule is piecewise linear over the messages; the payment rule is exponential in one's message and independent of the other's message when $m_1 + m_2 < 1$ (i.e., when the demands are compatible). Building on the techniques in Bergemann, Brooks, and Morris (2017b), Brooks and Du (2018) characterized an optimal mechanism for any finite number of bidders and an arbitrary distribution of common values. While the exponential price auction is not exactly optimal when $N > 1$, it has the merit of simplicity; in particular, its payment rule is much simpler and more intuitive than that of the mechanisms in Bergemann, Brooks, and Morris (2017b) and Brooks and Du (2018).

While this paper, along with Bergemann, Brooks, and Morris (2017a, 2017b) and Brooks and Du (2018), is motivated by the desire to avoid mechanisms that depend on fine details of bidders' information structure (the Wilson (1987) doctrine), the way we evaluate and construct mechanisms depends on the bidders having common knowledge of this information structure and reacting optimally to their information. McLean and Postlewaite (2018) constructed a detail-free, two-stage mechanism that fully extracts surplus in large markets when bidders have both common and private values and conditionally independent signals about the common value. Importantly, they did not assume bidders have common knowledge of the information structure. On the other hand, their revenue guarantee is weaker than mine, as it is over truthful revelation outcomes and not over all equilibrium outcomes. This partial-implementation approach was also taken by Chung and Ely (2007), Chen and Li (2018) and Yamashita (2017), who studied robust mechanism with pure private value.

2. MODEL

Information

The seller has one unit of good to sell. There are $N \geq 1$ bidders, who have a common value $v \in V = [0, \bar{v}]$ for the good and have quasi-linear utility. Let $p \in \Delta(V)$ be the prior

[3]In other words, in the case of one bidder, Roesler and Szentes (2017) characterized

$$\min_{\text{info. structure}} \quad \max_{\text{mechanism, equilibrium}} \quad \text{Revenue},$$

while I characterize

$$\max_{\text{mechanism}} \quad \min_{\text{info. structure, equilibrium}} \quad \text{Revenue},$$

and show it is equal to their minmax value. The exponential price auction and the Roesler–Szentes information structure thus form a saddle point in the zero-sum game in which the seller chooses mechanism to maximize revenue, while Nature chooses information structure to minimize revenue. Brooks and Du (2018) characterized this saddle point for any $N > 1$.

distribution of common values. The bidders know the prior p. The seller knows $\bar{v}$ but may or may not know p.

Each bidder i *may* possess some information $s_i \in S_i$ about the common value in addition to the prior, where S_i is a set of signals (or types), and there is a joint distribution over values and signals, $\tilde{p} \in \Delta(V \times \prod_{i=1}^N S_i)$, which is consistent with p: $\tilde{p}(B \times S) = p(B)$ for every measurable $B \subseteq V$. Bidder i thus has a posterior belief $\tilde{p}(\cdot \mid s_i) \in \Delta(V \times S_{-i})$ given a realization of his signal s_i. For conciseness, I refer to an information structure $(\tilde{p}, S_i)$ as $\tilde{p}$. The information structure $\tilde{p}$ is *not* known by the seller.

Mechanism

A mechanism is a set of allocation rules $q_i : M \to [0, 1]$ and payment/pricing rules $t_i : M \to \mathbb{R}$ satisfying $\sum_{i=1}^N q_i(m) \leq 1$, where M_i is the message space of bidder i, and $M = \prod_{i=1}^N M_i$ the space of message profiles. A mechanism defines a game in which the bidders simultaneously submit messages and have utility

$$U_i(v, m) = v \cdot q_i(m) - t_i(m). \tag{1}$$

The allocation $q_i(m)$ is the share of the good that bidder i receives in the case of a divisible good, and is the probability of getting the good in the case of an indivisible good.

I assume that a mechanism always has an opt-out option for each bidder i: there exists a message $m_i \equiv 0 \in M_i$ such that $q_i(0, m_{-i}) = t_i(0, m_{-i}) = 0$ for every $m_{-i} \in M_{-i}$.

Equilibrium

Given a mechanism (q_i, t_i) and an information structure $\tilde{p}$, we have a game of incomplete information. A *Bayes Nash Equilibrium* (BNE) of the game is a strategy profile $\sigma = (\sigma_i)$, $\sigma_i : S_i \to \Delta(M_i)$, such that σ_i is a best response to σ_{-i}: for any other strategy σ_i', we have

$$\int_{(v,s)\in V\times S} U_i\big(v, \big(\sigma_i(s_i), \sigma_{-i}(s_{-i})\big)\big)\,\tilde{p}(dv, ds) \geq \int_{(v,s)\in V\times S} U_i\big(v, \big(\sigma_i'(s_i), \sigma_{-i}(s_{-i})\big)\big)\,\tilde{p}(dv, ds),$$

where $U_i(v, (\sigma_i(s_i), \sigma_{-i}(s_{-i})))$ is the multilinear extension of U_i in Equation (1).

Revenue Guarantee

The expected revenue at an information structure $\tilde{p}$ and an equilibrium σ is

$$R(\tilde{p}, \sigma) = \int_{(v,s)\in V\times S} \sum_{i=1}^N t_i\big(\sigma_i(s_i), \sigma_{-i}(s_{-i})\big)\,\tilde{p}(dv, ds).$$

DEFINITION 1: A mechanism *guarantees* a revenue r if every information structure $\tilde{p}$ and every equilibrium σ has an expected revenue larger than or equal to r: $R(\tilde{p}, \sigma) \geq r$.

3. CLASSICAL AUCTIONS

In this section, I motivate the need for a new mechanism by arguing that the well-known auction formats give unsatisfactory revenue guarantees: even as the market gets large, there is no guarantee that the expected equilibrium revenue tends to the full surplus. In the auctions discussed below (with the exception of the last sentence of the final paragraph), each bidder simultaneously submits a bid which is a single number.

First, consider a standard auction that awards the good to the highest bidder, such as a kth price auction ($1 \leq k \leq N$, the winner pays the kth highest bid, the losers do not pay), an all pay auction (everyone pays his own bid), or an average price auction (the winner pays a (potentially weighted) average of all bids, the losers do not pay). Bergemann, Brooks, and Morris (2017a, 2017c) showed that in a standard auction, the revenue guarantee is bounded away from the full surplus as $N \to \infty$. Consider the following ("BBM") information structure: each bidder i receives an independent and identically distributed (i.i.d.) signal s_i from the cumulative distribution function (CDF) $F_N(s_i) = p(v \leq s_i)^{1/N}$, and the common value is $v = \max_i s_i$ (so the distribution of values is the prior p independent of N). Intuitively, in the BBM information structure with lots of bidders, most bidders have very little private information about the value as their signals are close to the lowest possible signal: if a signal s_i is not the lowest possible signal, that is, if $p(v \leq s_i) > 0$, then the probability that a bidder receives a signal less than or equal to s_i is $F_N(s_i) = p(v \leq s_i)^{1/N} \to 1$.[4] Bergemann, Brooks, and Morris (2017a) showed that the BBM information structure is the worst case information structure for the first price auction, and under this information structure the equilibrium revenue does not converge to the full surplus as $N \to \infty$. Bergemann, Brooks, and Morris (2017c) showed that in any standard auction, the BBM information structure has an equilibrium in which bidders behave as if the signals are their private values. But since all standard auctions generate the same expected revenue in equilibrium under independent private values (the revenue equivalence theorem), they also generate the same expected revenue in equilibrium under the BBM information structure. Therefore, that the equilibrium revenue under the BBM information structure does not tend to the full surplus for the first price auction implies the same statement for any standard auction.

Next, suppose the highest bidder does not get all of the good: the good is divided into $k_N \leq N$ equal parts (either probabilistically through randomization, or literally if the good is divisible), and the top k_N bidders each get a part of the good. There are two well-known auctions in this context: the discriminatory price auction, in which each of the k_N winners pays his own bid, and the uniform price auction, in which each of the k_N winners pays the $(k_N + 1)$th highest bid; in either auction, the losers do not pay. Jackson and Kremer (2007) showed that the revenue guarantee of the discriminatory price auction does not tend to the full surplus in large markets: when k_N/N tends to some constant between 0 and 1, in the limit the equilibrium revenue is strictly less than the full surplus for any conditionally independent information structure.

The situation is better with the uniform price auction. As shown by Pesendorfer and Swinkels (1997), if $k_N \to \infty$ and $N - k_N \to \infty$ as $N \to \infty$, then the equilibrium revenue tends to the full surplus in uniform price auction under conditionally independent information structure. Yet, it is easy to come up with an information structure whose equilibrium revenue does not converge to the full surplus in the uniform price auction: suppose there are k_N informed bidders who know the value, and the other $N - k_N$ bidders have no information beyond the prior. Then an equilibrium in the uniform price auction is that

[4]This feature is reminiscent of the information structure in Engelbrecht-Wiggans, Milgrom, and Weber (1983), in which one bidder knows the value, while the rest have no information. Engelbrecht-Wiggans, Milgrom, and Weber (1983) showed that, in the first price auction with this information structure, the equilibrium revenue also fails to converges to the full surplus as $N \to \infty$.

each of the informed bidders truthfully bids the value, while the rest bid 0; the equilibrium revenue is exactly zero, and all surplus is captured by the informed bidders. Note that the same equilibrium remains whether the uniform price auction is implemented as a sealed-bid or an ascending-bid auction.

4. MAIN RESULT

Recall the exponential price auction: $M_i = [0, 1]$, and given demands $1 \geq m_1 \geq m_2 \geq \cdots \geq m_N \geq 0 \equiv m_{N+1}$,

$$q_i(m_i, m_{-i}) = \sum_{j=i}^{N} \frac{m_j - m_{j+1}}{j}, \qquad t_i(m_i) = X_N\big(\exp(m_i/A_N) - 1\big), \tag{2}$$

and analogously for any other ordering of $(m_1, m_2, \dots, m_N)$.

THEOREM 1: *Suppose as $N \to \infty$, $A_N \log N \to 1$, $X_N = \frac{\bar{v}A_N}{K_N \exp(1/A_N)}$, where $\frac{\log K_N}{\log N} \to 0$, and $N \cdot X_N \to 0$. The revenue guarantee of the exponential price auction converges to the full surplus $\int_v vp(dv)$ as $N \to \infty$.*

The parameter X_N depends on the upper bound $\bar{v}$ of value; A_N and X_N do not depend on any other feature of the prior distribution of values. Of course, if the seller does know the prior distribution p, then he can optimize A_N and X_N with respect to p using the explicit lower bound in Equation (13) of the proof (page 1583).

Suppose the seller has mistaken beliefs about $\bar{v}$ and N; he might believe they are $\tilde{\bar{v}}$ and $\tilde{N}$ when $\bar{v}$ and N are the true values. If there exists a constant factor $C > 0$ independent of N such that $1/C \leq \tilde{\bar{v}}/\bar{v} \leq C$ and $1/C \leq \tilde{N}/N \leq C$, then the auction with the misspecified $\tilde{\bar{v}}$ and $\tilde{N}$ is still guaranteed to extract the full surplus when N is large; this is because the parameters $A_{\tilde{N}}$ and $X_{\tilde{N}}$ defined from $\tilde{\bar{v}}$ and $\tilde{N}$ are still within the range of parameters in Theorem 1.

In Theorem 1, the integer[5] $\lfloor K_N \rfloor$ is the equilibrium number of bidders who demand the entire allocation ($m_i = 1$) when there is common knowledge of the highest value ($v = \bar{v}$), since $\bar{v}/K_N = X_N e^{1/A_N}/A_N = t_i'(1)$. Thus, the condition $\log K_N/\log N \to 0$ in Theorem 1 says that the prices are sufficiently high so that even under the highest value, only a small number of bidders will demand the entire allocation: K_N must grow with N more slowly than any power function of N. On the other hand, the prices cannot be too high for Theorem 1, which is the condition $N \cdot X_N \to 0$. As I discuss following Equation (3), X_N is a fixed amount of information rent extracted by a bidder in equilibrium. The condition $N \cdot X_N \to 0$ thus says that the sum of such rent must go to zero as $N \to \infty$.

For some intuition about the limiting behavior in the exponential price auction and how it is connected with the parameters, consider an information structure and equilibrium such that the demand m_i is ex ante symmetrically distributed across i.[6] Suppose $A_N = 1/\log N$. Then, Theorem 1 says that

$$N \cdot \mathbb{E}\big[t_i(m_i)\big] = \frac{\bar{v}}{K_N \log N}\mathbb{E}\big[N^{m_i} - 1\big] \to \mathbb{E}[v], \quad \text{as } N \to \infty.$$

[5]For any real number x, $\lfloor x \rfloor$ is the largest integer that is smaller than or equal to x.

[6]I am grateful to a referee for this calculation.

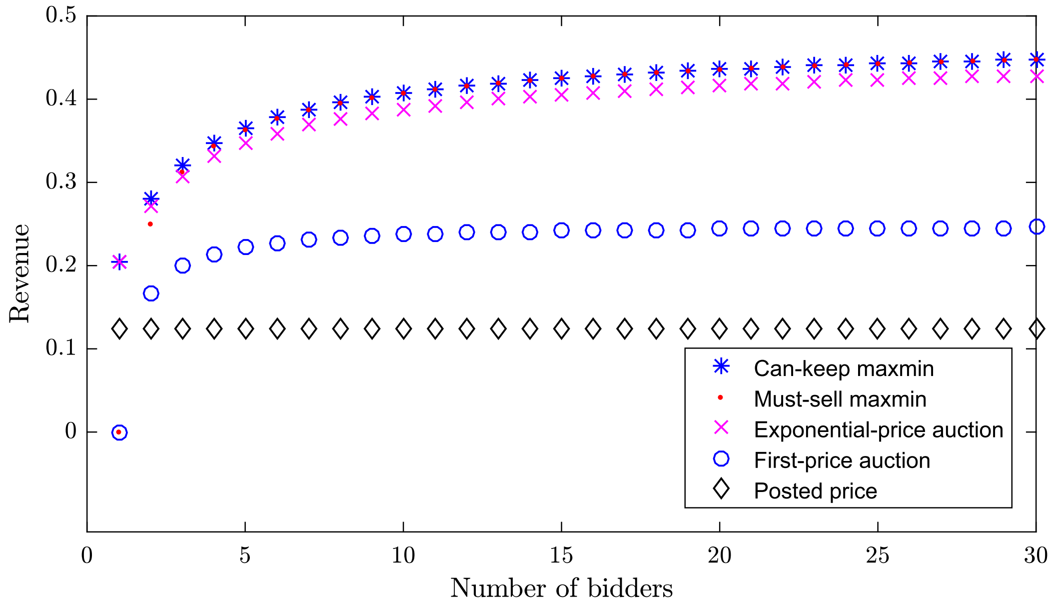

FIGURE 1.—Compare the revenue guarantees of various mechanisms for the uniform distribution on [0, 1].

Since $K_N \log N \to \infty$, which is the $N \cdot X_N \to 0$ condition in Theorem 1, $\mathbb{E}[N^{m_i}]$ grows with N at the slow rate of $K_N \log N$. In particular, by Jensen's inequality, we have $N^{\mathbb{E}[m_i]} \leq \mathbb{E}[N^{m_i}]$; thus $\mathbb{E}[m_i] \leq \frac{\log K_N + \log\log N}{\log N} \to 0$, by the $\frac{\log K_N}{\log N} \to 0$ condition in Theorem 1. That is, in a symmetric equilibrium, the demand of each individual bidder must vanish as $N \to \infty$.

The proof of Theorem 1 shows that (cf. Equation (13)) if $A_N = 1/\log N$ and K_N is a constant, then the revenue guarantee converges to the full surplus at a rate of $1/\log N$. Brooks and Du (2018) constructed mechanisms that maximize the revenue guarantee (maxmin mechanisms) with respect to a prior distribution of values when $N > 1$; they considered two cases: the must-sell case, which optimizes the revenue guarantee subject to the constraint that the allocation rule is efficient (the total allocation always adds up to 1), and the can-keep case, which gives the unconstrained optimum. They showed that in both cases, the optimal revenue guarantees converge to the full surplus at the rate of $1/\sqrt{N}$. Nevertheless, when the prior is the uniform distribution on [0, 1], the revenue guarantees of the exponential price auctions are fairly close to the optimum, as shown in Figure 1, which is taken from Brooks and Du (2018). In Figure 1, I optimize parameters A_N and X_N of the exponential price auction with respect to the uniform distribution. The full surplus in this case is 1/2. I also include the revenue guarantees of the first price auction (Bergemann, Brooks, and Morris (2017a)), which tends to 1/4 as $N \to \infty$, and the optimal revenue guarantee for a posted price (a posted price of 1/4, giving a revenue guarantee of 1/8), which is independent of N.

4.1. *Example*

For a given information structure $(\tilde{p}, S_i)$, the equilibrium strategy in the exponential price auction is straightforward to characterize. Without loss of generality, let us focus here on a pure-strategy equilibrium σ. The first-order condition, equating the marginal surplus with the marginal payment, is

$$\mathbb{E}\left[\frac{v}{\operatorname{rank}(\sigma_i(s_i);\sigma(s))}\,\Big|\,s_i\right]=t_i'(\sigma_i(s_i)),$$

where $\sigma_i(s_i)\in M_i$ is bidder i's equilibrium demand given the signal s_i, and $\operatorname{rank}(m_i;m)\in\{1,2,\dots,N\}$ is the rank of m_i in the message profile m; for example, if $m_i=0.5$ and $m=(0.2,0.5,0.3)$, then $\operatorname{rank}(m_i;m)=1$.[7] Thus, if a bidder's equilibrium demand is interior, that is, $\sigma_i(s_i)\in(0,1)$, then his equilibrium bid/payment is

$$t_i(\sigma_i(s_i))=A_N\cdot\mathbb{E}\left[\frac{v}{\operatorname{rank}(\sigma_i(s_i);\sigma(s))}\,\Big|\,s_i\right]-X_N, \tag{3}$$

since we have $t_i(m_i)=A_N\cdot t_i'(m_i)-X_N$. Thus, in equilibrium one's bid is a linear function of the conditional expectation of the marginal surplus, that is, the value divided by the rank of one's bid. The slope A_N represents the proportion of bid shading from the marginal surplus ($A_N<1$ in Theorem 1), while the intercept X_N is a fixed amount of information rent that a bidder keeps for himself and withholds from payment. Importantly, a bidder accounts for the winner's curse in the equilibrium strategy in (3): a low v is usually correlated with a high $\operatorname{rank}(\sigma_i(s_i);\sigma(s))$ in the conditional expectation, as the other bidders are likely to submit low demands in $\sigma_{-i}(s_{-i})$; and likewise for a high v and a low $\operatorname{rank}(\sigma_i(s_i);\sigma(s))$; thus, low v is "over-represented" and high v "under-represented" in the conditional expectation of the marginal surplus.

The conditional expectation of the marginal surplus in Equation (3) is often very simple in large markets. For example, suppose, conditional on a realization of value v, each bidder i receives an independent signal s_i from the CDF $F_N(\cdot\mid v)$. Then, in a symmetric monotone equilibrium, we have the following bidding strategy for a large N and an interior demand:

$$t_i(\sigma_i(s_i))\approx A_N\cdot\int_v\frac{v}{N(1-F_N(s_i\mid v))}\,\tilde{p}(dv\mid s_i)-X_N,$$

if $F_N(s_i\mid v)<1$ for all v: conditional on v, since the signals are i.i.d., the rank of $\sigma_i(s_i)$ is close to $N(1-F_N(s_i\mid v))$ with high probability when N is large.

To illustrate the full surplus extraction, let us take the simple case of independent signals. Consider the information structure in Bergemann, Brooks, and Morris (2017a, 2017c): the signal $s_i\in S_i=[0,1]$ is i.i.d. across the bidders, and $v=\max_i s_i$. The signal s_i has CDF $F_N(s_i)=p(v\le s_i)^{1/N}$. (See the discussion in Section 3.)

It is easy to check that $\mathbb{E}[\max_j s_j\mid s_i]\approx\mathbb{E}[\max_j s_j]=\mathbb{E}[v]$ when N is large, if $F_N(s_i)<1$. Therefore, the equilibrium bidding strategy for a large N and an interior demand is

$$t_i(\sigma_i(s_i))\approx A_N\cdot\frac{\mathbb{E}[v]}{N(1-F_N(s_i))}-X_N,$$

[7]When m_i ties with other demands in m, the left and right derivatives of $q_i(m_i,m_{-i})$ with respect to m_i are different; nevertheless, I can still write $\frac{\partial q_i}{\partial m_i}=1/\operatorname{rank}(m_i;m)$ if in determining $\operatorname{rank}(m_i;m)$ I break ties in favor of m_i for the right derivative (e.g., $\operatorname{rank}(0.3;(0.3,0.3,0.4,0.3))=2$), and break ties against m_i for the left derivative (e.g., $\operatorname{rank}(0.3;(0.3,0.3,0.4,0.3))=4$). Moreover, as the left and right derivatives of $t_i(m_i)$ coincide, in equilibrium there cannot be any tie in m_i with positive probability when $m_i\in(0,1)$; for otherwise bidder i must either have an incentive to demand a little more than m_i, or to demand a little less than m_i.

if $F_N(s_i) < 1$. As discussed in Section 3, when N is large, most bidders in the BBM information structure have very little private information about the value, as their signals are close to the lowest possible signal, so their conditional values are close to $\mathbb{E}[v]$. Therefore, not counting the payments of the bidders with $F_N(s_i) = 1$, the equilibrium revenue from the BBM information structure is close to the equilibrium revenue when all bidders are uninformed, which is at least

$$\sum_{i=1}^{N}\left(A_N \cdot \frac{\mathbb{E}[v]}{i} - X_N\right).$$

If $A_N \cdot \frac{\mathbb{E}[v]}{i} - X_N < 0$, then the ith highest bidder demands and pays 0 in equilibrium, so the above is a lower bound on the equilibrium revenue. Suppose $t_i'(1) = X_N e^{1/A_N}/A_N > \mathbb{E}[v]$, so no one demands the entire allocation in equilibrium. Since $\sum_j 1/j \approx \log N$, taking $A_N = 1/\log N$ and X_N such that $NX_N \to 0$ and $X_N e^{1/A_N}/A_N = X_N N \log N > \mathbb{E}[v]$, the above lower bound converges to the full surplus $\mathbb{E}[v]$.

The BBM information structure highlights an important difference between the exponential price auction and first price auction. In the first price auction with BBM information structure, bidders bid very small amount in equilibrium because they anticipate the winner's curse: one wins only when he has the highest signal, that is, only when the value is equal to his signal, which is small with an overwhelming probability. In other words, the equilibrium bid in the first price auction is contingent on the unlikely event that one's signal is the highest signal, which results in a pessimistic estimation of the value. In contrast, the equilibrium bid in the exponential price auction is contingent on the high probability event that one's signal is ranked according to its percentile, which does not result in any distortion in one's estimation of the value.

4.2. *Proof of Theorem 1 When $K_N < 1$*

I focus here on a simple case of Theorem 1 in which $K_N < 1$ (no one demands the entire allocation) and leave the general case to Section A.

Fix an information structure $\tilde{p}$ and an equilibrium σ. They induce a distribution $\rho \in \Delta(V \times M)$, which is known as a Bayes correlated equilibrium (BCE; see Bergemann and Morris (2013, 2016)).

I have the following first-order condition (ρ-almost-surely over m_i):

$$\int_{(m_{-i},v)} \frac{v}{\operatorname{rank}(m_i; m)} \rho(dm_{-i}, dv \mid m_i) \begin{cases} \le t_i'(m_i), & m_i = 0, \\ = t_i'(m_i), & 0 < m_i < 1, \\ \ge t_i'(m_i), & m_i = 1. \end{cases} \tag{4}$$

By the assumption of $K_N < 1$, $t_i'(1) = X_N e^{1/A_N}/A_N = \bar{v}/K_N > \bar{v}$. That is, the marginal surplus from a demand of 1 is strictly dominated by the marginal payment, so no bidder will demand $m_i = 1$ in equilibrium; that is,

$$\rho\big(V \times [0, 1)^N\big) = 1.$$

Integrating the first-order condition across $m_i \in [0, 1)$, with respect to $\rho(dm_i)$, I get a lower bound on the marginal payments:

$$\int_{(m,v)} t_i'(m_i)\rho(dm, dv) \ge \int_{(m,v)} \frac{v}{\operatorname{rank}(m_i; m)} \rho(dm, dv).$$

Since $t_i(m_i) = A_N \cdot t_i'(m_i) - X_N$, I can convert the above into a lower bound on the revenue:

$$\begin{aligned}\sum_{i=1}^{N}\int_{(v,m)} t_i(m_i)\rho(dv,dm) &\geq \sum_{i=1}^{N}\left(A_N\cdot\int_{(v,m)}\frac{v}{\operatorname{rank}(m_i;m)}\rho(dv,dm)-X_N\right)\\ &\geq \int_{(v,m)}\left(\sum_{i=1}^{N}\frac{A_N}{i}\right)v\rho(dv,dm)-N\cdot X_N\\ &= \int_{v}\left(\sum_{i=1}^{N}\frac{A_N}{i}\right)vp(dv)-N\cdot X_N,\end{aligned}$$

where in the second line I use the fact that $\sum_{i=1}^{N}\frac{1}{\operatorname{rank}(m_i;m)} \geq \sum_{i=1}^{N} 1/i$. Finally, as $A_N \log(N) \to 1$ and $N \cdot X_N \to 0$, the last line converges to the full surplus $\int_v vp(dv)$ as $N \to \infty$. This concludes the proof of Theorem 1 when $K_N < 1$.

REMARK 1: As I have only used the bidder's first-order condition in the proof of the revenue guarantee, the guarantee of full surplus extraction in Theorem 1 actually holds over all local equilibria, that is, strategy profile in which there is no incentive to deviate locally (deviations from m_i to $m_i + \varepsilon$ and to $m_i - \varepsilon$, when one's strategy is supposed to submit m_i, where $\varepsilon > 0$ is small). In fact, all I need is that there is no incentive to deviate locally upwards (deviation from m_i to $m_i + \varepsilon$); or in other words, all I have used in the proof is the first-order condition where the derivatives are defined by the right limits. Having a revenue guarantee over all local equilibria is nice from a robustness perspective because the set of local equilibria is a superset of the set of equilibria, and boundedly rational or inexperienced bidders are likely to end up at a local equilibrium as the local deviations tend to be more salient than the nonlocal deviations.

REMARK 2: Since the exponential price auction has continuous allocation and payment rules, an equilibrium exists under quite permissive conditions on the information structure. More specifically, an information structure $(\tilde{p}, S_i)$ is *product continuous* if $\operatorname{marg}_S \tilde{p}$ is absolutely continuous with respect to the independent product measure $\operatorname{marg}_{S_1} \tilde{p} \otimes \cdots \otimes \operatorname{marg}_{S_N} \tilde{p}$, where $\operatorname{marg}_S \tilde{p}$ is the marginal distribution of $\tilde{p}$ over $S = \prod_{i=1}^{N} S_i$, and similarly for $\operatorname{marg}_{S_i} \tilde{p}$ (cf. Milgrom and Weber (1985)). Notice that if S is a finite or countably infinite set, or if the signals are independent according to $\tilde{p}$, then the information structure is automatically product continuous. Theorem 1 of Carbonell-Nicolau and McLean (2017) implies that under any product continuous information structure, an equilibrium exists in the exponential price auction.

5. ONE BIDDER CASE

In this section, I suppose the seller knows the prior distribution of values. I show that the exponential price auction gives the best possible revenue guarantee when $N = 1$ by making a connection to a result of Roesler and Szentes (2017). Roesler and Szentes (2017) studied the optimal information structure for the bidder (and the worst for the seller) when the seller best responds to the information structure with a posted price. This information structure has the following CDF for the signals:

$$G_\pi^B(s) = \begin{cases} 1, & s \geq B, \\ 1 - \pi/s, & s \in [\pi, B), \\ 0, & s < \pi, \end{cases} \tag{5}$$

where s is an unbiased signal of the bidder about his value ($\mathbb{E}[v \mid s] = s$), $[\pi, B]$ is the support of the signal, $\pi > 0$, and there is an atom of size π/B at $s = B$. By construction, the prior p is a mean-preserving spread of G_π^B, that is, the following two conditions are satisfied:

$$\int_{v=0}^{\bar{v}} vp(dv) = \int_{s=0}^{\bar{v}} sG_\pi^B(ds) = \pi + \pi \log B - \pi \log \pi, \tag{6}$$

$$\begin{aligned} &\min_{s\in[\pi,B]} F(s, \pi, B) \geq 0, \quad \text{where} \\ &F(s, \pi, B) \equiv \int_{s'=0}^{s} p(v \leq s')\, ds' - \int_{s'=0}^{s} G_\pi^B(s')\, ds'. \end{aligned} \tag{7}$$

Equation (6) says that G_π^B has the same mean as p; let $B(\pi)$ be the unique B such that Equation (6) holds for a given π. Equation (7) says that G_π^B second-order stochastically dominates p. Let $F(s, \pi) \equiv F(s, \pi, B(\pi))$.

If $G_\pi^{B(\pi)}$ is the distribution of unbiased signals for the bidder, then the seller is clearly indifferent between every posted price in $[\pi, B]$ and has an expected revenue of π from an optimal mechanism. Roesler and Szentes (2017) proved that the best information structure for the bidder (and the worst for the seller) when the seller best responds to the information structure is $G_{\pi^*}^{B(\pi^*)}$, where π^* is the smallest π such that $\min_{s\in[\pi,B(\pi)]} F(s, \pi) \geq 0$; that is, π^* is the smallest π such that $G_\pi^{B(\pi)}$ second-order stochastically dominates p. Clearly, π^* is an upper bound on the revenue guarantee of any mechanism. On the other hand, the proof of Theorem 1 shows the following revenue guarantee for any information structure $\tilde{p}$ and equilibrium σ (cf. Equation (13)):

$$R(\tilde{p}, \sigma) \geq \int_v \min\big(A \cdot v - X, Xe^{1/A} - X\big)\, p(dv),$$

where, for notational simplicity, we drop the dependence of A and X on $N = 1$.

PROPOSITION 1: *Suppose $p(v \leq s) > 0$ for all $s \in (0, \bar{v}]$. There exist constants $A^* > 0$ and $X^* > 0$ such that*

$$\pi^* = \int_v \min\big(A^* \cdot v - X^*, X^* e^{1/A^*} - X^*\big)\, p(dv).$$

That is, when $N = 1$, the exponential price auction with parameters A^* and X^* gives the optimal revenue guarantee.

PROOF: Let s^* be an arbitrary selection from $\operatorname{argmin}_{s\in[\pi^*,B(\pi^*)]} F(s, \pi^*)$. Since $\min_{s\in[\pi,B(\pi)]} F(s, \pi)$ is a continuous function of π, we must have $F(s^*, \pi^*) = 0$, that is,

$$\int_{s=0}^{s^*} p(v \leq s)\, ds = \int_{s=0}^{s^*} G_{\pi^*}^{B(\pi^*)}(s)\, ds.$$

Moreover, we have $s^* \in (\pi^*, B(\pi^*)]$, so $\frac{\partial F}{\partial s}(s^*, \pi^*) = 0$ from the first-order condition,[8] that is,

$$p(v \le s^*) = G_{\pi^*}^{B(\pi^*)}(s^*). \tag{8}$$

Using integration by parts, these two equations imply

$$\int_{v=0}^{s^*} vp(dv) = \int_{s=0}^{s^*} sG_{\pi^*}^{B(\pi^*)}(ds) = \int_{s=\pi^*}^{s^*} sG_{\pi^*}^{B(\pi^*)}(ds). \tag{9}$$

Now consider the exponential price auction with parameters A^* and X^* such that

$$X^*/A^* = \pi^*, \qquad X^* e^{1/A^*}/A^* = s^*. \tag{10}$$

Suppose the bidder's information structure is $G_{\pi^*}^{B(\pi^*)}$. A bidder with a signal $s \in [\pi^*, s^*]$ optimally demands $\sigma(s) = A^* \log(s/\pi^*)$ and pays $A^*s - X^*$ in the auction; a bidder with a signal $s \in [s^*, B(\pi^*)]$ optimally demands $\sigma(s) = 1$ and pays $A^*s^* - X^* = X^* e^{1/A^*} - X^*$. The expected equilibrium revenue is thus

$$\begin{aligned} R\big(G_{\pi^*}^{B(\pi^*)}, \sigma\big) &= \int_{s=\pi^*}^{s^*} \big(A^*s - X^*\big) G_{\pi^*}^{B(\pi^*)}(ds) + \big(1 - G_{\pi^*}^{B(\pi^*)}(s^*)\big)\big(X^* e^{1/A^*} - X^*\big) \\ &= \int_{v=0}^{1} \min\big(A^* \cdot v - X^*, X^* e^{1/A^*} - X^*\big) p(dv), \end{aligned}$$

where, in the second line, the equilibrium revenue equals the revenue guarantee by Equations (8) and (9).

Finally, I show that $R(G_{\pi^*}^{B(\pi^*)}, \sigma) = \pi^*$; that is, the exponential price auction is an optimal mechanism when the seller knows the bidder has information structure $G_{\pi^*}^{B(\pi^*)}$. Recall that a bidder with an unbiased signal $s \ge \pi^*$ optimally chooses the allocation $\min(A^* \log(s/\pi^*), 1)$ and pays $\min(A^*s - X^*, A^*s^* - X^*)$ in the exponential price auction. This allocation and payment can also be obtained from the seller randomizing over posted price P with a CDF $G(P) = A^* \log(P/\pi^*)$, $P \in [\pi^*, s^*]$, and a bidder with an unbiased signal s accepts a posted price P if and only if $P \le s$: the equivalence in allocation is immediate, and the equivalence in payment follows from the fact that $\int_{P=\pi^*}^{s} Pg(P)\,dP = A^*s - X^*$, where $g(P)$ is the density of the distribution $G(P)$. When the bidder's information structure is $G_{\pi^*}^{B(\pi^*)}$, each posted price $P \in [\pi^*, s^*]$ leads to the optimal revenue of π^*; thus, the randomized posted price $G(P)$, and hence the exponential price auction as well, is an optimal mechanism. *Q.E.D.*

Proposition 1 is a generalization of Proposition 5 in Carrasco et al. (2018), where they have the result for a prior distribution with binary support (i.e., a mean constraint on the value distribution). When $N = 1$, the exponential price auction is essentially the randomized posted price mechanism in Proposition 5 of Carrasco et al. (2018) (cf. the last

[8]Since $\pi^* > 0$, $G_{\pi^*}^{B(\pi^*)}(\pi^*) = 0$, and $p(v \le s) > 0$ for all $s > 0$, we have $F(\pi^*, \pi^*) > 0$. Thus, we must have $s^* > \pi^*$.

Without loss, suppose $\bar{v} = \inf\{s : p(v \le s) = 1\}$, for otherwise we can replace $\bar{v}$ by the actual upper bound of the support. If $B(\pi^*) < \bar{v}$, we must have $F(B(\pi^*), \pi^*) > 0$, for otherwise we would have $\int_{s=0}^{\bar{v}} G_{\pi^*}^{B(\pi^*)}(s)\,ds > \int_{s=0}^{\bar{v}} p(v \le s)\,ds$, which would contradict the fact that $G_{\pi^*}^{B(\pi^*)}$ has the same mean as p; this implies $s^* < B(\pi^*)$, so $\frac{\partial F}{\partial s}(s^*, \pi^*) = 0$ because s^* is interior. If $B(\pi^*) = \bar{v}$, then either $s^* = \bar{v}$, in which case $\frac{\partial F}{\partial s}(s^*, \pi^*) = p(v \le \bar{v}) - G_{\pi^*}^{B(\pi^*)}(\bar{v}) = 0$, where the derivative is defined by the left limit, or $s^* < \bar{v}$, in which case $\frac{\partial F}{\partial s}(s^*, \pi^*) = 0$ because s^* is interior.

paragraph in the proof of Proposition 1), with a slight difference that the price is distributed over $[\pi^*, s^*]$ instead of $[\pi^*, \bar{v}]$ in their paper. When the support of the prior p is $\{0, \bar{v}\}$, we have $s^* = \bar{v}$. In general, we have $s^* < \bar{v}$.

Finally, when the bidder has information structure $G_{\pi^*}^{B(\pi^*)}$, the exponential price auction with A^* and X^*, while obtaining the optimal revenue, is not efficient: a positive fraction of the good can remain unallocated. This is in contrast to a posted price of π^*, which gives the optimal revenue as well as full efficiency. On the other hand, a posted price of π^* does not achieve the optimal revenue guarantee of π^*: for example, its equilibrium revenue is strictly lower than π^* when the bidder is perfectly informed of the value.

6. SUFFICIENT CONDITIONS FOR FULL SURPLUS EXTRACTION

In this section, I clarify the essential features of the allocation and payment rules that make Theorem 1 work. I focus on the case of Theorem 1 in which no one demands the entire allocation, since in this simple case, the sufficient conditions are easy to check for a given mechanism. For clarity, I make explicit the dependence of a mechanism (q^N, t^N) on the number N of bidders. Suppose $q_i^N : M \to [0, 1]$ and $t_i^N : M \to \mathbb{R}$ are differentiable functions, where $M_i = [0, 1]$, and $q_i^N(0, m_{-i}) = t_i^N(0, m_{-i}) = 0$ for all $m_{-i} \in M_{-i}$.

PROPOSITION 2: *Suppose the following two conditions hold for (q^N, t^N):*
1.
$$t_i^N(m) \geq A_N \cdot \frac{\partial t_i^N}{\partial m_i}(m) - X_N$$
for all $m \in [0, 1)^N$ and all bidder i, where
$$\frac{1}{A_N} = \inf_{m \in [0,1)^N} \sum_{i=1}^{N} \frac{\partial q^N}{\partial m_i}(m) > 0,$$[9]
and
$$N \cdot X_N \to 0, \quad \text{as } N \to \infty.$$
2.
$$\frac{\partial t_i^N}{\partial m_i}(1, m_{-i}) > \bar{v} \cdot \frac{\partial q_i^N}{\partial m_i}(1, m_{-i})$$
for all $m_{-i} \in M_{-i}$ and all bidder i.
Then the revenue guarantees of mechanisms (q^N, t^N) converge to the full surplus as $N \to \infty$, with a convergence rate of $O(N \cdot X_N)$.

Condition 1 ensures that one can lower bound the payment by the marginal surplus via the first-order condition. Condition 2 says that the marginal payment at the boundary is sufficiently high that no one will submit the boundary message in equilibrium. The proof of the proposition follows the same steps as the proof of Theorem 1 and is hence omitted.

The assumptions of Proposition 2 are restrictive. Condition 1 implies that
$$t_i^N(m) \leq X_N\big(\exp(m_i/A_N) - 1\big) \tag{11}$$

[9] Here $\frac{\partial q^N}{\partial m_i}(m)$ and $\frac{\partial t^N}{\partial m_i}(m)$ refer to the right derivatives.

for all $m \in M$; that is, the exponential functional form is the highest payment for the argument of Theorem 1 to work.[10]

To see the restriction on the allocation function, suppose $\max_{i,m_{-i}} \frac{\partial q_i^N}{\partial m_i}(1, m_{-i}) = C$ for a positive constant C independent of N. Condition 2 and Inequality (11) then imply

$$\frac{X_N e^{1/A_N}}{A_N} \geq \max_{i,m_{-i}} \frac{\partial t_i^N}{\partial m_i}(1, m_{-i}) > \bar{v} \cdot C.$$

But since we also have $X_N \cdot N \to 0$ from Condition 1, we conclude that as $N \to \infty$, $e^{1/A_N}/A_N$ must tend to infinity at a faster rate than N; in particular, $1/A_N = \inf_{m \in [0,1)^N} \sum_{i=1}^N \frac{\partial q_i^N}{\partial m_i}(m)$ itself must tend to infinity.

In fact, if $e^{1/A_N}/A_N$ tends to infinity at a faster rate than N, then the revenue guarantee of (q^N, t^N) converges to the full surplus at a rate of $O(\frac{N}{e^{1/A_N}/A_N})$, as one can set $X_N = \frac{\bar{v} \cdot C + \varepsilon}{e^{1/A_N}/A_N}$, $\varepsilon > 0$, and $t_i^N(m_i) = X_N(\exp(m_i/A_N) - 1)$ to satisfy the assumptions of Proposition 2. For example, in the exponential price auction, one has $1/A_N = \sum_{i=1}^N 1/i \approx \log N$, so the convergence rate to the full surplus is $O(1/\log N)$. It is an interesting open question if one could construct an allocation rule such that $1/A_N$ grows faster than $\log N$ and $\max_{i,m_{-i}} \frac{\partial q_i^N}{\partial m_i}(1, m_{-i}) = C$, which would give a faster convergence rate than $O(1/\log N)$ for the revenue guarantees.

As an application of Proposition 2, consider the following allocation rule: suppose $m_1 \geq m_2 \geq \cdots \geq m_N$, bidder i gets allocation

$$q_i^N(m) = \frac{1 - m_1}{N} + \sum_{j=i}^N \frac{m_j - m_{j+1}}{j}, \tag{12}$$

where $m_{N+1} \equiv 0$, and likewise for any other ordering of $(m_1, m_2, \ldots, m_N)$. That is, the previously unallocated fraction is uniformly distributed to the bidders, thus always resulting in an efficient allocation. The payment rule is exponential: $t_i^N(m_i) = X_N(\exp(m_i/A_N) - 1)$. One sees that

$$\sum_{i=1}^N \frac{\partial q_i^N}{\partial m_i}(m) \geq \sum_{i=1}^N \frac{1}{i} - \frac{1}{N}$$

and

$$\max_{i,m_{-i}} \frac{\partial q_i^N}{\partial m_i}(1, m_{-i}) = 1 - \frac{1}{N},$$

[10]Define

$$\xi_i(m) \equiv A_N \cdot \frac{\partial t_i^N}{\partial m_i}(m) - t_i^N(m) \leq X_N;$$

solving $t_i^N(m)$ in terms of $\xi_i(m)$ (together with the initial condition that $t_i^N(0, m_{-i}) = 0$) yields

$$\begin{aligned} t_i^N(m) &= \exp(m_i/A_N) \int_{x=0}^{m_i} \frac{\exp(-x/A_N)}{A_N} \xi_i(x, m_{-i})\, dx \\ &\leq \exp(m_i/A_N) \int_{x=0}^{m_i} \frac{\exp(-x/A_N)}{A_N} X_N\, dx = X_N\big(\exp(m_i/A_N) - 1\big). \end{aligned}$$

so one can take $1/A_N = \sum_{i=1}^{N-1} \frac{1}{i}$ and $X_N = \frac{\bar{v}}{e^{1/A_N}/A_N}$, and by Proposition 2, the revenue guarantee of this mechanism converges to the full surplus.

7. CONCLUSION

In this paper, I construct a new auction format, the exponential price auction, and show that for a common value good this auction can guarantee to extract the full surplus as the number of bidders tends to infinity, regardless of how information changes as more bidders are added.

As far as I can tell, the exponential price auction has not been used in practice. One possible reason might be that in the case of an indivisible good, the allocation is stochastic, so the bidders must trust that the seller is randomizing as specified in the allocation rule; there is no way for the bidders to check the randomization. Another reason might be that the auction is not ex post individually rational. In particular if the good is indivisible, then for all but one bidder, they pay the price but ex post get nothing in return; such an outcome might be embarrassing to explain to one's boss for a bidder in the real life. These two reasons suggest that the exponential price auction might be more practical when the good is divisible, in which case the bidders can check if the allocation is done right, and everyone gets some allocation. Whether/when the exponential price auction can be used in practice seems like an interesting question that deserves further exploration with both theory and experiments.

APPENDIX A: PROOF OF THE GENERAL CASE IN THEOREM 1

Fix an information structure $\tilde{p}$ and an equilibrium σ. Let $\rho \in \Delta(V \times M)$ be the distribution that $\tilde{p}$ and σ induce.

The first-order condition (4) implies for ρ-almost-every m_i:

$$t_i(m_i) \geq \min\biggl(\underbrace{A_N \cdot \int_{(m_{-i},v)} \frac{v}{\text{rank}(m_i; m)} \rho(dm_{-i}, dv \mid m_i) - X_N}_{\leq A_N \cdot t_i'(m_i) - X_N = t_i(m_i) \text{ if } m_i < 1}, t_i(1)\biggr).$$

By Jensen's inequality, I can move the min function inside the integral:

$$t_i(m_i) \geq \int_{(m_{-i},v)} \min\left(\frac{A_N \cdot v}{\text{rank}(m_i; m)} - X_N, t_i(1)\right) \rho(dm_{-i}, dv \mid m_i).$$

Integrating with respect to $\rho(dm_i)$ the above inequality across $m_i \in [0, 1]$, and then summing across i, I get a lower bound on the revenue:

$$\begin{aligned}
\sum_{i=1}^{N} \int_{(m,v)} t_i(m_i) \rho(dm, dv) &\geq \int_{(m,v)} \sum_{i=1}^{N} \min\left(\frac{A_N \cdot v}{\text{rank}(m_i; m)} - X_N, t_i(1)\right) \rho(dm, dv) \\
&\geq \int_{(m,v)} \sum_{i=1}^{N} \min\left(\frac{A_N \cdot v}{i} - X_N, t_i(1)\right) \rho(dm, dv) \\
&= \int_{v} \sum_{i=1}^{N} \min\left(\frac{A_N \cdot v}{i} - X_N, X_N e^{1/A_N} - X_N\right) p(dv) \qquad (13)
\end{aligned}$$

$$= \sum_{k=0}^{\lfloor K_N \rfloor} \int_{v=\nu_k}^{\nu_{k+1}} \left(k\left(X_N e^{1/A_N} - X_N\right) + \sum_{i=k+1}^{N} \left(\frac{A_N \cdot v}{i} - X_N \right) \right) p(dv)$$

$$= \sum_{k=0}^{\lfloor K_N \rfloor} \int_{v=\nu_k}^{\nu_{k+1}} \left(v\left(\sum_{i=k+1}^{N} \frac{A_N}{i} \right) + kX_N e^{1/A_N} - NX_N \right) p(dv),$$

where

$$\nu_k = kX_N e^{1/A_N}/A_N \quad \text{for } k \le \lfloor K_N \rfloor, \qquad \nu_{\lfloor K_N \rfloor + 1} = \bar{v}.$$

Note that when $K_N < 1$, this recovers the lower bound in the previous special case.

Since $k \le K_N$ and $\log K_N/\log N \to 0$, we have $\sum_{i=k+1}^{N} \frac{A_N}{i} \to 1$ as $N \to \infty$. Moreover, since $kX_N e^{1/A_N} \le \bar{v} A_N \to 0$ and $NX_N \to 0$, the last line of (13) converges to the full surplus $\int_v v p(dv)$. This completes the proof of Theorem 1.

REMARK 3: Define the following function on $[0, \bar{v}]$:

$$\gamma^N(v) = \sum_{i=k+1}^{N} \left(\frac{A_N \cdot v}{i} - X_N \right) + k\left(X_N e^{1/A_N} - X_N\right), \quad v \in \left[k\frac{X_N}{A_N} e^{1/A_N}, (k+1)\frac{X_N}{A_N} e^{1/A_N} \right].$$

The proof of Theorem 1 (Equation (13)) shows that $\int_v \gamma^N(v) p(dv)$ is a revenue guarantee for the exponential price auction. Suppose that p has full support over $[0, \bar{v}]$ and that the revenue guarantee $\int_v \gamma^N(v) p(dv)$ converges to the full surplus $\int_v v p(dv)$ as $N \to \infty$. For example, one might obtain A_N and X_N by maximizing $\int_v \gamma^N(v) p(dv)$. I claim that such (A_N, X_N) must satisfy the conditions in Theorem 1: as $N \to \infty$, $A_N \log N \to 1$, $X_N = \frac{\bar{v} A_N}{K_N \exp(1/A_N)}$, where $\frac{\log K_N}{\log N} \to 0$ and $NX_N \to 0$. That is, Theorem 1 covers all cases of full surplus extraction.[11]

I note that $\gamma^N(v)$ is closely related to the equilibrium revenue under the common knowledge that the value is v, which is

$$\tilde{\gamma}^N(v) = \sum_{i=k+1}^{N} \max\left(\frac{A_N \cdot v}{i} - X_N, 0 \right) + k\left(X_N e^{1/A_N} - X_N\right),$$

$$v \in \left[k\frac{X_N}{A_N} e^{1/A_N}, (k+1)\frac{X_N}{A_N} e^{1/A_N} \right],$$

where k is the number of bidders who demand $m_i = 1$ in equilibrium. The only difference between $\gamma^N(v)$ and $\tilde{\gamma}^N(v)$ is that for ith ranked bidder such that $\frac{A_N \cdot v}{i} - X_N < 0$, he demands and pays 0 in the equilibrium revenue $\tilde{\gamma}^N(v)$, but he contributes $\frac{A_N \cdot v}{i} - X_N < 0$ to the revenue guarantee $\gamma^N(v)$.

[11]To see this claim, first suppose $\gamma^N(v)$ pointwise converges to v as $N \to \infty$. For every N, $\gamma^N(v)$ is clearly a concave function of v. Therefore, at every v, the left and right derivatives of $\gamma^N(v)$ must converge to 1, that is, $A_N \log N \to 1$ and $\log K_N/\log N \to 0$. Since $\gamma^N(0) = -NX_N$, we also have $NX_N \to 0$.

Clearly, $\gamma^N(v) \le \tilde{\gamma}^N(v) \le v$ for every $v \in [0, \bar{v}]$ and N. Therefore, the convergence of the revenue guarantee to the full surplus implies that $\gamma^N(v)$ L1 converges to v. Standard L1 convergence result implies that along a subsequence of N, $\gamma^N(v)$ converges to v for p-almost-every v. Thus, for every subsequence of N, there is a further subsequence along which $A_N \log N \to 1$, $\log K_N/\log N \to 0$ and $NX_N \to 0$ (along this subsequence $\gamma^N(v)$ converges to v for p-almost-every v). Thus, we must have $A_N \log N \to 1$, $\log K_N/\log N \to 0$, and $NX_N \to 0$ as $N \to \infty$.

Suppose $X_N/A_N \le c/N$ for a constant $c > 0$, and $A_N \to 0$ as $N \to \infty$. Then for any $v > 0$,

$$\begin{aligned}\tilde{\gamma}^N(v) - \gamma^N(v) &\approx A_N\big(\log \tilde{N}(v) - \log N\big)v - \big(\tilde{N}(v) - N\big)X_N \\ &\le A_N\big(\log \tilde{N}(v) - \log N\big)v - \big(\tilde{N}(v) - N\big)A_N c/N \to 0, \quad \text{as } N \to \infty,\end{aligned}$$

where $\tilde{N}(v)$ is the equilibrium number of bidders who submit nonzero demand when v is commonly known, and this number satisfies $\tilde{N}(v) \ge \min(\frac{v}{c}N, N)$. Thus, for these parameters, everyone having complete information about v is asymptotically the worst case information structure in the exponential price auction (since its equilibrium revenue coincides with the revenue guarantee $\int_v \gamma^N(v)p(dv)$ as $N \to \infty$).

APPENDIX B: FINITE APPROXIMATION

In this section, I show that one can approximate the infinite exponential price auction with a sequence of finite mechanisms $(q^n, t^n)_{n\ge 1}$ whose revenue guarantees (over all finite information structures and all equilibria) converge to that of the infinite mechanism. Given a finite mechanism and a finite information structure, an equilibrium always exists.

Let us first discretize the values: suppose $V^n \subset V = [0, \bar{v}]$ is a finite subset, $p^n \in \Delta(V^n)$, and p^n weakly converges to p as $n \to \infty$, where we extend p^n to a measure on V by setting $p^n(B) = p^n(B \cap V^n)$ for every measurable $B \subseteq V$.

Fix the number N of bidders and the parameters A and X of the exponential price auction; here we suppress the dependence of the parameters on N. Let the message space be $M_i^n = \{0, 1/n, 2/n, \dots, 1\}$. The allocation function q_i^n is just the restriction of q_i in Equation (2) to M_i^n. The payment function is

$$t_i^n(m_i) = X\left(\left(1 + \frac{1}{A \cdot n}\right)^{m_i \cdot n} - 1\right).$$

For a fixed m_i, $t_i^n(m_i)$ clearly converges to $t_i(m_i)$ in Equation (2).

I will derive the revenue guarantee of mechanism (q^n, t^n) using linear programming duality, which gives some alternative perspectives and has proven useful in the methodology of Bergemann, Brooks, and Morris (2017b) and Brooks and Du (2018).

The revenue guarantee of a finite mechanism (q^n, t^n) is found by solving the following linear programming problem:

$$\min_{\mu \in \Delta(V^n \times M^n)} \sum_{(v,m)} \sum_i t_i^n(m)\mu(v, m)$$

subject to

$$\begin{aligned}&\sum_{(v,m_{-i})} \big[v\big(q_i^n(m_i, m_{-i}) - q_i^n\big(m_i', m_{-i}\big)\big) - \big(t_i^n(m_i, m_{-i}) - t_i^n\big(m_i', m_{-i}\big)\big)\big]\mu(v, m) \ge 0, \\ &\quad \forall i, \big(m_i, m_i'\big) \in M_i^n \times M_i^n, \\ &\sum_m \mu(v, m) = p^n(v), \quad \forall v \in V^n.\end{aligned} \tag{14}$$

That is, one minimizes the expected revenue from the distribution μ, subject to the constraint that μ is a Bayes correlated equilibrium (BCE), which always exists in a finite game.

As shown by Bergemann and Morris (2013, 2016), any information structure and equilibrium induce a BCE distribution over values and messages, and any BCE distribution is induced by some information structure and equilibrium.

The dual problem (cf. Vohra (2011)) to Problem (14) is

$$
\begin{aligned}
&\max_{(\alpha_i,\gamma)} \sum_v p^n(v)\gamma(v) \\
&\quad \text{subject to} \\
&\quad \gamma(v) + \sum_i \sum_{m_i'} \big[v\big(q_i^n(m) - q_i^n(m_i', m_{-i})\big) - \big(t_i^n(m) - t_i^n(m_i', m_{-i})\big)\big]\alpha_i(m_i' \mid m_i) \\
&\qquad \le \sum_i t_i^n(m), \\
&\qquad \forall v \in V^n, m \in M^n, \\
&\quad \alpha_i(m_i' \mid m_i) \ge 0, \quad \forall i, (m_i, m_i') \in M_i^n \times M_i^n,
\end{aligned}
\tag{15}
$$

where $\alpha_i(m_i' \mid m_i)$ is the multiplier on the incentive constraint in BCE that bidder i prefers m_i over m_i', and $\gamma(v)$ is the multiplier on the consistency constraint of $\sum_m \mu(v,m) = p^n(v)$. (See the previous version of this paper for an interpretation of $\alpha_i(m_i' \mid m_i)$ as the transition rates of a Markov process.) By the strong duality theorem, Problems (14) and (15) have the same optimal value. Therefore, the value of Problem (15) under any feasible multipliers (α_i, γ) is a lower bound on the optimal value of Problem (14), that is, a revenue guarantee.

Let the multipliers for the BCE incentive constraints be

$$
\alpha_i^n(m_i' \mid m_i) = \begin{cases} A \cdot n, & m_i' = m_i + 1/n, \\ 0, & m_i' \neq m_i + 1/n, \end{cases} \quad (m_i, m_i') \in M_i^n \times M_i^n. \tag{16}
$$

The above multipliers ignore all incentive constraints except local upward incentive constraint. This corresponds to the proof of Theorem 1 in which I only use the bidders' first-order conditions where all derivatives are defined by the right limits.

By construction,

$$
\begin{aligned}
&n \sum_i \big(q_i^n(m_i + 1/n, m_{-i}) - q_i^n(m)\big)\mathbf{1}_{m_i<1} \\
&\quad = \begin{cases} \displaystyle\sum_{j=k+1}^{N} \frac{1}{j}, & \big|\{i : m_i = 1\}\big| = k, \big|\{m_i : m_i < 1\}\big| = N - k, \\ \displaystyle > \sum_{j=k+1}^{N} \frac{1}{j}, & \big|\{i : m_i = 1\}\big| = k, \big|\{m_i : m_i < 1\}\big| < N - k, \end{cases}
\end{aligned}
$$

and

$$
t^n(m_i) - An\big(t^n(m_i + 1/n) - t^n(m_i)\big)\mathbf{1}_{m_i<1} = \begin{cases} -X, & m_i < 1, \\ X\left(\left(1 + \dfrac{1}{A \cdot n}\right)^n - 1\right), & m_i = 1, \end{cases} \tag{17}
$$

so I can take

$$\gamma^n(v) = \min_{0\le k\le N}\left(\sum_{i=k+1}^{N}\frac{A\cdot v}{i} - (N-k)X + kX\left(\left(1+\frac{1}{An}\right)^n - 1\right)\right).$$

By construction, the multipliers (α_i^n, γ^n) are feasible for the dual problem (15) of the mechanism (q^n, t^n). Therefore, the revenue guarantee of (q^n, t^n) for the prior p^n is at least

$$\sum_{v\in V^n} p^n(v)\cdot \min_{0\le k\le N}\left(\sum_{i=k+1}^{N}\frac{A\cdot v}{i} - (N-k)X + kX\left(\left(1+\frac{1}{An}\right)^n - 1\right)\right).$$

The above revenue guarantee clearly converges to the revenue guarantee in Equation (13) as $n\to\infty$.

REMARK 4: In a previous version of this paper, I show that to solve for a mechanism that maximizes the revenue guarantee, it is sufficient to restrict to the multipliers in Equation (16). That is, it is without loss to ignore all incentive constraints except the local upward incentive constraint, or equivalently, to focus on the bidders' first-order conditions with the right derivatives.

REFERENCES

ASKER, J. (2010): "A Study of the Internal Organization of a Bidding Cartel," *American Economic Review*, 100 (3) 724–762. [1570]

BALI, V., AND M. JACKSON (2002): "Asymptotic Revenue Equivalence in Auctions," *Journal of Economic Theory*, 106 (1), 161–176. [1569]

BERGEMANN, D., B. BROOKS, AND S. MORRIS (2017a): "First Price Auctions With General Information Structures: Implications for Bidding and Revenue," *Econometrica*, 85 (1), 107–143. [1569,1571,1573,1575,1576]

——— (2017b): "Informationally Robust Optimal Auction Design," Working Paper. [1571,1585]

——— (2017c): "Selling to Intermediaries: Optimal Auction Design in a Common Value Model," Working Paper. [1569,1573,1576]

BERGEMANN, D., AND S. MORRIS (2013): "Robust Predictions in Games With Incomplete Information," *Econometrica*, 81 (4), 1251–1308. [1577,1586]

——— (2016): "Bayes Correlated Equilibrium and the Comparison of Information Structures in Games," *Theoretical Economics*, 11 (2), 487–522. [1577,1586]

BROOKS, B., AND S. DU (2018): "Optimal Auction Design With Common Values: An Informationally-Robust Approach," Working Paper. [1571,1575,1585]

CARBONELL-NICOLAU, O., AND R. P. MCLEAN (2017): "On the Existence of Nash Equilibrium in Bayesian Games," *Mathematics of Operations Research*, 43 (1), 100–129. [1578]

CARRASCO, V., V. FARINHA LUZ, N. KOS, M. MESSNER, P. MONTEIRO, AND H. MOREIRA (2018): "Optimal Selling Mechanisms Under Moment Conditions," *Journal of Economic Theory*, 177, 245–279. [1570,1580]

CHEN, Y.-C., AND J. LI (2018): "Revisiting the Foundations of Dominant-Strategy Mechanisms," *Journal of Economic Theory* (forthcoming). [1571]

CHUNG, K.-S., AND J. C. ELY (2007): "Foundations of Dominant-Strategy Mechanisms," *Review of Economic Studies*, 74 (2), 447–476. [1571]

ENGELBRECHT-WIGGANS, R., P. R. MILGROM, AND R. J. WEBER (1983): "Competitive Bidding and Proprietary Information," *Journal of Mathematical Economics*, 11, 161–169. [1569,1573]

JACKSON, M. O., AND I. KREMER (2007): "On the Informational Inefficiency of Discriminatory Price Auctions," *Journal of Economic Theory*, 132 (1), 507–517. [1573]

LIBGOBER, J., AND X. MU (2017): "Informational Robustness in Intertemporal Pricing," Working Paper. [1571]

MCLEAN, R. P., AND A. POSTLEWAITE (2018): "A Very Robust Auction Mechanism," Working Paper. [1571]

MILGROM, P. R. (1979): "A Convergence Theorem for Competitive Bidding With Differential Information," *Econometrica*, 47 (3) 679–688. [1569]

MILGROM, P. R., AND R. J. WEBER (1985): "Distributional Strategies for Games With Incomplete Information," *Mathematics of Operations Research*, 10 (4), 619–632. [1578]

MOULIN, H. (1994): "Serial Cost-Sharing of Excludable Public Goods," *Review of Economic Studies*, 61 (2), 305–325. [1570]

PESENDORFER, W., AND J. M. SWINKELS (1997): "The Loser's Curse and Information Aggregation in Common Value Auctions," *Econometrica*, 65 (6), 1247–1281. [1569,1573]

ROESLER, A.-K., AND B. SZENTES (2017): "Buyer-Optimal Learning and Monopoly Pricing," *American Economic Review*, 107 (7), 2072–2080. [1570,1571,1578,1579]

VOHRA, R. V. (2011): *Mechanism Design: A Linear Programming Approach*. Cambridge: Cambridge University Press. [1586]

WILSON, R. (1977): "A Bidding Model of Perfect Competition," *Review of Economic Studies*, 44 (3) 511–518. [1569]

——— (1987): "Game-Theoretic Analyses of Trading Processes," in *Advances in Economic Theory: Fifth World Congress*, ed. by T. Bewley. Cambridge: Cambridge University Press, 33–70. [1571]

YAMASHITA, T. (2017): "Revenue Guarantee in Auction With a (Correlated) Common Prior and Additional Information," Working Paper. [1571]

Co-editor Dirk Bergemann handled this manuscript.

Manuscript received 9 January, 2017; final version accepted 25 April, 2018; available online 26 April, 2018.