6

Modal Propositional Logic

1. INTRODUCTION

In this book we undertook — among other things — to show how metaphysical talk of possible worlds and propositions can be used to make sense of the science of deductive logic. In chapter 1, we used that talk to explicate such fundamental logical concepts as those of contingency, noncontingency, implication, consistency, inconsistency, and the like. Next, in chapter 2, while resisting the attempt to identify propositions with sets of possible worlds, we suggested that identity-conditions for the constituents of propositions (i.e., for concepts), can be explicated by making reference to possible worlds and showed, further, how appeal to possible worlds can help us both to disambiguate proposition-expressing sentences and to refute certain false theories. In chapter 3 we argued the importance of distinguishing between, on the one hand, the modal concepts of the contingent and the noncontingent (explicable in terms of possible worlds) and, on the other hand, the epistemic concepts of the empirical and the a priori (not explicable in terms of possible worlds). In chapter 4 we showed how the fundamental methods of logic — analysis and valid inference — are explicable in terms of possible worlds and argued for the centrality within logic as a whole of modal logic in general and S5 in particular. Then, in chapter 5, we tried to make good our claim that modal concepts are needed in order to make sound philosophical sense even of that kind of propositional logic — truth-functional logic — from which they seem conspicuously absent. Now in this, our last chapter, we concentrate our attention on the kind of propositional logic — modal propositional logic — within which modal concepts feature overtly. Among other things, we try to show: (1) how various modal concepts are interdefinable with one another; (2) how the validity of any formula within modal propositional logic may be determined by the method of worlds-diagrams and by related reductio methods; and finally, (3), how talk of possible worlds, when suitably elaborated, enables us to make sense of some of the central concepts of inductive logic.

2. MODAL OPERATORS

Non-truth-functionality

The rules for well-formedness in modal propositional logic may be obtained by supplementing the rules (see p. 262) for truth-functional logic by the following:

R4: Any wff prefixed by a monadic modal sentential operator [e.g., "◇", "□", "∇", or "Δ"] is a wff.

R5: Any two wffs written with a dyadic modal operator [e.g., "∘", "ϕ", "→", or "↔"] between them and the whole surrounded by parentheses is a wff.

In light of these rules, it is obvious that from a *formal* point of view, modal propositional logic may be regarded as an accretion upon truth-functional logic: every wff of truth-functional propositional logic is likewise a wff of modal propositional logic, although there are wffs of modal propositional logic — those containing one or more modal operators — which are not wffs of truth-functional logic. (That modal propositional logic is formally constructed by adding onto a truth-functional base ought not, however, to be taken as indicating a parallel order as regards their conceptual priority. We have argued earlier in this book (chapters 4 and 5) that although modal concepts are not symbolized within truth-functional logic, one cannot adequately *understand* that logic without presupposing these very concepts.)

Modal operators are non-truth-functional. For example, the monadic operator "◇", unlike the truth-functional operator "∼", is non-truth-functional. Given the truth-value of P, one cannot, in general, determine the truth-value of the proposition expressed by the compound sentence "◇P", as one could in the case of "∼P". Obviously if "P" expresses a true proposition, then "◇P" must likewise express a true proposition. For "◇P" says nothing more than that the proposition expressed

P	◇P	□P	∇P	ΔP
T	T	I	I	I
F	I	F	I	I

P	Q	P ∘ Q	P ϕ Q	P → Q	P ↔ Q
T	T	T	F	I	I
T	F	I	I	F	F
F	T	I	I	I	F
F	F	I	I	I	I

TABLE *(6.a)*

Beware. Do not read the "I" which appears on these tables as if it were a 'third' truth-value. There are only *two* truth-values. Clearly, for every instantiation of constants for the variables in these sentence-forms, the resulting sentences will express propositions which bear one or other of the two truth-values, truth or falsity.

by "P" is true in some possible world, and if P is true, then it is true in some possible world. But the same can not be concluded for the case in which "P" expresses a falsehood. Suppose "P" expresses a false proposition. What, then, can we conclude about the truth-value of the proposition expressed by "$\Diamond$P"? This latter formula asserts that the proposition expressed by "P" is true in some possible world. This claim is true just in case the false proposition expressed by "P" is true in some possible world, i.e., just in case the false proposition, P, is contingent. But the proposition expressed by "$\Diamond$P" is false just in case the false proposition expressed by "P" is false in every possible world, i.e., just in case the false proposition P is noncontingent. Thus whether the proposition expressed by "$\Diamond$P" is false depends not simply on whether the proposition expressed by "P" is false, but in addition on whether that false proposition is contingent or noncontingent. It depends, that is, on the modality as well as the truth-value of P. This means that "$\Diamond$P" has a *partial* truth-table. In the case where "P" has been assigned "T", "$\Diamond$P" likewise is assigned "T"; but in the case where "P" is assigned "F", *neither* "T" *nor* "F" may be assigned to "$\Diamond$P". We fill in the gap with "I", where "I" stands for "the truth-value is indeterminate on the basis of the data (i.e., truth-value) specified." Like "$\Diamond$", every other modal operator will have only (at best) a partial truth-table.

It suffices for an operator to be non-truth-functional that there be a single "I" in its truth-table.

An important consequence of the fact that modal operators are non-truth-functional is that we will be unable — in contrast to the case of wffs in truth-functional logic — to ascertain the validity of modalized formulae by the truth-tabular techniques discussed in the previous chapter. How we might, instead, evaluate formulae containing modal operators is shown in sections 8 and 9.

Modal and nonmodal propositions; modalized and nonmodalized formulae

Any proposition at least one of whose constituent concepts is a modal concept is a *modal* proposition. All other propositions are *nonmodal.*

Any modal proposition can be represented in our conceptual notation by a wff containing one or more modal operators, e.g., "$\Box$", "$\Diamond$", etc. But of course a modal proposition need not be so represented. A modal proposition can also be represented in our symbolism by a wff containing no modal operators, e.g., by "A", or in the case of a conditional modal proposition, by "A $\supset$ B". In such a case the symbolism would fail to reveal as much as it could, viz., that a particular proposition is a modal one. But this is a characteristic of any system of symbolization. Every system of symbolization has the capacity to reveal different degrees of detail about that which it symbolizes. We have already seen, for example, how it is possible to symbolize the proposition *(5.75)*, viz., that today is Monday or today is other than the day after Sunday, as either "A$\vee\sim$A" or as "A$\vee$B". The latter symbolization reveals less information than the former, but neither is more or less 'correct' than the other.

A similar point can be made in the case of the symbolization of modal attributes. Consider the proposition

(6.1) It is necessarily true that all squares have four sides.

Clearly *(6.1)* is a modal proposition. Thus if we let "A" = "all squares have four sides", we can render this proposition in symbolic notation as

(6.2) "$\Box$A".

But we are not forced to do this. If for some reason we are not intent on conveying in symbols that *(6.1)* is a modal proposition, we can, if we like, represent it simply as, for example,

(6.3) "B".

Thus if we happen across a wff which does not contain a modal operator, we are not entitled to infer that the proposition represented by that wff is a nonmodal one. Being represented by a wff containing a modal operator is a sufficient condition for the proposition expressed being a modal one, but it is not a necessary condition.

A wff whose simple sentential components, i.e., capital letters, all occur as the arguments of or within the arguments of some modal operator or other will be said to be *fully modalized.* Thus, for example, the wffs

"$\Box(P \supset Q)$",

"$\Diamond P \supset \Box P$",

"$\Diamond\Box \sim P$", and

"$\Diamond(P \cdot \Box Q)$",

are all fully modalized.

A wff of whose simple sentential components some, but not all, occur as the arguments of or within the arguments of some modal operator or other will be said to be *partially modalized.* For example,

"$\Diamond P \supset P$",

"$P \supset \Box(P \vee Q)$", and

"$\Box P \vee (\Diamond Q \supset R)$",

are all partially modalized.

A wff of whose simple sentential components none occurs as the argument of or within the argument of a modal operator will be said to be *unmodalized.* For example,

"$P \supset P$",

"$P \cdot (P \supset Q)$", and

"P",

are all unmodalized.

Note that on these definitions, propositions, but not sentences, may be said to be *modal;* and sentences or sentence-forms, but not propositions, may be said to be modal*ized.*

EXERCISES

Write down all the forms of each of the following modalized formulae:

1. $\Box A \supset B$

2. $\Box(A \supset \Diamond B)$

3. $\Box(A \supset \Box A)$

4. $\Box\Box A \supset \Box A$

The interdefinability of the monadic and dyadic modal operators

We have introduced four monadic modal operators, viz., "$\Box$", "$\Diamond$", "∇" and "Δ", and four dyadic modal operators, viz., "$\circ$", "ϕ", "$\rightarrow$" and "$\leftrightarrow$". Each of these may be defined in terms of any one of the other seven. There are, then, a total of fifty-six such definitions. Some of these we have already seen; others are new. We list them all here for the sake of completeness.

Necessary truth

"$\Box P$" $=_{df}$ "$\sim\Diamond\sim P$"
"$\sim\nabla P \cdot P$"
"$\Delta P \cdot P$"
"$\sim(\sim P \circ \sim P)$"
"$\sim P \,\phi\, \sim P$"
"$\sim P \rightarrow P$"
"$(P \vee \sim P) \leftrightarrow P$"

Possibility

"$\Diamond P$" $=_{df}$ "$\sim\Box\sim P$"
"$\nabla P \vee P$"
"$\sim\Delta P \vee P$"
"$P \circ P$"
"$\sim(P \,\phi\, P)$"
"$\sim(P \rightarrow \sim P)$"
"$\sim[(P \cdot \sim P) \leftrightarrow P]$"

Contingency

"∇P" $=_{df}$ "$\sim\Box P \cdot \sim\Box \sim P$"
"$\Diamond P \cdot \Diamond\sim P$"
"$\sim\Delta P$"
"$(P \circ P) \cdot (\sim P \circ \sim P)$"
"$\sim(P \,\phi\, P) \cdot \sim(\sim P \circ \sim P)$"
"$\sim(P \rightarrow \sim P) \cdot \sim(\sim P \rightarrow P)$"
"$\sim[(P \cdot \sim P) \leftrightarrow P] \cdot \sim[(P \vee \sim P) \leftrightarrow P]$"

Noncontingency

"ΔP" $=_{df}$ "$\Box P \vee \Box \sim P$"
"$\sim \Diamond P \vee \sim\Diamond\sim P$"
"$\sim\nabla P$"
"$\sim(P \circ P) \vee \sim(\sim P \circ \sim P)$"
"$(P \,\phi\, P) \vee (\sim P \,\phi\, \sim P)$"
"$(P \rightarrow \sim P) \vee (\sim P \rightarrow P)$"
"$[(P \cdot \sim P) \leftrightarrow P] \vee [(P \vee \sim P) \leftrightarrow P]$"

Consistency

"$P \circ Q$" $=_{df}$ "$\sim\Box\sim(P \cdot Q)$"
"$\Diamond(P \cdot Q)$"
"$\nabla(P \cdot Q) \vee (P \cdot Q)$"
"$\sim\Delta(P \cdot Q) \vee (P \cdot Q)$"
"$\sim(P \,\phi\, Q)$"
"$\sim(P \rightarrow \sim Q)$"
"$\sim[(P \cdot \sim P) \leftrightarrow (P \cdot Q)]$"

Inconsistency

"$P \,\phi\, Q$" $=_{df}$ "$\Box\sim(P \cdot Q)$"
"$\sim \Diamond(P \cdot Q)$"
"$\sim\nabla(P \cdot Q) \cdot \sim(P \cdot Q)$"
"$\Delta(P \cdot Q) \cdot \sim(P \cdot Q)$"
"$\sim(P \circ Q)$"
"$P \rightarrow \sim Q$"
"$(P \cdot \sim P) \leftrightarrow (P \cdot Q)$"

Implication	*Equivalence*
"$P \rightarrow Q$" $=_{df}$ "$\Box(P \supset Q)$"	"$P \leftrightarrow Q$" $=_{df}$ "$\Box(P \equiv Q)$"
"$\sim \Diamond(P \cdot \sim Q)$"	"$\sim \Diamond \sim (P \equiv Q)$"
"$\sim \nabla \sim (P \cdot \sim Q) \cdot \sim (P \cdot \sim Q)$"	"$\sim \nabla (P \equiv Q) \cdot (P \equiv Q)$"
"$\Delta \sim (P \cdot \sim Q) \cdot \sim (P \cdot \sim Q)$"	"$\Delta (P \equiv Q) \cdot (P \equiv Q)$"
"$\sim (P \circ \sim Q)$"	"$\sim [\sim (P \equiv Q) \circ \sim (P \equiv Q)]$"
"$(P \,\phi\, \sim Q)$"	"$\sim (P \equiv Q) \,\phi\, \sim (P \equiv Q)$"
"$(P \cdot \sim P) \leftrightarrow (P \cdot \sim Q)$"	"$(P \rightarrow Q) \cdot (Q \rightarrow P)$"

For every wff in which there occurs some particular modal operator, there exist other, formally equivalent wffs,[1] in which that modal operator does not occur. More particularly, for every wff in which there occur one or more *dyadic* modal operators, there exist formally equivalent wffs in which only *monadic* modal operators occur. Thus, subsequently in this chapter, when we come to examine methods for determining the validity of modalized wffs, it will suffice to give rules for handling only *monadic* modal operators. Every wff containing one or more dyadic modal operators can be replaced, for the purposes of testing validity, with a formally equivalent wff containing only monadic modal operators. The above list of equivalences gives us the means of generating these formally equivalent wffs.

Elimination of "$\circ$": every wff of the form "$P \circ Q$" may be replaced by a wff of the form "$\Diamond(P \cdot Q)$"

Elimination of "ϕ": every wff of the form "$P \,\phi\, Q$" may be replaced by a wff of the form "$\sim \Diamond(P \cdot Q)$"

Elimination of "$\rightarrow$": every wff of the form "$P \rightarrow Q$" may be replaced by a wff of the form "$\Box(P \supset Q)$"

Elimination of "$\leftrightarrow$": every wff of the form "$P \leftrightarrow Q$" may be replaced by a wff of the form "$\Box(P \equiv Q)$".

Sometimes it will be useful, too, to eliminate the two monadic modal operators "∇" and "Δ" in favor of "$\Diamond$" and "$\Box$".

Elimination of "∇": every wff of the form "∇P" may be replaced by a wff of the form "$\Diamond P \cdot \Diamond \sim P$"

Elimination of "Δ": every wff of the form "ΔP" may be replaced by a wff of the form "$\Box P \vee \Box \sim P$"

1. Two sentence-forms are formally equivalent if and only if, for any uniform substitution of constants for the variables therein, there result two sentences which express logically equivalent propositions. (See section 10, this chapter.)

Examples:

"$P \rightarrow (P \circ Q)$"	may be replaced by	"$\Box[P \supset \Diamond(P \cdot Q)]$"
"$(P \phi Q) \circ R$"	may be replaced by	"$\Diamond[\sim \Diamond(P \cdot Q) \cdot R]$"
"$\Delta(P \circ Q)$"	may be replaced by	"$\Box \Diamond(P \cdot Q) \vee \Box \sim \Diamond(P \cdot Q)$"

EXERCISES

For each of the following formulae, find formally equivalent wffs in which the only modal operators are "$\Box$" and/or "$\Diamond$".

1. $P \rightarrow (Q \rightarrow P)$
2. $P \circ (\sim Q \phi P)$
3. $\Box P \rightarrow (Q \supset P)$
4. $\Box P \rightarrow \Delta(P \vee Q)$
5. $(\Box P \cdot \Box Q) \rightarrow (\Diamond P \leftrightarrow \Diamond Q)$

3. SOME PROBLEMATIC USES OF MODAL EXPRESSIONS

"It is possible that"

"It is possible that" has many uses in ordinary prose that one should distinguish from the use of that expression which is captured by the logical operator "$\Diamond$". Two of these uses are especially worthy of note.

Most of us have viewed television dramas in which a lawyer attempts to discredit a witness by asking, "Is it possible you might be mistaken?" Witnesses who are doing their utmost to be as fair as possible(!) are likely to take this question as if it were the question, "Is it logically possible that you might be mistaken?", to which they will answer, "Yes". But an overzealous lawyer might then pounce on this admission as if it were a confession of probable error. The point is that to say of a proposition that it is *possibly* false in the logical sense (i.e., is contingent or necessarily false) is not to say that it is *probably* false. A proposition which is *possibly* false, e.g., the contingent proposition that there is salt in the Atlantic Ocean, need *not* be *probably* false; quite the contrary, this particular proposition *is* true.

Secondly, consider the use of "It is possible that" in the sentence

(6.4) "It is possible that the Goldbach Conjecture (i.e., that every even number greater than two is the sum of two prime numbers) is true."

If we do not keep our wits about us, we might be inclined to think that this sentence is being used to express the proposition that the Goldbach Conjecture is possibly true. But a person who utters *(6.4)* and who accepts a possible-worlds analysis of the concept of possibility, ought to reject such a construal of his words. For clearly, he is trying to say something true; and yet, if the Goldbach Conjecture — unknown to us — is false (which is to say that it is necessarily false), then to say that it is possibly true is (as we shall see subsequently in this chapter) to say something which is itself necessarily false. The point can be put another way: the Goldbach Conjecture is either necessarily false or necessarily true. We do not know which it is. If, however, it *is* necessarily false, then to say of it that it is possibly true is to assert a proposition which itself is necessarily false.

What a person who asserts *(6.4)* is trying to say would seem to be something of the following sort:

> *(6.5)* "To the best of our knowledge, the Goldbach Conjecture is not false."

But this sentence, too, needs careful handling. We must beware not to take this as if it were synonymous with

> *(6.6)* "It is compatible with everything which we now know that the Goldbach Conjecture is true."

Clearly this won't do. If the Goldbach Conjecture is false (and hence necessarily false) then it is *not* compatible (i.e., is not consistent) with *anything* — let alone everything — we know, and to assert that it is, would be, once again, to assert a necessary falsehood.

The proper construal of sentences *(6.4)* and *(6.5)* would seem to lie in the abandonment of the attempt to capture in terms of the logical concept of possibility (or the logical concept of consistency) whatever sense of "possible" it is which *(6.4)* invokes. What *(6.4)* attempts to express can be stated nonparadoxically in the less pretentious-sounding sentence

> *(6.7)* "We do not know whether the Goldbach Conjecture is true and we do not know whether the Goldbach Conjecture is false."

In this paraphrase all talk of 'possibility' has fallen away. To try to re-introduce such talk in such contexts is to court logical disaster (i.e., the asserting of necessary falsehoods).

Problems with the use of "it is necessary that"; the modal fallacy; absolute and relative necessity

There is a widespread practice of marking the presence of an implication relation by the use of such expressions as:

"If . . . , then it is necessary that . . . "

"If . . . , then it must be that . . . "

"If . . . , then it has to be that . . . " and

" . . . Therefore, it is necessary that . . . "

Often enough, these locutions are harmless. But sometimes they beguile persons into the mistaken belief that what is being asserted is that the consequent of the implicative proposition (or the conclusion of an argument) is itself necessarily true. Of course, some conditional propositions expressed by sentences of these types *do* have necessarily true consequents, as for example, *(6.8)* "If the successor of an integer is equal to one plus that number, then it is necessary that the successor of an integer is greater than that number."

Here the expression "it is necessary that" in the consequent-clause of the sentence *(6.8)* is no cause for concern: the proposition expressed by the consequent-clause is indeed necessarily true. But often we use the very same grammatical construction in cases where the consequent-clause does *not* express a necessary truth: where the words "it is necessary that" are being used to signal the existence of an implication relation between the propositions expressed by the antecedent-clause and the consequent-clause and not being used to claim that the latter clause expresses a necessarily true proposition.

Consider the sentence

(6.9) "If Paul has three children and at most one daughter, then it is necessary that he has at least two sons."

The expression "it is necessary that" in *(6.9)* properly should be understood as marking the fact that the relation between the proposition expressed by the antecedent-clause and the proposition expressed by the consequent-clause (i.e., the relation of conditionality) holds necessarily; it should not be understood as asserting the (false) proposition that Paul's having two sons is a necessary truth.

Let "A" = "Paul has three children and at most one daughter", and "B" = "Paul has at least two sons". Using these sentential constants, the correct translation of *(6.9)* is

(6.10) "$\Box(A \supset B)$".

That is, the relation (viz., material conditionality) obtaining between the antecedent, A, and the consequent, B, holds necessarily. It would be incorrect to translate *(6.9)* as

(6.11) "$A \supset \Box B$".

To mistakenly transfer the modality of necessary truth to the consequent (as illustrated in *(6.11)*) of a true implicative proposition or to the conclusion of a deductively valid argument from the conditional relation which holds between the antecedent and consequent or between the premises and conclusion, is to commit what has come to be known as 'the' Modal Fallacy. Of course this is not the only way that one's thinking about modal concepts can go awry. There are, strictly speaking, indefinitely many modal fallacies one can commit. Yet it is this particular one which has been singled out by many writers for the title, 'the' Modal Fallacy.

We can hardly hope to reform ordinary prose so that it will accord with the niceties of our conceptual analysis. Even logicians are going to continue to say such things as

(6.12) "If today is Tuesday, then tomorrow must be Wednesday."

(6.13) "If a proposition is necessarily true, then it has to be noncontingent."

(6.14) "If Paul has three children among whom there is only one daughter, then he has to have two sons."

Let us adopt a special name for the kind of propositions expressed by the consequents of sentences utilizing this kind of construction. Let us call them "relative necessities".

A proposition, Q, then, will be said to be relatively necessary, or more exactly, to be necessary relative to the proposition P, if and only if Q is true in all possible worlds in which P is true; or, put another way, if and only if relative to all the possible worlds in which P is true, Q is true. Consider, for example, the nonmodal component of the consequent of the proposition expressed by *(6.12)*, viz., the proposition that tomorrow is Wednesday. The proposition that tomorrow is Wednesday is contingent in the *absolute* sense of "contingent", i.e., if we look at the set of all possible worlds. But this same proposition may be said to be noncontingent (specifically, necessarily true) *relative* to the set of all those possible worlds in which today is Tuesday.

There is a fallacy having no common name which is analogous to the Modal Fallacy and which sometimes arises in the use of epistemic concepts. In particular, some persons are wont to say that a proposition, P, can be known a priori if it can be validly inferred a priori from some proposition which is known to be true. This account, however, does not jibe with the explication of the concept of *aprioricity* which we gave in chapter 3. There we said that if a proposition is validly inferred from some proposition, the inferred proposition will be said to be known experientially if the proposition from which it is inferred is itself known experientially. Propositions which are known by the a priori process of inference may be said to instance the property of *relative* aprioricity; it is an open question of each of them whether it also instances the property of *absolute* aprioricity.

EXERCISES

Part A

Translate each of the following sentences into the symbolism of Modal Propositional Logic, taking care to avoid the modal fallacy.

1. *"If today is Tuesday, then tomorrow must be Wednesday."*

Let "A" = "Today is Tuesday"
"B" = "Tomorrow is Wednesday"

2. *"If today is Tuesday, then it is impossible that today is not Tuesday."*

Let "A" = "Today is Tuesday"

Part B

The modal fallacy can be very insidious. It occurs in both of the following arguments, yet some persons do not spot it. Try to see where it occurs. Then translate each of the arguments into the notation of Modal Propositional Logic in such a way that the fallacy is not committed.

3. *"If a proposition is true it can not be false. But if a proposition can not be false, then it is not only true but necessarily true. Thus if a proposition is true, it is necessarily true, and (consequently) the class of contingent truths is empty."*

4. *[It is not necessary to translate the part of this argument which is enclosed within the parentheses.]*

"If an event is going to occur, then it cannot not occur. But if an event cannot not occur, then it must occur. Therefore if an event is going to occur, it must occur. (We are powerless to prevent

what must happen. The future is pre-ordained and our thinking that we can affect it is mere illusion.)"

Part C

Reread in chapter 1, p. 25, exercise 4, the words of Lazarus Long concerning time travel. If we let "A" = "a goes back in time" and let "B" = "a shoots his grandfather before the latter sires a's father", which, if any, of the following do you think most closely capture(s) the point of Long's claim? Which, if any, of the following, do you think he is arguing against?

(a) $\sim\Diamond A \cdot \sim\Diamond B$

(b) $\sim\Diamond(A \cdot B)$

(c) $A \rightarrow \sim\Diamond B$

(d) $\Diamond A \cdot \Diamond B \cdot \sim\Diamond(A \cdot B)$

(e) $\sim\Diamond(A \cdot B) \rightarrow \sim\Diamond A$

(f) $\Diamond A \supset \Box \sim B$

(g) $\sim\Diamond A$

4. THE MODAL STATUS OF MODAL PROPOSITIONS

Every proposition is either necessarily true, necessarily false, or contingent. Since modal propositions form a proper subset of the class of propositions, every *modal* proposition must itself be either necessarily true, necessarily false, or contingent.

How shall we determine the modal status of modal propositions? So far as the methodology of Modal Propositional Logic is concerned, this question will be answered to the extent to which this logic can provide a means of ascertaining the validity of modalized formulae. A rigorous effective technique for that purpose will be presented in section 8 of this chapter. But as a step along the way toward developing that general technique, in this section we will lay the groundwork by examining only the very simplest cases, viz., those modalized formulae which consist of a single sentential variable which is the argument of (i.e., is modalized by) *one* of the operators "$\Box$", "$\Diamond$", "∇", or "Δ". Since all of these operators are definable in terms of one another, it will suffice to examine just one of them (see section 2). The one we choose is "$\Box$". Turning our attention to "$\Box$P", we can see that there are three cases requiring our attention:

1. The modal status of $\Box$P in the case where P is contingent;

2. The modal status of $\Box$P in the case where P is necessarily true; and

3. The modal status of $\Box$P in the case where P is necessarily false.

Case 1: What is the modal status of □P in the case where P is contingent?

Since, by hypothesis, P is contingent, there are some possible worlds in which P is true and others (all the others) in which it is false. Let us, then, divide the set of all possible worlds into two mutually exclusive and jointly exhaustive subsets, W_t and W_f, those possible worlds in which P is true and those possible worlds in which P is false.

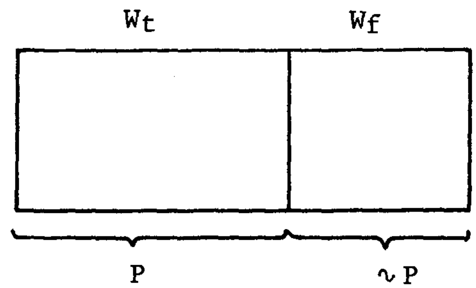


FIGURE *(6.b)*

Arbitrarily pick any possible world in W_t. Let us call that world "W_{t1}".

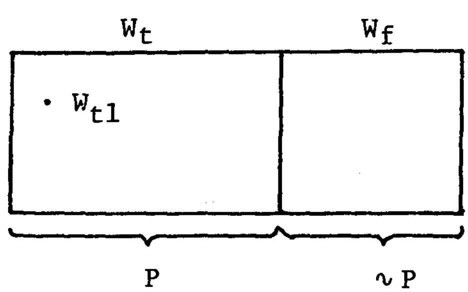


FIGURE *(6.c)*

What is the truth-value of □P in W_{t1}? Clearly □P is false in W_{t1}, for □P asserts that P is true in W_{t1} and in every other possible world as well. But P is false in every world in W_f. And if □P is false in W_{t1}, it is false throughout W_t, for whatever holds for any arbitrarily chosen member of a set (or more exactly, whatever holds of a member of a set irrespective of which member it is), holds for every other member of that set.

This leaves the possible worlds in W_f to be examined. What is the truth-value of □P in W_f? Arbitrarily pick any member of W_f. Call it "W_{f1}".

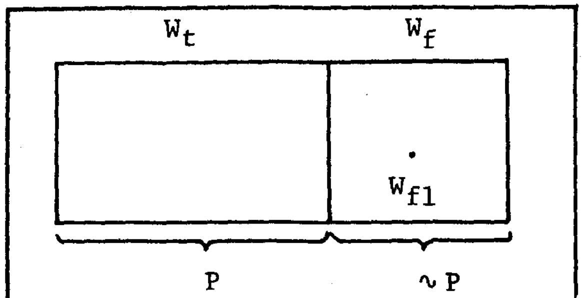


FIGURE *(6.d)*

Clearly $\Box P$ is false in W_{f1}, for $\Box P$ asserts that P is true in W_{f1} and in every other possible world as well. But P is false in W_{f1}. Therefore, since W_{f1} is but an arbitrarily chosen member of W_f, it follows that $\Box P$ is false in every member of W_f. Thus we have shown that $\Box P$ is false in every member of W_t and have now just shown that $\Box P$ is false, as well, in every member of W_f. But these are all the possible worlds there are. Hence in the case where P is contingent, $\Box P$ is false in every possible world, i.e., $\nabla P \rightarrow \Box \sim \Box P$. As a consequence, if P is contingent, then $\Box P$ is noncontingent, i.e.,

$\nabla P \rightarrow \Delta \Box P$. [i.e., if P is contingent, then the (modal) proposition that P is necessarily true is itself noncontingent.]

Case 2: What is the modal status of $\Box P$ in the case where P is necessarily true?

Arbitrarily pick any possible world whatever. Let us call that world "W_1".

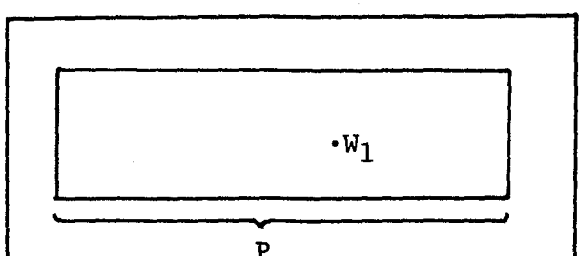


FIGURE *(6.e)*

What is the truth-value of $\Box P$ in W_1? Clearly $\Box P$ has the truth-value, truth, in W_1. For, in W_1, the proposition $\Box P$ asserts that P is true in W_1 and in every other possible world as well, and clearly this is the case. Hence $\Box P$ is true in W_1. But whatever is true of any arbitrarily selected possible world, is true of every possible world. Hence in the case where P is necessarily true, $\Box P$ is also necessarily true, i.e., $\Box P \rightarrow \Box\Box P$. As a consequence, if P is necessarily true, then $\Box P$ is (again) noncontingent, i.e.,

$\Box P \rightarrow \Delta \Box P$.

Case 3: What is the modal status of $\Box P$ in the case where P is necessarily false?

Again arbitrarily pick any possible world whatever. Again let us call that world "W_1".

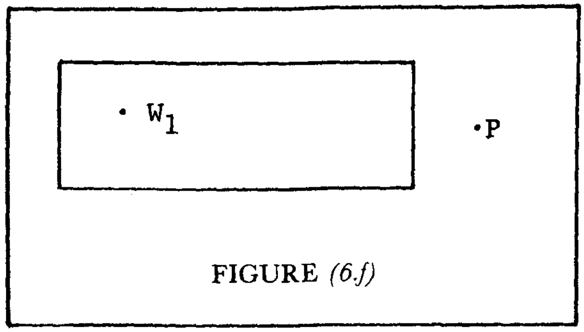


FIGURE *(6.f)*

What is the truth-value of $\Box P$ in W_1? Clearly $\Box P$ has the truth-value, falsity, in W_1. For in W_1, the proposition $\Box P$ asserts that P is true in W_1 and in every other possible world as well. But P is false in W_1. Hence $\Box P$ is false in W_1. But whatever is false in any arbitrarily selected possible world, is false in every possible world. Hence in the case where P is necessarily false, $\Box P$ is also necessarily false, i.e., $\Box \sim P \rightarrow \Box \sim \Box P$. As a consequence, if P is necessarily false, then $\Box P$ is (once again) noncontingent, i.e.,

$$\Box \sim P \rightarrow \Delta \Box P.$$

Conclusion: Every modal proposition expressible by a sentence of the form "$\Box P$" is noncontingent. If "P" expresses a necessary truth, then "$\Box P$" likewise expresses a necessary truth; if "P" expresses either a contingency or a necessary falsity, then "$\Box P$" expresses a necessary falsity. In short, a sentence of the form "$\Box P$" never expresses a contingency.

This last result holds as well for sentences of the form "$\Diamond P$", "∇P", and "ΔP"; i.e., no such sentence ever expresses a contingency.

We can sum up this section by saying that propositions ascribing the various members of the family of properties, necessary truth, necessary falsehood, possibility, impossibility, noncontingency, and contingency, to other propositions, are always themselves noncontingent. (Later, in section 6, we shall put the point by saying that these properties are *essential* properties of those propositions which instance them.)

EXERCISES

1. *Under what conditions of modal status for P will "$\Diamond P$" express a necessary truth? Under what conditions, a necessary falsity?*

2. *Under what conditions of modal status for P will "∇P" express a necessary truth? Under what conditions, a necessary falsity?*

3. *Under what conditions of modal status for P will "ΔP" express a necessary truth? Under what conditions, a necessary falsity?*

4. *Under what conditions of modal status for P will "$\sim \Box P$" express a necessary truth? Under what conditions, a necessary falsity?*

5. *Under what conditions of modal status for P will "$\Box \sim P$" express a necessary truth? Under what conditions, a necessary falsity?*

5. THE OPERATOR "IT IS CONTINGENTLY TRUE THAT"

In chapter 1 we introduced the concept of modal status and allowed various predicates to count as attributions of modal status, viz., "is possible", "is impossible", "is necessarily true", "is necessarily false", "is noncontingent", and "is contingent". However, we did not allow the predicates "is contingently true" and "is contingently false" to be so counted (p.13, fn. 10). We now have the conceptual techniques in hand to show how these two concepts differ from all the others just referred to. What, exactly, is it about the concepts of contingent truth and contingent falsity (as opposed to contingency itself) which sets them apart from the concepts of possibility, necessary truth, necessary falsity, and the like?

It is an easy matter to define operators representing the concepts of contingent truth and contingent falsity respectively in terms of the modal monadic operators already introduced. These definitions are, simply,

"⍒ P" $=_{df}$ "$\nabla P \cdot P$" [i.e., P is contingently true if and only if P is contingent and true]

"∀ P" $=_{df}$ "$\nabla P \cdot \sim P$" [i.e., P is contingently false if and only if P is contingent and false]

From a syntactical point of view there is nothing in these definitions to suggest that there is anything odd or peculiar about the concepts of contingent truth and contingent falsity. But when we come to examine the possible-worlds explication of these concepts the peculiarity emerges.

Let us ask, "What is the modal status of a proposition expressed by a sentence of the form "⍒ P" or "∀ P"? (We will here examine only the first of these two cases. The conclusions we reach in the one will apply equally to the other.) Concentrating our attention on "⍒ P", there are three cases to consider:

1. The modal status of ⍒ P in the case where P is contingent;
2. The modal status of ⍒ P in the case where P is necessarily true; and
3. The modal status of ⍒ P in the case where P is necessarily false.

Case 1: What is the modal status of ⍒ P in the case where P is contingent?

As before, we begin by dividing the set of all possible worlds into two mutually exclusive and jointly exhaustive subsets, W_t and W_f, those possible worlds in which P is true and those possible worlds in which P is false. Arbitrarily pick any world in W_t. Let us call that world "W_{t1}", (see figure *(6.c)*). What is the truth-value of $\bar{\nabla}$ P in W_{t1}? $\bar{\nabla}$ P will be true in W_{t1}, for $\bar{\nabla}$ P in W_{t1} asserts that P is true in W_{t1} and is false in some other possible world. Since both these conjuncts are true in W_{t1}, $\bar{\nabla}$ P is true in W_{t1}. But if $\bar{\nabla}$ P is true in W_{t1}, it is true throughout W_t. This leaves the possible worlds in W_f to be examined. What is the truth-value of $\bar{\nabla}$ P in W_f? Choose any arbitrary member of W_f. Call it "W_{f1}" (see figure *(6.d)*). What is the truth-value of $\bar{\nabla}$ P in W_{f1}? $\bar{\nabla}$ P will be false in W_{f1}, for $\bar{\nabla}$ P in W_{f1} asserts that P is true in W_{f1} and is false in some other possible world. (We do not know whether this second conjunct is true or false, that is, whether there is any other possible world besides W_{f1} in which P is false — perhaps there is just *one* possible world, W_{f1}, in which P is false. But, luckily, we do not have to pursue this question or worry about it. For we can confidently assert that irrespective of the truth or falsity of the second conjunct just mentioned, the first is determinately false.) The first conjunct asserts that P is true in W_{f1}, and this we know to be false since W_{f1} is a member of the set W_f, and P is false in every possible world in W_f. Therefore we may conclude that $\bar{\nabla}$ P is false in W_{f1} and in every other member (if there are any) of W_f. At this point we have shown that $\bar{\nabla}$ P is true throughout W_t and is false throughout W_f. It follows immediately, then, that in the case where P is contingent, $\bar{\nabla}$ P is also contingent; or in symbols,

$$\nabla P \rightarrow \nabla \bar{\nabla} P.$$

Case 2: What is the modal status of $\bar{\nabla}$ P in the case where P is necessarily true?

See figure *(6.e)*. Arbitrarily pick any possible world whatever. Let us call that world "W". What is the truth-value of $\bar{\nabla}$ P in W? Clearly $\bar{\nabla}$ P has the truth-value, falsity, in W. For, in W, the proposition $\bar{\nabla}$ P asserts that P is true in W and is false in some other world. But P is true in every possible world whatever and is false in none. Therefore any proposition which asserts that P is false in some possible world is false. Therefore $\bar{\nabla}$ P is false in W. But if $\bar{\nabla}$ P is false in W, it is false in every other possible world as well. Hence in the case where P is necessarily true, $\bar{\nabla}$ P is necessarily false, and is, ipso facto, noncontingent. In symbols, we have

$$\Box P \rightarrow \Delta \bar{\nabla} P.$$

Case 3: What is the modal status of $\bar{\nabla}$ P in the case where P is necessarily false?

See figure *(6.f)*. Arbitrarily pick any possible world, W. $\bar{\nabla}$ P is false in W. For, in W, $\bar{\nabla}$ P asserts that P is true in W and is false in some other possible world. But P is false in every possible world including W. Therefore $\bar{\nabla}$ P is false in W. But if $\bar{\nabla}$ P is false in W, it is false in every possible world. Therefore in the case where P is necessarily false, $\bar{\nabla}$ P is necessarily false, and is, ipso facto, noncontingent. In symbols, we have

$$\Box \sim P \rightarrow \Delta \bar{\nabla} P.$$

> *Conclusion:* Unlike ascriptions of necessary truth, necessary falsehood, possibility, impossibility, contingency, and noncontingency, which always yield propositions which are *noncontingent*, ascriptions of contingent truth (and contingent falsity) in some instances (viz., those in which the proposition being referred to is itself contingent) will yield propositions which are *contingent*. (See Case 1, this section.)

This difference between ascriptions of contingent truth and contingent falsity, on the one hand, and ascriptions of necessary truth, necessary falsity, possibility, impossibility, contingency, and noncontingency, on the other, is of the utmost importance for the science of logic.

To the extent that logic is an a priori science, and to the extent that an a priori science is incapable of gaining for us the truth-values of contingent propositions,[2] to that extent the truth-values of propositions attributing contingent truth (or contingent falsity) to contingent propositions will be unattainable within the science of logic.

When, earlier in this book, we gave various examples of contingently true propositions, e.g., *(3.10)*, the proposition that Krakatoa Island was annihilated by a volcanic eruption in 1883, we were not (nor did we claim to be) operating strictly within the methodology of the science of logic. For the property attributed to the proposition, viz., the property of being contingently true, is an *accidental* one (see the next section), and the determination that something has an accidental property lies outside the capabilities of the ratiocinative methodology of logic.

Such is not the case, however, when we attribute necessary truth, necessary falsity, possibility, noncontingency, or contingency to a proposition. Ascriptions of these properties, as we showed in the previous section, are always noncontingent. Thus when we attribute any of these latter properties to a proposition we can hope to determine, through the application of the a priori methodology of logic, whether they truly hold or not.

We took some pains to argue in chapter 1 that there are not two kinds of truth, contingent truth and noncontingent truth. There is but one kind of truth. And one should not be tempted to try to make the point of the present section by saying that logic is concerned with noncontingent truth and falsity but not with contingent truth and falsity. Rather we should prefer to put the point this way: Logic is concerned with contingency and noncontingency, and in the latter case, but not in the former, also with truth and falsity. Within Logic one can aspire to divide the class of propositions into three mutually exclusive and jointly exhaustive categories: the necessarily true; the necessarily false; and the contingent. Any attempt to divide further the latter category into true and false propositions, and then to determine which proposition resides in which subclass takes one outside the ratiocinative limits of Logic.

6. ESSENTIAL PROPERTIES OF RELATIONS

When we first introduced the distinction between *items* and *attributes* (chapter 1, p. 7) we said that an item was anything to which reference could be made, while an attribute is anything that can be ascribed to an item. Now it is clear that attributes can be referred to and that when we do refer to them we are regarding them as items to which still further attributes may be ascribed. For instance, we refer to the relation (two-place attribute) of implication when we say of it that it holds between propositions; we then treat the relation of implication as an item of which something can be said. And when we say of implication that it is a relation between propositions we are ascribing a property (one-place attribute) to it.

Some of the properties of relations are of little general significance. It is of little general significance, for instance, that the relation of *being older than* has the property of holding between the Tower of London and the Eiffel Tower. This, we want to say, is a purely 'accidental' feature of that relation insofar as it is not necessary that the relation have this property.[3]

Other properties of relations, however, *are* of general significance. For instance, it matters a great deal to our understanding the relation of *being older than* that this relation has the property of being

2. Although we have argued in chapter 3, section 6, that there probably are no contingent propositions knowable a priori, we are here leaving the question open in order to accommodate the views of such philosophers as Kant and Kripke.

3. **F** is said to be an accidental property of an item a if and only if in some possible world in which a exists (or has instances), a has the property **F**, and in some (other) possible world in which a exists, a does not have the property **F**.

asymmetrical, i.e., the property such that if *any* item (the Tower of London, the Sphinx, or anything else) stands in the relation of *being older than* to any other item (the Eiffel Tower, the Premier of British Columbia, or whatnot), then that other item does *not* stand in the same relation to it. Anyone who failed to understand that the relation of *being older than* has this property simply would not understand that relation. It is an 'essential' property of that relation insofar as it is a property which that relation cannot fail to have.[4]

It will help us a great deal in our understanding of relations quite generally, and in our understanding of modal relations in particular, if we get clear about some of the essential, noncontingent, properties that relations have. We shall consider just three sets of such properties of relations.

First, any relation whatever must be either *symmetrical* or *asymmetrical* or *nonsymmetrical.*

A relation, **R**, has the property of *symmetry* if and only if when *any* item *a* bears that relation to *any* item *b*, it follows that item *b* bears that relation to item *a*. The relation of *being true in the same possible world as* is an example of a symmetrical relation. For given any *a* and *b*, it is necessarily true that if *a* is true in the same possible world as *b* then *b* is true in the same possible world as *a*.

A relation, R, has the property of *asymmetry* if and only if when any item *a* bears that relation to any item *b,* it follows that *b* does *not* bear that relation to *a*. The relation of *being older than* is an example of an asymmetrical relation. For given any *a* and *b,* it is necessarily true that if *a* is older than *b,* then it is false that *b* is older than *a*.

A relation, R, has the property of *nonsymmetry* if and only if it is neither symmetrical nor asymmetrical. The relation of *being in love with* is nonsymmetrical. From a proposition which asserts that some item *a* is in love with some item *b*, it neither follows (although it may be true) that *b* is in love with *a,* nor does it follow that *b* is not in love with *a*.

Secondly, any relation whatever must be either *transitive* or *intransitive* or *nontransitive.*

A relation, R, has the property of *transitivity* if and only if when any item *a* bears that relation to any item *b* and *b* bears that relation to any item *c,* it follows that *a* bears that relation to *c*. The relation of *having the same weight as* is an example of a transitive relation. For given any *a, b,* and *c,* it is necessarily true that if *a* has the same weight as *b* and *b* has the same weight as *c,* then *a* has the same weight as *c.*

A relation, R, has the property of *intransitivity,* if and only if when any item *a* bears that relation to any item *b* and *b* bears that relation to any item *c,* it follows that *a* does *not* bear that relation to *c*. The relation of *being twice as heavy as* is an example of an intransitive relation. For given any *a, b,* and *c,* it is necessarily true that if *a* is twice as heavy as *b* and *b* is twice as heavy as *c,* then *a* is not twice as heavy as *c.*

A relation, R, has the property of *nontransitivity* if and only if it is neither transitive nor intransitive. The relation of *being a lover of* is an example of a nontransitive relation. From the propositions which assert that some item *a* is a lover of some item *b* and that *b* is a lover of some item *c,* it neither *follows* (although it may be true) that *a* is a lover of *c,* nor does it *follow* that *a* is not a lover of *c.*

4. **F** is said to be an essential property of an item *a* if and only if in every possible world in which *a* exists (or has instances), *a* has the property **F**.

Finally, any relation whatever must be either *reflexive* or *irreflexive* or *nonreflexive*.

A relation R, has the property of *reflexivity* if and only if when any item *a* bears that relation to any other item whatever, it follows that *a* bears that relation to itself. The relation of *being a graduate of the same university as* is reflexive. For given any item *a*, it is necessarily true that if *a* is a graduate of the same university as some other item, then *a* is a graduate of the same university as *a*.

A relation, R, has the property of *irreflexivity* if and only if it is impossible that anything should bear that relation to itself. The relation of *being better qualified than* is irreflexive. For given any item *a*, it is necessarily true that *a* is not better qualified than *a*.

A relation, R, has the property of *nonreflexivity* if and only if it is neither reflexive nor irreflexive. The relation of *being proud of* is a nonreflexive relation. From the proposition that *a* is proud of something or someone it does not *follow* (although it may be true) that *a* has self-pride and it does not *follow* that *a* lacks self-pride.[5]

In the light of this classificatory scheme for talking about the essential properties of relations, let us now consider the essential properties of each of the four modal relations we first singled out for attention. In each case we can determine what these properties are by attending once more to the way these relations have been defined.

We can easily prove that *consistency* is symmetrical, nontransitive, and reflexive, if we recall that a proposition, P, is consistent with a proposition, Q, just when there exists at least one possible world in which both are true, i.e., a possible world in which P is true and Q is true. But any possible world in which P is true and Q is true is also a possible world in which Q is true and P is true. Hence if P is consistent with Q, Q must also be consistent with P. That is to say, consistency is symmetrical. Suppose, now, not only that P is consistent with Q but further that Q is consistent with a proposition R. Then not only are there some possible worlds in which both P and Q are true, but also there are some possible worlds in which both Q and R are true. But must the set of possible worlds in which P and Q are true intersect with the set of possible worlds in which Q and R are true? Clearly we have no warrant for concluding either that they do intersect or that they do not. Hence from the suppositions that P is consistent with Q and that Q is consistent with R it does not *follow* that P is consistent with R. Nor does it *follow* that P is not consistent with R. Consistency, then, is a nontransitive relation. Suppose, finally, that P is consistent with at least one other proposition. Then there will exist at least one possible world in which P is true. But any possible world in which P is true will be a possible world in which both P and P itself will be true. Hence, if P is consistent with any other proposition, P is consistent with itself. Consistency, then, is a reflexive relation.[6]

Inconsistency is symmetrical, nontransitive and nonreflexive. If a proposition, P, is inconsistent with a proposition, Q, then not only is there no possible world in which both P and Q are true but also there is no possible world in which both Q and P are true. Hence if P is inconsistent with Q, Q is also

5. A relation, R, is sometimes said to be *totally reflexive* if and only if it is a relation which *every* item must bear to itself. An example is the relation *being identical with*. Likewise a relation, R, may be said to be *totally irreflexive* if and only if it is a relation which *nothing* can bear to itself. An example is the relation of *being non-identical with*. Plainly a relation which is not either totally reflexive or totally irreflexive will be *totally nonreflexive*. Most of the relations which come readily to mind have this latter property.

6. Note, however, that consistency is *not* a *totally* reflexive relation. As we have seen, if a proposition is necessarily false it is not consistent with itself but is self-inconsistent. Moreover, consistency, as we have defined it, is a relation which holds only between items which have a truth-value. Hence, unlike the relation of identity, it is *not* a relation which everything has to itself.

inconsistent with P. That is to say, inconsistency, like consistency, is a symmetrical relation. And, like the relation of consistency, the relation of inconsistency is nontransitive. Suppose not only that P is inconsistent with Q but also that Q is inconsistent with R. Then not only are there no possible worlds in which both P and Q are true but also there are no possible worlds in which both Q and R are true. Does this mean that there are no possible worlds in which both P and R are true? Does it mean that there *are* some possible worlds in which both P and R are true? Neither follows. Hence inconsistency is nontransitive. Where the relation of inconsistency differs from the relation of consistency is in respect of the property of reflexivity. Consistency, we saw, is reflexive. Inconsistency is not. Suppose a proposition, P, is inconsistent with some other proposition, Q. Then it does not follow (although it may be true) that P is inconsistent with itself (i.e., is self-inconsistent) nor does it follow (although it may be true) that P is not inconsistent with itself (i.e., is self-consistent). Inconsistency, then, is neither reflexive nor irreflexive, but nonreflexive.

Implication is nonsymmetrical, transitive, and reflexive. If a proposition, P, implies a proposition, Q, then there are no possible worlds in which P is true and Q is false. Does this mean that there are no possible worlds in which Q is true and P is false? Does it mean that there *are* some possible worlds in which Q is true and P is false? Neither follows. Hence implication is nonsymmetrical. Suppose, now, that P implies Q and that Q implies R. Then in any possible world in which P is true, Q is true; and likewise, in any possible world in which Q is true, R is true. But this means that in any possible world in which P is true, R is also true. Hence implication is transitive. (See Exercise on p. 218.) Further, implication is reflexive. If P implies Q then not only are there no possible worlds in which P is true and Q is false, but also there are no possible worlds in which P is true and P is false (any world in which P is true and P is false is an impossible world). That is, any proposition, P, implies itself. Hence implication is reflexive.

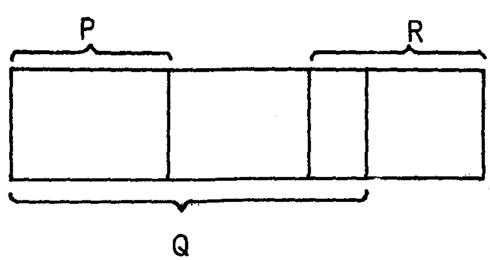
Finally, the relation of *equivalence* is symmetrical, transitive, and reflexive. If a proposition, P, is equivalent to a proposition, Q, then since P and Q are true in precisely the same possible worlds, Q is equivalent to P. Suppose, now, that P is equivalent to Q and Q is equivalent to R. Then not only are P and Q true in precisely the same possible worlds but also Q and R are true in precisely the same possible worlds. Hence P and R are true in precisely the same possible worlds. Equivalence, then, like implication, is transitive. Furthermore, it is reflexive. If P is true in precisely the same possible worlds as Q then P is also true in precisely the same possible worlds as itself. That is, any proposition, P, is equivalent to itself. Hence equivalence is reflexive.

EXERCISES

Part A

1. *Draw a worlds-diagram for three propositions, P, Q, and R, such that P is consistent with Q, Q is consistent with R, and P is inconsistent with R.*

Example:



This example is just one among several possible answers. Find another correct answer to this question.

2. *Draw a worlds-diagram for three propositions, P, Q, and R, such that P is consistent with Q, Q is consistent with R, and P is consistent with R.*

3. *Draw a worlds-diagram for three propositions, P, Q, and R, such that P is inconsistent with Q, Q is inconsistent with R, and P is inconsistent with R.*

4. *Draw a worlds-diagram for three propositions, P, Q, and R, such that P is inconsistent with Q, Q is inconsistent with R, and P is consistent with R.*

5. *See figure* (1.i) *(p. 51). (a) Which worlds-diagrams represent cases in which P stands to Q in a symmetrical relation? (b) Which are cases of an asymmetrical relation? (c) Which are cases of a nonsymmetrical relation? (d) Which are cases of a reflexive relation? (e) Which are cases of an irreflexive relation? (f) Which are cases of a nonreflexive relation?*

6. *For each of the relations below, tell whether it is (a) symmetrical, asymmetrical, or nonsymmetrical; (b) transitive, intransitive, or nontransitive; (c) reflexive, irreflexive, or nonreflexive.*

   i. *extols the virtues of*

   ii. *is not the same age as*

   iii. *is the same age as*

   iv. *is heavier than*

   v. *is twice as heavy as*

7. *Suppose that Adams employs Brown, that Brown employs Carter, and that Adams also employs Carter. May we say, then, that in this instance the relation of* employs *is a transitive one? Explain your answer.*

Part B

(The following three questions are more difficult than those in Part A.)

8. *It is possible to define a vast number of different dyadic modal relations in terms of the fifteen worlds-diagrams of figure* (1.i). *Suppose we were to single out one among this vast number, let us say, the modal relation which we will arbitrarily name "R#": the relation, R#, holds between two propositions, P and Q, if and only if P and Q are related to one another as depicted in worlds-diagram 1, worlds-diagram 2, worlds-diagram 3, or worlds-diagram 4. What are the essential properties of the relation R#?*

9. *What are the essential properties of the relation R!, where "R!" is defined as that relation which holds between any two propositions, P and Q, when P and Q are related as depicted in worlds-diagram 9 or worlds-diagram 10?*

10. *What are the essential properties of the relation R+, where "R+" is defined as that relation which holds between any two propositions, P and Q, when P and Q are related as depicted in worlds-diagram 5 or worlds-diagram 6?*

Part C

On the definitions given here of "transitivity" and "intransitivity", it turns out that some relations are both transitive and intransitive, i.e., the classificatory scheme, transitive/intransitive/nontransitive, while exhaustive of the class of all dyadic relations, is not exclusive.

For example, the relation depicted in worlds-diagram 2 (see figure (1.i)*) is both transitive and intransitive. This is so because it is logically impossible that there should be three propositions such that the first stands in just this relation to the second and the second in this same relation to the third. But since this is so, the antecedent conditions of the definitions of both "transitivity" and "intransitivity" are unsatisfiable for any propositions whatever, and thus — in the case of this relation — the two definitions are themselves (vacuously) satisfied. (I.e., any conditional proposition with a necessarily false antecedent is itself necessarily true.)*

11. *Find all the worlds-diagrams in figure* (1.i) *which depict relations which are both transitive and intransitive.*

12. *Consider the relation R$, where "R$" is defined as that relation which obtains between any two integers, x and y, when x is twice y, and y is even. Is R$ transitive, intransitive, or nontransitive?*
13. *Let R% be that relation which holds between any two integers, x and y, when x is twice y, and y is odd. Is R% transitive, intransitive, or nontransitive?*

14. *Is the classificatory scheme, reflexive/irreflexive, exclusive?*

15. *Is the classificatory scheme, symmetric/asymmetric, exclusive?*

Part D (discussion questions)

In trying to ascertain the essential properties of relations we must take care not to be conceptually myopic. We must be sure to consider possible worlds other that the actual one, worlds in which natural laws and commonplace events are radically different from those in the actual world. In the actual world, so far as we can tell, travel into the past is physically impossible. But to answer the question whether the relation, for example, being the father of, *is intransitive or not, it is insufficient to consider only the actual world. If in some possible worlds, time travel into the past occurs — as described by Heinlein — then in some of these worlds we will have instances in which a person goes back in time and fathers himself or his father. If we are to admit the existence of such possible worlds, then it follows that the relation of* being the father of *is not, as we might first think, intransitive, but is, we see after more thought about the matter, nontransitive.*

If we assume that Heinlein-type worlds in which travel into the past occurs are possible (See chapter 1, section 1), what, then, would we want to say are the essential properties of the following relations?

- *i.* *is the mother of*
- *ii.* *is an ancestor of*
- *iii.* *was born before*

If a person should father himself and then wait around while the child he has fathered grows up, what then becomes of the often heard claim that one person cannot be in two places at the same time?

7. TWO CASE STUDIES IN MODAL RELATIONS: A Light-hearted Interlude

Case study 1: The pragmatics of telling the truth

There is a well known saying which goes, "It is easier to tell the truth than to lie." But in what sense of "easier" is it easier to tell the truth than to lie? Some persons find it psychologically or morally very difficult to lie, and when they try to do so are very unconvincing. Other persons can lie blithely and yet appear sincere. From the *psychological* viewpoint, it is simply false that all persons find it difficult to lie. But there is another sense in which lying can be said to be "difficult", and in this sense lying is difficult for *everyone,* saint and sinner alike.

Lying is *logically* difficult. To tell the truth a person need only report the facts; the facts are always consistent. Of course a person may falter in his recollection of them or in his reporting of them, but if he tries to report them honestly he stands a greater chance of relating a consistent story than if he tries to lie. If a person succeeds in relating the facts as they occurred, then consistency is assured; in a metaphorical sense we can say that the facts themselves look after the matter of consistency. But when a person sets out to lie, then his task is very much more difficult. For not only must he bear in mind what actually happened, he must also bear in mind what he has said falsely about those matters, and must try to preserve consistency in everything he says. But to look after the matter of consistency he will need a fair amount of logical prowess, especially if his story is long. The difficulty does not increase linearly with increasing numbers of propositions: it grows, as the mathematician would say, exponentially.

Suppose we wish to check an arbitrary set of propositions for inconsistency. How might we go about it? We would probably begin with the easiest case: checking each individual proposition in the set to see whether it is self-inconsistent or not. Failing to find any self-inconsistent propositions, we would then proceed to the next easiest case, that of searching for inconsistency among all possible pairs of propositions in the set. If we fail to find inconsistency among the pairs of propositions in the set, we would then proceed to all possible triples; and should we happen not to detect inconsistency among the triples, we would pass on to the foursomes, etc. In general, if there are n propositions in a set, then there are m distinct non-empty subsets constructible on that set, where m is given by the formula:

$$m = 2^n - 1$$

Thus in the case where there is one proposition in a set ($n = 1$), the number of distinct non-empty subsets is 1; for a set of two propositions, 3; for a set of three propositions, 7; for a set of four propositions, 15; for a set of five propositions, 31; for a set of six propositions, 63, etc. For a set of only ten propositions, which is a fairly short story — far, far less than one would be called upon to relate in, for example, a typical courtroom encounter — there are no less than 1023 distinct non-empty subsets. And for a still small set of twenty propositions, the number of distinct non-empty subsets jumps to a staggering 1,048,575.

It must be pointed out however, that the person who is telling the story has a somewhat easier task in looking after consistency than does the person hearing the story who is looking for inconsistency. After all, a person telling a story in which he deliberately lies, presumably knows which of his own propositions are true and which false. In order for the teller of the story to ensure that his story is consistent, he need only check for consistency among those subsets which include at least one false proposition. All those other subsets which consist entirely of true propositions he knows to be consistent and he can safely disregard them.

What is the measure of this difference between the difficulty of the tasks of the speaker and of the listener? The speaker is in a slightly more favorable position, but by how much? Let's try an example. Suppose a person were to assert twenty propositions. We already know that there are 1,048,575 non-empty subsets constructible on this set. Also suppose that just *one* of the twenty propositions asserted is false, and of course, that that one is known to be false by the speaker. How many subsets will the speaker have to check? Our naive intuitions tell us that this false proposition will occur in only one-twentieth of all the subsets. But our naive intuitions are wildly wrong in this regard. For the case where there is only one false proposition in a set, the number of subsets which contain that false proposition is always *at least half* the total number of subsets. This quite unexpected result can be made plausible by examining a short example. Suppose we have four propositions, A, B, C, and D, one of which, namely C, is false. In how many subsets does C occur? We list all the non-empty subsets. By the formula above we know that there are exactly 15 distinct subsets:

1. A	5. AB	11. ABC	15. ABCD
2. B	6. AC	12. ABD	
3. C	7. AD	13. ACD	
4. D	8. BC	14. BCD	
	9. BD		
	10. CD		

Proposition C, the single false proposition, occurs in no fewer than eight of these subsets, viz., nos. 3, 6, 8, 10, 11, 13, 14, and 15.

The generalized formula is given as follows, where "q" is the number of subsets which contain one or more false propositions, "f" is the number of false propositions in the set, and "n", as above, is the number of propositions in the set:

$$q = 2^n - 2^{(n-f)}$$

Thus in the case where a liar asserts twenty propositions only one of which is false, he is presented with the task of checking 524,288 subsets for consistency. And if two of the twenty propositions are false the number of subsets containing at least one false proposition, and hence possible sources of inconsistency which would reveal that his story was not entirely true, would jump to 786,432, a number not remarkably smaller than the number of subsets (1,048,575) which his listener would theoretically have to check.

Small wonder, then, that we say that it is difficult to lie. And this fact explains, in part, the wisdom of judicial procedures in which witnesses can be cross-examined. While it is possible for a witness to prepare beforehand a false but consistent story, it is difficult to add to that story or to elaborate it in a short time without falling into inconsistency. Truth-tellers do not have this worry: they merely have to relate the facts and their stories will be consistent. Thus to a certain extent, logical principles, not only moral ones, underpin our judicial system. Indeed, if lying were logically as simple as telling the truth, our legal practice of cross-examination probably would not work at all.

EXERCISES

1. *Consider a set of propositions consisting only of true contingent propositions. Is the set consisting of all and only the negations of those propositions also consistent? Give reasons, and, if possible, illustrations for your answer.*

2. *What difference, if any, is there between lying and not telling the truth?*

Note: The following questions are for mathematically sophisticated students only.

3. *Derive the formula for* m.

4. *Derive the formula for* q.

5. *If a set of propositions is known to contain* s *necessarily true propositions, how shall we modify the formulae for* m *and* q*?*

* * * * *

Case study 2: An invalid inference and an unwitting impossible description

In a recent book of so-called "mental exercises" the following puzzle is posed:

> It took 20 days for all of the leaves to fall from a tree. If the number of leaves that fell each day was twice that of the previous day, on which day was the tree half bare?[7]

Most persons, including the author of the book in which this puzzle appears, say that the answer is "on the nineteenth day". The author replies this way:

> If the number of leaves that fell doubled each day, the tree must have been half bare on the 19th day.[8]

This answer, in spite of its initial plausibility, has been reached by a faulty inference. The fact that each number in a series is the double of its immediate predecessor, does *not* imply that it is double the sum of all its immediate predecessors. For a tree to be half bare implies that the number of leaves remaining on it is equal to the sum of the numbers of the leaves which fell on each of the preceding days. For any series of numbers in which each number after the first is double its predecessor, the sum of all of them up to but not including the last is always *less* than the last.[9] The tree cannot be half bare on the nineteenth day: more leaves remain than the sum of all the numbers of leaves which have fallen on each day up to that point. The tree will become half bare only sometime during the last day, the twentieth. (Moreover, the tree will be *exactly* half bare at some time only if there was an even number of leaves on the tree to begin with.)

It would be easy to leave the puzzle at this point, thinking that with this repair all now is in order. All is not in order, however. The puzzle harbors a still more subtle and crippling flaw provided we take the description given of the tree *absolutely literally*. Ask yourself this question: if all the leaves fell from the tree within a twenty-day period, and if on each day the number of leaves which fell was twice

7. Alfred G. Latcha, *How Do You Figure It?: Modern Mental Exercises in Logic and Reasoning*, Cranbury, N.J., A.S. Barnes and Co., Inc., 1970, p. 19.

8. *Ibid.*, p. 53.

9. The series at issue is of this sort: n, 2n, 4n, 8n, 16n, . . . One can terminate this series at any point one likes, and one will find that the last term of the terminated series will always be greater than the *sum* of all the previous terms. Thus if one sums through all members of the series up to, but not including, the last term, one will *not* reach one-half of the total sum.

that of the previous day, how many, then, fell on the *first* day? The answer we are forced to give is: "zero". For if all the leaves fell within a twenty-day period, it follows logically that none fell during any time before that period. But if none fell any time before that twenty-day period began, then it also follows logically that none could have fallen on the day before that twenty-day period began. Let's call that day, "Day Zero"; let's call the first day of the twenty-day period, "Day One"; the second, "Day Two"; etc. Since no leaves whatever fell on Day Zero, none fell on Day One; for Day One, like every other day in the twenty-day period, is a day in which twice as many leaves fell as on the previous day. But zero leaves fell on Day Zero, and since twice zero is zero, no leaves fell on Day One. But if no leaves fell on Day One, then no leaves fell on Day Two, for we are told (that is, the description of the tree implies) that on Day Two twice as many leaves fell as on Day One, but again, twice zero is zero. Continuing this line of reasoning (that is, tracing out this line of implications), we can easily show that no leaves fell on Day Three, none on Day Four, and so on, right through and including Day Twenty. In sum, at no time during the twenty-day period did any leaves fall from the tree.

Something (to say the least) is seriously amiss. By two impeccable lines of reasoning we have shown in the first place that the tree was half bare sometime during the twentieth day and in the second place that at no time during that twenty-day period was it half bare. What precisely is wrong?

There is no flaw whatever in any of the implications we have just asserted. The description of the tree *does* imply that the tree will be (at least) half bare sometime during the twentieth day and *does* also imply that the tree will never be half bare anytime during that period. Yet, these conclusions, taken together, are impossible. It is logically impossible both that a tree should be half bare during the course of a certain day and that it should not be half bare at any time during the course of that day.

The trouble with this case lies in the original description of the tree: the description is itself logically impossible, or as we might say, logically self-inconsistent. *Just because* this description implies an impossibility, we know that it itself is impossible. It is logically impossible that there should be a tree which is both half bare and not half bare at a certain time. Yet this is the kind of tree which has been described in the statement of the puzzle. Obviously the author of the puzzle book didn't see this implication; he didn't see that the description implied two logically inconsistent propositions.

Many inconsistent descriptions are of this sort. To the untrained eye, and oftentimes to the trained one as well, the inconsistency does not stand out. And indeed it may take a very long time for the inconsistency to be revealed—if it ever is. Cases are known where inconsistency has escaped detection for many, many years. Classical probability theory invented by Pierre Simon LaPlace was inconsistent. But this inconsistency went unnoticed for seventy-five years until pointed out by Bertrand in 1889.[10] Even now, many teachers of probability theory do not know that this theory is inconsistent and still persist in teaching it in much the form that LaPlace himself stated it.

Every inconsistent set of propositions shares with the case being examined here the feature of implying a contradiction. Indeed, that a set of propositions implies a contradiction is both a necessary and sufficient condition for that set's being inconsistent.

It is easy to underestimate the grievousness of an inconsistent description. We can imagine a person following the two lines of reasoning we have gone through which lead to two different, incompatible answers to the puzzle, and then asking naively, "Well, which one is the *correct* answer?"

What are we to make of such a question? How are we to answer it?

Our reply is to reject a presupposition of the question, viz., that there *is* a correct answer to this question. Not all apparent questions have 'correct' answers. 'The' answer to the puzzle posed is no more the first (repaired) one given (viz., "on the twentieth day") than it is the latter (viz., "the tree is

10. A detailed treatment of the so-called "Bertrand Paradox" occurs in Wesley Salmon, *The Foundations of Scientific Inference*, University of Pittsburgh Press, 1967, pp. 65–68.

never half bare"). Both answers follow logically from the description of the tree; but neither is true, simply because there can be no such tree answering to the description given.[11]

Finally, before we turn our attention away from these case studies, let us glean one further point from our discussion. We have said that the original description as quoted is self-inconsistent: it is logically impossible that there should be a tree which is both half bare and not half bare at a certain time. But note carefully: the original description of the tree did not say *precisely* this. Indeed, most persons, unless they are prompted, would not see that this latter description also fits the tree as originally described. The latter, the obviously impossible description, is *implied* by what was written, but was not stated explicitly. Yet, for all that, any person who subscribes to the original description is committed to the explicitly contradictory one. We are, in a quite straightforward sense, committed to *everything* that is logically implied by what we say. This is not to say that we *know* everything that is implied by what we say, or even that we are dimly aware of these things. The point is, rather, that if we are shown that something does logically follow from what we say or believe, then we are logically committed to it also. If an explicit contradiction logically *follows* from something we've asserted, then we can be accused of having asserted a contradiction just as though we had asserted that contradiction explicitly in the first instance.

EXERCISES

1. *Finding that a set of propositions implies a contradiction suffices to show that that set is inconsistent. But failure to show that a set of propositions implies a contradiction does* not *suffice to show that that set is consistent. Why?*

2. *Repair the description of the tree in the quotation from the puzzle book so that it is consistent and so that the correct answer to the question will be, "Sometime during the twentieth day."*

3. *(This question is somewhat more difficult than question 2.) Repair the description of the tree in the quotation from the puzzle book so that it is consistent and the correct answer will be, as the author suggested, "At the end of the nineteenth day."*

4. *The following paragraph is inconsistent. Proceeding in a stepwise fashion (as we have done in the preceding discussion), validly infer from it two obviously inconsistent propositions.*

> *John is Mary's father. John has only two children, one of whom is unmarried and has never been married. Mary has no brothers. Mary is married to Simon who is an only child. Mary's son has an uncle who has borrowed money from John.*

11. Upon analysis, it turns out that these two answers are contraries of one another, and although they are inconsistent with one another, it is not the case that one is true and the other false; they are both false. The pertinent logical principle involved is the following: any proposition which ascribes a property to an impossible item is necessarily false. Clearly we can see this principle illustrated in the present case. Since there is no possible world in which a tree such as the one described exists, it follows that there is no possible world in which such a tree exists and has the property **F**, *and* it follows that there is no possible world in which such a tree exists and has the property **G**.

5. *Is the following description consistent or inconsistent?*

> *It took twenty days for all of the leaves to fall from a tree. The number of leaves which fell each day was 100 more than fell the previous day.*

8. USING WORLDS-DIAGRAMS TO ASCERTAIN THE VALIDITY OF MODALIZED FORMULAE

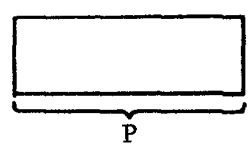
The results of section 4 — in which we proved that every propositional-variable modalized by any one of the operators, "□", "◇", "∇", or "Δ" can be instantiated to express only a noncontingent proposition — provide the opportunity to state some additional rules for the interpretation of worlds-diagrams so as to allow these diagrams to be used in intuitively appealing ways to demonstrate whether a modalized formula, either fully modalized or partially modalized, is valid, contravalid, or indeterminate.

These additional rules for the width ("W") of brackets are:

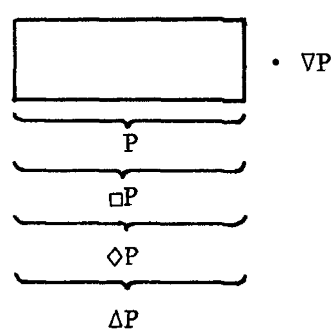
Rule WA:

Whenever a bracket for a proposition, P, spans *all* (hence the "A" in "WA") of the rectangle representing the set of all possible worlds, i.e., whenever P is necessarily true, we may add additional brackets for □P, ◇P, and ΔP each also spanning the entire rectangle. If we wish to add ∇P to the diagram, it will have to be relegated to the external point representing the impossible worlds.

From



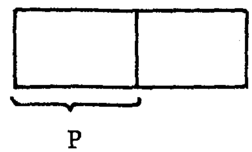
one may derive



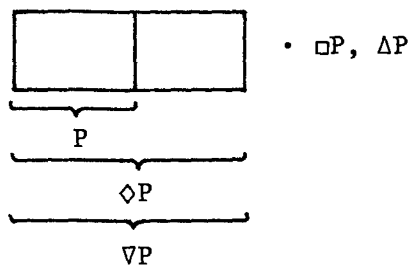
Rule WS:

Whenever a bracket for P spans only part (i.e., *some* but not all) of the rectangle, i.e., whenever P is contingent, we may add additional brackets for ◇P and ▽P each also spanning the entire rectangle. □P and △P will each have to be relegated to the external point.

From



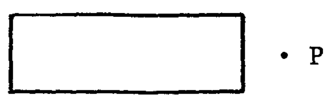
one may derive



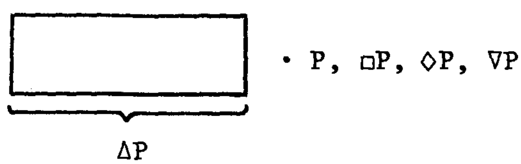
Rule WN:

Whenever P spans *none* of the rectangle representing all possible worlds, i.e., whenever P is necessarily false, we may add a bracket for △P spanning the entire rectangle. □P, ◇P, and ▽P will each have to be relegated to the external point.

From



one may derive



Let us see, now, how the addition of these rules facilitates our use of worlds-diagrams in the ascertaining of the validity of modalized formulae.

Applications

Case 1: Determine the validity of "□P ⊃ ◇P".

Since there is but one sentence-variable type instanced in the formula "□P ⊃ ◇P", we need examine only three worlds-diagrams. They are:

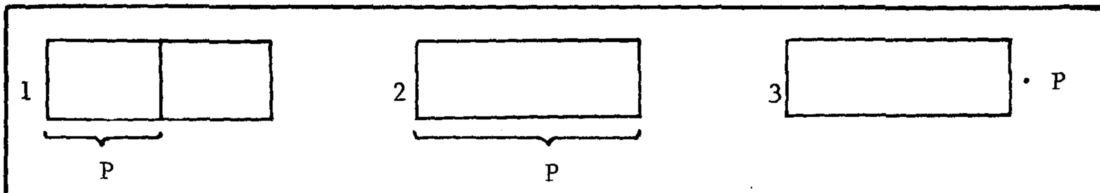


FIGURE *(6.g)*

Rule WA allows us to place a bracket for $\Box P$ spanning the entire rectangle in diagram 2. Rule WS allows us to place $\Box P$ on an external point in diagram 1, and Rule WN allows us to place $\Box P$ on an external point in diagram 3.

By Rules WA and WS we may place a bracket for $\Diamond P$ spanning the entire rectangle in diagrams 1 and 2, and by Rule WN we place $\Diamond P$ on the external point in diagram 3.

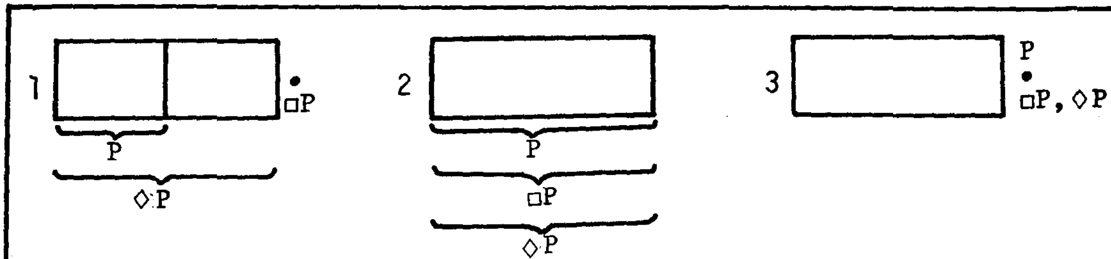


FIGURE *(6.h)*

Now we are in a position to place $\Box P \supset \Diamond P$ on our worlds-diagrams. To do this all we need do is remember the rule (from chapter 5) for the placement of material conditionals: represent a material conditional by a bracket spanning all possible worlds except those in which the antecedent of the conditional is true and the consequent false. Immediately we may write

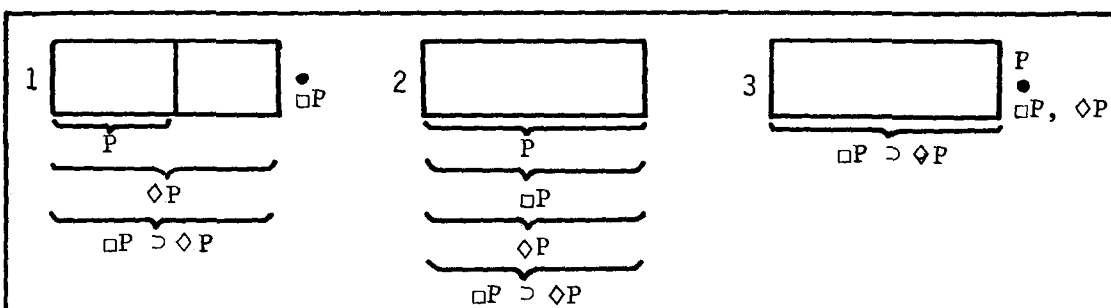


FIGURE *(6.i)*

By inspection we can see that $\Box P \supset \Diamond P$ spans all possible worlds in every possible case. Therefore "$\Box P \supset \Diamond P$" is *valid:* every possible substitution-instance of this formula expresses a necessary truth.

Case 2: Determine the validity of "◇P ⊃ □P".

In the previous example we have already placed □P and ◇P on the relevant three worlds-diagrams (see figure *(6.h)*). It remains only to add ◇P ⊃ □P.

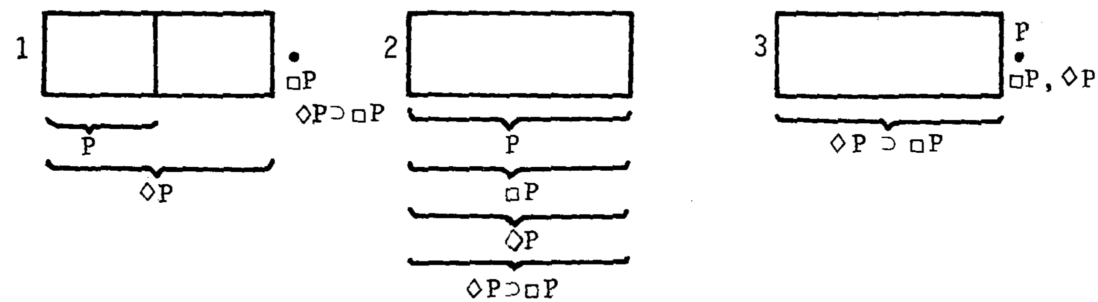


FIGURE *(6.j)*

By inspection we can see that "◇P ⊃ □P" is *not* a valid formula: some of its substitution-instances will express necessary falsehoods (see diagram 1 in figure *(6.j)*) while others will express necessary truths (see diagrams 2 and 3). It is, then, an indeterminate form. (However, as one would expect in the case of a fully modalized formula, none of its substitution-instances can express a contingency.)

Case 3: Determine whether "(□(P ⊃ Q) · ◇P) ⊃ ◇Q" is valid.

Since there are two sentence-variable types instanced in this formula, we shall have to begin by constructing the fifteen worlds-diagrams required for the examination of the modal relations obtaining between two arbitrarily selected propositions. On each of these we shall have to add a bracket depicting the possible worlds in which P ⊃ Q is true. This we have already done in the previous chapter in figure *(5.i)* (p. 265). In seven of these cases, viz., 1, 3, 4, 6, 8, 9, and 11, the bracket for P ⊃ Q spans the entire rectangle and hence, by Rule WA above, we may add a bracket for □(P ⊃ Q) which also spans the entire rectangle. In all other cases, viz., 2, 5, 7, 10, 12, 13, 14, and 15, by either Rule WS or Rule WN we place □(P ⊃ Q) on the external point.

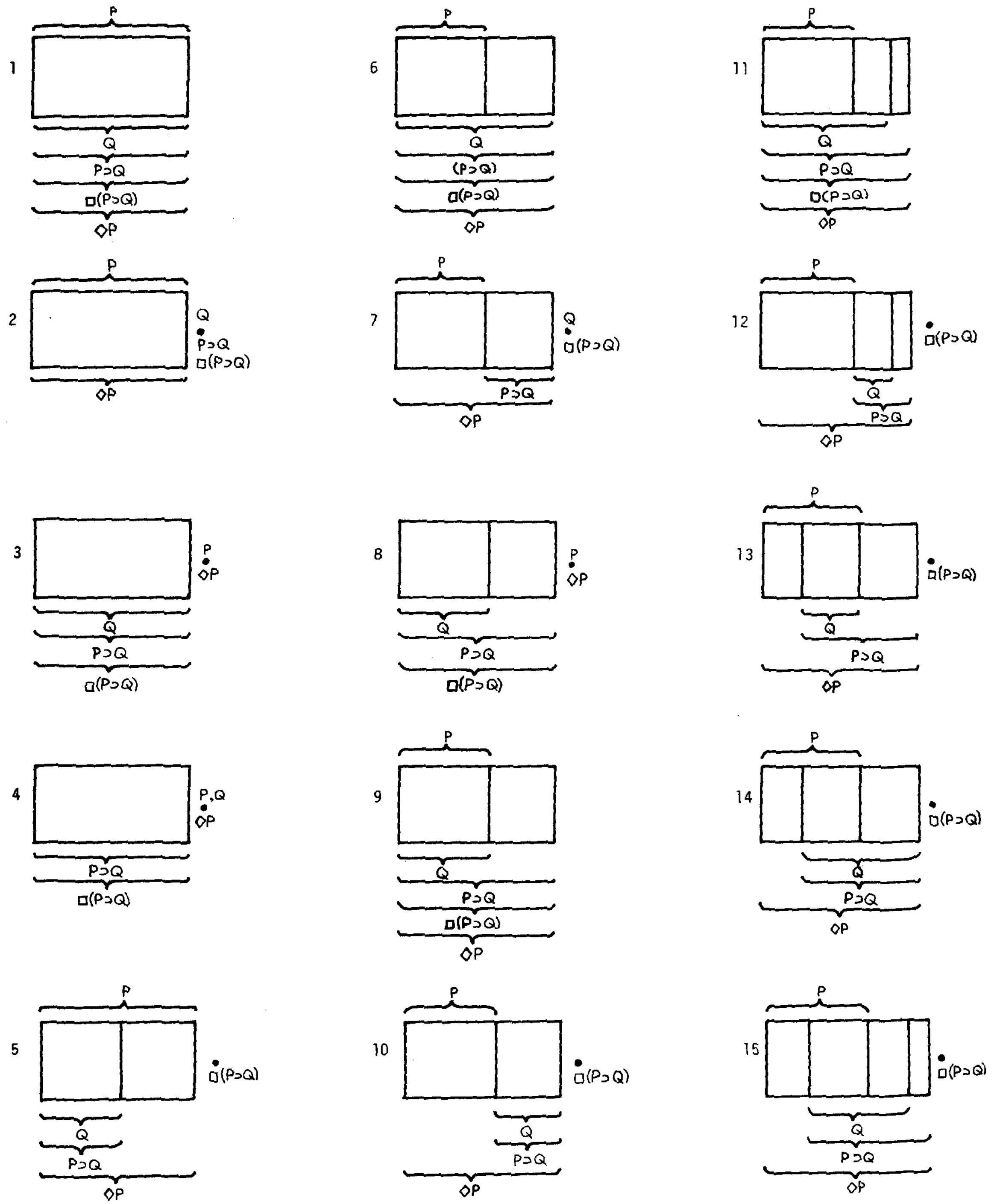
Next we add a bracket for ◇P. Rules WA and WS allow us to add brackets for ◇P spanning the entire rectangle in diagrams 1, 2, 5, 6, 7, 9, 10, 11, 13, 14, and 15. Only in diagrams 3, 4, and 8 (in accordance with Rule WN) will we place ◇P on the external point.

The placement of these first five formulae on the set of fifteen worlds-diagrams is shown in figure *(6.k)* on p. 354.

Next we are in a position to add □(P ⊃ Q) · ◇P to our diagrams. We recall from chapter 5 that the rule for placing a conjunctive proposition on a worlds-diagram is to have its bracket span just those worlds *common* to the brackets representing its conjuncts. Thus the bracket for □(P ⊃ Q) · ◇P will span the entire rectangle in cases 1, 6, 9, and 11, and will be relegated to the external point in all other cases, viz., 2, 3, 4, 5, 7, 8, 10, 12, 13, 14, and 15.

Now we add the bracket for ◇Q. By WA and WS, this bracket will span the entire rectangle in cases 1, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, and 15. In accordance with WN, it will be assigned to the external point in cases 2, 4, and 7.

Finally we are in a position to add a bracket for (□(P ⊃ Q) · ◇P) ⊃◇Q to each of our diagrams by invoking the rule for placing material conditionals on a worlds-diagram. The completed figure appears on p. 355.



1
P
Q
P⊃Q
□(P⊃Q)
◇P
2
P
Q
P⊃Q
□(P⊃Q)
◇P
3
P
◇P
Q
P⊃Q
□(P⊃Q)
4
P,Q
◇P
P⊃Q
□(P⊃Q)
5
P
□(P⊃Q)
Q
P⊃Q
◇P
6
P
Q
(P⊃Q)
□(P⊃Q)
◇P
7
P
Q
□(P⊃Q)
P⊃Q
◇P
8
P
◇P
Q
P⊃Q
□(P⊃Q)
9
P
Q
P⊃Q
□(P⊃Q)
◇P
10
P
□(P⊃Q)
Q
P⊃Q
◇P
11
P
Q
P⊃Q
□(P⊃Q)
◇P
12
P
□(P⊃Q)
Q
P⊃Q
◇P
13
P
□(P⊃Q)
Q
P⊃Q
◇P
14
P
□(P⊃Q)
Q
P⊃Q
◇P
15
P
□(P⊃Q)
Q
P⊃Q
◇P

FIGURE *(6.k)*

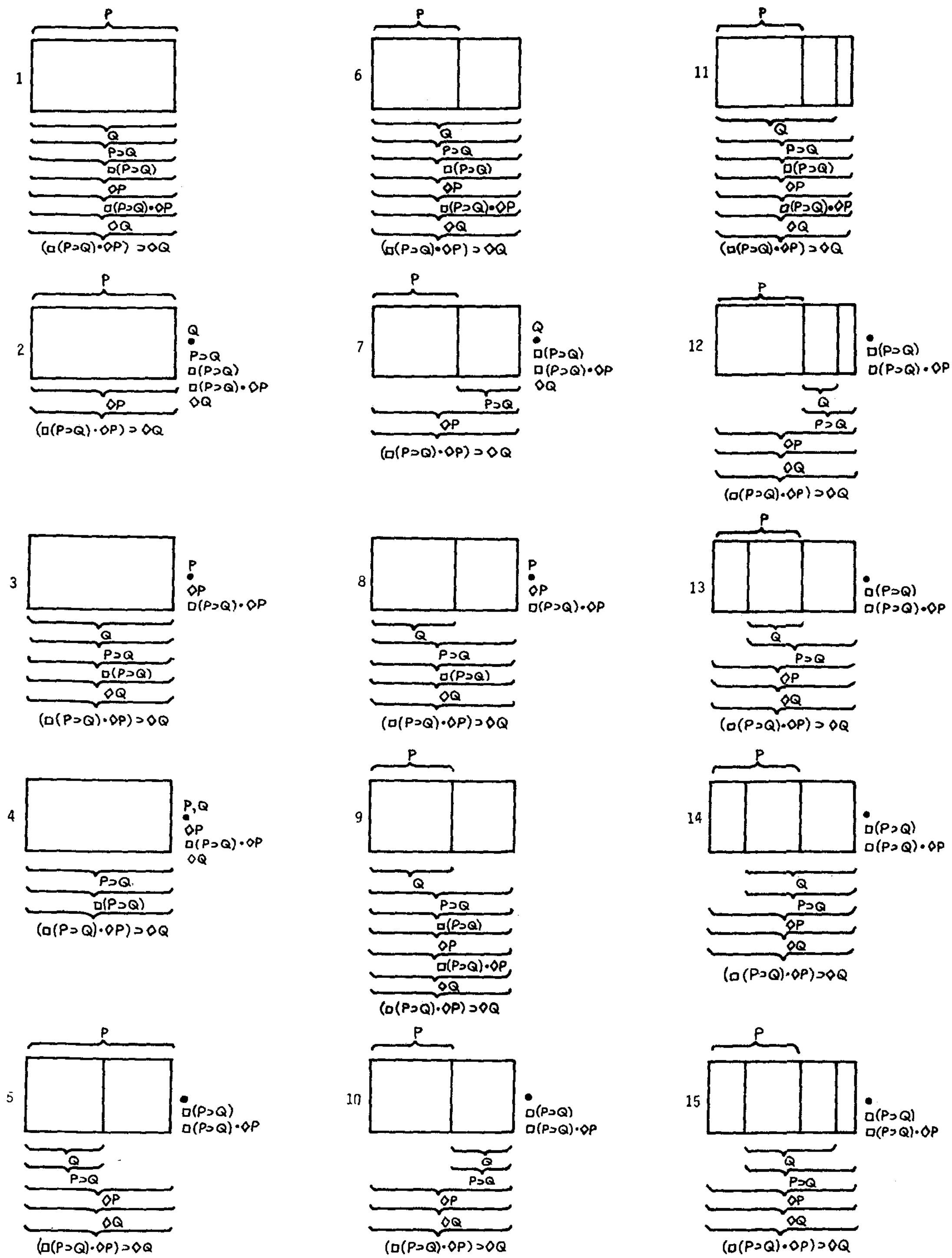


FIGURE *(6.l)*

On examining each of the 15 worlds-diagrams we find that in every case the bracket for $(\Box(P \supset Q) \cdot \Diamond P) \supset \Diamond Q$ spans the entire rectangle. This shows that $(\Box(P \supset Q) \cdot \Diamond P) \supset \Diamond Q$ is valid, i.e., every possible instantiation of it is necessarily true.

The validity of the axioms of S5.

We have spoken in chapter 4 of the modal system S5. Let us now use the methods just established to test the validity of its axioms. One axiomatization of S5 (that provided by Gödel) consists of any set of axioms of Truth-functional Propositional Logic[12] subjoined to the following three:

(A1) $\Box P \supset P$

(A2) $\Box(P \supset Q) \supset (\Box P \supset \Box Q)$

(A3) $\sim \Box P \supset \Box \sim \Box P$

It is a trivial matter (using truth-tables, for example) to demonstrate the validity (tautologousness) of any axiom-set for Truth-functional Propositional Logic. It remains only to determine the validity of (A1)–(A3).

Axiom 1: $\Box P \supset P$

Since there is only one sentence-variable type instanced in this formula, we need examine only three worlds-diagrams. It is a simple matter, invoking only the rules WA, WS, and WN, and the rule for placing material conditionals on worlds-diagrams to add brackets first for $\Box P$ and then for $\Box P \supset P$.

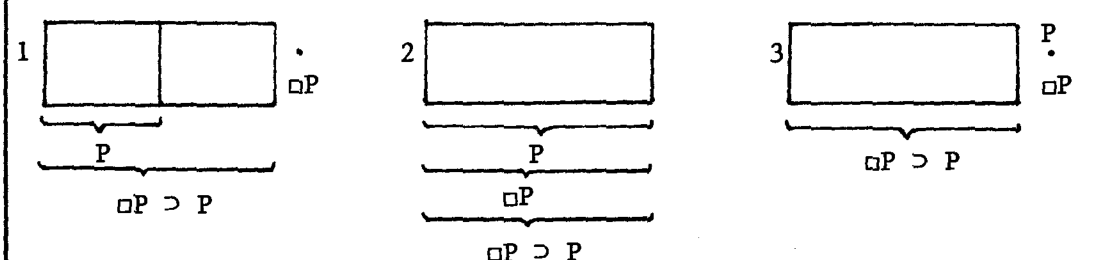


FIGURE *(6.m)*

By inspection one can see that every possible instantiation of $\Box P \supset P$ is necessarily true. Hence $\Box P \supset P$ is valid.

12. For example, the following axioms, due to Whitehead and Russell, are sufficient (along with their rules of inference) to generate every valid wff of Truth-functional Propositional Logic.

(PC1) $(P \vee P) \supset P$

(PC2) $Q \supset (P \vee Q)$

(PC3) $(P \vee Q) \supset (Q \vee P)$

(PC4) $(Q \supset R) \supset ([P \vee Q] \supset [P \vee R])$

Axiom 2: $\Box(P \supset Q) \supset (\Box P \supset \Box Q)$

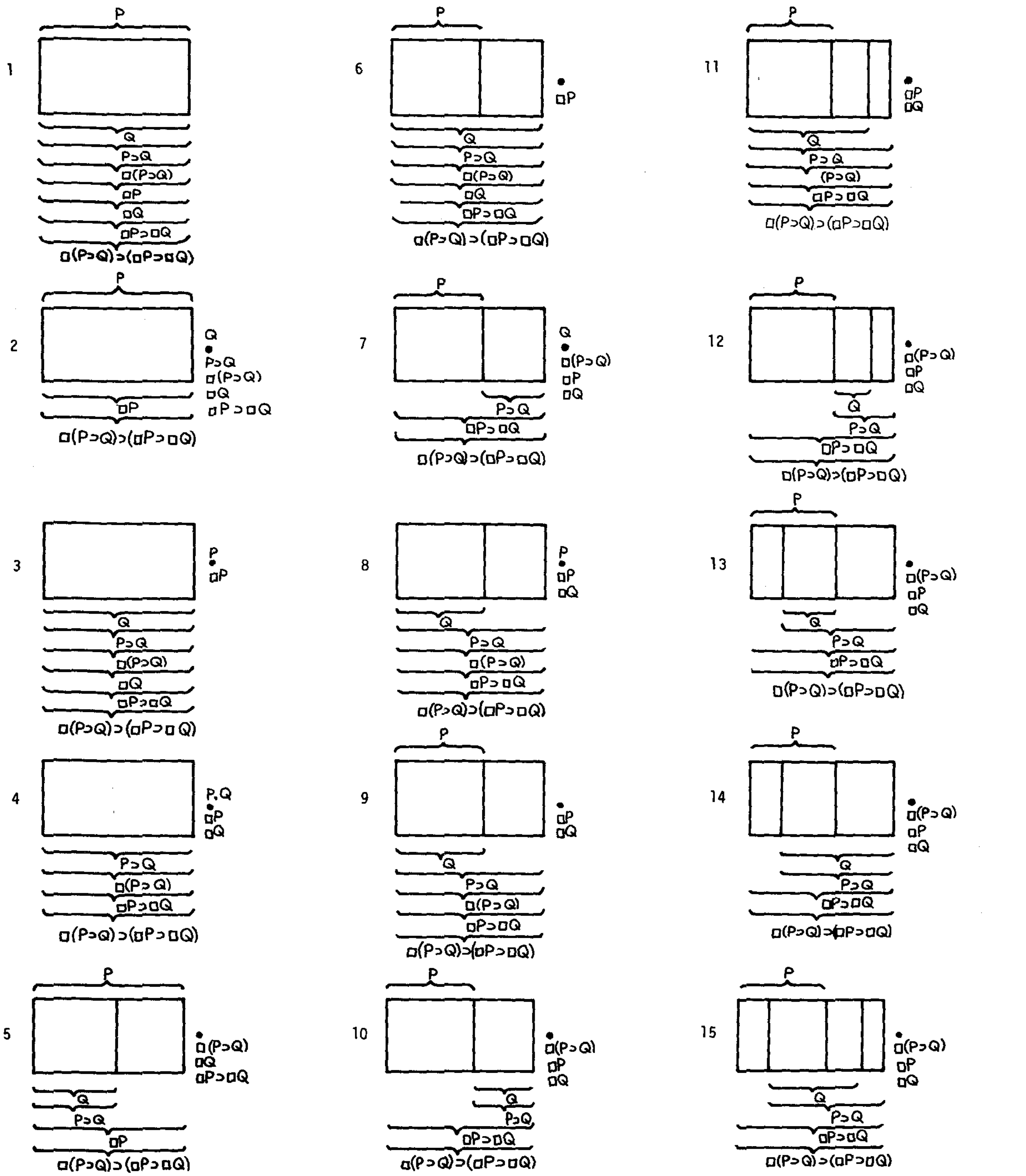


FIGURE *(6.n)*

Figure *(6.n)* reveals, as expected, that Axiom 2 is valid.

Axiom 3: $\sim\Box P \supset \Box\sim\Box P$

As with Axiom 1, only three worlds-diagrams need be considered.

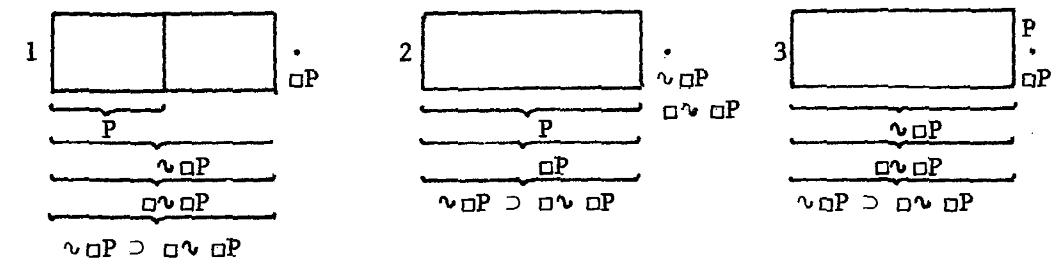


FIGURE *(6.o)*

Just as was the case in testing Axioms 1 and 2, we find that Axiom 3 is also valid.

The nonvalidity of the axiom set for S6

The modal system S6 can be obtained by subjoining a certain subset of the theses of S5 to the single axiom, $\Diamond\Diamond P$. Let us examine the validity of this axiom. Immediately we may write down:

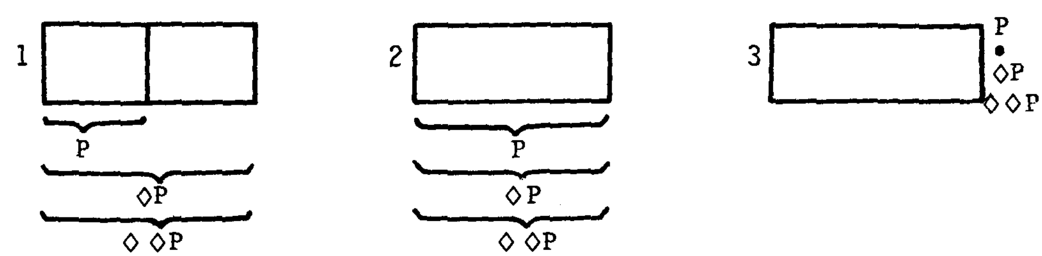


FIGURE *(6.p)*

Here we can see that the axiom $\Diamond\Diamond P$ is *not* valid on that interpretation of "□" and "◇" which interprets "□" as "it is true in all possible worlds that" and "◇" as "it is true in some possible worlds that". This is not to deny that on some *alternative* interpretation (e.g., reading "◇" as "it is possibly known by God whether", or "it is possibly believed that"), this formula may be valid. (And similar conclusions hold for the distinctive axioms of S7 and S8.)

EXERCISES

Use worlds-diagrams to determine of each of the formulae 1 through 5 whether it is valid, contravalid, or indeterminate.

If a formula contains a dyadic modal operator, first find a formally equivalent formula (using the methods of section 2) containing no modal operators other than "$\Diamond$" and/or "$\Box$".

1. $P \supset \Box P$

2. $\Box(P \supset Q) \supset (P \supset \Box Q)$ *[the so-called 'modal fallacy']*

3. $(\Diamond P \cdot \Diamond Q) \supset (P \circ Q)$

4. $\Diamond(P \cdot Q) \supset (\Diamond P \cdot \Diamond Q)$

5. $(P \supset Q) \supset (P \rightarrow Q)$

6. *Consider the S6 axiom, $\Diamond\Diamond P$. A substitution-instance of this axiom is $\Diamond\Diamond(P \cdot \sim P)$, which is the negation of $\sim\Diamond\Diamond(P \cdot \sim P)$. Use worlds-diagrams to show that this latter wff is S5-valid.*

   (Note that if $\Diamond\Diamond P$ were, contrary to fact, S5-valid, then it would be possible to derive in S5 both $\Diamond\Diamond(P \cdot \sim P)$ and its negation $\sim\Diamond\Diamond(P \cdot \sim P)$, and thus S5, contrary to fact, would be inconsistent.)

9. A SHORTCUT FORMAL METHOD FOR DETERMINING THE VALIDITY OF MODALIZED FORMULAE: Modal reductios[13]

The method of utilizing worlds-diagrams, as outlined in the previous section, is *effective*: by the mechanical application of its rules, one can determine the validity of *any* well-formed modalized formula. In this regard it is the analog of truth-table techniques in Truth-functional Propositional Logic. And like truth-table techniques, it suffers from the fault of being excessively burdensome. Indeed it is a more aggravated case. In truth-functional logic, in cases where there is only one sentence-type instanced in a formula, we require a 2-row truth-table; two sentence-types, a 4-row table; 3 sentence-types, an 8-row table; and 4 sentence-types, a 16-row table. But when we come to examine modalized formulae, the complexity explodes. For if we wish to ascertain the validity of a modalized formula instancing one sentence-type, we require 3 worlds-diagrams; 2 sentence-types, 15 worlds-diagrams; 3 sentence-types, 255 worlds-diagrams; and 4 sentence-types, 65,535. Small wonder, then, that logicians have sought other methods to ascertain the validity of modalized formulae.

One of these methods may be regarded as the modal version of the Reductio Ad Absurdum method we have already examined. Like the earlier Reductio method, it works well for some cases, allowing us

13. The general method described in this section is the product of many years' work by many persons, some heralded and some not. Among its pioneers must be numbered Beth, Hintikka, and Kripke. Our own method owes much to some unpublished notes of M.K. Rennie. Stylistic variants, closely resembling ours, are to be found in M.K. Rennie, "On Postulates for Temporal Order", in *The Monist* (July 1969) pp. 457–468, and in G.E. Hughes and M.J. Cresswell, *Introduction to Modal Logic,* London, Methuen & Co. Ltd., 1968.

to ascertain the validity of some formulae very easily and rapidly; but for some other cases it works poorly and cumbersomely, at best. Nonetheless, it is so much easier to use in certain instances than is the effective method of worlds-diagrams, that it is useful to pursue it, in spite of its shortcomings.

The strategy of Reductio methods has already been described. One makes an initial assignment to a formula and then looks to see what its consequences are: whether in any possible world that assignment leads to assigning both truth and falsity to one proposition. If it does, then the initial assignment was an impossible one.

To construct the method we need to call upon all those earlier rules we used for making assignments to the components of truth-functional sentences on the basis of assignments having been made to the compound sentences themselves, e.g., if "T" is assigned to "P · Q", then "T" should be assigned to "P" and "T" should be assigned to "Q"; if "F" is assigned to "P ⊃ Q", then "T" should be assigned to "P" and "F" to "Q"; etc.

Since all dyadic modal operators can be 'defined away' in terms of the monadic modal operators, "◇" and "□" (see section 2, this chapter), it will suffice to stipulate rules for handling formulae modalized by just these two operators.

We require rules which tell us how to make assignments to the components of modalized formulae on the basis of assignments having been made to the modalized formulae themselves. For example, suppose the formula "□P" has been assigned "T"; what, then, shall we assign to "P"? There are in all, four cases. Let us examine the appropriate rule in each case. We shall call the rules, "RA-rules", where the "RA" stands for "Reduction to Absurdity".

Rule RA1

If □P is true in W_n, then P is true in W_n and in all other possible worlds as well. Thus we assign "T" to "P" in W_n, and record the fact that this latter assignment is to persist throughout all other possible worlds we examine as well, both those previously examined and those yet to be. To show this, we write immediately below this assignment, the symbol,

$$\text{“}\underline{T}\Updownarrow\text{”}$$

The double-stroke arrow indicates that this assignment is to persist throughout *all* possible worlds. Thus RA1 may be stated this way:

$$RA1:\quad \textit{If in } W_n \textit{ we have, } \text{“}\underset{T}{\Box}P\text{”}, \textit{ then we may write, } \text{“}\underset{T}{\Box}\underset{\underline{T}\Updownarrow}{P}\text{”}.$$

Rule RA2

If □P is false in W_n, then P is false in some possible world (it need not be W_n, however.) Given just the information that □P is false in W_n, the truth-value of P is indeterminate in W_n. (This is not to say, however, that some other, additional information might not determine P's truth-value in W_n.) In sum, Rule RA2 may be stated this way:

$$RA2:\quad \textit{If in } W_n \textit{ we have, } \text{“}\underset{F}{\Box}P\text{”}, \textit{ then we may write, } \text{“}\underset{F}{\Box}\underset{\overline{\downarrow F}}{P}\text{”}.$$

The 'weak' arrow under "P" indicates that the assignment "F" is to be made in at least one possible world to be examined subsequently. Note that no assignment has been made in W_n itself to "P" and we do not assign "F" to P in a world previously examined. Nothing sanctions that, since we know *only* that P is false in *some* world.

Rule RA3

If ◇P is true in W_n, then P is true in some possible world (it need not be W_n itself.)

RA3: If in W_n we have, "◇P", then we may write, "◇P"

RA3: If in W_n we have,	"◇P"	*then we may write,* "◇P"
	T	*T*
		↓T

Rule RA4

If ◇P is false in W_n, then P is false in W_n and in all other possible worlds as well.

RA4: If in W_n we have,	"◇P"	*then we may write,* "◇P".
	F	*F* *F*⇕

Example 1: Is the formula "□(P ⊃ ◇P)" valid?

We begin by assigning "F" to this formula in possible world W_1.

	□	(P	⊃	◇P)
W_1	F			
			↓F	
	(1)		(2)	

The assignment at step (2) was made in accordance with RA2. Step (2) is as far as we can go in W_1: no further assignments are determined in W_1. But one assignment *is* determined for some *other* possible world; for we have written down " ↓F " at step (2). So let us now examine such a world. We will call it "W_2".

	□	(P	⊃	◇	P)
W_1	F				F
			↓F		
W_2		T	F	F	F⇕
		(2)	(1)	(2)	(3)

The assignments at step (2) were made in accordance with the truth-functional rule for material conditionals. The assignments at step (3) [in W_1 and in W_2] were made in accord with Rule RA4. At this point we discover an inconsistent assignment in W_2: "P" has been assigned both "T" (at step (2)) and "F" (at step (3)). Thus the initial assignment of "F" to "□(P ⊃ ◇P)" is an impossible one, and we may validly conclude that this formula is *valid*.

Example 2: Is the formula "$[(P \rightarrow Q) \cdot \Box P] \rightarrow \Box Q$" valid?

The first step must here consist of replacing the two occurrences of dyadic modal operators with monadic modal operators. This is easily done and we may rewrite the formula this way:

"□ ([□ (P ⊃ Q) · □P] ⊃ □Q)"

Just as in Example 1, not a great deal is revealed about this formula in W_1:

	□ (	[□	(P ⊃ Q)	·	□ P]	⊃	□ Q)
W_1	F					↓F	
	(1)					(2)	

We turn, then, to W_2.

	□	([□	(P	⊃	Q)	·	□	P]	⊃	□	Q)
W_1	F			T				T	↓F		
W_2		T	T	T↕	T	T	T	T↕	F	F	↓F
		(3)	(5)	(4)†	(6)	(2)	(3)	(4)†	(1)	(2)	(3)*

* in accord with Rule RA2 † in accord with Rule RA1

No inconsistent assignment occurs in W_2, nor was one necessitated in W_1 by the upward pointing arrows at step (4); but there are conditions in W_2 laid down for some subsequent world. In particular, we have not yet examined the consequences of having written " ↓F " under the last occurrence of "Q" in the formula. Let us now turn to a possible world in which Q is false:

	□	([□	(P	⊃	Q)	·	□	P]	⊃	□	Q)
W_1	F			T				T	↓F		
W_2		T	T	T↕	T	T	T	T↕	F	F	↓F
W_3			T	T	T			T			F
			(3)	(2)	(4)*			(2)			(1)

* "T" has been assigned to "(P ⊃ Q)" and to "P", hence "T" must be assigned to "Q". But "Q" has already been assigned "F" in step (1).

In W_3 an inconsistent assignment is necessitated for "Q". Hence we may validly infer that the initial assignment of "F" to the formula represents an impossible assignment, and thus may infer that the formula is *valid.*

Example 3: Is the formula "◇P ⟶ P" valid?

Again we begin by replacing the dyadic operator with a monadic one: "□(◇P ⊃ P)". The assignments in W_1 are straightforward.

	□	(◇ P	⊃	P)
W_1	F			
			↧F	

We turn, next, to W_2.

	□	(◇	P	⊃	P)
W_1	F				
				↧F	
W_2		T	F*	F	F
			↧T†		
		(2)	(3)	(1)	(2)

* required by our having assigned "F" to "P" in step (2)
† required by our having assigned "T" to "◇P" in step (2) [Rule RA3.]

At this point all assignments have been made in W_2 and no inconsistent assignments have been made. But a condition has been laid down in W_2 for some other possible world: the " ↧T " which occurs under the first occurrence of "P" requires that we examine a possible world in which "P" is assigned "T". We call that world "W_3".

	□	(◇	P	⊃	P)
W_1	F				
				↧F	
W_2		T	F	F	F
			↧T		
W_3		T	T	T	T
		(3)	(1)	(4)	(2)

In W_3 no inconsistent assignment has been necessitated. Moreover, in W_3, all earlier downward pointing arrows have been satisfied or 'discharged'. Our test is at its end and no inconsistent assignment has emerged. We may validly infer, then, that the initial assignment of "F" to the formula does not represent an impossible assignment. The formula, thus, is *not* valid. However, it remains an open question whether it is contravalid or indeterminate. To choose between these two alternatives we would have to examine the consequences of assigning "T" to the formula. If that assignment leads to a subsequent inconsistent assignment, then the formula is contravalid; if it does not lead to a subsequent inconsistent assignment, the formula is indeterminate.

Example 4: Is the formula "$(\Box P \vee \Box Q) \supset \Box(P \vee Q)$" valid?

$$\begin{array}{cccccccc} & (\Box P & \vee & \Box Q) & \supset & \Box(P & \vee & Q) \\ W_1 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & & \begin{array}{c} T \\ \\ (2) \end{array} & & \begin{array}{c} F \\ \\ (1) \end{array} & \begin{array}{c} F \\ \\ (2) \end{array} & \begin{array}{c} \\ \downarrow F \\ (3) \end{array} & \end{array}$$

In W_1 we have three choice points: having assigned "T" to "$\Box P \vee \Box Q$" we can assign "T" to "$\Box P$" and "T" to "$\Box Q$"; "T" to "$\Box P$" and "F" to "$\Box Q$"; or "F" to "$\Box P$" and "T" to "$\Box Q$". Only if *each* of these assignments leads to an inconsistent assignment in some world or other can we validly infer that the formula is valid. At this point, other methods, e.g., worlds-diagrams, seem more attractive as a means to test this particular formula.[14]

EXERCISES

Part A

Using the method of Modal Reductio, determine which of the following formulae are valid.

1. $\Box(P \vee \sim P)$
2. $\Box(P \vee Q)$
3. $\Diamond(P \cdot \sim P)$
4. $(P \supset Q) \supset (P \rightarrow Q)$
5. $(P \supset Q) \rightarrow (P \supset Q)$
6. $(P \supset Q) \rightarrow (P \rightarrow Q)$
7. $(P \rightarrow Q) \supset (P \supset Q)$
8. $(P \rightarrow Q) \supset (P \rightarrow Q)$
9. $(P \rightarrow Q) \rightarrow (P \supset Q)$
10. $(P \rightarrow Q) \rightarrow (P \rightarrow Q)$
11. $\sim P \rightarrow \sim \Diamond P$
12. $\Box P \rightarrow P$
13. $[(P \rightarrow Q) \cdot \sim \Diamond Q] \rightarrow \sim \Diamond P$
14. $[(P \rightarrow Q) \cdot \sim \Diamond P] \rightarrow \sim \Diamond Q$
15. $\sim \Diamond P \rightarrow (P \rightarrow Q)$
16. $\Box P \rightarrow (Q \rightarrow P)$

14. Natural deduction techniques for S5 (as well as for the systems T and S4) are to be found in Hughes and Cresswell, *An Introduction to Modal Logic*, pp. 331 – 334.

17. $\Box\Diamond P \rightarrow \Diamond P$

18. $\Diamond\Box P \rightarrow \Box P$

19. $\Box P \rightarrow \Box\Box P$

20. $\sim\Diamond P \rightarrow (P \phi Q)$

21. $(\Diamond P \cdot \Diamond Q) \rightarrow \Diamond(P \cdot Q)$

22. $(\Box P \cdot \Box Q) \rightarrow \Box(P \cdot Q)$

23. $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$

24. $(P \rightarrow Q) \rightarrow (P \circ Q)$

25. $\Delta\Diamond P$

Part B

26. *Determine whether the formula in Example 3 is contravalid or indeterminate.*

27. *Construct a Modal Reductio which proves that the Augmentation Principle (viz. $(P \rightarrow Q) \rightarrow [(P \cdot R) \rightarrow Q])$ cited in chapter 4, section 5, is valid.*

28. *Construct a Modal Reductio which proves that the Collapse Principle (viz., $(P \rightarrow Q) \rightarrow [(P \cdot Q) \leftrightarrow P])$ cited in chapter 4, section 5, is valid. Note that there is a two-pronged branch-point in this reductio. It will be necessary to examine both branches.*

10. THE NUMBER OF FORMALLY NON-EQUIVALENT SENTENCE-FORMS CONSTRUCTIBLE ON *N* SENTENCE-VARIABLES[15]

Two sentence-forms will be said to be formally equivalent if and only if, for any uniform substitution of constants for the variables therein, there result two sentences which express logically equivalent propositions. Sentence-forms which are not formally equivalent are said to be formally non-equivalent.[16] For example, according to these definitions, the two sentence-forms, "$P \vee \sim P$" and "$P \supset P$", are formally equivalent, while the two sentence forms, "P" and "$P \vee \sim P$", are formally non-equivalent.

The formation rules of propositional logics allow us to concatenate symbols into strings of indefinite length. We may have a wff containing as few as one symbol (e.g., "P" standing alone) or as many as a trillion or more. Clearly some of these strings will be formally equivalent to others and will be formally non-equivalent to all the remaining ones. The question arises whether the number of distinct formal equivalence-sets of sentence-forms is finite or infinite. As we shall now see, the answer to this question depends on the number of sentence-variables one has in one's symbolic language.

Let us begin with the simplest case, that in which we construct sentence-forms, α, in which there appear sentence-variable tokens of one and only one sentence-variable type. These would include such wffs as

15. Instructors may find that the material in this section is best reserved for their mathematically more proficient students.

16. Note that equivalence tout court (or logical equivalence or 'strict' equivalence) is a property of propositions. *Formal* equivalence is a property of sentence-forms.

"P"

"$P \supset (P \vee \sim P)$"

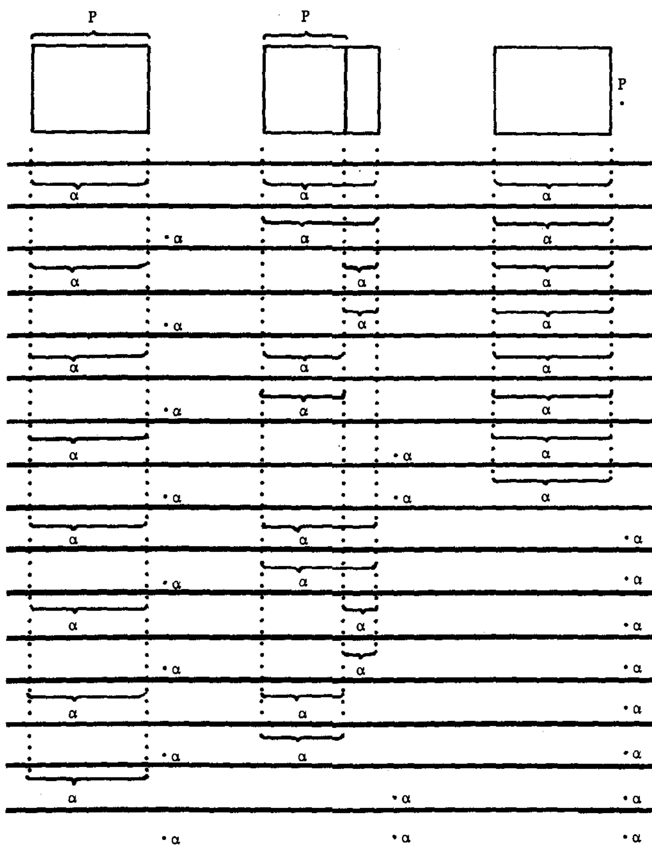
"$\nabla P \cdot P$"

"$\Box P \supset P$"

Into how many distinct formal equivalence-classes may this (in principle) infinite list be subdivided? Interestingly, the answer is: a mere 16. Let us see how we arrive at this figure.

When we wish to put a formula, α, on a set of worlds-diagrams, that formula must be placed on each rectangle so that it spans none of its segments, some but not all of them, or all of them. This fact immediately sets an upper limit to the number of formally non-equivalent formulae which may be depicted on a set of worlds-diagrams: this maximum number is simply the number of ways one can distribute brackets over the total number of segments in the set of worlds-diagrams.

In the case of one sentence-variable type (as we saw in chapter 1) there are three worlds-diagrams comprising a total of four segments. The number of ways that brackets may be distributed over four segments is 2^4, i.e., 16. Each of these ways is shown below and some members from the equivalence-class defined by that particular distribution of brackets are written alongside.



1 $\alpha = P \vee \sim P$; $\Box(P \supset P)$; $\Box P \supset P$; etc.

2 $\alpha = \sim\Box P$; $\nabla P \vee \Box\sim P$; etc.

3 $\alpha = \Box P \vee \sim P$; $\sim(\nabla P \cdot P)$; etc.

4 $\alpha = \sim P$; $(\nabla P \cdot \sim P) \vee (\Box\sim P)$; etc.

5 $\alpha = P \vee \Box\sim P$; $\sim(\nabla P \cdot \sim P)$; etc.

6 $\alpha = (\nabla P \cdot P) \vee \Box\sim P$; etc.

7 $\alpha = \Delta P$; $(\Delta P \cdot P) \vee (\Delta P \cdot \sim P)$; etc.

8 $\alpha = \Box\sim P$; $\sim P \cdot \sim\nabla P$; etc.

9 $\alpha = \Diamond P$; $P \vee (\nabla P \cdot \sim P)$; etc.

10 $\alpha = \nabla P$; $\Diamond P \cdot \Diamond\sim P$; etc.

11 $\alpha = \Box P \vee (\nabla P \cdot \sim P)$; etc.

12 $\alpha = \nabla P \cdot \sim P$; $\sim(\Diamond P \supset P)$; etc.

13 $\alpha = P$; $\Box P \vee P$; etc.

14 $\alpha = \nabla P \cdot P$; etc.

15 $\alpha = \Box P$; $\Delta P \cdot P$; etc.

16 $\alpha = P \cdot \sim P$; $\sim\Box(P \supset P)$; $(P \cdot \sim P) \cdot \Diamond P$; etc.

FIGURE *(6.q)*

The rules which have been given in the course of the preceding and present chapters for depicting formulae on worlds-diagrams allow us to generate a set of brackets for any arbitrarily chosen wff in truth-functional and modal propositional logic. But we have not given any rules for passing the other way. How can one generate an appropriate formula, as has obviously been done in figure *(6.q)*, to match any arbitrarily drawn set of brackets on a set of worlds-diagrams? Here we are faced with a problem, for the number of distinct ways of distributing brackets is finite (in this particular case, sixteen), while the number of distinct formally non-equivalent formulae corresponding to each of these sets of brackets is infinite. While various rules can be given for the generation of at least *some* formula for each set of brackets, no set of rules can generate all the formulae, nor is any simple set of rules known to us which in each case generates the *shortest* formula appropriate for a given set of brackets. In the case of figure *(6.q)* the formulae appearing in the right-hand column were not generated by the application of an effective method, but were instead found by insight, understanding, and trial and error — in short, by a 'feel' for the material.[17]

Each of the rows of figure *(6.q)* defines a class of formally equivalent sentence-forms; these classes are mutually exclusive of one another and are jointly exhaustive of the entire class of sentence-forms which contain only one sentence-variable type. Each formula occurring in the right-hand column of figure *(6.q)* is formally equivalent to every other formula occurring in the same row, and is formally non-equivalent to each formula occurring in each of the other rows. Any wff, α, which contains variables of only one type, and which does not occur explicitly on figure *(6.q)*, can be placed in one and only one row of that figure. Consider, for example, the formula "$P \supset \Box P$". Depicting this formula on a set of worlds-diagrams gives us:

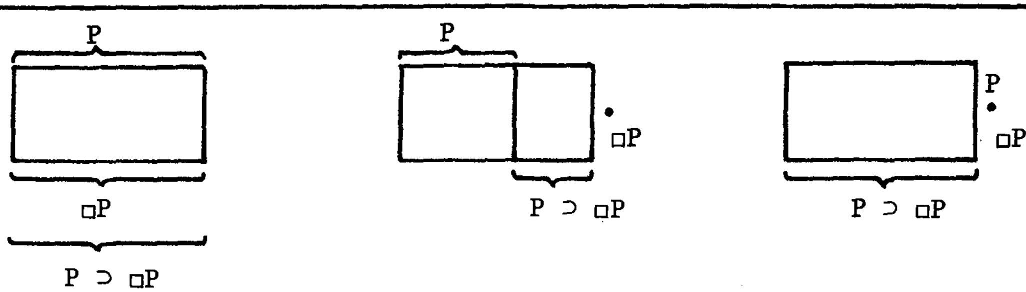


FIGURE *(6.r)*

By inspection, we can see that the brackets for "$P \supset \Box P$" in figure *(6.r)* are distributed exactly as are the brackets in row 3 of figure *(6.q)*. This tells us immediately that the formula "$P \supset \Box P$" and the formulae occurring on row 3 of figure *(6.q)*, viz., "$\Box P \vee \sim P$" and "$\sim(\nabla P \cdot P)$", are formally equivalent. Similarly, any other wff, α, containing propositional variables of only one type, must prove to be a member of exactly one of the sixteen equivalence-classes defined on figure *(6.q)*.

Glancing down the right-hand column of figure *(6.q)*, we notice that four rows, viz., 1, 4, 13, and 16, contain wffs which are unmodalized (i.e., are formulae of truth-functional logic). The question arises: Are these all the rows which contain at least one unmodalized formula? Do any of the other

17. A great deal, if not indeed the bulk, of advanced work in both logic and mathematics is precisely of this sort in that it demands insight and creativity and is not attainable by the rote following of recipes. The generation of proofs, the finding of axiom sets, the solving of partial differential equations, etc. etc., lie, like most of logic and mathematics, in the realm of creativity, not in the realm of assembly-line procedures. Textbooks — since they are usually geared to displaying solved problems and effective procedures — tend to obscure this point.

classes, defined by the remaining 12 rows, contain any unmodalized formulae? The answer is: No. That there are exactly four classes of formally non-equivalent truth-functional formulae containing one propositional variable follows immediately from the fact that the truth-table for any unmodalized formula, μ, containing one propositional variable-type, contains exactly two rows.

P	μ
T	
F	

"μ" is any unmodalized wff containing one propositional variable type, "P".

FIGURE *(6.s)*

There are exactly four distinct ways that truth-values can be assigned to "μ". These are

T	F	T	F
T	T	F	F

These assignments represent rows 1, 4, 13, and 16 respectively in figure *(6.q)*. In general, the number, u, of classes of formally distinct non-equivalent truth-functional formulae containing n propositional variable types will be equal to the number of ways that "T"s and "F"s may be distributed in the last column of a truth-table for n variable-types. Thus if there are 2 rows in the truth-table (i.e., one variable-type), there will be four ways to assign "T" and "F" to a compound wff, α; if 4 rows (i.e., two variable-types), then 16 ways; etc. In short, the number of classes, u, is equal to 2^m, where m equals the number of rows in a truth-table for n propositional-variable types. In chapter 5, m was defined equal to 2^n; thus $u = 2^{2^n}$.

Eight rows (viz., 1, 2, 7, 8, 9, 10, 15, and 16) of figure *(6.q)* contain wffs which are fully modalized. Looking at the configurations of brackets on each of these rows, it is easy to see what property it is on the worlds-diagrams by virtue of which a formula is itself (or is formally equivalent to) a formula which is wholly modalized: the brackets for such a formula will, on each rectangle in a set of worlds-diagrams, span all or none of that rectangle. (This fact follows from the rules WA, WS, and WN. [See section 8.]) That fully modalized formulae map onto worlds-diagrams in this fashion allows us immediately to calculate the maximum number of distinct classes of fully modalized formulae: this number is simply the maximum number of ways brackets may be distributed so that on each rectangle in a set of worlds-diagrams the bracket spans all or none of that rectangle. Letting "k" equal the number of rectangles in a set of worlds-diagrams, the maximum number of ways of distributing brackets in this fashion is, simply, 2^k. Of this number, two configurations are found to be appropriate for two truth-functional formulae as well: the case where the brackets span every rectangle in the set (corresponding to "$P \vee \sim P$"); and the case where the brackets span no rectangle in the set (corresponding to "$P \cdot \sim P$"). Thus the number of distinct classes, f, of formally equivalent wffs which contain at least one wholly modalized formula and no unmodalized formulae is $2^k - 2$.

The total number of distinct classes, t, of formally equivalent formulae — that is, the number of distinct classes without regard being paid to whether those classes contain any unmodalized or any fully modalized, formulae — is equal to the maximum number of ways that brackets may be distributed over the total number of *segments* occurring in a set of worlds-diagrams. In a set of 3 diagrams, there are 4 segments; in a set of 15 worlds-diagrams, there are 32 segments; in a set of 255 worlds-diagrams, there are 1024 segments. Thus the total number of distinct classes, t, of formally equivalent sentence-forms constructible on one sentence-variable type is 2^4 (i.e., 16, as we have already seen); on two sentence-variable types, 2^{32} (i.e., 4,294,967,296); and on three sentence-variable types, 2^{1024}.

We may generalize on these results. If we let n = number of sentence-variable types, we have

Total number of worlds-diagrams	$k = 2^{2^n} - 1$	[See chapter 1, p. 57]
Total number of segments in a set of k worlds-diagrams[18]	$s = 2^{2^n} \times 2^{n-1}$	
Total number of ways which brackets may be distributed over s segments	$t = 2^s$	
Total number of distinct classes of formally equivalent sentence-forms such that the class contains at least one unmodalized formula (truth-functional formulae)	$u = 2^{2^n}$	[See chapter 5, p. 272; $u = 2^m$]
Total number of distinct classes of formally equivalent sentence-forms such that no member of the class is unmodalized and at least one member is fully modalized	$f = 2^k - 2$	

The total number of formally distinct classes of sentence-forms constructible on n sentence-variables is t. Of this number t, a certain number, viz., u, of these classes contain at least one unmodalized formula, and a different number, f, of these classes contain at least one wholly modalized formula. For every value of n, the sum of u and f is smaller than t. This means that for every value of n, there must exist a number of distinct classes, p $(= t - u - f)$, which contain neither an unmodalized formula nor a wholly modalized formula, i.e., which contain *only* partially modalized formulae. As n increases, this number, p, approaches closer and closer to t. What this means is that by far the greater number of classes of formally equivalent wffs are classes whose members are all partially modalized, and hence are classes whose members are formally indeterminate, i.e., neither valid nor contravalid formulae.

We may see some of these results more clearly by actually calculating these various parameters for the first few values of n:

18. The derivation for the formula for s is not given here. Mathematically adept students are invited to try to derive it themselves.

n Sentence-variable types	k Worlds-diagrams	s Total segments	u Truth-Functional formulae	f Fully modalized formulae	p Partially modalized formulae	t Formally distinct formulae
1	3	4	4	6	6	16
2	15	32	16	32,766	4,294,934,514	4,294,967,296*
3	255	1,024	256	$2^{255}-2$	$\approx 2^{1024}$	2^{1024}
4	65,535	524,288	65,536	$2^{65,535}-2$	$\approx 2^{524,288}$	$2^{524,288}$
5	etc.					

FIGURE *(6.t)*

By the time we have reached sentence-forms containing as few as four sentence-variable types we can construct (in principle, if not in fact,) $2^{524,288}$ formally non-equivalent sentence-forms. When we pass on to five, six, and seven variables, the numbers become so large as to beggar the imagination.

EXERCISE

How many of the rows of figure (6.q) *represent classes of formally equivalent* valid *formulae?*

11. LOOKING BEYOND MODAL LOGIC TO INDUCTIVE LOGIC

Modal logic, as presently conceived, concerns itself with those modal attributes which can be explicated in terms of the concepts: (1) *being true (false) in* all *possible worlds;* (2) *being true (false) in* no *possible worlds;* and (3) *being true (false) in* some *possible worlds.*

Inductive logic tries to refine the latter of these three concepts. For intuitively we have the idea that the notion of "some possible worlds" admits of further elaboration: that there is some sense of "size" which allows us to say, of some pairs of contingent propositions — each of which is true in some but not all possible worlds — that one is true in a set of possible worlds which is larger in 'size' than the set of possible worlds in which the other is true. For example, we have a natural disposition to say that the set of possible worlds in which it is true that today is Tuesday is greater in 'size' than the set of possible worlds in which it is true that today is the second Tuesday in November.

* Gerald Massey, in his book, *Understanding Symbolic Logic* (New York, Harper & Row, 1970, pp. 188 - 190), derives through matrix methods this same number as the total number of formally non-equivalent formulae containing propositional variables of two types. He, like the present authors, remarks in effect that 16 of these formulae are formally equivalent to truth-functional formulae. But he, unlike the present authors, does not further subdivide the remaining class into those subclasses in which every member is a partially modalized formula, and those in which at least one member is not partially modalized, i.e., in which at least one member is a fully modalized formula.

The cardinality of a class and other concepts of class size

Our first inclination probably would be to identify this notion of 'size' with the number of members (i.e., possible worlds) in each class, with what mathematicians call the "cardinality" of the class. But this simple notion won't do. It comes to grief on the fact that contingent propositions may be true, not in finite classes of possible worlds, but in infinite classes. Consider, for example, the two propositions, (a) that today is Tuesday, and (b) that today is Tuesday or Wednesday. Intuitively we might feel inclined to say that the former is true in a *fewer* number of possible worlds than the latter. But we are barred from saying this. Each of these propositions is true in an infinite number of possible worlds and moreover, even though the former is true in a proper subset of the latter, the two sets have the *same* cardinality.[19]

If the requisite sense of the 'size' of a class cannot, then, for present purposes, be identified with the cardinality of the class, with what property *can* it be identified? This is no easy question, and one which has no obvious answer. Many solutions have suggested themselves to researchers in inductive logic.

In certain ways the problem is reminiscent of a problem in geometry. In geometry, we want to be able to say of two lines, for example, that they differ in 'size' even though (of necessity) each of the two lines contains exactly the same number of points. Happily, geometers have sought and found ways which allow us to do just this; to invoke a notion of the 'length' of a line which does *not* depend on the number of points in that line.[20]

The goal in inductive logic is to define a measure which stands to an infinite set of possible worlds much as the notion of length stands to the set of points which comprise a line: two lines containing the same number of points may yet differ in length (size). Similarly we should like to find a way to say that two sets of possible worlds each containing an infinite number of worlds may yet differ in "size". As just one example of how this measure might be constructed consider the following. The cardinality of the class of all integers is $\aleph_0$ (read "Aleph-nought"). Similarly the cardinality of the class of all *even* integers is $\aleph_0$. In *one* sense of "size"; viz., that in which cardinalities are compared, the class of all integers is *equal* in size to the class of even integers. But we can define a *different* notion of "size" which makes the latter class half the size of the former. Consider the two classes {even integers less than n} and {integers less than n} and let "N{α}" stand for "the cardinality of the class {α}". For any even integer n, the ratio

$$\frac{\text{N}\{\text{even integers less than } n\}}{\text{N}\{\text{integers less than } n\}}$$

is less than ½. As n increases, the value of this ratio approaches ever closer to ½. The value, ½, is the *limit* of this ratio as n approaches infinity:

$$\lim_{n \to \infty} \frac{\text{N}\{\text{even integers less than } n\}}{\text{N}\{\text{integers less than } n\}} = ½.$$

We can use this latter formula to define a second notion of 'size' such that — using this latter notion — it is correct to say that the 'size' of the class of even integers is half the 'size' of the class of all integers.

19. Recall that in chapter 3, pp. 146–47, it was shown that an infinite set and a proper subset of that infinite set may each have the same number of members, i.e., the same cardinality.

20. To be more specific: The length "L" of a line which lies in a two-dimensional orthogonal coordinate system and which has end points at (x_1, y_1) and at (x_2, y_2) is given by the formula:

$$L = \left| [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2} \right|$$

Difficult as it may be to give a rigorous explication of the precise sense of 'the size of a class' which we presuppose when we say, e.g., that the 'size' of the class of possible worlds in which it is true that today is Tuesday is greater than the 'size' of possible worlds in which it is true that today is the second Tuesday in November, the concept of 'size' nonetheless figures as the intuitive foundation of much thinking in inductive logic.

The concept of contingent content

Every proposition satisfies both the Law of the Excluded Middle and the Law of Noncontradiction. The first says that every proposition is either true or false, that there is no 'middle' or third truth-value. The second law says that no proposition is both true and false. Together these two laws say that the properties of truth and falsehood are mutually exclusive and jointly exhaustive of the entire class of propositions.

Corresponding to each of these two laws just cited we can state two analogues for modal status. In the first place we can say that every proposition is either contingent or noncontingent. And in the second, we can say that no proposition is both contingent and noncontingent. The two properties, contingency and noncontingency, are mutually exclusive and jointly exhaustive of the class of propositions.

Between contingency and noncontingency there is no 'middle' or third category. Contingency and noncontingency, like truth and falsehood, do not come in degrees. No proposition is 'half contingent' or 'three-quarters noncontingent' or any other fractional measure, just as no proposition is half or three-quarters true (or false). No contingent proposition is more contingent or less contingent than any other contingent proposition; and no noncontingent proposition is more noncontingent or less noncontingent than any other noncontingent proposition.

None of this means, however, that we cannot talk cogently of one proposition being *closer to being necessarily true* than another. To explicate this latter concept we shall introduce the concept of the *contingent content* of a proposition. And to do this we begin by noticing a curious fact about necessary truths.

In a memorable passage in *Through the Looking Glass*, Alice and the White Knight have the following conversation:

> "You are sad," the Knight said in an anxious tone: "let me sing a song to comfort you."
> "Is it very long?" Alice asked, for she had heard a good deal of poetry that day.
> "It's long," said the Knight, "but it's very *very* beautiful. Everybody that hears me sing it — either it brings *tears* into their eyes, or else — "
> "Or else what?" said Alice, for the Knight had made a sudden pause.
> "Or else it doesn't, you know. . . . "[21]

Although Lewis Carroll doesn't tell us Alice's reaction to this piece of 'information', we can well imagine that Alice would have been somewhat annoyed in being told it. There is a certain sense in which being told that a particular song brings tears to everyone's eyes or it doesn't, is vacuous. Like all necessary truths, in being true of all possible worlds, this proposition of the Knight's tells us nothing specific about his world, about how his song is usually met in his world, which makes that song different from any other song. In its bringing or not bringing tears to the eyes of everyone who hears it, it shares a property in common with every song everywhere, in the past as well as the future and in this, the actual world, in the imaginary world of *Through the Looking Glass*, and in every other possible world as well.

21. Lewis Carroll, *Alice's Adventures in Wonderland & Through the Looking Glass*, New York, Signet Classics, 1960, pp. 211 - 212.

Although what the Knight said to Alice is true, it lacks what has come to be called, "contingent content". The contingent information, or the contingent content, of the Knight's declaration is nil or zero.

Philosophers have gone a long way in inductive logic toward constructing measures for the amount of contingent content in a proposition. The basic idea is this: *the* smaller *the 'size' of the set of possible worlds in which a proposition is true, the* greater *its amount of contingent content.* A more specific version of this would be: the contingent content of a proposition is *inversely proportional* to the 'size' of the set of possible worlds in which that proposition is true.

Suppose someone were to ask us how many stars there are. If we were to reply,

> *(6.15)* "There are a million or fewer, or between one million and a billion, or a billion or more",

what we would express would be true. Indeed it would be necessarily true, and we would have succeeded not at all in telling our questioner specifically how many stars there are. Our answer would be true in all possible worlds and we would run no risk of being wrong in giving it. If, however, we were to omit one of the disjuncts, asserting only the remaining two, our answer would no longer be necessarily true. It could be false. For example, if we were to say,

> *(6.16)* "There are a million or fewer, or between one million and a billion",

then if in fact there were 10 billion stars, we would speak falsely. But whether we in fact end, in this latter case, speaking truly or falsely, our listener would be in receipt of contingent information. He would be entitled to infer that we are asserting (the contingent fact) that there are no more than a billion stars. As we reduce the number of alternatives in our answer, the contingent content (as well as our risk of being wrong) correspondingly increases. The proposition that there are a million or fewer stars, or between one million and a billion, or a billion or more, is true in all possible worlds and contains the least amount of contingent content. The proposition that there are a million or fewer stars or between a million and a billion, is not true in all possible worlds and contains a considerable amount of contingent content. And the proposition that there are a million or fewer stars is true in a set of possible worlds of yet smaller 'size' and contains still more contingent content. The more a proposition *excludes* (rather than includes), the *greater* its amount of contingent content. (Our naive intuitions might have suggested that the relation would be 'the other way round', but it is not.) The more a proposition excludes, the greater is the 'size' of the set of possible worlds in which it is false, and the greater is our risk, in the absence of other information, in holding to it.

From an epistemic point of view the most useful contingent truths are those that are most risky (in the sense just mentioned), for they carry the greatest amount of contingent content. Just notice how we prefer answers with as few disjuncts, with as much contingent content, as possible. When we ask someone, "Where are the scissors?", we would prefer to be told something of the sort, "They are in the cutlery drawer", than to be told something of the sort, "They are in the cutlery drawer, or beside the telephone, or in the desk, or in the sewing basket, or in the woodshed among the gardening tools." And when we ask someone what time it is, we would prefer to be told something of the sort, "It is three minutes past nine" than to be told something of the sort, "It is either three minutes past nine, or six minutes past eleven, or twenty minutes before eight."

Our intuitions in these matters can be captured by appeal to worlds-diagrams. Consider once more figure *(5.f)* (p. 258) which illustrated the relation of disjunction. It can there be seen that the bracket representing the disjunction of two propositions is never smaller than the bracket representing either one of those propositions. What that shows is that a disjunctive proposition (which as we have just seen generally has *less* contingent content than either of its disjuncts) is true in a set of possible worlds which is equal to, or greater in size, than either of the sets of possible worlds in which its disjuncts are true. Choosing one diagram (no. 15) as illustration from among all of those of figure *(5.f)* gives us:

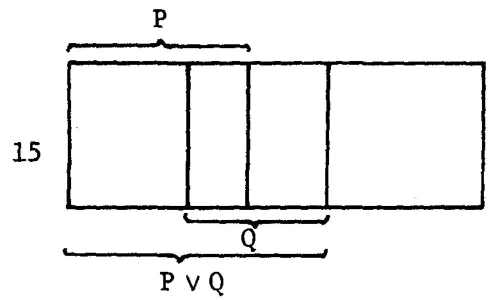


FIGURE *(6.u)*

Note that in this and all the other worlds-diagrams in figure *(5.f)*, the bracket representing $P \vee Q$ must be at least as long as the bracket representing P and must be at least as long as the bracket representing Q.

As we disjoin more alternatives onto the proposition that there are a million or fewer stars, the content of the proposition systematically decreases. It reaches its lowest point when we say that there are a million or fewer stars, between a million and a billion, or a billion or more. At this point the proposition ceases to have *any* contingent content whatever. This latter proposition is necessarily true, and we can view the process by which we passed from the highly contentful proposition that there are a million or fewer stars, to this latter one in which the contingent content is nil, as passing through an ordered list of propositions each one of which is systematically closer to being a necessary truth. Of course only the last in this list *is* a necessary truth, but the others can be thought to be close or far from that proposition in the list.

The two concepts, closeness to necessary truth and contingent content, can be defined in terms of the size of the set of possible worlds in which a proposition holds.

1. The greater the size of the set of possible worlds in which a proposition is true, the closer it is to being necessarily true.
2. The greater the size of the set of possible worlds in which a proposition is true, the smaller its amount of contingent content.

Closeness to being necessarily true can be seen to vary inversely with the contingent content of a proposition.[22]

22. Let us mention one point which is a source of potential confusion. In recent years there has been a remarkable growth in the science of cybernetics or information theory. In cybernetics, a certain parameter has been defined which bears the name, "information content". But it should be pointed out explicitly that this latter concept is distinct from the concept of contingent content which has here been defined. For one thing, information content is a measure of a property of *sentences*, while contingent content is a measure of a property of *propositions*. This being so allows the information content of a sentence-token expressing a noncontingent truth, on occasion, to be quite high, while the contingent content of the corresponding proposition would, as we shall see, in all cases remain precisely zero.

Monadic modal functors

The contingent content of a proposition is a property of a proposition which comes in various degrees. It cannot, therefore, be symbolized by a single fixed symbol, after the fashion of "$\Box$", "$\Diamond$", "∇", and "Δ". Instead, in order to symbolize the concept of contingent content and allow for the fact that propositions may have varying degrees of contingent content we use a *functor* rather than a sentential operator.

A functor, like an operator, takes as its argument a wff; but unlike an operator, the result of applying a functor to a wff is not the generation of a sentential wff, but rather the generation of a *numerical* wff, i.e., a wff which stands for a number.

The functor we shall introduce to signify the concept of contingent content will be "$\mathfrak{C}$" (German "C"). Its argument is to be written in parentheses immediately to the right of it, e.g.,

$$\text{“}\mathfrak{C}(P)\text{”},$$
$$\text{“}\mathfrak{C}(P \supset (Q \cdot R))\text{”}.$$

The expression "$\mathfrak{C}(P)$" is to be read as: "The contingent content of P". Both the expressions, "$\mathfrak{C}(P)$" and "$\mathfrak{C}(P \supset (Q \cdot R))$", represent numbers. Such numerical wffs may be used in arithmetical sentential wffs in the standard way that *any* symbol designating a number may be used, for example:

$$\text{“}\mathfrak{C}(P \supset Q) = \mathfrak{C}(\sim P \vee R)\text{”}$$
$$\text{“}\mathfrak{C}(A \supset B) > 0.67\text{”}$$

The first of these is the sentential wff which says that the contingent content of $P \supset Q$ is the same as the contingent content of $\sim P \vee R$. The second says that the contingent content of $A \supset B$ is greater than 0.67.

The amount of contingent content which a proposition has is measured on a scale of 0 to 1, with 0 being the contingent content of the least contentful proposition, and 1, the greatest. On this scale it is obvious that noncontingent truths rate a value of 0. For example,

$$\mathfrak{C}\ (\text{It is raining or it is not raining}) = 0, \text{ and}$$

$$\mathfrak{C}\ (\text{All aunts are females}) = 0$$

Contingent propositions will assume a value between 0 and 1.

$$0 < \mathfrak{C}\ (\text{It is raining}) < 1$$

But what value do we assign to noncontingent *falsities?* In accordance with the above so-called 'basic idea' (p. 373), the amount of contingent content in necessarily false propositions would seem to be 1. Does this make sense? Or should the amount of contingent content of all noncontingent propositions (both those that are true as well as those that are false) be the same, i.e., zero?

While philosophers assert that necessarily true propositions are contingently empty, they assert in contrast that necessarily false propositions are full.

Consider these two propositions:

(6.17) It is raining or it is not raining. [necessarily true]

(6.18) It is raining and it is not raining. [necessarily false]

From *(6.17)* nothing logically follows about the distinctive state of the weather in this or any other possible world — it does not follow that it is raining nor does it follow that it is not raining. *(6.17)* is a useless piece of information if we want to know how today's weather conditions differ from those of any other day or any other place or any other possible world for that matter. *(6.18)*, on the other hand, does contain the information we desire. For from *(6.18)* it follows that it is raining. Unfortunately, where *(6.17)* had a dearth of contingent content, *(6.18)* is afflicted with a surfeit of it. For not only does *(6.18)* imply that it is raining; it also implies that it is not. Be that as it may, *(6.18)* certainly does have contingent content. How much exactly is dictated by a fairly standard condition that is imposed on the numerical values for measures of contingent content. This condition is specifically:

$$(6.19) \quad \mathfrak{C}(P) = 1 - \mathfrak{C}(\sim P)$$

or alternatively,

$$(6.20) \quad \mathfrak{C}(P) + \mathfrak{C}(\sim P) = 1$$

Roughly, what this condition says is that whatever contingent content one proposition lacks, any of its contradictories has. Since we have already assigned the value of zero to necessarily true propositions, we must assign the value of one to their contradictories, which are, of course, all those propositions which are necessarily false. In symbols we have:

$$(6.21) \quad \Box P \rightarrow [\mathfrak{C}(P) = 0]$$

$$(6.22) \quad \Box \sim P \rightarrow [\mathfrak{C}(P) = 1]$$

$$(6.23) \quad \nabla P \rightarrow [0 < \mathfrak{C}(P) < 1]$$

If we allow the 'sizes' of sets of possible worlds to range from zero (for the case of necessarily false propositions) to one (for the case of necessarily true propositions), then it seems perfectly natural to identify the contingent content of a proposition with the 'size' of the set of possible worlds in which that proposition is *false*.

It is sometimes useful to have available a second functor which measures the size of the set of possible worlds in which a proposition is *true*. Its definition, in terms of the functor "$\mathfrak{C}$", is trivial. We shall call this second functor "$\mathfrak{M}$":

$$\text{“}\mathfrak{M}(P)\text{”} =_{df} \text{“}1 - \mathfrak{C}(P)\text{”, or alternatively,}$$

$$\text{“}\mathfrak{M}(P)\text{”} =_{df} \text{“}\mathfrak{C}(\sim P)\text{”}$$

We may read "$\mathfrak{M}(P)$" as "P's closeness to necessary truth", or alternatively, "the size of the set of possible worlds in which P is true".

The problem of finding an appropriate sense for the concept of "size" being invoked in this context comes down to devising a suitable formula for assigning numerical-values to the $\mathfrak{M}$-functor. Intuitively we can represent the $\mathfrak{M}$-value of a proposition (the size of the class of possible worlds in which that proposition is true) by a segment on a worlds-diagram whose *width* is proportional to that $\mathfrak{M}$-value.

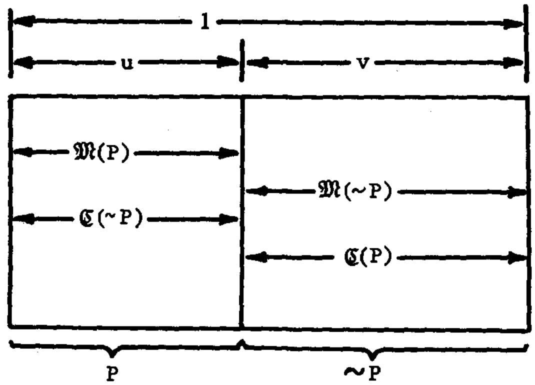


FIGURE *(6.v)*

In this instance, $\mathfrak{M}(P) = u$ and $\mathfrak{M}(\sim P) = v$. Of course, $u + v = 1$.

Much of what we have been saying about contingent content, closeness to necessary truth, $\mathfrak{C}$-values and $\mathfrak{M}$-values, may be organized on one illustrative figure.

Proposition is necessarily false	There are ten stars and it is not the case that there are ten stars	Maximum Contingent Content $\mathfrak{C}(P) = 1$ $\mathfrak{M}(P) = 0$
Propositions are contingent	 There are ten stars There are ten stars or there are nine stars There are ten stars or there are nine stars or there are eight stars	Increasing closeness to necessary truth Decreasing contingent content
Proposition is necessarily true	There are no stars or there is one star or there are two stars or .... or .... (etc.)	Minimum Contingent Content $\mathfrak{C}(P) = 0$ $\mathfrak{M}(P) = 1$

FIGURE *(6.w)*

Note that the measure of the contingent content of a *contingent* proposition is *independent* of that proposition's truth-value. A false proposition may have a greater contingent content than a true one.

Propositions having $\mathfrak{C}$-values close to 0 ($\mathfrak{M}$-values close to 1) are true in 'large' sets of possible worlds and are closer to being necessary truths (even if they are false) than are other propositions having higher $\mathfrak{C}$-values. Propositions having $\mathfrak{C}$-values close to 1 ($\mathfrak{M}$-values close to 0) are true in 'small' sets of possible worlds and are closer to being necessary falsehoods (even if they are true) than are other propositions having lower $\mathfrak{C}$-values.

Note that while contingent propositions may vary among themselves as to their respective 'distances' from being necessarily true (or false), i.e., in their $\mathfrak{M}$-values, there is no corresponding feature for noncontingent propositions. Necessarily true propositions do *not* vary among themselves as to their respective 'distances' from being contingent. They are all 'equi-distant' from contingency. No necessarily true proposition is any closer to being contingent than is any other. (And mutatis mutandis for necessarily false propositions.)

A few rather important theses about contingent content might profitably be noted. The first of these we have already explained, viz.,

(6.24) $\mathfrak{C}(P \vee Q) \leq \mathfrak{C}(P)$ and $\mathfrak{C}(P \vee Q) \leq \mathfrak{C}(Q)$

That is, the contingent content of a disjunction is always equal to or less than the contingent content of either of its disjuncts. To this theorem we may add the following ones:

(6.25) $\mathfrak{C}(P \cdot Q) \geq \mathfrak{C}(P)$ and $\mathfrak{C}(P \cdot Q) \geq \mathfrak{C}(Q)$ [A conjunction tells us the same as or more than either of its conjuncts]

(6.26) $\mathfrak{M}(P \cdot Q) \leq \mathfrak{M}(P)$ and $\mathfrak{M}(P \cdot Q) \leq \mathfrak{M}(Q)$

and

(6.27) $(P \rightarrow Q) \rightarrow [\mathfrak{C}(P) \geq \mathfrak{C}(Q)]$ [If P implies Q, then Q has the same or less contingent content than P]

Each of these theses may easily be proved by inspecting the fifteen worlds-diagrams (figure *(1.i)*) in chapter 1.

The last of these theses is particularly important: it tells us that in cases where one proposition implies another, the former has the same or more contingent content than the latter, i.e., that the relation of implication can at most preserve contingent content, but that there can never be more contingent content in the consequent than in the antecedent.

These latter facts present a seeming puzzle. Why should we be interested in examining the consequences of propositions once we realize that these consequences can, at best, have the same contingent content, but will, in a great many cases, have less contingent content than the propositions which imply them? Why should we be interested in passing from propositions with high contingent content to propositions with lesser contingent content?

The answer lies wholly in the character of human knowledge. Although a person may know a proposition P, which implies another proposition Q, it does not follow that that person knows Q. Remember (from chapter 3) that four conditions must be satisfied in order for a person to know a proposition. In the case where P implies Q, one's having knowledge that P satisfies one and only one of the four conditions necessary for knowing that Q: one's knowing that P guarantees the *truth* of Q, for as we have earlier seen (1) one cannot know a false proposition, and (2) the relation of implication preserves truth. But one's knowing that P does not guarantee any of the other three conditions, viz., that one believe Q, that one have good evidence that Q, and that this justified belief be indefeasible. If, however, one does know both that P and that P implies Q, then one is in a position to have an indefeasibly justified true belief that Q, i.e., is in a position to know that Q. When a person learns that Q, on the basis of having inferred Q from the known proposition P, even though Q may have less contingent content than P, he adds a further item of knowledge to his store of knowledge.

What are the prospects for a fully developed inductive logic?

Inductive *propositional* logic is a going concern and has been for many years. Vast numbers of important theses in this logic are easily provable. Although rigorous proofs can be given for all of its theses, many of them, virtually by inspection, can be 'read off' worlds-diagrams. It is, for example, a trivial matter to establish any of the following theses simply by examining the set of fifteen worlds-diagrams for two propositions.

(6.28) $\mathfrak{M}(P \vee \sim P) = 1$

(6.29) $\mathfrak{M}(P \cdot \sim P) = 0$

(6.30) $(P \phi Q) \rightarrow [\mathfrak{M}(P \vee Q) = \mathfrak{M}(P) + \mathfrak{M}(Q)]$

(6.31) $(P \phi Q) \rightarrow [\mathfrak{M}(P \cdot Q) = 0]$

(6.32) $(P \rightarrow Q) \rightarrow [\mathfrak{M}(P \cdot Q) = \mathfrak{M}(P)]$

(6.33) $(P \rightarrow Q) \rightarrow [\mathfrak{M}(P \vee Q) = \mathfrak{M}(Q)]$

(6.34) $(P \leftrightarrow Q) \rightarrow [\mathfrak{M}(P) = \mathfrak{M}(Q)]$

(6.35) $[\nabla P \cdot (P \rightarrow Q) \cdot \sim (Q \rightarrow P)] \rightarrow [\mathfrak{C}(P) > \mathfrak{C}(Q)]$

(6.36) $(P \circ Q) \rightarrow [\mathfrak{M}(P \vee Q) = \mathfrak{M}(P) + \mathfrak{M}(Q) - \mathfrak{M}(P \cdot Q)]$

So far as it goes, inductive propositional logic is a very attractive logic. But the trouble is that it does not go very far. To be more specific, it never assigns $\mathfrak{M}$- or $\mathfrak{C}$-values to propositions expressed by *simple* sentences, but only to propositions — and then only to some, not all — which are expressed by *compound* sentences.[23] In the above examples we can see that inductive propositional logic sometimes assigns $\mathfrak{M}$-values to propositions expressed by compound sentences solely on the basis of the *forms* of those sentences (see, for example, theses *(6.28)* and *(6.29)*). In other cases, such assignments can be made only upon the specification of the $\mathfrak{M}$- or $\mathfrak{C}$-values of the propositions expressed by their simple sentential components *together with* a specification of certain information about the modal attributes of the latter propositions (see, for example, theses *(6.30)* – *(6.35)*). And in still other cases, inductive propositional logic is unable to do even the latter as can be seen in thesis *(6.36)* which expresses the $\mathfrak{M}$-value of $P \vee Q$ in terms of the $\mathfrak{M}$-value of a proposition (viz., $P \cdot Q$) which is itself expressed by a *compound* sentence and whose own $\mathfrak{M}$-value is *not* calculable within this logic from the $\mathfrak{M}$-values of P and of Q.

It would seem, then, that the next development one would want to see in an inductive logic would be a means of assigning $\mathfrak{M}$- and $\mathfrak{C}$-values to propositions expressed by *simple* sentences, and to those compound sentences the calculation of whose $\mathfrak{M}$- and $\mathfrak{C}$-values apparently lies beyond the capabilities of a propositional logic.

How can these assignments be made?

On the interpretation which has here been suggested for $\mathfrak{M}$- and $\mathfrak{C}$-values, it would appear that the only way to make such assignments would be a priori. Empirical techniques confined to the actual

23. Note that in its inability to assign $\mathfrak{M}$- and $\mathfrak{C}$-values to propositions expressed by simple sentences, inductive propositional logic is analogous to truth-functional propositional logic which give us no logical grounds for assigning truth-values to propositions expressed by simple sentences.

world are not going to be able to tell us, for example, the size of the set of all the *other* possible worlds in which some particular contingent proposition is true. But how, exactly, is this a priori program to be carried out?

In the early 1950s Rudolf Carnap made a valiant attempt at constructing a logic to do just this.[24] His, of course, was a logic of *analyzed* propositions; for without analyzing propositions, there can be no basis for assigning one proposition one value, and another proposition some other value. To make these assignments, Carnap constructed what he called "state-descriptions". Although he did not use the possible-worlds idiom, we may regard a state-description as a description of a unique possible world (or as a set of possible worlds which share in common all of a stipulated set of attributes).

The trouble with Carnap's pioneering work, however, was that he was never able to extend his analysis to the entire set of possible worlds. He found it necessary, in each case, to examine only those possible worlds which were describable by very impoverished languages. Within small, highly artifically restricted sets of possible worlds, he was able to assign $\mathfrak{M}$- and $\mathfrak{C}$-values to individual propositions relative to those restricted sets of possible worlds. He was not able to extend his analysis to the unrestricted, full set of all possible worlds.

To date, no completely satisfactory solution has been found to the general problem. The construction of a fully-developed inductive logic remains a challenging, tantalizing goal.

In their search for an inductive logic of analyzed propositions, logicians are guided by a number of paradigm cases, i.e., examples about whose appropriate $\mathfrak{M}$- and $\mathfrak{C}$-values many logicians have shared and strong convictions. For example, we would want our logic to assign much higher $\mathfrak{C}$-values to so-called 'positive' propositions, than to 'negative'. Virtually all of us intuitively feel that the set of possible worlds in which it is false that there are 31 persons in room 2b, is greater in size than the set of possible worlds in which it is true: there are vast numbers of ways for it *not* to be the case that there are 31 persons in room 2b (e.g., there are none; there are 2; there are 17; there are 78; there are 455,921; etc., etc.); but there is only one way for there to *be* 31 persons in that room.

There are, of course, countless numbers of other propositions about whose $\mathfrak{M}$- and $\mathfrak{C}$-values our intuitions fail us. Consider, for example, the two propositions, A, that oranges contain citric acid, and B, that the Greek poet Homer wrote two epics. Is the set of possible-worlds in which A is true, larger, equal to, or smaller in size than the set in which B is true? This question has no *obvious* answer. But this is not a cause for despairing of the possibility of an inductive logic. If an inductive logic can be satisfactorily achieved which yields the 'right' $\mathfrak{M}$- and $\mathfrak{C}$-values for the paradigm cases, then we can simply let it dictate the $\mathfrak{M}$- and $\mathfrak{C}$-values for those propositions about which we have no firm intuitions. Indeed this is one of the motivating factors in searching for any logic, whether it be an inductive logic or any other kind: the constructing of a new powerful logic holds out the promise of providing new knowledge, knowledge beyond knowledge of the paradigm cases which are used to test its mettle.

EXERCISES

1. a. *Which, if either, has more contingent content: the proposition that today is Sunday or the proposition that today is not Sunday?*

   b. *Which, if either, has more contingent content: the proposition that Mary Maguire is over twenty-one years of age, or the proposition that Mary Maguire is forty-two years of age?*

24. Rudolf Carnap, *The Logical Foundations of Probability*, 2nd ed., Chicago, The University of Chicago Press, 1962.

2. *Find a* contingent *proposition which has* more *contingent content than the proposition that today is Sunday.*

3. *Suppose that A implies B, that* $\mathfrak{M}(A) = 0.3$ *and* $\mathfrak{M}(B) = 0.5$. *What is the value of* $\mathfrak{M}(A \supset B)$? *the value of* $\mathfrak{M}(A \cdot B)$? *the value of* $\mathfrak{M}(A \vee B)$? *and the value of* $\mathfrak{M}(A \equiv B)$? *[Hint: Study figure* (6.u) *and reread pp. 313–315.]*

4. *Suppose that* $A \Phi B$, *that* $\mathfrak{M}(A) = 0.12$ *and* $\mathfrak{M}(B) = 0.43$. *What is the value of* $\mathfrak{M}(A \supset B)$?

5. *Philosophers often talk about the 'absolute probability' of a proposition, and by this they mean the probability of a proposition in and of itself without regard to* any *other contingent information whatever. Which do you think is the more appropriate concept with which to identify this notion of absolute probability: the contingent content of the proposition or its complement, closeness to necessary truth? Explain your answer. In seeking answers to questions do we want answers with high or low absolute probability? Again, explain your answer.*

6. *A question to ponder: We have said that necessary truths have no* contingent *content. Does this mean that they all lack* informative *value? Can some* different *sense of "content" be devised such that necessary truths will have some sort of informational (epistemic) value?*

* * * * *

The concept of probabilification

Whether a proposition P implies a proposition Q, is an 'all-or-nothing-affair'; that is, either P implies Q or it is not the case that P implies Q. Implication — like consistency, like truth, like falsity, etc. — does not come in degrees. No proposition *partially* implies another; no implicative proposition is partially true or, for that matter, partially false.

Nonetheless it would be a boon to logical analysis if we could define a somewhat weaker notion than implication, a notion which shares various features in common with that concept, but which *does* 'come in degrees'. Intuitively we can distinguish among various cases of non-implication: some of them *do* seem to 'come closer' to being cases of implication than others. For example, neither of the following cases is a case of implication:

(6.37) If repeated, diligent searches have failed to find a Himalayan Snowman, then Himalayan Snowmen do not exist;

(6.38) If Admiral Frank's July 1923 expedition did not find a Himalayan Snowman, then Himalayan Snowmen do not exist.

Many of us would intuitively feel that *(6.37)* is somehow 'closer' to being an instance of implication than is *(6.38)*.

In the case where the relation of implication obtains between two propositions, the truth of the former *guarantees* the truth of the latter. But might there not be a somewhat weaker logical relation such that if *it* were to obtain between two propositions the truth of the former would — if not guarantee — at least *support* the latter?

Philosophers have christened this latter relation "probabilification" (alternatively, "confirmation"). It is allegedly illustrated in the example *(6.37)* above: the proposition that repeated, diligent searches have failed to find a Himalayan Snowmen, is thought to 'probabilify' (to support, warrant, or confirm)— even though it does not imply — the proposition that Himalayan Snowmen do not exist.[25]

A dyadic modal functor for the concept of probabilification

Let us introduce a functor, "$\mathfrak{P}$" (German "P"), to symbolize the concept of probabilification. The functor, "$\mathfrak{P}$", is dyadic: it takes two arguments, written in parentheses and separated by a comma.

$$\text{“}\mathfrak{P}(P,Q)\text{”} =_{df} \text{“the degree to which P probabilifies Q”}$$

To construct a wff using such an expression, one may use it in any way in which one would use any other symbol in arithmetic which expresses a numerical value, for example,

$$(6.39) \quad \text{“}\mathfrak{P}(A \vee B, A \supset B) = \mathfrak{P}(A, A \equiv C) - 0.45\text{”}$$

The probabilification-functor is to assume numerical values between 0 and 1 (inclusive). If P provides utterly no support for Q, as would be the case, for example, if P and Q were both contingent and inconsistent with one another, then the corresponding $\mathfrak{P}$-value would be zero. If, on the other hand, P implies Q, then the $\mathfrak{P}$-functor is to have the maximum value possible, viz., one. All other cases will be assigned numerical values between these two limits.

In terms of worlds-diagrams, how is the $\mathfrak{P}$-functor to be interpreted?

Let us begin with an example. We will choose two logically independent propostions: A, the proposition that there are fewer than 30 persons in room 2A, and B, the proposition that there are at least 25 but fewer than 40 persons in room 2A. The relevant worlds-diagram is the fifteenth.

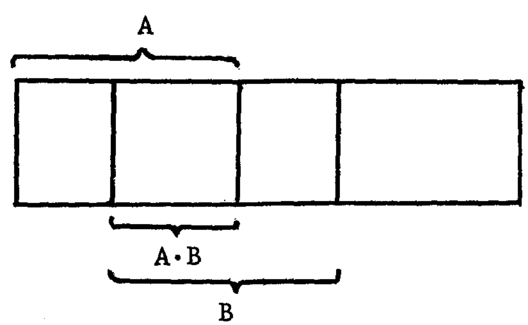


FIGURE *(6.x)*

Arbitrarily pick a world, W, in the set of possible worlds in which A is true. What are the 'chances' that this will be a world in which B is also true? The 'chances' will depend on what proportion of the segment representing A overlaps the segment representing B. More specifically, the 'chances' of W's lying in the segment representing A • B is simply the 'width' of that segment compared to the total width of the segment representing A itself. But the 'widths' of segments are nothing other than the associated $\mathfrak{M}$-values. Hence in this case we may immediately write down the following formula:

25. Rudolf Carnap argued at length that there are *at least* two *different* concepts standardly designated by the term "probability": what he called "confirmation" and "relative frequency" respectively. It is the first alone of the two different concepts which we are examining. For more on Carnap's views see esp. pp. 19 - 36, *op. cit.*

$$(6.40)\quad \mathfrak{P}(P, Q) = \frac{\mathfrak{M}(P \cdot Q)}{\mathfrak{M}(P)}$$

This formula is not fully general, however. It cannot be applied to all cases. Recall that division by zero is a disallowed operation in arithmetic. Thus we must not allow $\mathfrak{M}(P)$ in the above formula to assume the value zero. While *(6.40)* can be used in cases in which P is possible, it cannot be used in cases where P is impossible, for when P is impossible, $\mathfrak{M}(P)=0$. Thus we need a formula different from *(6.40)* to cover the cases where P is necessarily false. What that formula should be is obvious. In cases where P is necessarily false, P *implies* Q, and we have already said that in cases of implication, the $\mathfrak{P}$-functor is to bear the numerical value, one. Thus we replace formula *(6.40)* with the following two formulae:

$$(6.41)\quad \Diamond P \rightarrow [\mathfrak{P}(P, Q) = \frac{\mathfrak{M}(P \cdot Q)}{\mathfrak{M}(P)}],$$

and

$$(6.42)\quad \sim\Diamond P \rightarrow [\mathfrak{P}(P, Q) = 1]$$

It is interesting to calculate the value of $\mathfrak{P}(P,Q)$ in the case where P is necessarily true. The formula we use is *(6.41)*. In cases where P is necessarily true, $\mathfrak{M}(P \cdot Q)=\mathfrak{M}(Q)$, and $\mathfrak{M}(P)=1$. Substituting these values in formula *(6.41)* we find:

$$(6.43)\quad \Box P \rightarrow [\mathfrak{P}(P, Q) = \mathfrak{M}(Q)]$$

The $\mathfrak{M}$-value of a proposition may be considered its 'absolute' or 'degenerate' probability, i.e., the degree to which it is probabilified by a proposition having *no* contingent content. Or putting this another way, the absolute probability of a proposition is its probability in the absence of any contingent information about that proposition.[26]

Given the above explications of probabilification, we can see that there are no further problems in incorporating such a notion into inductive logic than those already mentioned in regard to the calculation of $\mathfrak{M}$- and $\mathfrak{C}$-values. For the problem of actually assigning numerical values to the $\mathfrak{P}$-functor comes down to the problem of assigning $\mathfrak{M}$-values. If we are able to solve that problem, we will automatically be able to assign probabilification-measures.

EXERCISE

Let "C"= "There are at least 25, but fewer than 100, persons in room 2A". Add C to figure (6.x). *Does A probabilify C less than, equal to, or more than the degree to which it probabilifies B?*

26. What we are here calling the "absolute" probability, is sometimes called the "a priori" probability. We eschew this particular use of the latter term. Although the absolute probability of a proposition can be known *only* a priori, it seems to us misleading to favor this particular probability-measure with that name. For on our account, *all* measures of degree of probabilification, whether absolute or relative (i.e., whether on the basis of propositions having no, or some, contingent content) are — if knowable at all — knowable a priori.

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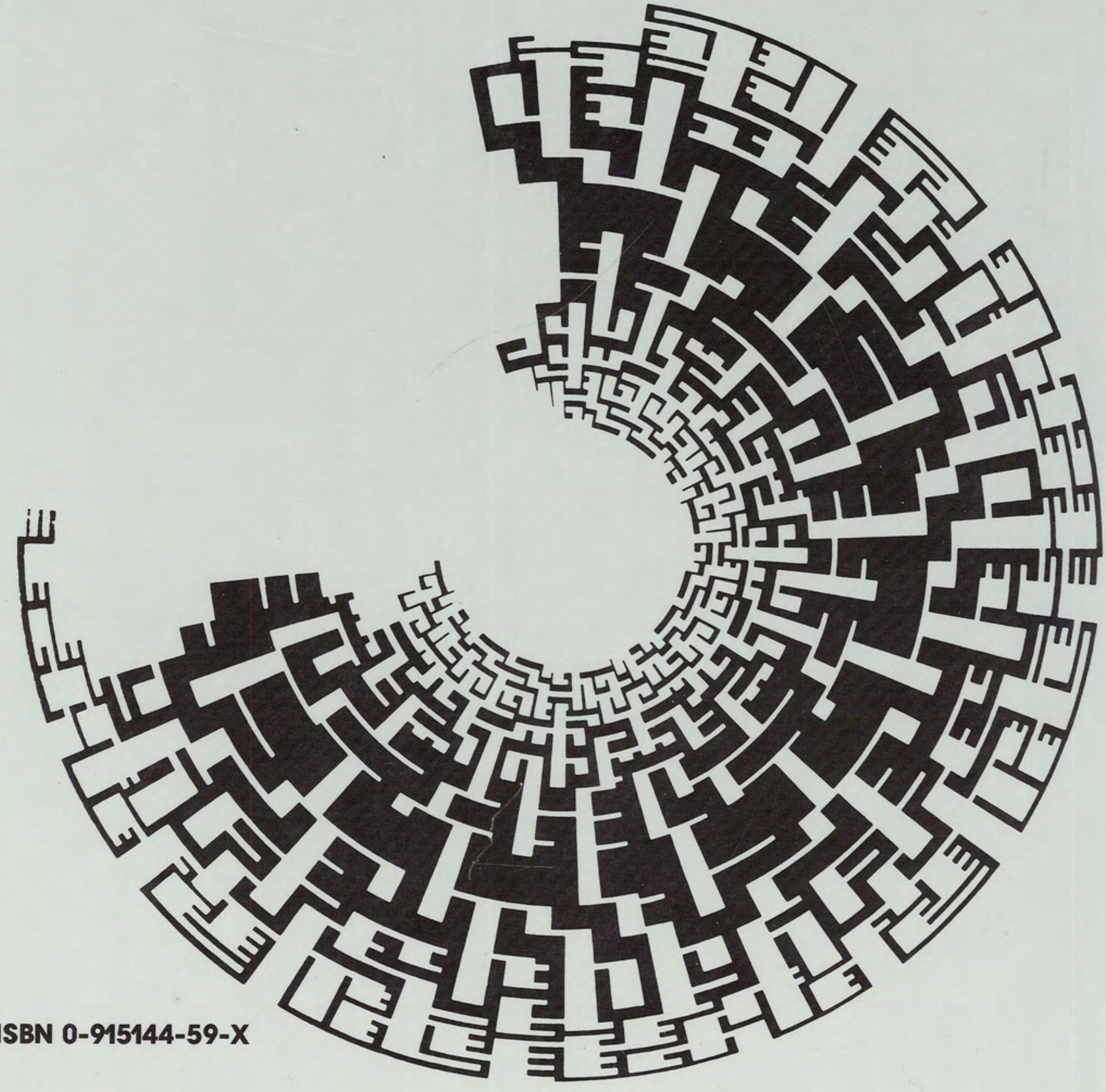

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