Robust Mechanisms Under Common Valuation

Songzi Du*
Simon Fraser University

January 7, 2017

Abstract

We study robust mechanisms to sell a common-value good. We assume that the mechanism designer knows the prior distribution of the buyers' common value but is unsure of the buyers' information structure about the common value. We use linear programming duality to derive mechanisms that guarantee a good expected revenue for all information structures and all equilibria. Our mechanism maximizes the revenue guarantee when there is one buyer. As the number of buyers tends to infinity, the revenue guarantee of our mechanism converges to the full surplus. We numerically demonstrate that the revenue guarantees of our mechanisms are generally close to optimal when there are two buyers.

Keywords: robust mechanism, common value, full surplus extraction, Bayes correlated equilibrium
JEL Codes: D44, D82

*First draft: March 20, 2016. I thank Gabriel Carroll, Vitor Farinha Luz, Ben Golub, Wei Li, Michael Ostrovsky, and participants at Canadian Economic Theory Conference and NBER Market Design workshop for comments and discussions. Email: songzid@sfu.ca

1 Introduction

In this paper we study robust mechanism design for selling a common-value good. A robust mechanism is one that works well under a variety of circumstances, in particular under weak assumptions about participants' beliefs. The goal of robust mechanism design is to reduce the "base of common knowledge required to conduct useful analyses of practical problems," as envisioned by Wilson (1987).

The literature on robust mechanism design has so far largely focused on private value settings[1], with a notable exception of Bergemann, Brooks, and Morris (2016b) which we discuss below. Common value is of course important in many real-life markets (particularly financial markets) and has a long tradition in auction theory. We suppose only the prior distribution of the buyers' common value is known by the seller. We allow the buyers to have arbitrary beliefs (and higher order beliefs) about the common value, as long as it is consistent with the prior; such a distribution of beliefs is called an information structure. In particular, the buyers' beliefs in an information structure may be generated by multi-dimensional signals that are arbitrarily correlated.

We want to design a selling mechanism that guarantees a good expected revenue for every information structure and every equilibrium in the information structure. Such mechanism is clearly useful when the seller does not know the nature of information/beliefs held by the strategic buyers, as it is commonly the case in practice. Moreover, such mechanism can be adopted as the default trading protocol that works well in a variety of circumstances with minimal customization. We are inspired by Bergemann, Brooks, and Morris (2016a) who work out the revenue guarantee of the first price auction (among other results).

We come up with a mechanism that guarantees a better revenue than the first price auction. The mechanism is simple to implement and can be described as follows. Suppose there are $I \geq 1$ buyers with quasi-linear utility. Let the message space for each buyer i be the interval $[0, 1]$. We think of a message $z_i \in [0, 1]$ as the demand of buyer i. Buyer i gets $q_i(z_i, z_{-i})$ quantity of allocation (q_i could also be a probability of getting the good if the good is indivisible) and pays $P_i(z_i, z_{-i})$. If $z_1 \geq z_2 \geq \cdots \geq z_I$, then

$$q_i(z_i, z_{-i}) = \sum_{j=i}^{I-1} \frac{z_j - z_{j+1}}{j} + \frac{z_I}{I}, \qquad P_i(z_i, z_{-i}) = X(\exp(z_i/A) - 1), \tag{1}$$

[1] See Chung and Ely (2007), Brooks (2013), Frankel (2014), Carroll (2015, 2016a), Yamashita (2015, 2016), Carrasco, Farinha Luz, Monteiro, and Moreira (2015), Kos and Messner (2015), Chen and Li (2016), Hartline and Roughgarden (2016), among others; we follow this literature by adopting a max-min approach to robust mechanism.

and analogously for any other ordering of $(z_1, z_2, \ldots, z_I)$. That is, the lowest buyer gets $1/I$ of his demand, the second lowest buyers gets that plus $1/(I-1)$ of the difference between his and the lowest demand, and so on. Thus, the total quantity of allocation is equal to the highest demand. Moreover, each buyer's payment depends only on his demand and is independent of his final allocation, like an all-pay auction. Finally, $A > 0$ and $X > 0$ in Equation (1) are constants that are optimized for the prior distribution of value; intuitively, A and X are choices of units for the demand and the payment, respectively. We call the mechanism in Equation (1) the *exponential price mechanism*.

We prove that the exponential price mechanism gives the optimal revenue guarantee when there is one buyer ($I = 1$). In this case we know sharp upper bound on the revenue guarantee. For example, if the prior is the uniform distribution on $[0, 1]$, then the seller can guarantee (among all information structures and all equilibria) a revenue of at most $1/4$: fix any mechanism, there is the private-value information structure, and its equilibrium revenue must be less than $1/4$ which is obtained by the private-value optimal mechanism (a posted price of $1/2$). Roesler and Szentes (2016) study the optimal information structure for a buyer when the seller is best responding to this information structure. Roesler and Szentes's optimal information structure gives a subtle upper bound on the seller's revenue guarantee. For example, when the prior is the uniform distribution on $[0, 1]$, the seller can guarantee a revenue of at most 0.2036. We prove that the exponential price mechanism exactly guarantees the Roesler-Szentes upper bound for any prior distribution when there is one buyer.[2] In contrast, any posted price does not give the optimal revenue guarantee; for example, when the prior is the uniform $[0, 1]$ distribution, the optimal posted price guarantees a revenue of only $1/8$.[3]

As the number I of buyers increases, the exponential price mechanism guarantees a better revenue, as we numerically demonstrate in Figure 1 (page 10). We prove that as the number of buyers tends to infinity, the revenue guarantee of the exponential price mechanism (over

[2] In other words, Roesler and Szentes (2016) characterize for one buyer:

$$\min_{\text{info. structure}} \quad \max_{\text{mechanism, equilibrium}} \quad \text{Revenue},$$

while we characterize:

$$\max_{\text{mechanism}} \quad \min_{\text{info. structure, equilibrium}} \quad \text{Revenue},$$

and show it is equal to their min-max value. Equilibrium here is a mapping from signals of the information structure to messages in the mechanism, such that there is no incentive to deviate.

[3] When the prior is the uniform $[0, 1]$ distribution, a posted price of $p \leq 1/2$ guarantees a revenue of $(1 - 2p)p$: suppose the buyer's information about his value is the partition $\{[0, 2p), [2p, 1]\}$, an equilibrium is to buy at price p if and only if $[2p, 1]$ is realized.

all information structures and all equilibria) becomes arbitrarily close to the full surplus (the expectation of the common value), i.e., asymptotic guarantee of full surplus extraction. In fact, this asymptotic revenue guarantee can be achieved in a prior-free way: there is one set of parameters for the exponential price mechanism that simultaneously guarantees asymptotic full surplus extraction for all prior distributions of value on $[0, 1]$. Asymptotic guarantee of full surplus extraction is not obtained by the first price auction (as shown by Engelbrecht-Wiggans, Milgrom, and Weber (1983) and Bergemann, Brooks, and Morris (2016a)), second price auction[4], all-pay auction[5], or with a posted price (consider the case when all I buyers have symmetric information about the common value). Unlike the mechanisms of Crémer and McLean (1985, 1988), our mechanism is detail free and relies on leveraging competition among buyers to extract full surplus.

To analyze the revenue guarantee of mechanisms we introduce a duality approach. We want to minimize the expected revenue over the set of information structures and equilibria for a given mechanism, and then maximize the minimized revenue over the set of mechanisms. Bergemann and Morris (2016) give the powerful insight that we can combine information structure and equilibrium into a single entity called *Bayes Correlated Equilibrium*, which is a joint distribution over actions and value subject to incentive and consistency constraints. Minimizing revenue over Bayes correlated equilibria for any fixed mechanism is a linear programming problem, and we can equivalently solve the dual problem which is a maximization problem over the Lagrange multipliers of constraints associated with Bayes correlated equilibrium. We can combine the maximization over the multipliers with the maximization over the allocations and payments, so we have a single maximization problem which is equivalent to but more tractable than the original max-min problem. Moreover, the multipliers have the interpretation as transition rates for a continuous-time Markov process over the message space of mechanism[6]; this Markov process interpretation and its ergodic theorem play a crucial role in showing that for maximizing revenue guarantee it suffices to consider one dimensional message space and a single multiplier for all local incentive constraints. This sufficiency leads to efficient numerical computation of the optimal revenue guarantee, and

[4]For a second price auction with a reserve price (potentially zero), suppose there is one informed buyer who knows the common value v, and $I-1$ uninformed buyers who only knows the prior. The following is an equilibrium: the informed buyer truthfully bids v, and all uninformed buyers bid 0. Clearly, this equilibrium does not obtain the full surplus in revenue as $I \to \infty$.

[5]The minimum-revenue information structure in Bergemann, Brooks, and Morris (2016a) for the first price auction also fails to extract the full surplus in revenue for an all-pay auction as $I \to \infty$.

[6]This interpretation is similar to that of Myerson (1997), which has transition probabilities as multipliers for complete-information correlated equilibrium, and is also used in Carroll (2016b).

we use such computation to show that when there are two buyers the revenue guarantees of the exponential price mechanisms are generally close to optimal.

Our duality framework has proven useful in a new paper by Bergemann, Brooks, and Morris (2016b), who work out the optimal mechanism that maximizes the revenue guarantee when there are two buyers and binary common values ($V = \{0, 1\}$). Bergemann, Brooks, and Morris (2016b) construct an intriguing information structure with independent, one-dimensional types that gives an upper bound on the revenue guarantee. Moreover, they build on our lower-bound method to construct a mechanism that exactly obtains their upper bound when there are two buyers and binary values. Our paper and theirs are clearly complementary.

In the case of one buyer, our exponential price mechanism is a generalization of the mechanisms of Carrasco, Farinha Luz, Monteiro, and Moreira (2015) and Kos and Messner (2015) that maximize the revenue guarantee of a seller who only knows the mean of the buyer's value. The seller in our model knows the entire prior distribution of the buyer's value (the buyer may have information beyond the prior), which enables a higher revenue guarantee. That the optimal mechanism has exponential payment when the seller knows the mean and when the seller knows the entire prior distribution speaks to the robustness of the exponential functional form.

Our robustness exercise is conceptually similar to that of Chung and Ely (2007), Chen and Li (2016) and Yamashita (2016); while they focus on private values, we study common value. Moreover, they argue for the maxmin optimality of belief-free dominant strategy mechanism, while the mechanisms obtained by us (and by Bergemann, Brooks, and Morris (2016b)) make use of beliefs in Bayes correlated equilibrium.

2 Model

Information

The mechanism designer has one unit of good to sell. Let $\mathcal{I} = \{1, 2, \ldots, I\}$ be a finite set of buyers, $I \geq 1$. The buyers have a common value $v \in V = \{0, \nu, 2\nu, \ldots, 1\}$ for the good and have quasi-linear utility, where $\nu > 0$ is a constant. Let $p \in \Delta(V)$ be the prior distribution of common value; the prior p is known by the designer as well as by the buyers. (The designer only knows the prior p about the value.)

Each buyer i *may* possess some additional information $s_i \in S_i$ about the common value beyond the prior, where S_i is a finite set of signals (or types), and there is a distribution

$\tilde{p} \in \Delta(V \times \prod_{i\in\mathcal{I}} S_i)$ such that $\text{marg}_V\, \tilde{p} = p$.[7] Thus buyer i has belief $\tilde{p}(\,\cdot \mid s_i) \in \Delta(V \times S_{-i})$ for each realization of the signal s_i. As discussed in the introduction, the information structure $(S_i, \tilde{p})_{i\in\mathcal{I}}$ is *not* known by the designer.

Mechanism

A mechanism is a set of allocation rules $q_i : M \to [0,1]$ and payment rules $P_i : M \to \mathbb{R}$ satisfying $\sum_{i\in\mathcal{I}} q_i(m) \leq 1$, where M_i is the message space of buyer i and is a finite set, and $M = \prod_{i\in\mathcal{I}} M_i$ the space of message profiles. A mechanism defines a game in which the buyers simultaneously submit messages and have utility

$$U_i(v, m) = v \cdot q_i(m) - P_i(m). \tag{2}$$

The allocation $q_i(m)$ is the share of the good that buyer i receives in the case of a divisible good, and is the probability of getting the good in the case of an indivisible good.

We assume that a mechanism always has an opt-out option for each buyer i: there exists a message $m_i \equiv 0 \in M_i$ such that $q_i(0, m_{-i}) = P_i(0, m_{-i}) = 0$ for every $m_{-i} \in M_{-i}$.

In this paper we focus on *symmetric* mechanism, which satisfies

$$\begin{aligned} q_i(m'_i, m'_{-i}) &= q_1(m_1 = m'_i, m_{-1} = m'_{-i}) \equiv q(m'_i, m'_{-i}) \\ P_i(m'_i, m'_{-i}) &= P_1(m_1 = m'_i, m_{-1} = m'_{-i}) \equiv P(m'_i, m'_{-i}) \end{aligned} \tag{3}$$

for every $i \in \mathcal{I}$ and $m' \in M$. By $m_{-1} = m'_{-i}$ we mean that m_{-1} and m'_{-i} have the same elements but not necessarily the same ordering of elements; for example we may have $m_{-1} = (a, b, c)$ and $m'_{-i} = (c, b, a)$. Intuitively, in a symmetric mechanism every buyer is treated in the same way. For a symmetric mechanism we abbreviate $q_1(m)$ to $q(m)$ and $P_1(m)$ to $P(m)$. Restricting attention to symmetric mechanism is without loss of generality; see Section 5.1.

Equilibrium

Given a mechanism $(q_i, P_i)_{i\in\mathcal{I}}$ and an information structure $(S_i, \tilde{p})_{i\in\mathcal{I}}$, we have a game of incomplete information. A *Bayes Nash Equilibrium* (BNE) of the game is defined by strategy $\sigma_i : S_i \to \Delta(M_i)$ for each buyer i such that for every $s_i \in S_i$, the support of $\sigma_i(s_i)$ is among

[7] Let $\text{marg}_V\, \tilde{p}$ be the marginal distribution of $\tilde{p}$ over V.

the best responses to others' strategies:

$$\operatorname{supp}\sigma_i(s_i) \subseteq \operatorname*{argmax}_{m_i \in M_i} \sum_{(v,s_{-i})\in V\times S_{-i}} U_i(v,(m_i,\sigma_{-i}(s_{-i})))\tilde{p}(v,s_{-i} \mid s_i), \tag{4}$$

where $U_i(v,(m_i,\sigma_{-i}(s_{-i})))$ is linearly extended from Equation (2).

The ex ante distribution $\mu \in \Delta(V \times M)$ generated by any BNE $(\sigma_i)_{i\in\mathcal{I}}$ of any information structure $(S_i,\tilde{p})_{i\in\mathcal{I}}$ satisfies the following two conditions:

$$\sum_{m\in M} \mu(v,m) = p(v), \quad v \in V, \tag{Consistency}$$

$$\sum_{(v,m_{-i})\in V\times M_{-i}} \mu(v,m)\left(U_i(v,(m_i,m_{-i})) - U_i(v,(m_i',m_{-i}))\right) \geq 0, \quad i \in \mathcal{I}, (m_i,m_i') \in M_i \times M_i. \tag{Incentive}$$

A distribution $\mu \in \Delta(V \times M)$ that satisfies the above two conditions is called a *Bayes Correlated Equilibrium* (BCE) of the mechanism $(q_i,P_i)_{i\in\mathcal{I}}$. For any BCE μ, there exists an information structure and a BNE of that information structure that generates μ. See Bergemann and Morris (2016) for more details.

Designer's problem

The mechanism designer wants to solve:

$$\sup_{(q_i,P_i)_{i\in\mathcal{I}}} \min_{\mu\in\Delta(V\times M)} \sum_{(v,m)} \sum_i \mu(v,m)P_i(m) \tag{5}$$
$$\text{such that } \mu \text{ is a BCE of } (q_i,P_i)_{i\in\mathcal{I}}.$$

For notational brevity, we sometimes omit the set to which a summation variable belongs when it is obvious; for example, summing over (v,m) means summing over $(v,m) \in V \times M$.

Definition 1. A mechanism *guarantees* a revenue R if every BCE of this mechanism has an expected revenue larger than or equal to R.

3 Main Results

Our main results are a class of mechanisms that give good revenue guarantee. Consider a symmetric mechanism with k messages besides the opt-out message: $M_i = \{0,1,\ldots,k\}$, for every buyer $i \in \mathcal{I}$. The allocation $q(m_1, m_{-1})$ is given by:

$$q(0, m_{-1}) = 0, \qquad m_{-1} \in M_{-1}, \tag{6}$$

$$q(m_1+1, m_{-1}) - q(m_1, m_{-1}) = \left(\frac{1}{|\operatorname{rank}(m_1, m_{-1})|} \sum_{j \in \operatorname{rank}(m_1, m_{-1})} \frac{1}{j} \right) \cdot \frac{1}{k}, \quad 0 \le m_1 \le k-1,$$

where $\operatorname{rank}(m_1, m_{-1}) \subseteq \{1, 2, \dots, I\}$ is the set of ranks (from the top) of m_1 in $(m_1, m_2, \dots, m_I)$; for example, $\operatorname{rank}(20, 10, 20, 40, 30) = \{3, 4\}$, because $m_1 = 20$ and $m_3 = 20$ are tied for the third and the fourth place in this list; and $\operatorname{rank}(20, 10, 30, 40, 30) = \{4\}$ because in this list $m_1 = 20$ is unambiguously ranked fourth, even though there is a tie for the second and the third rank. We think of a message m_i as the demand of a fraction m_i/k of the good; the allocation in Equation (6) is increasing with the demand at a rate equal to the reciprocal of the demand's rank: a rate of 1 for the highest demand, of 1/2 for the second highest demand, of 1/3 for the third highest demand, and so on. Moreover, we break tie in a symmetric way and randomize over all "justifiable" ranks, in the case when $|\operatorname{rank}(m_1, m_{-1})| > 1$. It is easy to check that Equation (6) uniquely defines an allocation function (i.e., the feasibility condition is always satisfied); the total amount of allocation is at most $\max(m_1, m_2, \dots, m_I)/k$.

The payment of our mechanism is:

$$P(m_1, m_{-1}) = X \left(\left(1 + \frac{1}{a} \right)^{m_1} - 1 \right), \tag{7}$$

where $X > 0$ and $a > 0$ are constants that are optimized for a given prior distribution p. That is, the payment of every buyer depends only on his message and is independent of his final allocation.

As $k \to \infty$ and $a = A \cdot k$, the mechanism from Equations (6) and (7) converges to Equation (1), where we reparametrize $m_i \in \{0, 1, \dots, k\}$ to $z_i \equiv m_i/k \in [0, 1]$, where z_i is buyer i's demand. Thus, we abuse the terminology and refer to the mechanism from Equations (6) and (7) as the exponential price mechanism as well.

Intuitively, the exponential price mechanism tries to be egalitarian and allocate some quantity of the good to every buyer. Of course, a buyer with a higher demand gets more quantity because such buyer is paying more. If $z_1 > z_2 > \cdots > z_I$, then buyer i gets exactly $(z_i - z_{i+1})/i$ more than the allocation of buyer $i+1$; we have the factor $1/i$ because the quantity $(z_i - z_{i+1})/i$ is also acquired by all buyer $j > i+1$, and by definition there are i of them. The intuition for the exponential functional form of the payment rule is best illustrated when there is a single buyer and is presented in Section 4.2.1.

When there is a single buyer ($I = 1$), the exponential price mechanism becomes:

$$q(m_1) = m_1/k, \qquad P(m_1) = X\left(\left(1 + \frac{1}{a}\right)^{m_1} - 1\right), \qquad m_1 \in \{0, 1, \ldots, k\}, \tag{8}$$

since $\text{rank}(m_1) = \{1\}$ by definition. In fact, this mechanism achieves the optimal revenue guarantee:

Theorem 1. *Suppose there is one buyer, and as $\nu \to 0$ the prior p converges to a distribution with a positive density. There exist constants $A > 0$ and $X > 0$ such that the exponential price mechanism with $a = A \cdot k$ and the given X achieves the optimal revenue guarantee (i.e., solution to Problem (5)) as $k \to \infty$ and $\nu \to 0$.*

That is, for any $\epsilon > 0$, there exist $\bar{\nu}$ and $\bar{k}$ such that for any $\nu \leq \bar{\nu}$ and $k \geq \bar{k}$, the exponential price mechanism with $a = A \cdot k$ and the given X guarantees a revenue within ϵ of the best possible from Problem (5). We defer the intuition of Theorem 1 to Section 4.2.1, after presenting a basic framework to study revenue guarantee.

Our second result states the exponential price mechanism guarantees in expected revenue the full surplus (the expected common value) as the number of buyers tends to infinity. In this sense the mechanism is asymptotically optimal.

Theorem 2. *Let $a = \frac{k}{\log(I)}$ and $X = \frac{1}{2I\log(I)}$. The exponential price mechanism guarantees a revenue of $\sum_v v \cdot p(v)$ as $k \to \infty$ and $I \to \infty$.*

That is, for any $\epsilon > 0$, there exist $\bar{k}$ and $\bar{I}$ such that for any $k \geq \bar{k}$ and $I \geq \bar{I}$, the exponential price mechanism with $a = \frac{k}{\log(I)}$ and $X = \frac{1}{2I\log(I)}$ guarantees a revenue within ϵ of $\sum_v v \cdot p(v)$. The intuition for this result is that we combine exponential price which is optimal for any individual buyer with fractional allocations. In standard auctions with common value, the winner's curse is the main obstacle for the buyers to bid aggressively and thus for the auctioneer to obtain high revenue; for example in a second price auction, if there is one expert who knows the common value v (and who has a dominant strategy to bid v), and $I - 1$ uninformed buyers who only know the prior of v, then given the dominant strategy of the expert the winner's curse makes it sensible for an uniformed buyer to bid zero, since with any other bid $b > 0$ the uninformed buyer wins only if $v < b$, in which case winning is bad news and the winning surplus is always zero. Fractional allocations mitigate the effect of the winner's curse, since with fractional allocations winning more allocation is less indicative that others have pessimistic signals about the common value.

The mechanism (and the values of a and X) in Theorem 2 depend only on the upper bound of the support of value (which is normalized to be 1) and is independent of other detail of the prior. Thus, the convergence of the mechanism's revenue guarantee to the full surplus is robust even to mis-specification of the prior. While such prior-free, asymptotic guarantee of full surplus extraction is useful in practice, it also comes at a cost of a slow rate of convergence: one can see in the proof of Theorem 2 that the revenue guarantee converges to the full surplus at a rate of $O(1/\log(I))$, independent of the prior. We conjecture that by leveraging detail of the prior (e.g., generalizing the exactly optimal mechanism of Bergemann, Brooks, and Morris (2016b)) one gets a better convergence rate.

We illustrate the revenue guarantee of the exponential price mechanism as a function of the number of buyers in Figure 1; we also compare with the first price auction with the reserve price chosen to maximize the revenue guarantee (Bergemann, Brooks, and Morris, 2016a). For this figure we take the prior to be the uniform distribution on $[0, 1]$, $\nu \to 0$, and optimize the constants a and X for the uniform distribution. We see that the revenue guarantee of the exponential price mechanism is fairly close to the full surplus of 0.5 when there are 20 buyers.

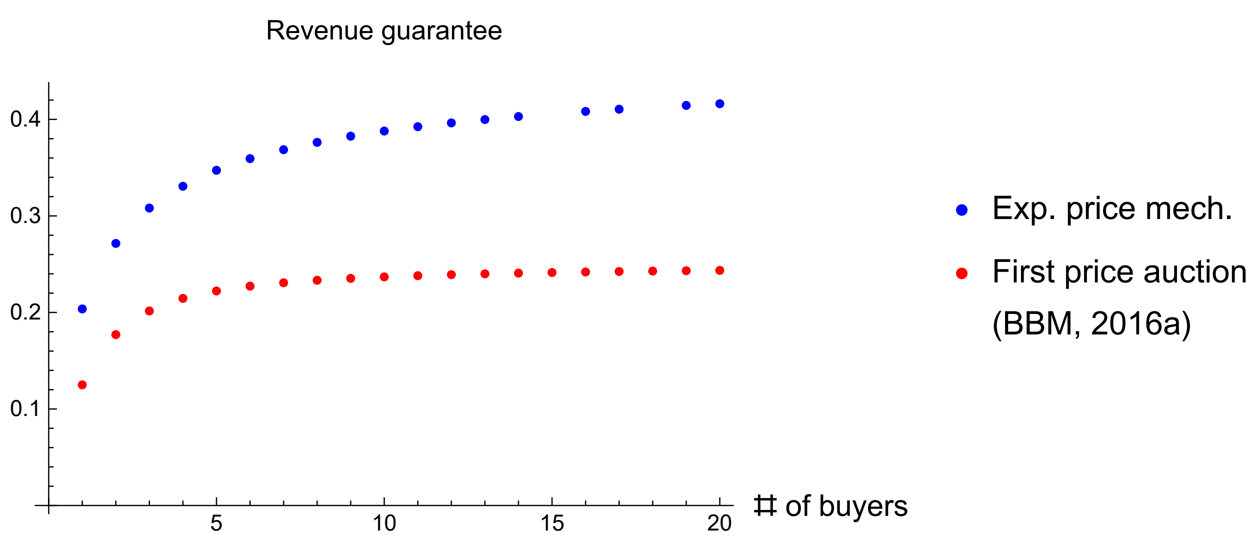


Figure 1: Revenue guarantees of the exponential price mechanism and of the first price auction with reserve price, when the prior is the uniform distribution on $[0, 1]$ and $\nu \to 0$.

4 Duality Approach to Robust Mechanism

To prove Theorem 1 and Theorem 2, we introduce a duality approach. For a given mechanism $(q_i, P_i)_{i\in\mathcal{I}}$, its revenue guarantee is found by the following problem:

$$\min_{\mu} \sum_{(v,m)} \sum_{i} P_i(m)\mu(v,m) \tag{9}$$

subject to:

$$\sum_{(v,m_{-i})} (U_i(v,m) - U_i(v,(m_i', m_{-i})))\mu(v,m) \geq 0, \quad i \in \mathcal{I}, (m_i, m_i') \in M_i \times M_i,$$

$$\sum_{m} \mu(v,m) = p(v), \quad v \in V,$$

$$\mu(v,m) \geq 0, \quad v \in V, m \in M,$$

where U_i is the utility function defined by Equation (2).

The dual problem to Problem (9) is:

$$\max_{(\alpha_i,\gamma)_{i\in\mathcal{I}}} \sum_{v} p(v)\gamma(v) \tag{10}$$

subject to:

$$\gamma(v) + \sum_{i} \sum_{m_i'} [U_i(v,m) - U_i(v,(m_i', m_{-i}))]\alpha_i(m_i' \mid m_i) \leq \sum_{i} P_i(m), \quad v \in V, m \in M,$$

$$\alpha_i(m_i' \mid m_i) \geq 0, \quad i \in \mathcal{I}, (m_i, m_i') \in M_i \times M_i,$$

where $\alpha_i(m_i' \mid m_i)$ is the multiplier on the incentive constraint of preferring m_i over m_i' (when "recommended" to play m_i), and $\gamma(v)$ is the multiplier on the consistency constraint of $\sum_m \mu(v,m) = p(v)$. By the linear programming duality theorem, Problems (9) and (10) have the same optimal value; their solutions are characterized by the complementary slackness conditions. See Vohra (2011) for details on linear programming and its classical applications to mechanism design.

Therefore, the mechanism designer's problem in (5) to solve for the optimal revenue guarantee is equivalent to:

$$
\begin{aligned}
&\sup_{(P_i,q_i,\alpha_i,\gamma)_{i\in\mathcal{I}}} \sum_v p(v)\gamma(v) \qquad (11)\\
&\text{subject to:}\\
\gamma(v) \leq \sum_i P_i(m) + \sum_i \sum_{m_i'} &[v(q_i(m_i', m_{-i}) - q_i(m)) - P_i(m_i', m_{-i}) + P_i(m)]\alpha_i(m_i' \mid m_i), \quad v \in V, m \in M,\\
&q_i(m) \geq 0, \quad \sum_{i'} q_{i'}(m) \leq 1, \quad q_i(0, m_{-i}) = P_i(0, m_{-i}) = 0, \qquad i \in \mathcal{I}, m \in M\\
&\alpha_i(m_i' \mid m_i) \geq 0, \qquad (m_i, m_i') \in M_i \times M_i, i \in \mathcal{I},
\end{aligned}
$$

where we label the opt-out message as $0 \in M_i$.

The advantage of Problem (11) over the equivalent Problem (5) is that we work with a maximization problem instead of a max-min problem. Moreover, we work with $(\alpha_i)_{i\in\mathcal{I}}$, where each α_i has $|M_i \times M_i|$ dimensions, instead of μ which has $|V \times \prod_{i\in\mathcal{I}} M_i|$ dimensions; the reduction in dimensions is significant if $|V|$ is large. Lastly, if we find a tuple $(q_i, P_i, \alpha_i)_{i\in\mathcal{I}}$ that satisfies the constraints of Problem (11), then the value of (11) under such $(q_i, P_i, \alpha_i)_{i\in\mathcal{I}}$ is by definition a lower bound on the optimal revenue guarantee. On the other hand, finding a feasible tuple $(q_i, P_i, \mu)_{i\in\mathcal{I}}$ for Problem (5) (i.e., μ is a BCE of $(q_i, P_i)_{i\in\mathcal{I}}$) does not yield by itself any conclusion about the revenue guarantee, since there may exist another BCE μ' of $(q_i, P_i)_{i\in\mathcal{I}}$ with a lower expected revenue than μ.

Problem (11) can be summarized as:

$$
\sup_{(q_i,P_i,\alpha_i)_{i\in\mathcal{I}}} \sum_v p(v) \cdot \min_m \text{Rev}(v, m), \tag{12}
$$

subject to the feasibility and opt-out constraints, where

$$
\text{Rev}(v, m) \equiv \sum_i \left(P_i(m) + \sum_{m_i'} (U_i(v, m_i', m_{-i}) - U_i(v, m))\alpha_i(m_i' \mid m_i) \right). \tag{13}
$$

We call $\text{Rev}(v, m)$ the *virtual revenue*. Since $U_i(v, m)$ is a linear function of v, so is $\text{Rev}(v, m)$ for any fixed m. We interpret $\alpha_i(m_i' \mid m_i)$ as buyer i's rate of deviation from message m_i to m_i', and $\text{Rev}(v, m)$ as the revenue generated by the message profile m, *plus* the incentive to deviate from m given value v and rates of deviation $(\alpha_i)_{i\in\mathcal{I}}$. By minimizing $\text{Rev}(v, m)$ over m, we are ignoring message profile m that either (1) has a large revenue, or (2) there is a large incentive to deviate from m by a buyer. Intuitively, (1) and (2) combines to give equilibrium message profile with minimum revenue.

Problem (11) is bounded above by the full surplus $\sum_v v \cdot p(v)$, as the following lemma shows:

Lemma 1. *For every $v \in V$, we have:*

$$\min_m \text{Rev}(v, m) \leq v. \tag{14}$$

4.1 A Lower Bound

We work with symmetric mechanism $q(m) \equiv q_1(m)$ and $P(m) \equiv P_1(m)$ (cf. Equation (3)).

Instead of directly solving Problem (12), we make some educated guess on (q_i, P_i, α_i) to get a lower bound on the maximum value of Problem (12). Suppose $M_i = \{0, 1, \ldots, k\}$ for every buyer $i \in \mathcal{I}$, where $q(0, m_{-1}) = 0 = P(0, m_{-1})$ for every $m_{-1} \in M_{-1}$.

We assume for every buyer i

$$\alpha_i(j' \mid j) = \begin{cases} a & j' = j+1 \\ 0 & j' \neq j+1 \end{cases}, \quad (j, j') \in \{0, 1, \ldots, k\}^2. \tag{15}$$

Condition (15) says that the local upward incentive constraint in BCE is binding: if the above α satisfies the complementary slackness condition with a BCE μ, then a buyer is indifferent between messages j and $j+1$ if he is "recommended" to submit j in the BCE μ. In Section 5.1 we show that for maximizing revenue guarantee Condition (15) is without loss of generality.

Condition (15) implies that there are two kinds of messages: the "interior" message $j \in \{0, 1, \ldots, k-1\}$, and the "boundary" message $j = k$. Thus there are $I + 1$ kinds of message profiles $m \in M = \{0, 1, \ldots, k\}^I$, depending on the number of boundary messages in m. For $0 \leq n \leq I$, define the class of message profiles:

$$M(n) = \{m \in M : |\{i \in \mathcal{I} : m_i = k\}| = n\}. \tag{16}$$

The sets $M(n)$, $0 \leq n \leq I$, form a partition of M. Our second assumption is that

$$\text{Rev}(v, m) = \text{Rev}(v, m') \ \forall v \in V, \qquad \text{if } m \text{ and } m' \text{ belong to the same } M(n). \tag{17}$$

Condition (17) attempts to make the virtual revenue $\text{Rev}(v, m)$ over m as redundant as possible (treating all interior messages as equivalent), to minimize the number of items inside the min function in Equation (12).

We now go to the exponential price mechanism defined by Equations (6) and (7). Clearly, if there are n boundary messages in a message profile m, then the rest (the interior messages) have ranks among $\{n+1, n+2, \ldots, I\}$, and by Equation (6) we have:

$$\sum_{i: m_i < k} q(m_i + 1, m_{-i}) - q(m_i, m_{-i}) = \frac{1}{k} \sum_{j=n+1}^{I} \frac{1}{j}. \tag{18}$$

Moreover, for an interior m_i we have:

$$P(m_i, m_{-i}) - a(P(m_i + 1, m_{-i}) - P(m_i, m_{-i})) = -X \tag{19}$$

by Equation (7).[8] Therefore, under Condition (15) we have:

$$\text{Rev}(v, m) = \frac{a\,v}{k} \sum_{j=n+1}^{I} \frac{1}{j} + nX((1 + 1/a)^k - 1) - (I - n)X, \quad \text{if } m \in M(n). \tag{20}$$

Thus, Condition (17) is satisfied, and we have the following lower bound on the optimal revenue guarantee:

$$\Pi_I^e \equiv \sup_{k \geq 1,\, a \geq 0,\, X} \left(\sum_v p(v) \cdot \min_{0 \leq n \leq I} \left(\frac{a\,v}{k} \sum_{j=n+1}^{I} \frac{1}{j} + nX((1 + 1/a)^k - 1) - (I - n)X \right) \right). \tag{21}$$

Proposition 1. *The exponential price mechanism guarantees a revenue of* Π_I^e *defined in Equation* (21).

Proof. The proof is given by the construction above. ☐

We prove Theorem 1 and Theorem 2 by studying Π_1^e and $\lim_{I \to \infty} \Pi_I^e$.

[8] In fact, Equation (7) is the solution to the difference equation

$$P(m_i, m_{-i}) - a(P(m_i + 1, m_{-i}) - P(m_i, m_{-i})) = -X$$

for $m_i \in \{0, 1, \ldots, k-1\}$, with the initial condition of $P(0, m_{-i}) = 0$.

4.2 Proof of Theorem 1

Let $I = 1$. Suppose as $\nu \to 0$, the prior p converges to a distribution with density ρ, where $\rho : [0, 1] \to [0, \infty)$ is positive almost everywhere.

To prove Theorem 1, we need to discuss a revelant result in Roesler and Szentes (2016). Roesler and Szentes (2016) study the optimal information structure for the buyer (and the worst for the seller) when the seller best responds to the information structure. Such information structure has the following cumulative distribution function for the signals:

$$G^B_\pi(s) = \begin{cases} 1 & s \geq B \\ 1 - \pi/s & s \in [\pi, B) \\ 0 & s < \pi \end{cases}, \tag{22}$$

where $s \in [0, 1]$ is an unbiased signal of the buyer for his value ($\mathbb{E}[v \mid s] = s$), $0 < \pi \leq B$ are two free parameters, and there is an atom of size π/B at $s = B$. If the buyer has this distribution of unbiased signals and observes the realization of the signal, then the seller is clearly indifferent between every posted price in $[\pi, B]$ and has a revenue of π from the optimal mechanism (which is a posted price).[9] Thus, π is an upper bound on the seller's revenue guarantee.

Given the density $\rho(v)$, $G^B_\pi(s)$ is a distribution of an unbiased signal on v if and only if ρ is a mean-preserving spread of $G^B_\pi(s)$, which holds if and only if:

$$\int_0^1 v\,\rho(v)\,dv = \int_0^1 s\,dG^B_\pi(s) = \pi + \pi\log B - \pi\log\pi \tag{23}$$

$$\min_{s\in[\pi,B]} F(s,\pi) \geq 0, \text{ where} \tag{24}$$

$$F(s,\pi) \equiv \int_{s'=0}^{s}\int_{v=0}^{s'} \rho(v)\,dv\,ds' - \int_0^s G^B_\pi(s')\,ds' = \int_{s'=0}^{s}\int_{v=0}^{s'} \rho(v)\,dv\,ds' - (s - \pi - \pi\log s + \pi\log\pi),$$

i.e., G^B_π has the same mean as ρ and second-order stochastically dominates ρ. Let $B = B(\pi)$ be defined from π by Equation (23).

Roesler and Szentes (2016) prove that the best information structure for the buyer (and the worst for the seller) when the seller best responds to the information structure is $G^{B^*}_{\pi^*}$, where π^* is the smallest π such that $\min_{s\in[\pi,B(\pi)]} F(s,\pi) \geq 0$, and $B^* \equiv B(\pi^*)$; that is, π^*

[9] Intuitively, if the seller has a strict incentive over the posted price, then we can slightly change the buyer's information structure to lower the seller's optimal revenue and to increase the buyer's surplus, while preserving the seller's best response in posted price.

is the smallest π such that ρ is a mean-preserving spread of $G_{\pi}^{B(\pi)}(s)$. For our purpose, by making π small we tighten the upper bound on the seller's revenue guarantee.

We now show that the exponential price mechanism can obtain the upper bound π^*. In the case of one buyer, Problem (21) simplifies to:

$$\Pi_1^e \equiv \max_{k\geq 1,\, a\geq 0,\, X} \sum_v \min\left(\frac{a\,v}{k} - X,\, X\left(\left(1+\frac{1}{a}\right)^k - 1\right)\right) p(v), \tag{25}$$

Set $a = A\cdot k$. As $\nu \to 0$ and $k\to\infty$, we have

$$\sum_v \min(av/k, X(1+1/a)^k)\, p(v) - X \longrightarrow \Pi_1 \equiv \int_0^1 \min(Av, X\exp(1/A))\,\rho(v)dv - X. \tag{26}$$

We maximize Π over A and X. Suppose $\frac{X\exp(1/A)}{A}\in[0,1]$, the first order condition is:

$$\begin{aligned} \frac{\partial \Pi_1}{\partial X} &= \int_{\frac{X\exp(1/A)}{A}}^1 \exp(1/A)\,\rho(v)\,dv - 1 = 0, \\ \frac{\partial \Pi_1}{\partial A} &= \int_0^{\frac{X\exp(1/A)}{A}} v\,\rho(v)\,dv - \int_{\frac{X\exp(1/A)}{A}}^1 \frac{X\exp(1/A)}{A^2}\,\rho(v)\,dv = 0. \end{aligned} \tag{27}$$

If the above first order condition holds and $\frac{X\exp(1/A)}{A}\in[0,1]$, then we have $\Pi_1 = X/A$.

Going back to the construction of Roesler-Szentes, let s^* be an arbitrary selection from $\operatorname{argmin}_{s\in[\pi^*,B^*]} F(s,\pi^*)$. Since $\min_{s\in[\pi,B(\pi)]} F(s,\pi)$ is a continuous function of π, we have $F(s^*,\pi^*)=0$. Moreover, s^* must be interior[10], so we have $\frac{\partial F}{\partial s}(s^*,\pi^*)=0$.

Therefore, we have (the first line is $\frac{\partial F}{\partial s}(s^*,\pi^*)=0$, and the second line is $F(s^*,\pi^*)=0$):

$$\begin{aligned} &\int_{v=0}^{s^*} \rho(v)\,dv - 1 + \pi^*/s^* = 0, \\ &\int_{s=0}^{s^*}\int_{v=0}^{s} \rho(v)\,dv\,ds - (s^* - \pi^* - \pi^*\log s^* + \pi^*\log\pi^*) = -\int_0^{s^*} v\,\rho(v)\,dv + \pi^*\log s^* - \pi^*\log\pi^* = 0, \end{aligned} \tag{28}$$

where in the second equality of the second line we use integration by parts and substitute in the first line. Clearly, there exist unique $A>0$ and $X>0$ such that $s^* = X\exp(1/A)/A$

[10] If $B^*<1$, we must have $F(B^*,\pi^*)>0$, for otherwise we would have $\int_0^1 G_{\pi^*}^{B^*}(s)\,ds > \int_{s=0}^1\int_{v=0}^s \rho(v)\,dv\,ds$, which would contradict the fact that $G_{\pi^*}^{B^*}$ has the same mean as p. If $B^*=1$, then we have $\frac{\partial F}{\partial s}(B^*,\pi^*) = \frac{\pi^*}{B^*} > 0$. In any case $s^*\neq B^*$. Since $F(\pi^*,\pi^*)>0$, we also have $s^*\neq\pi^*$.

and $\pi^* = X/A$. (We have $s^* < B^* < 1$.) Then the above equations become:

$$\int_{\frac{X\exp(1/A)}{A}}^{1} \rho(v)\, dv = \exp(-1/A), \qquad \int_0^{\frac{X\exp(1/A)}{A}} v\,\rho(v)\, dv - \frac{X}{A^2} = 0,$$

which is clearly equivalent to Equation (27).

Therefore, for any $\epsilon > 0$, when k is sufficiently large and ν sufficiently small, we have $\Pi_1^e \geq \pi^* - \epsilon$. When ν is sufficiently small, the revenue guarantee must be smaller than $\pi^* + \epsilon$ by the ν-discretization of the Roesler-Szentes construction. This concludes the proof.

4.2.1 Intuition on Theorem 1.

We note that as $k \to \infty$ and $a = A \cdot k$, the exponential price mechanism in Equation (8) becomes:

$$q(z) = z, \qquad P(z) = X(\exp(z/A) - 1), \tag{29}$$

where $z \equiv m_1/k \in [0, 1]$ is the demand of the buyer.

Fix an unbiased information structure (S, G) for the buyer: $\mathbb{E}[v \mid s] = s$ for every $s \in S \subseteq [0, 1]$, and $s \in S$ has the cumulative distribution function $G(s)$.

Given a realization of signal s, the buyer solves:

$$\max_z s \cdot z - X(\exp(z/A) - 1).$$

If $s \leq \pi^* \equiv X/A$, then the buyer's optimal demand is $z = 0$, and he pays 0; if $s \geq s^* \equiv X\exp(1/A)/A$, then the buyer's optimal demand is $z = 1$, and he pays $X(\exp(1/A) - 1))$. If $s \in [\pi^*, s^*]$, then the optimal demand is given by the first order condition $s = X\exp(z/A)/A$, and the buyer pays $X(\exp(z/A) - 1)|_{s = X\exp(z/A)/A} = As - X$. Thus, the equilibrium revenue under the unbiased information structure (S, G) is:

$$\Pi_1(G) \equiv \int_{s=\pi^*}^{1} \min(As - X, X(\exp(1/A) - 1))\, dG(s), \tag{30}$$

which is similar to Π_1 in Equation (26), but with a different lower limit in the integral.

In general, we have:

$$\Pi_1(G) \geq \int_{s=0}^{1} \min(As - X, X(\exp(1/A) - 1))\, dG(s) \geq \int_{s=0}^{1} \min(As - X, X(\exp(1/A) - 1))\, \rho(s)\, ds = \Pi_1, \tag{31}$$

since ρ is a mean-preserving spread of G, and $\min(As - X, X(\exp(z/A) - 1))$ is a concave function of s; thus, Π_1 is a lower bound on the equilibrium revenue over all information structures, confirming Proposition 1 when $I = 1$.

The proof of Theorem 1 shows that $\Pi_1 = \Pi_1(G)$ when $G = G_{\pi^*}^{B^*}$ as constructed by Roesler and Szentes. This can be seen in Equation (31) as follows: the first inequality in (31) is an equality when $G = G_{\pi^*}^{B^*}$ because $G_{\pi^*}^{B^*}$ is supported on the interval $[\pi^*, B^*]$; the second inequality in (31) is an equality when $G = G_{\pi^*}^{B^*}$ because $\int_{s=\pi^*}^{s^*} s\, dG_{\pi^*}^{B^*}(s) = \int_{v=0}^{s^*} v\, \rho(v)\, dv$ and $G_{\pi^*}^{B^*}(s^*) = \int_{v=0}^{s^*} \rho(v)\, dv$ by Equation (28).

4.3 Proof of Theorem 2

For each $0 \leq n \leq I$, define

$$\mathrm{Rev}_n(v) = \frac{a\, v}{k} \sum_{j=n+1}^{I} \frac{1}{j} + nX((1 + 1/a)^k - 1) - (I - n)X, \tag{32}$$

which is $\mathrm{Rev}(v, m)$ for any $m \in M(n)$. Let

$$v(n) = \frac{(n+1)kX}{a}(1 + 1/a)^k, \tag{33}$$

By construction, we have $\mathrm{Rev}_n(v(n)) = \mathrm{Rev}_{n+1}(v(n))$ for each $0 \leq n \leq I - 1$. Set $v(-1) = 0$ and $v(I) = \infty$. Clearly, if $X > 0$, then $\mathrm{Rev}_n(v) = \min_{0 \leq n' \leq I} \mathrm{Rev}_{n'}(v)$ if and only if $v \in [v(n-1), v(n)]$.

We want to approximate the identity function v by $\min_{0 \leq n \leq I} \mathrm{Rev}_n(v)$. To do so, we set $A = 1/\log(I)$, $X = 1/(2I\log(I))$ and $a = Ak$. We have $\lim_{k\to\infty}(1 + 1/a)^k = I$, $\lim_{k\to\infty} v(1) = 1$. Thus,

$$\begin{aligned}\lim_{k\to\infty} \sum_{v\in V} p(v) \min_{0\leq n\leq I} \mathrm{Rev}_n(v) = &\sum_{v\leq v(0), v\in V} p(v) \left(\frac{v}{\log(I)} \sum_{j=1}^{I} \frac{1}{j} - \frac{1}{2\log(I)} \right) \\ &+ \sum_{v > v(0), v\in V} p(v) \left(\frac{v}{\log(I)} \sum_{j=2}^{I} \frac{1}{j} - \frac{1}{\log(I)} \right)\end{aligned} \tag{34}$$

Clearly, the above equation converges to $\sum_v v \cdot p(v)$ as $I \to \infty$. This completes the proof.

5 Generalization and Numerical Computation

Since they play a central role in the derivation of exponential price mechanism, we try to better understand Conditions (15) and (17) in this section. We first show that to maximize the revenue guarantee it is sufficient to assume Condition (15) on the multipliers and to focus on symmetric mechanism, for any prior distribution and any number of buyers. Given Condition (15) and suppose two buyers, we then show that Condition (17) is fully characterized by a simple tweak of exponential price mechanism. Finally, we numerically demonstrate that the revenue guarantees of the exponential price mechanisms are generally close to optimal when there are two buyers.

5.1 Simplifying the Multipliers

Proposition 2. *For any $\epsilon > 0$, in a tuple $(q_i, P_i, \alpha_i)_{i\in\mathcal{I}}$ that obtains within ϵ of the optimal revenue guarantee (Problem (12)), the mechanism (q_i, P_i) is symmetric, the message space $M_i = \{0, 1, \ldots, k\}$ for some integer k, and the multiplier α_i satisfies Equation (15).*

5.1.1 Proof of Proposition 2

Transition probability matrix

Fix an arbitrary tuple $(q_i, P_i, \alpha_i)_{i\in\mathcal{I}}$. Recall that there exists an opt-out message $0 \in M_i$ for every buyer i. We define a new $(\tilde{q}_i, \tilde{P}_i, \tilde{\alpha}_i)_{i\in\mathcal{I}}$ as follows:

(i) Define

$$a \equiv \max_{i\in\mathcal{I}} \max_{m_i\in M_i} \sum_{m_i'\neq m_i} \alpha_i(m_i' \mid m_i) + c, \tag{35}$$

where $c > 0$ is an arbitrary constant, and

$$\mathcal{A}_i(m_i' \mid m_i) \equiv \begin{cases} \alpha_i(m_i' \mid m_i)/a & m_i' \neq m_i \\ 1 - \sum_{m_i''\neq m_i} \mathcal{A}_i(m_i'' \mid m_i) & m_i' = m_i \end{cases}, \qquad (m_i, m_i') \in M_i^2 \tag{36}$$

If we interpret α_i as the transition rates of a continuous-time Markov process over M_i, then $\mathcal{A}_i$ is a transition probability matrix embedded in the process. Because of the positive constant c, $\mathcal{A}_i(m_i \mid m_i) > 0$ for every $m_i \in M_i$, so every m_i is aperiodic in $\mathcal{A}_i$ (Stroock (2013), Section 3.1.3)

(ii) Define a new message space $\tilde{M}_i \equiv \{0, 1, \ldots, k\}$ for every buyer i. Message $0 \in \tilde{M}_i$ is still the opt-out message: $\tilde{q}_i(0, m_{-i}) = \tilde{P}_i(0, m_{-i}) = 0$ for each $m_{-i} \in M_{-i}$. Message $j \in \tilde{M}_i$ is the "mixed-strategy" message given by $(\mathcal{A}_i)^j(\cdot \mid 0) \in \Delta(M_i)$:

$$\tilde{q}_i(j, m_{-i}) \equiv \sum_{m_i \in M_i} (\mathcal{A}_i)^j(m_i \mid 0)\, q_i(m_i, m_{-i}), \qquad m_{-i} \in M_{-i}, \tag{37}$$
$$\tilde{P}_i(j, m_{-i}) \equiv \sum_{m_i \in M_i} (\mathcal{A}_i)^j(m_i \mid 0) P_i(m_i, m_{-i}),$$

where $(\mathcal{A}_i)^j$ is $\mathcal{A}_i$ raised to the j-th power (the j-step transition probability matrix). Moreover, we extend $\tilde{q}_i(\tilde{m}_i, m_{-i})$ and $\tilde{P}_i(\tilde{m}_i, m_{-i})$ linearly to $\tilde{q}_i(\tilde{m}_i, \tilde{m}_{-i})$ and $\tilde{P}_i(\tilde{m}_i, \tilde{m}_{-i})$ for each $\tilde{m}_{-i} \in \tilde{M}_{-i}$, according to the mixed-strategy $(\mathcal{A}_l)^{\tilde{m}_l}(\cdot \mid 0) \in \Delta(M_l)$, $l \neq i$.

(iii) Define

$$\tilde{\alpha}_i(\tilde{m}'_i \mid \tilde{m}_i) \equiv \begin{cases} a & \tilde{m}'_i = \tilde{m}_i + 1, \\ 0 & \tilde{m}'_i \neq \tilde{m}_i + 1. \end{cases} \tag{38}$$

Let $\tilde{U}_i(v, \tilde{m}) \equiv v \cdot \tilde{q}_i(\tilde{m}) - \tilde{P}_i(\tilde{m})$. We have:

$$\begin{aligned} &\sum_{m_i \in M_i} (\mathcal{A}_i)^j(m_i \mid 0) \sum_{m'_i \in M_i} [U_i(v, (m'_i, m_{-i})) - U_i(v, (m_i, m_{-i}))]\alpha_i(m'_i \mid m_i) \qquad (39)\\ &= \sum_{m_i \in M_i} (\mathcal{A}_i)^j(m_i \mid 0) \sum_{m'_i \in M_i} a \cdot [U_i(v, (m'_i, m_{-i})) - U_i(v, (m_i, m_{-i}))]\mathcal{A}_i(m'_i \mid m_i) \\ &= a \sum_{m'_i \in M_i} \left[U_i(v, (m'_i, m_{-i}))\, (\mathcal{A}_i)^{j+1}(m'_i \mid 0) - U_i(v, (m'_i, m_{-i}))\, (\mathcal{A}_i)^j(m'_i \mid 0)\right] \\ &= a\, [\tilde{U}_i(v, (j+1, m_{-i})) - \tilde{U}_i(v, (j, m_{-i}))] \\ &\begin{cases} = \tilde{\alpha}_i(j+1 \mid j)\, [\tilde{U}_i(v, (j+1, m_{-i})) - \tilde{U}_i(v, (j, m_{-i}))] & j < k, \\ \leq \epsilon & j = k, \end{cases} \end{aligned}$$

where in the last line, k is chosen to be sufficiently large so that $a\,||(\mathcal{A}_i)^{k+1}(\cdot \mid 0) - (\mathcal{A}_i)^k(\cdot \mid 0)||_{\text{v}} \cdot \max_{v,m} |U_i(v, m)| \leq \epsilon$, $||\cdot||_{\text{v}}$ is the total variation norm, and $\lim_{k \to \infty} (\mathcal{A}_i)^k$ is well defined because every m_i is aperiodic (Stroock (2013), Section 4.1.7). Clearly, $\epsilon > 0$ can be made arbitrarily small.

Consequently, the virtual revenue $\tilde{\text{Rev}}(v, \tilde{m})$ given by $(\tilde{q}_i, \tilde{P}_i, \tilde{\alpha}_i)_{i\in\mathcal{I}}$ satisfies

$$\tilde{\text{Rev}}(v,\tilde{m}) \begin{cases} = \sum_{m\in M}\prod_i (\mathcal{A}_i)^{\tilde{m}_i}(m_i \mid 0)\cdot \text{Rev}(v,m) \geq \min_{m\in M}\text{Rev}(v,m) & \tilde{m}_i < k, \forall i \in \mathcal{I}, \\ \geq \sum_{m\in M}\prod_i (\mathcal{A}_i)^{\tilde{m}_i}(m_i \mid 0)\cdot \text{Rev}(v,m) - I\cdot\epsilon \geq \min_{m\in M}\text{Rev}(v,m) - I\cdot\epsilon & \text{otherwise,} \end{cases} \tag{40}$$

for every $v \in V$ and every $\tilde{m} \in \tilde{M}$. Since ϵ is arbitrarily small, to maximize the revenue guarantee in Equation (12) we can ignore $(q_i, P_i, \alpha_i)_{i\in\mathcal{I}}$ and focus on $(\tilde{q}_i, \tilde{P}_i, \tilde{\alpha}_i)_{i\in\mathcal{I}}$.

Symmetric mechanism

Now fix a mechanism $(q_i, P_i)_{i\in\mathcal{I}}$ such that $M_i = \{0, 1, \ldots, k\}$, for every $i \in \mathcal{I}$, and suppose Equation (15) holds for α_i. We symmetrize the mechanism:

$$\tilde{q}_1(m'_1, m'_{-1}) \equiv \frac{1}{I!}\sum_{\sigma} q_{\sigma(1)}(m_{\sigma(1)} = m'_1, m_{\sigma(2)} = m'_2, \ldots, m_{\sigma(I)} = m'_I), \qquad m' \in M \tag{41}$$

$$\tilde{P}_1(m'_1, m'_{-1}) \equiv \frac{1}{I!}\sum_{\sigma} P_{\sigma(1)}(m_{\sigma(1)} = m'_1, m_{\sigma(2)} = m'_2, \ldots, m_{\sigma(I)} = m'_I),$$

where we sum over all permutations σ over $\{1, 2, \ldots, I\}$. Likewise for the allocations and payments of buyers $2, 3, \ldots, I$. Since α_i is symmetric, the virtual revenue $\tilde{\text{Rev}}(v, m)$ given by $(\tilde{q}_i, \tilde{P}_i, \alpha_i)_{i\in\mathcal{I}}$ is the average, over all permutations σ, of $\text{Rev}(v, (m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(I)}))$ given by $(q_i, P_i, \alpha_i)_{i\in\mathcal{I}}$; thus, we have $\tilde{\text{Rev}}(v, m) \geq \min_{m'} \text{Rev}(v, m')$. Therefore, to maximize the revenue guarantee we can focus on symmetric mechanism $(\tilde{q}_i, \tilde{P}_i)_{i\in\mathcal{I}}$. This completes the proof of Proposition 2.

5.1.2 Discussion on Proposition 2

Proposition 2 greatly simplifies the numerical computation of optimal revenue guarantee, since it justifies considering a single multiplier; we present numerical computation in Section 5.3. Proposition 2 is also true if there are interdependent values: each buyer has a potentially different value $v_i \in V_i$, and the designer knows the prior $p \in \Delta(\prod_i V_i)$; the setup in Section 2 and Section 4, and the proof of Proposition 2 can all be obviously adapted by changing v to v_i. Thus, the optimal revenue guarantee with general interdependent values can be numerically computed using the method in Section 5.3.

Proposition 2 also implies that the message space of the optimal mechanism has a strict ordering, i.e., it is one-dimensional. This is consistent with the conjecture of Bergemann, Brooks, and Morris (2016b) that the critical worst-case information structure always has independent, one dimensional types.

5.2 Generalized Exponential Price Mechanism

Given Condition (15) on the multipliers and a symmetric mechanism, Condition (17) is a natural assumption on the virtual revenue. In this subsection we fully characterize the implications of Condition (17) for the revenue guarantee; from the characterization we arrive at a generalization of the exponential price mechanism.

For simplicity, suppose $I = 2$. We consider a symmetric mechanism: $M_1 = M_2 = \{0, 1, \ldots, k\}$. The feasibility and opt-out conditions on the mechanism are:

$$q(0, j) = 0 = P(0, j), \quad q(j, l) \geq 0, \quad q(j, l) + q(l, j) \leq 1, \quad (j, l) \in \{0, 1, \ldots, k\}^2. \tag{42}$$

Assume Condition (15). Condition (17) holds under $I = 2$ if and only if

$$\begin{aligned}
&2q(1, 0) = q(j+1, l) - q(j, l) + q(l+1, j) - q(l, j), \quad (j, l) \in \{0, 1, \ldots, k-1\}^2, \\
&q(1, k) = q(j+1, k) - q(j, k), \quad j \in \{0, 1, \ldots, k-1\},
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
-2aP(1, 0) &= P(j, l) + P(l, j) - a(P(j+1, l) - P(j, l)) - a(P(l+1, j) - P(l, j)), \\
&\qquad\qquad (j, l) \in \{0, 1, \ldots, k-1\}^2, \\
P(k, 0) - aP(1, k) &= P(j, k) + P(k, j) - a(P(j+1, k) - P(j, k)), \quad j \in \{0, 1, \ldots, k-1\}.
\end{aligned} \tag{44}$$

It is without loss to assume that the feasibility constraint $q(j, k) + q(k, j) \leq 1$ binds for every j (if not, we can increase $q(k, j)$, which strictly increases $\mathrm{Rev}(v, (k-1, j))$, without decreasing any other $\mathrm{Rev}(v, m)$ or violating any feasibility constraint):

$$q(j, k) + q(k, j) = 1, \quad j \in \{0, 1, \ldots, k\}. \tag{45}$$

Lemma 2. *For any allocation q that satisfies Conditions (43) and (45), we have $q(1, 0) = (3k+1)/(4k^2)$ and $q(1, k) = 1/(2k)$.*

Lemma 3. *For any payment P that satisfies Condition (44), we have:*

$$P(k,k) = \left((1+1/a)^k - 1\right)^2 aP(1,0) + \left((1+1/a)^k - 1\right)(aP(1,k) - P(k,0)). \tag{46}$$

Define,

$$Y_0 \equiv 2aP(1,0), \qquad Y_1 \equiv -P(k,0) + aP(1,k), \tag{47}$$

i.e., $-Y_0$ is equal to the first line of (44), and $-Y_1$ is equal to the second line of (44).

Lemma 2 and Lemma 3 lead us unambiguously to the following problem:

$$\Pi_2^g \equiv \sup_{k\geq 1,\, a\geq 0,\, Y_0,\, Y_1} \sum_v \min\left(\frac{3k+1}{2k^2}av - Y_0,\ \frac{av}{2k} - Y_1,\ Y_0\left((1+1/a)^k - 1\right)^2 + 2Y_1\left((1+1/a)^k - 1\right)\right)p(v). \tag{48}$$

Comparing Π_2^g above with Π_2^e in Equation (21), the main difference is that in Π_2^g there are two variables Y_0 and Y_1, instead of a single variable X in Π_2^e; the difference between the coefficient of $\frac{3k+1}{2k^2}$ in Π_2^g and of $\frac{3}{2k}$ in Π_2^e is unimportant, since $k \to \infty$ in both maximization problems. In fact, as $k \to \infty$, Π_2^e becomes a special case of Π_2^g with $Y_0 = 2X$ and $Y_1 = X - X((1+1/a)^k - 1)$. Thus, we have $\Pi_2^g \geq \Pi_2^e$ as $k \to \infty$.

Proposition 3. *Suppose there are two buyers. There exists a symmetric mechanism that guarantees a revenue of Π_2^g defined in Equation (48).*

Proof. The proof is given by the construction above. □

We now specify a mechanism for Proposition 3 (the generalized exponential price mechanism). Consider the following allocation rule:

$$q(0,l) = 0, \qquad l \in \{0,1,\ldots,k\}, \tag{49}$$

$$q(j+1,l) - q(j,l) = \begin{cases} (2k+1)/(4k^2) & j < l \\ (3k+1)/(4k^2) & j = l \\ (4k+1)/(4k^2) & j > l \end{cases}, \qquad (j,l) \in \{0,1,\ldots,k-1\}^2, \tag{50}$$

$$q(j+1,k) - q(j,k) = 1/(2k), \qquad j \in \{0,1,\ldots,k-1\}.$$

It is easy to check that the above allocation rule satisfies the feasibility constraint, and Conditions (43) and (45). In the optimum $k \to \infty$, so the above allocation rule becomes identical to the allocation rule of exponential price mechanism in Equation (6).

Given any values of Y_0 and Y_1, we can choose the following solution to Equation (44):

$$P(j,l) - a(P(j+1,l) - P(j,l)) = \begin{cases} -Y_0/2 & 0 \le l < k, \\ -Y_1 - P(k,j) & l = k, \end{cases}, \quad j \in \{0,1,\ldots,k-1\}, \tag{51}$$

which is equivalent to (see footnote 8):

$$P(j,l) = \begin{cases} ((1+1/a)^j - 1)\,\frac{Y_0}{2} & 0 \le l < k \\ ((1+1/a)^j - 1)\left(Y_1 + \left((1+1/a)^k - 1\right)\frac{Y_0}{2}\right) & l = k \end{cases}, \qquad (j,l) \in \{0,1,\ldots,k\}^2. \tag{52}$$

The above payment rule is identical to the payment rule of exponential price mechanism in Equation (7), except when the other player submits the boundary message k. Here $\frac{Y_0}{2}$ and $Y_1 + \left((1+1/a)^k - 1\right)\frac{Y_0}{2}$ are constants and can be interpreted as units of payment, thus this mechanism is an exponential price mechanism in which a buyer 1 pays in one unit (e.g., Canadian dollar) when the other buyer has a message $m_2 < k$, and pays in another, more expensive unit (e.g., US dollar) when $m_2 = k$. Indeed at the optimum we generally have $Y_1 + \left((1+1/a)^k - 1\right)\frac{Y_0}{2} > \frac{Y_0}{2}$. Intuitively, $m_2 = k$ indicates that buyer 2 is very optimistic about the value; in this case buyer 1 is committed to pay with a higher monetary unit. In the next section we show that this change of unit is useful when the prior is heavily concentrated on the high values.

5.3 Numerical Solution

For fixed values of $a > 0$ and $k > 0$, let $\Pi_I^*(a,k)$ be the optimal revenue guarantee given Assumption (15) on multipliers, i.e.,

$$\Pi_I^*(a,k) \equiv \max_{(q_i,P_i)} \sum_v p(v) \cdot \min_m \mathrm{Rev}(v,m), \tag{53}$$

subjective to feasibility and opt-out constraints. When $|V|$ and k are not too large, $\Pi_I^*(a,k)$ can be efficiently and reliably computed, since it is a linear programming problem.[11] For the optimal revenue guarantee, we solve $\sup_{a,k} \Pi_I^*(a,k)$, which is facilitated by the following

[11] When $I = 2$, solving $\Pi_I^*(a,k)$ involves $2k(k+1)$ variables and $(|V|+1) \times (k+1)(k+2)/2$ constraints after restricting to symmetric mechanism.

lemma.

Lemma 4. *For any fixed $a > 0$, $\Pi_I^*(a, k)$ is weakly increasing in k.*

Thus, to calculate the optimal revenue guarantee, we can fix a large value of k and vary a, computing $\Pi_I^*(a, k)$ for each a by linear programming.

Suppose there are two buyers ($I = 2$), $k = 500$ and $\nu = 1/10$ (so the set of values is $V = \{0, 0.1, 0.2, \ldots, 1\}$). Let the prior distribution be $p(v) = \rho(v)/\sum_{v' \in V} \rho(v')$ for each $v \in V$, where $\rho(v)$ is a density function. In Table 1 we report the optimal revenue guarantee $\max_a \Pi_I^*(a, k)$ for various density functions, and compare with the revenue guarantees of the exponential and generalized exponential price mechanisms (i.e., Π_2^e of Equations (21) and Π_2^g of Equation (48), fixing $k = 500$). The computation is performed on the NEOS Server[12].

Table 1 shows the revenue guarantee of the generalized exponential price mechanism is close to optimal for these priors. Moreover, the revenue guarantee of the exponential price mechanism is generally close to that of the generalized mechanism, though the latter is noticeably better when the prior is heavily concentrated on the high values (the Beta(8, 1) distribution, and Beta(4, 2) to a less extent).

In Figure 3 we compare the revenue guarantees of our mechanisms with the optimal revenue guarantee of Bergemann, Brooks, and Morris (2016b) when there are two buyers and $V = \{0, 1\}$. We see that for these reasonable values of $p(1)$ the exponential price mechanism guarantees over 90% of the optimal revenue. As before, the generalized mechanism picks up more of the slack left by the exponential price mechanism as the prior becomes more concentrated on the high value of 1.

6 Conclusion

We propose a new class of mechanisms (the exponential price mechanisms) to sell a common value good. The mechanisms are simple to implement, and guarantee a good expected revenue over all information structures and equilibria. The revenue guarantee is provably optimal when there is one buyer, and converges to the full surplus as the number of buyers tends to infinity. Moreover, we numerically demonstrate that the exponential price mechanisms are generally close to optimal in revenue guarantee when there are two buyers. To derive these mechanisms we introduce a linear programming duality approach, which we

[12] https://neos-server.org/neos/

Figure 2: Prior distributions. Beta(b,c) has a pdf of $v^{b-1}(1-v)^{c-1} \cdot \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}$.

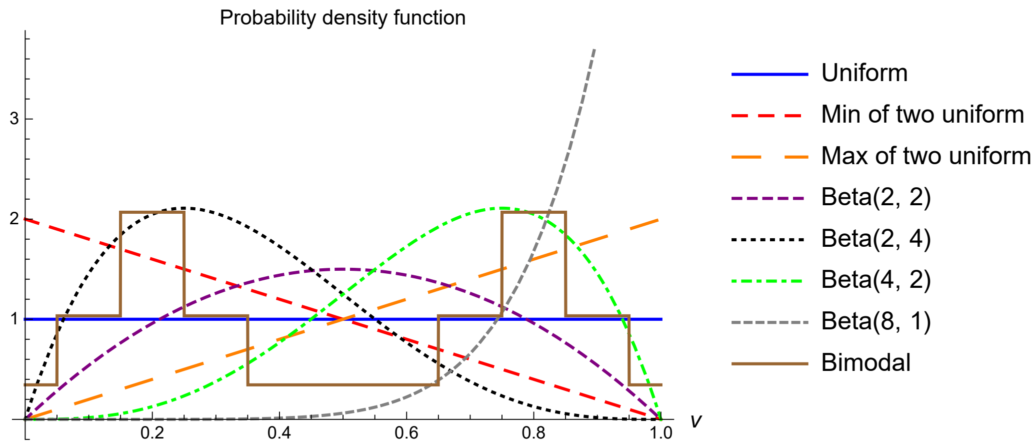
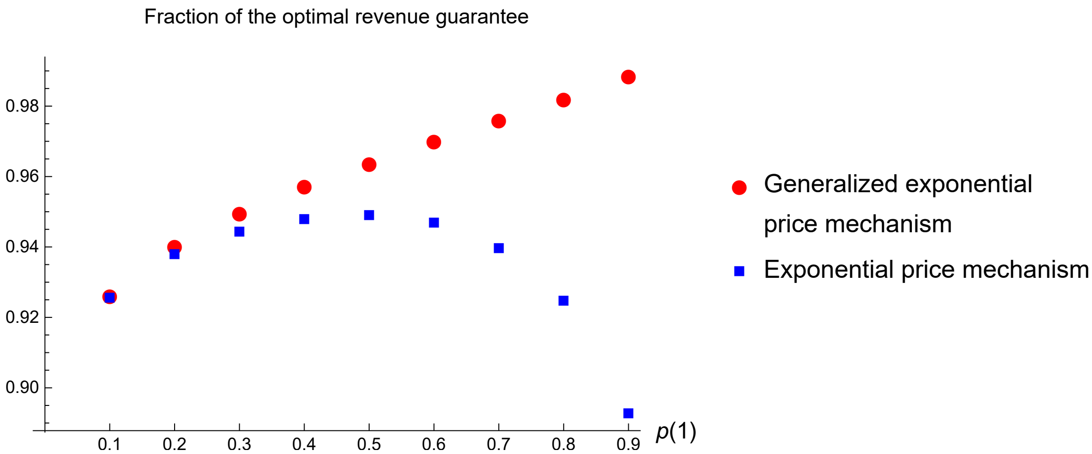


Table 1: Revenue guarantees for two buyers, $k = 500$, and $\nu = 1/10$.

Prior distribution	Full surplus	Optimal revenue guarantee ($\max_a \Pi_2^*(a,k)$)	Generalized exponential (Π_2^g)	Exponential (Π_2^e)
Uniform	0.500	0.271	0.264	0.262
Min of two uniform	0.300	0.146	0.141	0.141
Max of two uniform	0.700	0.472	0.461	0.455
Beta(2, 2)	0.500	0.314	0.305	0.304
Beta(2, 4)	0.339	0.207	0.196	0.196
Beta(4, 2)	0.661	0.488	0.474	0.463
Beta(8, 1)	0.928	0.807	0.792	0.747
Bimodal	0.500	0.273	0.265	0.260

believe is useful for other robust mechanism design problems as well, e.g., for studying the revenue guarantee when buyers have both common and private values.

Figure 3: $V = \{0, 1\}$ and two buyers, comparison with the optimal revenue guarantee of Bergemann, Brooks, and Morris (2016b)



Appendix

A Miscellaneous Proofs

Proof of Lemma 1. Fix an arbitrary $v \in V$. Consider the problem:

$$\max_{\gamma, (\alpha_i)_{i \in \mathcal{I}}} \gamma \tag{54}$$

subject to:

$$\gamma + \sum_i \sum_{m_i'} (U_i(v, m) - U_i(v, (m_i', m_{-i})))\alpha_i(m_i' \mid m_i) \leq \sum_i P_i(m), \quad m \in M,$$

$$\alpha_i(m_i' \mid m_i) \geq 0, \quad i \in \mathcal{I}, (m_i, m_i') \in M_i \times M_i.$$

The dual to the above problem is:

$$\min_{\mu} \sum_{m} \mu(m) \sum_{i} P_i(m) \tag{55}$$

subject to:

$$\sum_{m_{-i}} \mu(m)(U_i(v,m) - U_i(v,(m_i', m_{-i}))) \geq 0, \quad i \in \mathcal{I}, (m_i, m_i') \in M_i \times M_i,$$

$$\sum_{m} \mu(m) = 1,$$

$$\mu(m) \geq 0, \quad m \in M,$$

which is minimizing the revenue over *complete-information* correlated equilibria μ (for the fixed v). For any μ satisfying the constraints, we have $\sum_{m_{-i}} \mu(m)U_i(v,m) = \sum_{m_{-i}} \mu(m)(vq_i(m) - P_i(m)) \geq 0$ for every $i \in \mathcal{I}$ and $m_i \in M_i$ because of the presence of the opt-out message $0 \in M_i$. Therefore, $\sum_m \mu(m) \sum_i (vq_i(m) - P_i(m)) \geq 0$, and $\sum_m \mu(m) \sum_i P_i(m) \leq \sum_m \mu(m) \sum_i vq_i(m) \leq v$. Thus the optimal solution of (54) is bounded above by v. □

Proof of Lemma 2. By (45) we have $q(k,k) = 1/2$. By the second line of (43) this implies that $q(j,k) = j/(2k)$ and $q(k,j) = 1 - j/(2k)$, $j = 0, 1, \ldots, k$. Then we have

$$k - \frac{(k-1)k}{4k} = \sum_{j=0}^{k-1} q(k,j) = \sum_{j=0}^{k-1} \sum_{l=0}^{k-1} q(l+1,j) - q(l,j) = k^2 q(1,0) \tag{56}$$

where the last equality follows from the first line of (43). Thus, $q(1,0) = (3k+1)/(4k^2)$. □

Proof of Lemma 3. Fix an arbitrary P that satisfies Condition (44).

From the second line of (44) ($\mathrm{Rev}(v,(j-1,k)) = \mathrm{Rev}(v,(j,k))$), we have

$$P(j+1,k) - P(j,k) = (1+1/a)(P(j,k) - P(j-1,k)) + (P(k,j) - P(k,j-1))/a, \tag{57}$$

for $j = 1, 2, \ldots, k-1$. Equation (57) implies that

$$P(j+1,k) - P(j,k) = (1+1/a)^j P(1,k) + \sum_{j'=1}^{j} (1+1/a)^{j-j'} (P(k,j') - P(k,j'-1))/a, \tag{58}$$

and as a consequence, for any $j = 0, 1, \ldots, k$:

$$P(j,k) = a((1+1/a)^j - 1)P(1,k) + \sum_{j'=1}^{j-1}((1+1/a)^{j-j'} - 1)(P(k,j') - P(k,j'-1)). \quad (59)$$

We claim that

$$\begin{aligned} X(l) &\equiv \sum_{j=1}^{l-1}(1+1/a)^{l-j}(P(l,j) - P(l,j-1)) \\ &= P(l,l-1) + a((1+1/a)^l - 1)^2 P(1,0) - (1+1/a)^l P(l,0), \end{aligned} \quad (60)$$

for every $l = 1, 2, \ldots, k$. Equation (60) for $l = k$ and Equation (59) together imply Equation (46), which proves the lemma.

Clearly, (60) is true for $l = 1$. Suppose (60) is true for $l = \kappa < k$ as an induction hypothesis; we prove that this implies (60) is true for $l = \kappa + 1$.

From $\mathrm{Rev}(v, (\kappa, j-1)) = \mathrm{Rev}(v, (\kappa, j))$ we have:

$$\begin{aligned} &P(\kappa+1, j) - P(\kappa+1, j-1) \quad (61) \\ =&(1+1/a)(P(\kappa,j) - P(\kappa,j-1)) + (1+1/a)(P(j,\kappa) - P(j-1,\kappa)) \\ &- (P(j+1,\kappa) - P(j,\kappa)), \end{aligned}$$

summing the above equation across $j = 1, 2, \ldots, \kappa - 1$ gives:

$$\begin{aligned} &\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+1-j}(P(\kappa+1,j) - P(\kappa+1,j-1)) \quad (62) \\ =&\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(\kappa,j) - P(\kappa,j-1)) + \sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(j,\kappa) - P(j-1,\kappa)) \\ &- \sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+1-j}(P(j+1,\kappa) - P(j,\kappa)) \\ =&\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(\kappa,j) - P(\kappa,j-1)) + (1+1/a)^{\kappa+1}P(1,\kappa) - (1+1/a)^2(P(\kappa,\kappa) - P(\kappa-1,\kappa)). \end{aligned}$$

That is,

$$
\begin{aligned}
&X(\kappa+1) \qquad (63)\\
&=\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(\kappa,j)-P(\kappa,j-1))+(1+1/a)^{\kappa+1}P(1,\kappa)\\
&\quad-(1+1/a)^2(P(\kappa,\kappa)-P(\kappa-1,\kappa))+(1+1/a)(P(\kappa+1,\kappa)-P(\kappa+1,\kappa-1))\\
&=(1+1/a)^2[P(\kappa,\kappa-1)+a((1+1/a)^{\kappa}-1)^2P(1,0)-(1+1/a)^{\kappa}P(\kappa,0)]+(1+1/a)^{\kappa+1}P(1,\kappa)\\
&\quad-(1+1/a)^2(P(\kappa,\kappa)-P(\kappa-1,\kappa))+(1+1/a)(P(\kappa+1,\kappa)-P(\kappa+1,\kappa-1)),
\end{aligned}
$$

where in the last equality we have used the induction hypothesis (60) for $l=\kappa$.

From $\text{Rev}(v,(\kappa,0))=\text{Rev}(v,(1,0))$ we have $(1+1/a)P(\kappa,0)-P(1,\kappa)=P(\kappa+1,0)-2P(1,0)$. Therefore, the previous equation is equivalent to:

$$
\begin{aligned}
&X(\kappa+1) \qquad (64)\\
&=(1+1/a)^2P(\kappa,\kappa-1)+a(1+1/a)^2((1+1/a)^{\kappa}-1)^2P(1,0)-(1+1/a)^{\kappa+1}(P(\kappa+1,0)-2P(1,0))\\
&\quad-(1+1/a)^2(P(\kappa,\kappa)-P(\kappa-1,\kappa))+(1+1/a)(P(\kappa+1,\kappa)-P(\kappa+1,\kappa-1))\\
&=(1+1/a)^2P(\kappa,\kappa-1)+[a(1+1/a)^2((1+1/a)^{\kappa}-1)^2+2(1+1/a)^{\kappa+1}]P(1,0)-(1+1/a)^{\kappa+1}P(\kappa+1,0)\\
&\quad-(1+1/a)^2(P(\kappa,\kappa)-P(\kappa-1,\kappa))+(1+1/a)(P(\kappa+1,\kappa)-P(\kappa+1,\kappa-1))
\end{aligned}
$$

From $\text{Rev}(v,(\kappa,\kappa))=\text{Rev}(v,(1,0))$ we have $(1+1/a)P(\kappa,\kappa)-P(\kappa+1,\kappa)=-P(1,0)$. Therefore, the previous equation is equivalent to:

$$
\begin{aligned}
&X(\kappa+1) \qquad (65)\\
&=(1+1/a)^2P(\kappa,\kappa-1)+[a(1+1/a)^2((1+1/a)^{\kappa}-1)^2+2(1+1/a)^{\kappa+1}+(1+1/a)]P(1,0)\\
&\quad-(1+1/a)^{\kappa+1}P(\kappa+1,0)+(1+1/a)^2P(\kappa-1,\kappa)-(1+1/a)P(\kappa+1,\kappa-1).
\end{aligned}
$$

From $\text{Rev}(v,(\kappa-1,\kappa))=\text{Rev}(v,(1,0))$ we have $(1+1/a)P(\kappa,\kappa-1)+(1+1/a)P(\kappa-1,\kappa)-P(\kappa+1,\kappa-1)=P(\kappa,\kappa)-2P(1,0)$, Therefore, the previous equation is equivalent to:

$$
\begin{aligned}
&X(\kappa+1) \qquad (66)\\
&=[a(1+1/a)^2((1+1/a)^{\kappa}-1)^2+2(1+1/a)^{\kappa+1}-(1+1/a)]P(1,0)\\
&\quad-(1+1/a)^{\kappa+1}P(\kappa+1,0)+(1+1/a)P(\kappa,\kappa).
\end{aligned}
$$

Finally, using $(1+1/a)P(\kappa,\kappa) - P(\kappa+1,\kappa) = -P(1,0)$ again we get:

$$\begin{aligned} & X(\kappa+1) \\ = & [a(1+1/a)^2((1+1/a)^\kappa - 1)^2 + 2(1+1/a)^{\kappa+1} - (1+1/a) - 1]P(1,0) \\ & - (1+1/a)^{\kappa+1}P(\kappa+1,0) + P(\kappa+1,\kappa). \end{aligned} \tag{67}$$

Since $a(1+1/a)^2((1+1/a)^\kappa - 1)^2 + 2(1+1/a)^{\kappa+1} - (1+1/a) - 1 = a((1+1/a)^{\kappa+1} - 1)^2$, this proves (60) when $l = \kappa + 1$. □

Proof of Lemma 4. Suppose $k < \tilde{k}$. We can replicate any mechanism $(q_i, P_i)_{i\in\mathcal{I}}$ with $M_i = \{0, 1, \ldots, k\}$ in a larger message space $\tilde{M}_i = \{0, 1, \ldots, \tilde{k}\}$:

$$\tilde{q}_i(m) \equiv q_i(\xi(\tilde{m})), \qquad \tilde{m} \in \tilde{M}, \qquad \xi(\tilde{m}) = (\xi(\tilde{m}_1), \ldots, \xi(\tilde{m}_I)),$$

where $\xi : \tilde{M}_i \to M_i$ is $\xi(j) = j$ if $j \leq k$ and $\xi(j) = k$ if $j > k$. And likewise for $\tilde{P}_i$. Clearly, the virtual revenue $\widetilde{\text{Rev}}(v, \tilde{m})$ given by $(\tilde{\alpha}_i, \tilde{q}_i, \tilde{P}_i)_{i\in\mathcal{I}}$, where

$$\tilde{\alpha}_i(\tilde{m}_i' \mid \tilde{m}_i) = \begin{cases} a & \tilde{m}_i' = \tilde{m}_i + 1 \\ 0 & \tilde{m}_i' \neq \tilde{m}_i + 1 \end{cases},$$

is equal to the virtual revenue $\text{Rev}(v, \xi(\tilde{m}))$ given by $(\alpha_i, q_i, P_i)_{i\in\mathcal{I}}$, where

$$\alpha_i(m_i' \mid m_i) = \begin{cases} a & m_i' = m_i + 1 \\ 0 & m_i' \neq m_i + 1 \end{cases}.$$

□

References

Bergemann, D., B. Brooks, and S. Morris (2016a): "First Price Auctions with General Information Structures: Implications for Bidding and Revenue," *Econometrica*, forthcoming.

——— (2016b): "Informationally Robust Optimal Auction Design," Working paper.

Bergemann, D. and S. Morris (2016): "Bayes Correlated Equilibrium and The Comparison of Information Structures in Games," *Theoretical Economics*.

Brooks, B. (2013): "Surveying and Selling: Belief and Surplus Extraction in Auctions," Working paper.

Carrasco, V., V. Farinha Luz, P. Monteiro, and H. Moreira (2015): "Robust Selling Mechanisms," Working paper.

Carroll, G. (2015): "Robustness and Linear Contracts," *American Economic Review*, 105, 536–63.

——— (2016a): "Informationally Robust Trade and Limits to Contagion," *Journal of Economic Theory*, forthcoming.

——— (2016b): "Robustness and Separation in Multidimensional Screening," *Econometrica*, Forthcoming.

Chen, Y.-C. and J. Li (2016): "Revisiting the Foundations of Dominant-Strategy Mechanisms," Working paper.

Chung, K.-S. and J. C. Ely (2007): "Foundations of Dominant-Strategy Mechanisms," *The Review of Economic Studies*, 74, 447–476.

Crémer, J. and R. P. McLean (1985): "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 53, 345–361.

——— (1988): "Optimal Selling Strategies under Uncertainty for a Discriminating Monopolist when Demands are Interdependent," *Econometrica*, 56, 1247–1257.

Engelbrecht-Wiggans, R., P. R. Milgrom, and R. J. Weber (1983): "Competitive bidding and Proprietary Information," *Journal of Mathematical Economics*, 11, 161–169.

Frankel, A. (2014): "Aligned Delegation," *American Economic Review*, 104, 66–83.

Hartline, J. and T. Roughgarden (2016): "Optimal Platform Design," Working paper.

Kos, N. and M. Messner (2015): "Selling to the Mean," Working paper.

Myerson, R. B. (1997): "Dual Reduction and Elementary Games," *Games and Economic Behavior*, 21, 183–202.

Roesler, A.-K. and B. Szentes (2016): "Buyer-Optimal Learning and Monopoly Pricing," Working paper.

Stroock, D. (2013): *Introduction to Markov Processes*, Springer, 2 ed.

Vohra, R. V. (2011): *Mechanism Design: A Linear Programming Approach*, Cambridge University Press.

Wilson, R. (1987): "Game-Theoretic Analyses of Trading Processes," in *Advances in Economic Theory: Fifth World Congress*, ed. by T. Bewley, Cambridge University Press, 33–70.

Yamashita, T. (2015): "Implementation in Weakly Undominated Strategies: Optimality of Second-Price Auction and Posted-Price Mechanism," *The Review of Economic Studies*, 82, 1223–1246.

——— (2016): "Revenue Guarantee in Auction with Common Prior," Working paper.