Nestings in partitions

Sophie Burrill

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\[ B_n = \sum_{k=1}^{n} \binom{n}{k} B_k = 1, 1, 2, 5, 15, 52, \ldots \]

Counts the number of partitions of a set \( \{1, 2, \ldots, n\} \).

**Figure:** All partitions of the set \( \{1, 2, 3\} \).
A *nesting* in a partition:

![Diagram of a nesting in a partition](image)

**Goal**

Count the number of partitions according to both length and number of nestings.
\[ \mathcal{P}_n := \text{set of partitions on } \{1, 2, \ldots, n\}. \]
\[ \mathcal{M}_n^w := \text{set of bicolored Motzkin paths of length } n \text{ with weight vector } w. \]

**Observation**

\[ |\mathcal{P}_n| = |\mathcal{M}_n^w| \]

Using a bijection \( \Phi : \mathcal{P}_n \rightarrow \mathcal{M}_n^w \), we can accomplish our goal.
Nestings in partitions

**Regular Motzkin path**

- Legal steps:
- length $n$,
- may touch but not cross $x$ axis.

Include continued fractions here?**
Nestings in partitions

$M_n^w$

- Legal steps:
- length $n$,
- may touch but not cross x axis and,
- weight vector $w = (w_1, w_2, \ldots, w_n)$ assigned to each step.

Figure: $w := (0,0,0,0,0,0,0,0,0,4,2,0,3,2,1,0,0,0,0)$
The bijection $\Phi : \mathcal{P}_n \rightarrow \mathcal{M}_n^w$: 

<table>
<thead>
<tr>
<th>Vertex Type</th>
<th>Name</th>
<th>Step</th>
<th>Weight $w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="opener" /></td>
<td>Opener</td>
<td>/</td>
<td>0</td>
</tr>
<tr>
<td><img src="image" alt="closer" /></td>
<td>Closer</td>
<td>|</td>
<td>$0 &lt; w_i &lt; i-1$</td>
</tr>
<tr>
<td><img src="image" alt="transitory" /></td>
<td>Transitory</td>
<td>-</td>
<td>$0 &lt; w_i &lt; i-1$</td>
</tr>
<tr>
<td><img src="image" alt="singleton" /></td>
<td>Singleton</td>
<td>-</td>
<td>0</td>
</tr>
</tbody>
</table>
Weighting: \( w = (w_0, w_1, \ldots, w_j, \ldots, w_n) \)

Let step \( j \) be of type \( \leftarrow \) or \( \rightarrow \). Then \( w_j \) is found as follows:

- Step \( j \) corresponds to vertex \( j \) in the arc annotated diagram.
- Vertex \( j \) is a closer or transitory.
- Label all available openers and transitory vertices in increasing order from the left, starting with 0.
- The label of the vertex connected on the left to \( j \) is the weight \( w_j \) assigned to step \( j \).
Example

Consider the partition \( P_1 := \{1, 5, 6, 8\}, \{2, 9\}, \{3, 4\}, \{7\} \). First, the arc annotated sequence is:

Next we find the bicolored shape of the corresponding Motzkin path \( M_1 \):
Finally, we use the arc annotated sequence to get the weight vector for $M_1$:

So our corresponding weight vector is $(0, 0, 0, 2, 0, 1, 0, 0, 0, 0)$.

Include final MP?**
A nesting corresponds to a $w_i > 0$ in the Motzkin path.

We can count weighted Motzkin paths using continued fractions.

Let $a_i$ denote north steps.

Let $b_i$ denote south steps.

Let $c_i$ denote east steps.

Note

$i$ tracks starting height.
We use *specification* to count our Motzkin paths.

- $M_{n}^{h,w} := \text{bicoloured weighted Motzkin paths of maximum height } h.$
- $M_{n}^{0,w} := SEQ(c_{0}) = \frac{1}{1-c_{0}}.$
- For $M_{n}^{1,w}$ we get $c_{0} \rightarrow c_{0} + a_{1}SEQ(c_{1})b_{1}.$
- $M_{n}^{1,w} = \frac{1}{1-c_{0}-a_{0}b_{1}} \frac{a_{0}b_{1}}{1-c_{1}}.$
Following this method, the generalized continued fraction is:

\[
M_n^w = \frac{1}{1 - c_0 - \frac{a_0 b_1}{1 - c_1 - \frac{a_1 b_2}{1 - c_2 - \frac{a_2 b_3}{\ddots}}}}
\]

Using \(x\) to count length and \(y\) to count nestings we do the following substitution:

\[
\begin{align*}
a_i &\rightarrow x \\
b_i &\rightarrow (1 + y + y^2 + \ldots, y^{i-1})x \\
c_i &\rightarrow x + (1 + y + y^2 + \ldots, y^{i-1}x
\end{align*}
\]
Now we have our continued fraction, and using Maple, we can get the series that results. Here is the triangular table for the series. The maximum height being used here is 10:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>y</th>
<th>y^2</th>
<th>y^3</th>
<th>y^4</th>
<th>y^5</th>
<th>y^6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x^2</td>
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<tr>
<td>x^4</td>
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<tr>
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<td>6</td>
<td>3</td>
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<td>x^8</td>
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<td>28</td>
<td>28</td>
<td>20</td>
<td>10</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>
Solving for the average number of nestings according to maximum heights gives the following plot of averages:
Conclusion

- We gain a lot of information by using this bijection, \( \Phi \).
- “Crossings and nestings of matchings and partitions” by Chen et al.
- The following combinatorial objects can also be examined this way:
  - matchings,
  - permutations,
  - graphs