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The Riemann problem for the Leray–Burgers equation

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ABSTRACT

For Riemann data consisting of a single decreasing jump, we find that the Leray regularization captures the correct shock solution of the inviscid Burgers equation. However, for Riemann data consisting of a single increasing jump, the Leray regularization captures an unphysical shock. This behavior can be remedied by considering the behavior of the Leray regularization with initial data consisting of an arbitrary mollification of the Riemann data. As we show, for this case, the Leray regularization captures the correct rarefaction solution of the inviscid Burgers equation. Additionally, we prove the existence and uniqueness of solutions of the Leray-regularized equation for a large class of discontinuous initial data. All of our results make extensive use of a reformulation of the Leray-regularized equation in the Lagrangian reference frame. The results indicate that the regularization works by bending the characteristics of the inviscid Burgers equation and thereby preventing their finite-time crossing.

1. Introduction

Consider the following regularization of the Burgers equation:

$$u_t + u^\alpha u_x = 0, \tag{1a}$$

$$u^\alpha = \psi^\alpha * u. \tag{1b}$$

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Subscripts denote differentiation. We let $\psi^\alpha(x) = \alpha^{-1}\psi(x/\alpha)$, where $\psi(x)$ is a smoothing kernel. We take ψ to be a smooth, even, integrable function normalized to have total integral equal to one ($\int_{\mathbb{R}} \psi(x)\,dx = 1$). Valid choices of ψ include

$$\psi(x) = \frac{1}{\sqrt{\pi}} \exp(-x^2) \qquad \text{Gaussian, and} \tag{2}$$

$$\psi(x) = \begin{cases} C_0 \exp[1/(x^2-1)], & |x| < 1, \\ 0, & |x| \geqslant 1, \end{cases} \qquad \text{bump,} \tag{3}$$

where in the last equation we set $C_0 = 1/\int_{-1}^{1} \exp[1/(x^2-1)]\,dx$. An important class of smoothing kernels,

$$c_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{c}_n(k) e^{-ikx}\,dk = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ikx}}{(1+k^2)^n}\,dk, \tag{4}$$

is realized by taking the inverse Fourier transform of $\hat{c}_n(k) = (1+k^2)^{-n}$ for integers $n \geqslant 1$. One checks that $c_n(x)$ has $2n-2$ continuous derivatives; the $(2n-1)$st derivative exists but is discontinuous at $x = 0$. Suppose $\psi(x) = c_n(x)$. Using a change of variable, it is easy to show that the Fourier transform of $\psi^\alpha(x)$ is $(1+\alpha^2 k^2)^{-n}$. Applying standard theorems, we then find that (1b) is equivalent to

$$\left(1 - \alpha^2 \partial_x^2\right)^n u^\alpha(x,t) = u(x,t). \tag{5}$$

We may now use (5) to eliminate u from (1a). The result is

$$\left(1 - \alpha^2 \partial_x^2\right)^n u_t^\alpha(x,t) + u^\alpha \left(1 - \alpha^2 \partial_x^2\right)^n u_x^\alpha(x,t) = 0, \tag{6}$$

where n is any positive integer. The PDE obtained for $n = 1$,

$$u_t^\alpha + u^\alpha u_x^\alpha - \alpha^2 u_{txx}^\alpha - \alpha^2 u^\alpha u_{xxx}^\alpha = 0,$$

was studied by the authors in [1,2]; details about this work will be given later.

Even though (1) may appear to be innocuous, the above remarks demonstrate that (1) includes as a special case (6), an equation that contains a term with a mixed space–time derivative and the nonlinear term $u^\alpha \partial_x^{2n+1} u^\alpha$. For large values of n, Eq. (6) would be considered a high-order, exotic PDE; the mixed and nonlinear terms would necessitate a delicate analysis. Our analysis begins with (1), circumventing these issues.

1.1. Motivation

It is well known that, for smooth initial data $u(x,0)$ that is decreasing at least at one point (so there exists y such that $u_x(y,0) < 0$), the classical solution $u(x,t)$ of the inviscid Burgers equation

$$u_t + u u_x = 0 \tag{7}$$

fails to exist beyond a certain finite break time $T > 0$. The reason this breakdown occurs is that the characteristics of (7) cross in finite time. System (1) seeks to remedy this finite-time breakdown by filtering the convective velocity, i.e., by replacing the term uu_x by $u^\alpha u_x$, where u^α is smoother than u. This idea was first employed in 1934 by Leray [10] to treat the incompressible Navier–Stokes equations, so we refer to system (1) as the Leray-regularized Burgers equation. As we show in this paper, the reason the Leray regularization works for the Burgers equation is that the characteristics of (1) bend slightly out of the way of one another, avoiding any finite-time intersection.

The study of the weak form of (7) with discontinuous initial data is completely standard. Take for instance Riemann data, where

$$u(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases}$$

with real constants u_L and u_R. It is well known that the unique entropy solution of (7) with this Riemann initial data consists of either a shock wave that propagates with the speed $(u_L + u_R)/2$ if $u_L > u_R$ or a rarefaction wave when $u_L < u_R$.

Prior studies of (1) have focused on continuous initial data. This has left open the question of how the Leray regularization behaves with Riemann data, let alone more general types of discontinuous initial data. More specifically, one would like to know whether, in the $\alpha \to 0$ limit, solutions of (1) with Riemann data converge to the entropy solutions for (7) mentioned in the previous paragraph.

To study (1) with discontinuous initial data, we require a reformulation of (1) that allows $u(x,t)$ to be non-smooth in x and t. Note that (1a) may not be in conservation law form, so the standard notion of a weak solution is not applicable here. Our approach is to instead derive from (1) an equivalent system where the *only* dynamical variables are the particle position map $\eta(X,t)$ and its derivative $\eta_X(X,t)$. This system, which lives entirely in the Lagrangian coordinate frame, explicitly allows the initial data $u(x,0) = u_0(x)$ to be discontinuous and non-vanishing at infinity, permitting the study of Riemann problems.

1.2. Statement of results

Using the Lagrangian approach mentioned above, we arrive at the following results:

- For a general class of kernels ψ, we establish the global existence and uniqueness of solutions of (1) for initial data u_0 such that $u_0 = v_0 + w_0$, where v_0 is bounded and Lipschitz and w_0 is bounded and absolutely integrable. This result allows for many types of discontinuous initial data u_0, including Riemann data.
- We study explicit solutions of (1) for Riemann data. When $u_L > u_R$, we find that, in the $\alpha \to 0$ limit, solutions of the Leray regularization converge to weak entropy solutions of the inviscid Burgers equation. These weak solutions consist of shocks that satisfy the Rankine–Hugoniot condition. The solutions show that for any $\alpha > 0$, the Leray regularization bends the characteristics of the inviscid Burgers equation (7) so that they avoid a finite-time collision. The bending is done so that the characteristics approach the shock line as time passes.
- When $u_L < u_R$, we find that, in the $\alpha \to 0$ limit, solutions of the Leray regularization with Riemann initial data converge to weak solutions of inviscid Burgers that violate the Lax entropy condition, i.e., unphysical shocks. However, an arbitrary smoothing of the Riemann initial data changes this behavior completely. To state this result, let us define $u_0^\delta = \lambda^\delta * u_0$, where u_0 is Riemann data with $u_L < u_R$, $\lambda^\delta(x) = \delta^{-1}\lambda(x/\delta)$, and $\lambda(x)$ is a mollifier. Suppose we solve (1) with the smoothed Riemann data u_0^δ; we show that for all $\delta > 0$, the $\alpha \to 0$ limit of this solution is a rarefaction wave that coincides with the entropic solution of (7) with initial data u_0^δ. In other words, for certain arbitrarily small perturbations of the $u_L < u_R$ Riemann data, the Leray-regularized solution captures, in the $\alpha \to 0$ limit, the entropy solution of the inviscid Burgers equation.

1.3. Historical remarks

We are aware of only two prior studies of system (1) with a general kernel ψ. The first preprint [13] establishes existence and uniqueness for an n-dimensional version of (1) with continuously differentiable initial data. The second preprint [14] establishes that solutions $u^\alpha(x,t)$ of (1) converge strongly, in the $\alpha \to 0$ limit, to a weak solution of the inviscid Burgers equation (7). Under further hypotheses that the initial data is unimodal and continuously differentiable, it is shown that the limit solution is the unique entropy solution. However, neither study [13,14] provides a reformulation of (1) that lifts the requirement that $u(x,t)$ be classically differentiable in x and t. Consequently, the statements and arguments provided there are insufficient to rigorously prove existence and uniqueness with non-smooth initial data.

Eq. (1) with the Helmholtz mollifier

$$\psi(x) = \frac{1}{2}\exp\bigl(-|x|\bigr) \tag{8}$$

has been studied by the authors in [1] and [2]. Note that (8) is just the $n=1$ case of (4), i.e., $\psi(x) \equiv c_1(x)$. As demonstrated in [1], system (1) with kernel (8) is globally well-posed with initial data $u_0(x)$ in the Sobolev space $W^{2,1}(\mathbb{R})$. It is also proved rigorously in [1] that the solutions of (1) converge strongly, as $\alpha \to 0$, to weak solutions of the inviscid Burgers equation (7). The numerics suggest that the limiting weak solution is the correct entropy solution of (7). In the second paper [2], we examined smooth, monotone traveling wave solutions, or "front" solutions, for (1). We proved the stability of monotone decreasing fronts and the instability of monotone increasing fronts. These two types of traveling fronts correspond, respectively, to viscous shocks and rarefaction waves.

The literature on system (1) includes, in addition to the works already cited, papers on water wave models, traveling waves, and integrable/Hamiltonian structures [5–9,12]. These prior works have not exploited the Lagrangian coordinate frame, a cornerstone of the present work. We believe that for the Leray-regularized Burgers equation, Lagrangian coordinates may yield important new results that are more difficult to obtain using Eulerian or hybrid Eulerian–Lagrangian coordinates.

2. Lagrangian framework, global existence and uniqueness

2.1. Lagrangian framework

Consider system (1) on the real line with an initial condition

$$u(x,0) = u_0(x). \tag{9}$$

The particle paths are defined by the trajectories $\eta(X,t)$ which emanate from position X at time $t=0$ (X is also the particle label) and satisfy

$$\frac{d\eta}{dt}(X,t) = u^\alpha\bigl(\eta(X,t),t\bigr). \tag{10}$$

Along the particle paths, the solution u is constant:

$$u\bigl(\eta(X,t),t\bigr) = u_0(X), \quad \text{for all } t. \tag{11}$$

Using (1b), (10) can be written as

$$\frac{d\eta}{dt}(X,t) = \int_{\mathbb{R}} \psi^\alpha\bigl(\eta(X,t)-y\bigr)u(y,t)\,dy$$

and after the change of variables $y = \eta(Y,t)$,

$$\frac{d\eta}{dt}(X,t) = \int_{\mathbb{R}} \psi^\alpha\bigl(\eta(X,t)-\eta(Y,t)\bigr)u_0(Y)\eta_Y(Y,t)\,dY. \tag{12}$$

Here we used (11) and worked under the assumption that $\eta(X,t)$ is a diffeomorphism in X for all t so that the change of variables is justified. Taking a derivative with respect to X in (12), we get

$$\frac{d\eta_X}{dt}(X,t) = \eta_X(X,t) \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(X,t) - \eta(Y,t)\big) u_0(Y) \eta_Y(Y,t)\, dY, \tag{13}$$

where the prime sign $'$ denotes differentiation of ψ^α with respect to its argument. Eqs. (12) and (13) are the primary objects of study in this work. Both of these equations explicitly allow the function u_0 to be discontinuous and/or non-zero at infinity. Also, it is worth noting that the only difference between (10)–(11) and the characteristic equations for the inviscid Burgers equation (7) is that, for the Leray regularization, the velocity field of the characteristics in (10) is the smoothed velocity field u^α, not u.

In the following paragraphs, we show the global existence and uniqueness of solutions of (12)–(13) with $\eta(X,0) = X$ and $\eta_X(X,0) = 1$. The regularity theory also shows that $\eta(X,t)$ remains a diffeomorphism in X for all time. This justifies the change of variable used in deriving (12), which in turn proves that (12)–(13) is in fact equivalent to (10)–(11).

2.2. Local existence and uniqueness

We substitute

$$\eta_X(X,t) = 1 + f(X,t) \tag{14}$$

into (13). Note that the fundamental theorem of calculus and (14) together imply

$$\eta(X,t) - \eta(Y,t) = \int_Y^X \eta_Z(Z,t)\, dZ = \int_Y^X \big(1 + f(Z,t)\big)\, dZ. \tag{15}$$

Hence from (13) we derive the following evolution equation for $f(X,t)$:

$$\frac{df}{dt}(X,t) = \big(1 + f(X,t)\big) \int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X \big(1 + f(Z,t)\big)\, dZ\right) u_0(Y)\big(1 + f(Y,t)\big)\, dY. \tag{16}$$

We take $f(X,t)$ as the fundamental variable, and will show that (16) with the initial condition $f(X,0) = 0$ is locally well-posed in a certain Banach space.

Let us first define the norm

$$\|f\|_{\mathbf{E}} = \|f\|_\infty + \sup_{X \in \mathbb{R}} \left| \int_{-\infty}^X f(s)\, ds \right|. \tag{17}$$

Consider the normed vector space of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\|f\|_{\mathbf{E}} < \infty$. Let $\mathbf{E}$ denote the completion of this normed vector space. Then $\mathbf{E}$ is a Banach space.

Let $U \subset \mathbf{E}$ be the open set given by all $f \in \mathbf{E}$ with $\|f\|_{\mathbf{E}} < 1 - \gamma$, where $\gamma \in (0,1)$ is a fixed real number. Let $V[f]$ be the vector field defined by the right-hand side of (16), namely,

$$V[f] := \big(1 + f(X)\big) \int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X \big(1 + f(Z)\big)\, dZ\right) u_0(Y)\big(1 + f(Y)\big)\, dY. \tag{18}$$

The local existence result is given by the following theorem:

Theorem 2.1 *(Local existence). Suppose that ψ^α satisfies $\psi^\alpha, \psi^{\alpha\prime} \in L^1(\mathbb{R})$ and $\psi^{\alpha\prime\prime} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Also assume that the initial condition $u_0 \in L^\infty(\mathbb{R})$ can be written as $u_0(X) = v_0(X) + w_0(X)$, where v_0 is bounded and Lipschitz continuous on $\mathbb{R}$ with Lipschitz constant K and $w_0 \in L^1(\mathbb{R})$. Then there exist $\epsilon > 0$ and a unique C^1 integral curve $t \mapsto f(t) \in U$, defined on the interval $t \in (-\epsilon, \epsilon)$, satisfying the initial-value problem*

$$\frac{df}{dt} = V\big[f(t)\big], \qquad f(0) = 0.$$

Proof. Under our hypotheses, it is trivial to show that V maps U to $\mathbf{E}$. Next we show that the map V is Lipschitz on U in the $\mathbf{E}$-norm, i.e., there exists $L > 0$ such that for all $f, g \in U$,

$$\big\|V[f] - V[g]\big\|_{\mathbf{E}} \leqslant L\|f - g\|_{\mathbf{E}}.$$

Then we apply the standard Picard theorem on a Banach space (see Theorem 4.1 in [11] or Theorem II.D.2 in [4]) to conclude the short time existence of a unique solution of (16).

The proof of the Lipschitz condition is rather long, so we defer it to the end of the paper—see Section 5 for the complete proof. □

Remarks.

1. Under the hypotheses of the above theorem, we obtain $f(X,t)$. We may then define $\eta(X,t)$ using

$$\eta(X) = X + \int_{-\infty}^{X} f(Z)\,dZ \tag{19}$$

and $\eta_X(X,t)$ using (14). In this way, we obtain unique solutions for (12)–(13). The theorem guarantees that $f(\cdot,t)$ stays in U for $t \in (-\epsilon, \epsilon)$, and from the proof we know this is sufficient for $\eta(\cdot,t)$ to be a diffeomorphism for each t. This legitimates the change of variable used in deriving (12) from (10), (1b) and (11).

2. For discontinuous initial conditions u_0, a "weak" solution $u(x,t)$ of the original system (1) has to be understood in the sense of (10)–(11), with the values of u simply being transported by the particle maps η. Note that this is not the usual concept of a weak solution for hyperbolic conservation laws that utilizes test functions and integration by parts.
3. In the subsequent sections we consider the Leray-regularized Burgers equation (1) with Riemann and Riemann-like initial data. The assumption from Theorem 2.1 that the initial condition u_0 can be written as a sum of a Lipschitz function v_0 and an integrable function w_0 was made precisely to include these types of initial conditions. Note, for instance, that the Riemann initial data

$$u_0(X) = \begin{cases} u_L, & X < 0, \\ u_R, & X > 0, \end{cases} \tag{20}$$

can be written as $u_0 = v_0 + w_0$, with

$$v_0(X) = \begin{cases} u_L, & X < 0, \\ u_L + (u_R - u_L)X, & 0 < X < 1, \\ u_R, & X \geqslant 1, \end{cases}$$

and

$$w_0(X) = \begin{cases} u_R - u_L - (u_R - u_L)X, & 0 < X < 1, \\ 0 & \text{otherwise}. \end{cases}$$

4. To prove Theorem 2.1, we assumed that the kernel ψ has a bounded second derivative. The assumption, which is needed for handling the L^1-component w_0 of the initial data, includes kernels such as (2), (3), and (4) for $n \geqslant 2$. However, the assumption excludes certain non-smooth kernels such as the Helmholtz kernel given by (8), which is the $n = 1$ case of (4).
   However, in Section 3 we solve exactly the Riemann problem with the Helmholtz kernel (8) and therefore show that such a solution exists. This calculation indicates that the assumption $\psi'' \in L^\infty(\mathbb{R})$ may either be removed or replaced by a weaker condition. For the smoothed Riemann initial data considered in Section 4, this assumption on the kernel is no longer needed, since the L^1-component w_0 of the initial data is simply 0.

2.3. Global existence

We now move from local to global well-posedness. Inspired by the work of R. Camassa [3], we prove that crossing of characteristics cannot occur in the system (12)–(13). Therefore, the solution exists and retains its smoothness for arbitrarily large finite times.

Theorem 2.2 *(Global existence). The solution of* (12)–(13) *exists for arbitrarily large finite times.*

Proof. As noted above, the fact that characteristics do not cross in finite time imply global existence. We argue by contradiction. Suppose there is a time T when the crossing of characteristics first occurs. Denote by $\overline{X}$ the location from which one of these characteristics emanates at $t = 0$. Crossing of characteristics at $(\overline{X}, T)$ implies $\eta_X(\overline{X}, T) = 0$. Also, since $\eta_X(X, 0) = 1 > 0$ initially, for all X, we have $\eta_X(X, t) > 0$, for all X and $0 \leqslant t < T$.

Evaluate (13) at $\overline{X}$ for $t \in [0, T)$ and divide by $\eta_X(\overline{X}, t)$. We obtain

$$\frac{1}{\eta_X(\overline{X}, t)} \frac{d\eta_X}{dt}(\overline{X}, t) = \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(\overline{X}, t) - \eta(Y, t)\big) u_0(Y) \eta_Y(Y, t)\, dY.$$

The left-hand side can be written as $d/dt \log|\eta_X(\overline{X}, t)|$. Using $\eta_X(\overline{X}, 0) = 1$ we obtain after integration from $t = 0$ to $t = T$,

$$\big|\eta_X(\overline{X}, T)\big| = \exp\left(\int_0^T \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(\overline{X}, t) - \eta(Y, t)\big) u_0(Y) \eta_Y(Y, t)\, dY dt\right).$$

For every $t \in [0, T)$, the following estimates are immediate:

$$\begin{aligned}
\left| \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(\overline{X}, t) - \eta(Y, t)\big) u_0(Y) \eta_Y(Y, t)\, dY \right| &\leqslant \|u_0\|_\infty \int_{\mathbb{R}} \big|\psi^{\alpha\prime}\big(\eta(\overline{X}, t) - \eta(Y, t)\big)\big| \eta_Y(Y, t)\, dY \\
&= \|u_0\|_\infty \int_{\mathbb{R}} \big|\psi^{\alpha\prime}\big(\eta(\overline{X}, t) - y\big)\big|\, dy \\
&= \|u_0\|_\infty \big\|\psi^{\alpha\prime}\big\|_{L^1}.
\end{aligned}$$

The change of variable $y = \eta(Y, t)$ is justified, since $\eta(Y, t)$ is a diffeomorphism in Y for all $t \in [0, T)$. Next we infer that

$$\big|\eta_X(\overline{X}, T)\big| \geqslant \exp\big(-\|u_0\|_\infty \big\|\psi^{\alpha\prime}\big\|_{L^1} T\big) > 0,$$

which contradicts the assumption that $\eta_X(\overline{X}, T) = 0$. $\square$

3. Riemann problem

Having established the well-posedness of (12)–(13) with possibly discontinuous $u_0(X)$, we proceed to solve the Riemann problem. Before proceeding, let us note that the kernels ψ can be defined in a piecewise fashion

$$\psi(x) = \begin{cases} \psi^-(x), & x < 0, \\ \psi^+(x), & x > 0. \end{cases}$$

For smooth kernels like (2), ψ^- and ψ^+ are identical, but for non-smooth kernels like (8) they are different. Let ϕ denote the piecewise anti-derivative of ψ:

$$\phi(x) = \begin{cases} \phi^-(x), & x < 0, \\ \phi^+(x), & x > 0, \end{cases}$$

where

$$\phi^-(x) = \int_{-\infty}^{x} \psi^-(y)\,dy \quad \text{and} \quad \phi^+(x) = -\int_{x}^{\infty} \psi^+(y)\,dy.$$

Clearly, $\phi(x) \to 0$ as $x \to \pm\infty$. Note also that ϕ is discontinuous at $x = 0$. Since we assumed that ψ is even,

$$\phi^-(0) = \frac{1}{2} \quad \text{and} \quad \phi^+(0) = -\frac{1}{2}. \tag{21}$$

Therefore, $\phi(x)$ has a jump at $x = 0$:

$$\phi(0+) - \phi(0-) = \phi^+(0) - \phi^-(0) = -1.$$

Note that this jump does not depend at all on $\psi(x)$ being defined piecewise for $x < 0$ and $x > 0$. For example, for $\psi(x)$ given by the Gaussian (2), one may check that (21) remains true. As another example, for $\psi(x)$ given by (8), we have

$$\psi^+(x) = \frac{1}{2}e^{-x} \quad \Longrightarrow \quad \phi^+(x) = -\frac{1}{2}e^{-x}, \tag{22a}$$

$$\psi^-(x) = \frac{1}{2}e^{x} \quad \Longrightarrow \quad \phi^-(x) = \frac{1}{2}e^{x}. \tag{22b}$$

Now consider (10)–(11), or equivalently (12)–(13), with Riemann initial data $u_0(X)$ given by (20). In this case, (12) reduces to

$$\frac{\partial \eta}{\partial t}(X,t) = \frac{u_L}{\alpha}\int_{-\infty}^{0} \psi\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY + \frac{u_R}{\alpha}\int_{0}^{\infty} \psi\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY. \tag{23}$$

3.1. An exact solution to the Riemann problem

There are three cases depending on where X is situated relative to 0. For all three cases, we know from Section 2 that $\eta(X,t)$ is a strictly increasing function of X for each t fixed.

Case I: $X<0$. Then (23) becomes

$$\begin{aligned}\frac{\partial\eta}{\partial t}(X,t) &= \frac{u_L}{\alpha}\int_{-\infty}^{X}\psi^{+}\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY \\ &\quad + \frac{u_L}{\alpha}\int_{X}^{0}\psi^{-}\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY \\ &\quad + \frac{u_R}{\alpha}\int_{0}^{\infty}\psi^{-}\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY.\end{aligned}$$

Using the anti-derivative of $\psi^{\pm}$ we may write the previous equation as

$$\begin{aligned}\frac{\partial\eta}{\partial t}(X,t) = {}&-u_L\left[\phi^{+}\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_{-\infty}^{X} - u_L\left[\phi^{-}\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_{X}^{0} \\ &- u_R\left[\phi^{-}\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_{0}^{\infty}.\end{aligned}$$

Using the fact that $\eta(Y,t)\to\pm\infty$ as $Y\to\pm\infty$, the decay-at-infinity of ϕ and (21) we have

$$\frac{\partial\eta}{\partial t}(X,t) = u_L + (u_R-u_L)\phi^{-}\left(\frac{\eta(X,t)-\eta(0,t)}{\alpha}\right). \tag{24}$$

Case II: $X=0$. Now (23) becomes

$$\begin{aligned}\frac{\partial\eta}{\partial t}(0,t) &= \frac{u_L}{\alpha}\int_{-\infty}^{0}\psi^{+}\left(\frac{\eta(0,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY \\ &\quad + \frac{u_R}{\alpha}\int_{0}^{\infty}\psi^{-}\left(\frac{\eta(0,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY.\end{aligned}$$

Using the anti-derivative of $\psi^{\pm}$ we write the previous equation as

$$\frac{\partial\eta}{\partial t}(0,t) = -u_L\left[\phi^{+}\left(\frac{\eta(0,t)-\eta(Y,t)}{\alpha}\right)\right]_{-\infty}^{0} - u_R\left[\phi^{-}\left(\frac{\eta(0,t)-\eta(Y,t)}{\alpha}\right)\right]_{0}^{\infty}.$$

Using $\eta(Y,t)\to\pm\infty$ as $Y\to\pm\infty$, the decay-at-infinity of ϕ and (21), we have

$$\frac{\partial\eta}{\partial t}(0,t) = \frac{1}{2}(u_L+u_R). \tag{25}$$

Case III: $X > 0$. Eq. (23) becomes

$$\frac{\partial \eta}{\partial t}(X,t) = \frac{u_L}{\alpha}\int_{-\infty}^{0} \psi^+\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY + \frac{u_R}{\alpha}\int_{0}^{X} \psi^+\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY + \frac{u_R}{\alpha}\int_{X}^{\infty} \psi^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY.$$

A similar calculation to that of Case I leads to

$$\frac{\partial \eta}{\partial t}(X,t) = u_R + (u_R - u_L)\phi^+\left(\frac{\eta(X,t)-\eta(0,t)}{\alpha}\right). \tag{26}$$

Eq. (25) can be readily solved:

$$\eta(0,t) = \frac{1}{2}(u_L + u_R)t. \tag{27}$$

The trajectory that emanates at $X = 0$ is a line of slope $\frac{1}{2}(u_L + u_R)$, which is the Rankine–Hugoniot shock speed.

This value of $\eta(0,t)$ can be plugged into (24) and (26). Now everything amounts to solving a differential equation for $\eta(X,t)$—for $X < 0$ use (24) and for $X > 0$ use (26), with initial condition $\eta(X,0) = X$. In either case $\eta(0,t)$ is given by (27).

Recall (10). Comparing this equation with (24), (25), and (26), it is clear that the exact solution of the Riemann problem in the u variable is

$$u^\alpha(x,t) = \begin{cases} u_L + (u_R - u_L)\phi^-(\frac{x-\frac{1}{2}(u_L+u_R)}{\alpha}), & x < \frac{t}{2}(u_L+u_R), \\ \frac{1}{2}(u_L+u_R), & x = \frac{t}{2}(u_L+u_R), \\ u_R + (u_R - u_L)\phi^+(\frac{x-\frac{1}{2}(u_L+u_R)}{\alpha}), & x > \frac{t}{2}(u_L+u_R). \end{cases} \tag{28}$$

The values of u are simply transported along characteristics—see (11). At every t, $\eta(X,t)$ is a smooth monotone increasing map on the real line. The particle originating at $X = 0$ follows the straight line given by (27). Therefore, trajectories $\eta(X,t)$ starting from positions $X > 0$ are mapped into $x > \frac{1}{2}(u_L + u_R)$, to the right of $\eta(0,t)$. Similarly, trajectories $\eta(X,t)$ with $X < 0$ are mapped into $x < \frac{1}{2}(u_L + u_R)$. From (11) and (20) we conclude

$$u(x,t) = \begin{cases} u_L, & x < \frac{t}{2}(u_L+u_R), \\ u_R, & x > \frac{t}{2}(u_L+u_R). \end{cases} \tag{29}$$

It is interesting to note that the solution $u(x,t)$ has no dependence on α and that makes the limit $\alpha \to 0$ in (29) trivial. Also note that it does not depend on the choice of the mollifier ψ either. The pointwise limit of u^α given by (28) can also be computed easily, using the decay-at-infinity of ϕ^- and ϕ^+:

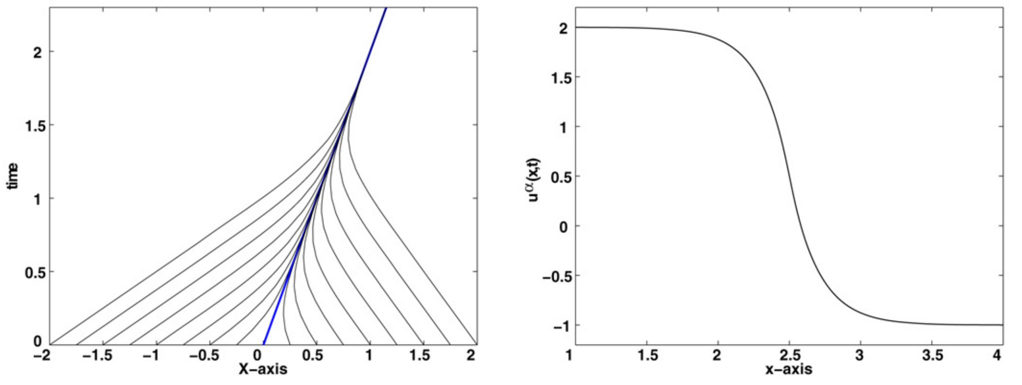

Fig. 1. Shock initial condition with $u_L = 2$, $u_R = -1$. Left: plot of the particle trajectories $\eta(X,t)$ given by (32), with $\alpha = 0.2$. Note that all particle trajectories approach the shock (thick line) as $t \to \infty$. Right: plot of the regularized velocity field u^α from (33) at $t = 5$. As $\alpha \to 0$ the profile steepens and $u^\alpha(x,t)$ converges pointwise to the entropic shock solution $u(x,t) = 2$ for $x < \frac{1}{2}t$ and $u(x,t) = -1$ for $x > \frac{1}{2}t$.

$$\lim_{\alpha\to 0} u^\alpha(x,t) = \begin{cases} u_L, & x < \frac{t}{2}(u_L + u_R), \\ \frac{1}{2}(u_L + u_R), & x = \frac{t}{2}(u_L + u_R), \\ u_R, & x > \frac{t}{2}(u_L + u_R). \end{cases}$$

3.2. *Shock and rarefaction solutions*

It is well known that for $u_L > u_R$, the unique entropy solution of the Burgers equation with initial condition (20) is a piecewise constant solution that has a discontinuity across the shock line $x = \frac{1}{2}(u_L + u_R)$. The solution has values u_L and u_R to the left and to the right of the shock, respectively. From (29) we conclude that the Leray-regularized Burgers equation captures the correct entropy solution of the Burgers equation. The Leray smoothing bends the characteristics (see Fig. 1) so that no shock is formed. All characteristics approach the shock line as $t \to \infty$.

For $u_L < u_R$, the correct entropy solution of the Burgers equation with initial condition (20) is a rarefaction wave,

$$u(x,t) = \begin{cases} u_L, & x < tu_L, \\ x/t, & tu_L < x < tu_R, \\ u_R, & x > tu_R. \end{cases}$$

It is clear from (29) that the Leray regularization of the Burgers equation fails to capture the rarefaction wave. What it does capture is the unphysical shock solution, which in textbooks is referred to as the solution where characteristics emanate from the shock line. See Fig. 2 for a plot of the characteristics in the case $u_L < u_R$.

3.3. *Specializing to the case of the Helmholtz kernel (8)*

Let us fix ψ to be the Helmholtz kernel (8) and write down the exact solution of the Riemann problem. We use (22) and (27) to rewrite (24) and (26). We obtain two ordinary differential equations where X appears as a parameter:

$$\frac{\partial \eta}{\partial t}(X,t) = u_L - \frac{1}{2}(u_L - u_R)\exp\left(\frac{\eta(X,t) - \frac{t}{2}(u_L + u_R)}{\alpha}\right), \quad X < 0, \tag{30}$$

$$\frac{\partial \eta}{\partial t}(X,t) = u_R - \frac{1}{2}(u_R - u_L)\exp\left(\frac{\frac{t}{2}(u_L + u_R) - \eta(X,t)}{\alpha}\right), \quad X > 0. \tag{31}$$

Subject to the initial condition $\eta(X,0)=X$, the exact solution is

$$\eta(X,t)=\begin{cases}\frac{t}{2}(u_L+u_R)-\alpha\log(1+(-1+\exp[-X/\alpha])\exp[\frac{t(u_R-u_L)}{2\alpha}]), & X<0,\\ \frac{t}{2}(u_L+u_R), & X=0,\\ tu_R+\alpha\log(-1+\exp[\frac{t(u_L-u_R)}{2\alpha}]+\exp[X/\alpha]), & X>0.\end{cases} \tag{32}$$

From (28) and (22) we get

$$u^\alpha(x,t)=\begin{cases}u_L-\frac{1}{2}(u_L-u_R)\exp([x-\frac{t}{2}(u_L+u_R)]/\alpha), & x<\frac{t}{2}(u_L+u_R),\\ \frac{1}{2}(u_L+u_R), & x=\frac{t}{2}(u_L+u_R),\\ u_R-\frac{1}{2}(u_R-u_L)\exp([\frac{t}{2}(u_L+u_R)-x]/\alpha), & x>\frac{t}{2}(u_L+u_R).\end{cases} \tag{33}$$

We illustrate the smoothing mechanism of the Leray regularization by inspecting the exact solution (32) and (33).

Suppose we have $u_L>u_R$, the shock case. It is easy to perform the following limits:

- α fixed. All trajectories $\eta(X,t)$ given by (32) approach the shock line $x=\frac{t}{2}(u_L+u_R)$ as t increases, that is

$$\lim_{t\to\infty}\left\{\eta(X,t)-\frac{t}{2}(u_L+u_R)\right\}=0.$$

- X and t fixed, take $\alpha\to 0$. For $X<0$ for instance (the case $X>0$ can be treated similarly), we have

$$\lim_{\alpha\to 0}\eta(X,t)=\begin{cases}X+tu_L, & t<-2X/(u_L-u_R),\\ \frac{t}{2}(u_L+u_R), & t>-2X/(u_L-u_R).\end{cases}$$

  Hence, the limit trajectory follows the line of slope u_L originating from X until the time when this line meets the shock. Beyond this time, the limit trajectory is the shock line. See also Fig. 1. This behavior as $\alpha\to 0$ is in perfect agreement with the solution of the inviscid Burgers equation.

In the rarefaction case, when $u_L<u_R$, we have

- α fixed. The trajectories $\eta(X,t)$ given by (32) have the following asymptotic behavior as t increases:

$$\lim_{t\to\infty}\left\{\eta(X,t)-X-tu_L\right\}=-\alpha\log\bigl(1-\exp[X/\alpha]\bigr),\quad \text{for } X<0, \tag{34a}$$

$$\lim_{t\to\infty}\left\{\eta(X,t)-X-tu_R\right\}=\alpha\log\bigl(1-\exp[-X/\alpha]\bigr),\quad \text{for } X>0. \tag{34b}$$

  Therefore, the trajectories approach lines of slopes u_L and u_R originating from X, shifted by a quantity that depends on α and X only. The closer the particle X is to the origin 0, the larger the shift is. For large X, the shift is negligible. This behavior can be observed in Fig. 2.
- For X and t fixed and $\alpha\to 0$, we have

$$\lim_{\alpha\to 0}\eta(X,t)=\begin{cases}X+tu_L, & X<0,\\ X+tu_R, & X>0.\end{cases}$$

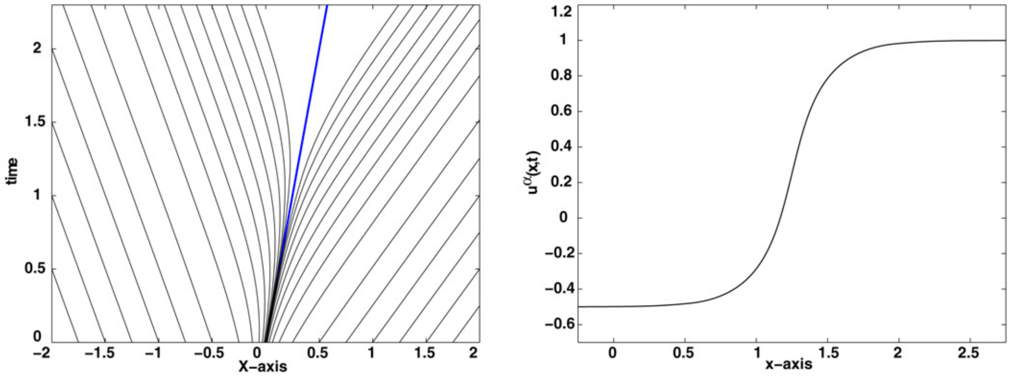

Fig. 2. Rarefaction initial condition with $u_L = -0.5$, $u_R = 1$. Left: plot of the particle trajectories $\eta(X,t)$ given by (32), with $\alpha = 0.2$. Note that trajectories emanating from particles close to the origin follow the shock (thick line) for a short time, then approach lines of slopes u_L or u_R as $t \to \infty$ (see (34)). Right: plot of the regularized velocity field u^α from (33) at $t = 5$. As $\alpha \to 0$ the profile steepens and $u^\alpha(x,t)$ converges pointwise to the unphysical shock solution with $u(x,t) = -0.5$ for $x < \frac{1}{4}t$ and $u(x,t) = 1$ for $x > \frac{1}{4}t$.

4. Smoothed rarefaction Riemann data

In the previous section we found that the Leray regularization (10)–(11) does not capture the entropy solution of the inviscid Burgers equation for Riemann initial data (20) with $u_L < u_R$. Instead of the rarefaction solution, the regularized system captures an unphysical shock.

In this section we consider smoothed Riemann data with $u_L < u_R$. We show that the resulting solution of the Leray-regularized equation does indeed rarefact, and that it converges as $\alpha \to 0$ to the entropic solution of the Burgers equation corresponding to the smoothed Riemann initial condition.

Consider the Riemann data (20) with $u_L < u_R$ smoothed by convolution with the mollifier λ:

$$u_0^\delta(x) = \lambda^\delta * u_0(x), \tag{35}$$

where $\delta > 0$ is a fixed small real number different from α. We require λ to be even, non-negative and have total integral 1. Define λ in a piecewise fashion

$$\lambda(x) = \begin{cases} \lambda^-(x), & x < 0, \\ \lambda^+(x), & x > 0. \end{cases}$$

Associated with λ is its piecewise anti-derivative θ, defined by

$$\theta(x) = \begin{cases} \theta^-(x) = \int_{-\infty}^{x} \lambda^-(y)\,dy, & x < 0, \\ \theta^+(x) = -\int_{x}^{\infty} \lambda^+(y)\,dy, & x > 0. \end{cases} \tag{36}$$

Similar to (21), we have

$$\theta^-(0) = \frac{1}{2} \quad \text{and} \quad \theta^+(0) = -\frac{1}{2}. \tag{37}$$

To proceed with our calculations, we must prove a simple lemma:

Lemma 4.1. *Consider the initial condition* (35)*, where u_0 is given by* (20)*, with $u_L < u_R$. Suppose the two mollifiers ψ and λ are even, non-negative and have total integral* 1*. Then the solution $\eta(X,t)$ of* (10)–(11) (*or equivalently* (12)–(13)) *satisfies $\eta_X(X,t) \geqslant 1$, for all X and t.*

Proof. Indeed, from (13) and integration by parts we have

$$\frac{d\eta_X}{dt}(X,t) = -\eta_X(X,t)\int_{\mathbb{R}} \frac{d}{dY}\psi^\alpha\big(\eta(X,t)-\eta(Y,t)\big)u_0^\delta(Y)\,dY$$
$$= \eta_X(X,t)\int_{\mathbb{R}} \psi^\alpha\big(\eta(X,t)-\eta(Y,t)\big)\frac{du_0^\delta}{dY}\,dY.$$

From (35) and (20) it is easy to show that

$$\frac{du_0^\delta}{dY} = \lambda^\delta(Y)(u_R-u_L).$$

For $u_L < u_R$, $\frac{du_0^\delta}{dY} \geqslant 0$, as the mollifier λ is non-negative. We showed in Theorem 2.2 that $\eta_X(X,t) > 0$ for all X and finite t. Therefore,

$$\frac{d\eta_X}{dt}(X,t) \geqslant 0,$$

and since $\eta_X(X,0) = 1$ for all X, the statement of the lemma follows. □

Theorem 4.2. *Consider the Cauchy problem consisting of the Leray-regularized Burgers equation* (1) *with the initial condition $u(x,0) = u_0^\delta(x)$, where u_0^δ given by* (35) *represents smoothed rarefaction Riemann data* (u_0 *given by* (20) *with $u_L < u_R$). Assume that the two mollifiers ψ and λ are even and non-negative. Then, as $\alpha \to 0$, the solution of this Cauchy problem converges to the solution of the inviscid Burgers equation* (7) *with the same initial data u_0^δ.*

Proof. Note that the two smoothing parameters, α and δ are independent of each other. In fact, we keep δ fixed and send $\alpha \to 0$.

Calculate u_0^δ from (35) and (20) to get

$$u_0^\delta(Y) = u_L\int_Y^\infty \lambda^\delta(Z)\,dZ + u_R\int_{-\infty}^Y \lambda^\delta(Z)\,dZ.$$

We look again at the Lagrangian formulation (10)–(11). Using (12), the differential equation for the characteristics becomes

$$\frac{\partial\eta}{\partial t}(X,t) = \int_{-\infty}^{\infty} \psi^\alpha\big(\eta(X,t)-\eta(Y,t)\big)\left[u_L\int_Y^\infty \lambda^\delta(Z)\,dZ + u_R\int_{-\infty}^Y \lambda^\delta(Z)\,dZ\right]\eta_Y(Y,t)\,dY.$$

Suppose $X < 0$. We get

$$\frac{\partial\eta}{\partial t}(X,t) = \frac{1}{\alpha}\int_{-\infty}^{X} \psi^+\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\left[u_L\left(1-\theta^-\left(\frac{Y}{\delta}\right)\right) + u_R\theta^-\left(\frac{Y}{\delta}\right)\right]\eta_Y(Y,t)\,dY$$
$$+\frac{1}{\alpha}\int_{X}^{0} \psi^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\left[u_L\left(1-\theta^-\left(\frac{Y}{\delta}\right)\right) + u_R\theta^-\left(\frac{Y}{\delta}\right)\right]\eta_Y(Y,t)\,dY$$

$$+\frac{1}{\alpha}\int_0^\infty \psi^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\left[-u_L\theta^+\left(\frac{Y}{\delta}\right)+u_R\left(1+\theta^+\left(\frac{Y}{\delta}\right)\right)\right]\eta_Y(Y,t)\,dY.$$

Using the anti-derivative of $\psi^\pm$ we may write the previous equation as

$$\begin{aligned}\frac{\partial\eta}{\partial t}(X,t)={}&-u_L\left[\phi^+\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_{-\infty}^X-u_L\left[\phi^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_X^0\\&-u_R\left[\phi^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_0^\infty-(u_R-u_L)\left\{\int_{-\infty}^X\frac{d}{dY}\phi^+\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\theta^-\left(\frac{Y}{\delta}\right)dY\right.\\&\left.+\int_X^0\frac{d}{dY}\phi^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\theta^-\left(\frac{Y}{\delta}\right)dY+\int_0^\infty\frac{d}{dY}\phi^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\theta^+\left(\frac{Y}{\delta}\right)dY\right\}.\end{aligned}$$

To handle the three integrals on the right-hand side, we perform integration by parts. Using the fact that $\eta(Y,t)\to\pm\infty$ as $Y\to\pm\infty$, the decay-at-infinity of ϕ and (21) we have

$$\begin{aligned}\frac{\partial\eta}{\partial t}(X,t)={}&u_L+(u_R-u_L)\phi^-\left(\frac{\eta(X,t)-\eta(0,t)}{\alpha}\right)-(u_R-u_L)\left\{\left[\phi^+\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\theta^-\left(\frac{Y}{\delta}\right)\right]_{-\infty}^X\right.\\&\left.+\left[\phi^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\theta^-\left(\frac{Y}{\delta}\right)\right]_X^0+\left[\phi^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\theta^+\left(\frac{Y}{\delta}\right)\right]_0^\infty\right\}\\&+(u_R-u_L)\left\{\int_{-\infty}^X\phi^+\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\frac{1}{\delta}\lambda^-\left(\frac{Y}{\delta}\right)dY\right.\\&\left.+\int_X^0\phi^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\frac{1}{\delta}\lambda^-\left(\frac{Y}{\delta}\right)dY+\int_0^\infty\phi^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\frac{1}{\delta}\lambda^+\left(\frac{Y}{\delta}\right)dY\right\}.\end{aligned}$$

We want to invoke Lebesgue's dominated convergence theorem and pass to the limit $\alpha\to 0$ in the last three integrals on the right-hand side. Note that $\phi^\pm$ are bounded functions and $\lambda^\pm$ are integrable, so the integrands are dominated by integrable functions. The integrands approach zero as $\alpha\to 0$; this is due to the decay-at-infinity of $\phi^\pm$ and Lemma 4.1, which implies that at each time t, $\eta(X,t)$ and $\eta(Y,t)$ are separated by at least $|X-Y|$, the initial separation.

Therefore, in the limit $\alpha\to 0$,

$$\begin{aligned}\frac{\partial\eta}{\partial t}(X,t)\to{}&u_L+(u_R-u_L)\phi^-\left(\frac{\eta(X,t)-\eta(0,t)}{\alpha}\right)\\&-(u_R-u_L)\left[\phi^+(0)\theta^-\left(\frac{X}{\delta}\right)+\phi^-\left(\frac{\eta(X,t)-\eta(0,t)}{\alpha}\right)\theta^-(0)\right.\\&\left.-\phi^-(0)\theta^-\left(\frac{X}{\delta}\right)-\phi^-\left(\frac{\eta(X,t)-\eta(0,t)}{\alpha}\right)\theta^+(0)\right].\end{aligned}$$

Finally, using (21) and (37) we obtain that, as $\alpha\to 0$,

$$\frac{\partial\eta}{\partial t}(X,t)\to u_L+(u_R-u_L)\theta^-\left(\frac{X}{\delta}\right),\quad \text{for } X<0. \tag{38}$$

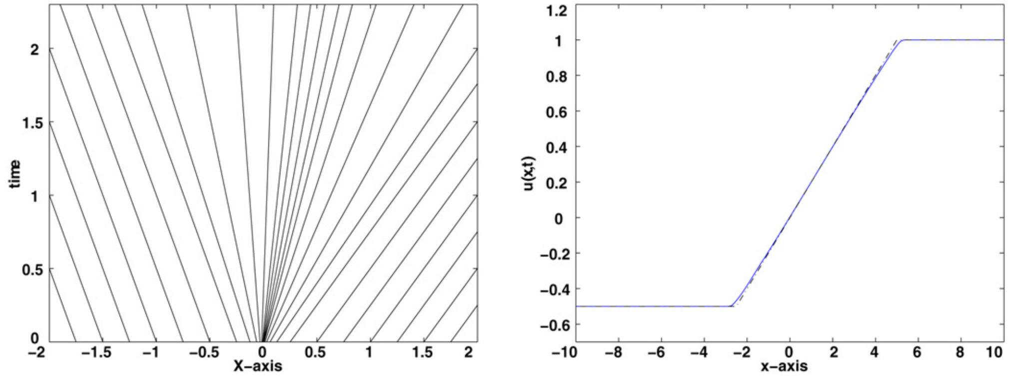

Fig. 3. Smoothed rarefaction initial condition with $u_L = -0.5$, $u_R = 1$ and $\delta = 0.05$. Left: plot of the $\alpha \to 0$ limit of the particle trajectories $\eta(X,t)$ of the Leray-regularized system—see (39). Note that all trajectories follow straight lines, producing the well-known rarefaction fan. Right: the solid line represents the plot at $t = 5$ of the $\alpha \to 0$ limit of the velocity field u^α of the Leray-regularized Burgers equation with smoothed rarefaction initial condition. The dash–dot line is the Burgers solution at $t = 5$ with unsmoothed rarefaction initial condition ($\delta = 0$).

A similar calculation can be performed for $X > 0$. We conclude that the $\alpha \to 0$ limit of the solutions of (10) with the initial condition (35) is given by

$$\eta(X,t) = \begin{cases} X + [u_L + (u_R - u_L)\theta^-(\frac{X}{\delta})]t, & X < 0, \\ X + [u_R + (u_R - u_L)\theta^+(\frac{X}{\delta})]t, & X > 0. \end{cases} \tag{39}$$

The straight lines given by (39) are precisely the characteristics of the inviscid Burgers equation (7) with the smoothed Riemann initial condition (35).[1] The proof concludes with the observation that the solutions of both the regularized equation (1) and the inviscid Burgers equation simply transport the initial condition u_0^δ along characteristics. □

In Fig. 3, we plot the characteristics (39) for the special case when the mollifier is given by the Helmholtz kernel (8) and recover the well-known rarefaction fan.

5. Proof of Theorem 2.1

We want to show that $V : U \subset \mathbf{E} \to \mathbf{E}$ is a Lipschitz map, i.e. there exists $L > 0$ such that for all $f, g \in U$,

$$\|V[f] - V[g]\|_{\mathbf{E}} \leqslant L\|f - g\|_{\mathbf{E}}.$$

[1] This calculation is standard. Let $u(x,t)$ solve the inviscid Burgers equation with initial data (35) and $u_L < u_R$; then u is smooth and constant along the characteristics. For $X < 0$:

$$\begin{aligned} \frac{d\eta(X,t)}{dt} &= u_0^\delta(X) = \frac{u_L}{\delta}\int_{-\infty}^{X} \lambda^+\left(\frac{X-Y}{\delta}\right)dY + \frac{u_L}{\delta}\int_X^0 \lambda^-\left(\frac{X-Y}{\delta}\right)dY + \frac{u_R}{\delta}\int_0^\infty \lambda^-\left(\frac{X-Y}{\delta}\right)dY \\ &= -\left\{u_L\theta^+\left(\frac{X-Y}{\delta}\right)\Big|_{-\infty}^X + u_L\theta^-\left(\frac{X-Y}{\delta}\right)\Big|_X^0 + u_R\theta^-\left(\frac{X-Y}{\delta}\right)\Big|_0^\infty\right\} \\ &= u_L + (u_R - u_L)\theta^-\left(\frac{X}{\delta}\right), \end{aligned}$$

in agreement with (38).

Take $f, g \in U$. The **E**-norm (17) has two components; we start by estimating $\|V[f] - V[g]\|_\infty$.

Part I: Estimate $\|V[f] - V[g]\|_\infty$. A simple manipulation leads to

$$\begin{aligned} V[f] - V[g] = {} & \big(f(X) - g(X)\big) \int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X \big(1 + f(Z)\big)\,dZ\right) u_0(Y)\big(1 + f(Y)\big)\,dY \\ & + \big(1 + g(X)\big)\Bigg[\int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X \big(1 + f(Z)\big)\,dZ\right) u_0(Y)\big(1 + f(Y)\big)\,dY \\ & - \int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X \big(1 + g(Z)\big)\,dZ\right) u_0(Y)\big(1 + g(Y)\big)\,dY\Bigg]. \end{aligned} \tag{40}$$

We deal in turn with the first and second terms on the right-hand side of (40). For the first term,

$$\begin{aligned} & \left\|\big(f(X) - g(X)\big) \int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X \big(1 + f(Z)\big)\,dZ\right) u_0(Y)\big(1 + f(Y)\big)\,dY\right\|_\infty \\ & \leqslant \|f - g\|_\infty \left\|\int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X \big(1 + f(Z)\big)\,dZ\right) u_0(Y)\big(1 + f(Y)\big)\,dY\right\|_\infty. \end{aligned}$$

Note that $f \in \mathbf{E}$ implies that $\eta(X)$ in (19) is well defined for all X. Moreover, with this definition, η is differentiable and $\eta_X(X) = 1 + f(X)$. By the fundamental theorem of calculus,

$$\eta(X) - \eta(Y) = \int_Y^X \big(1 + f(Z)\big)\,dZ. \tag{41}$$

Now note that $f \in U$ implies $\|f\|_\infty < 1 - \gamma$. Hence $1 + f(Z) > \gamma > 0$ almost everywhere. So, for $X > Y$, both sides of (41) are positive. We conclude that η is a monotonically increasing differentiable function. By (19), the range of η is all of $\mathbb{R}$, so η is in fact a diffeomorphism of $\mathbb{R}$. Then

$$\begin{aligned} \left\|\int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X 1 + f(Z)\,dZ\right) u_0(Y)\big(1 + f(Y)\big)\,dY\right\|_\infty & = \operatorname*{ess\,sup}_{X\in\mathbb{R}} \left|\int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(X) - \eta(Y)\big) u_0(Y)\eta_Y(Y)\,dY\right| \\ & = \operatorname*{ess\,sup}_{X\in\mathbb{R}} \left|\int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(X) - y\big) u_0\big(\eta^{-1}(y)\big)\,dy\right| \\ & \leqslant \left\|\psi^{\alpha\prime}\right\|_{L^1} \|u_0\|_\infty. \end{aligned}$$

We used the change of variable $y = \eta(Y)$ to pass from the second to the third line. To summarize, we have shown that

$$\left\|\big(f(X) - g(X)\big) \int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X 1 + f(Z)\,dZ\right) u_0(Y)\big(1 + f(Y)\big)\,dY\right\|_\infty \leqslant \|f - g\|_\infty \left\|\psi^{\alpha\prime}\right\|_{L^1} \|u_0\|_\infty, \tag{42}$$

which takes care of the first term on the right-hand side of (40). We now estimate the second term:

$$\begin{aligned}
&\left\| \big(1+g(X)\big)\left[\int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X 1+f(Z)\,dZ\right)u_0(Y)\big(1+f(Y)\big)\,dY \right.\right.\\
&\qquad \left.\left. - \int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X 1+g(Z)\,dZ\right)u_0(Y)\big(1+g(Y)\big)\,dY\right]\right\|_\infty \\
&\quad \leqslant \|1+g\|_\infty \left\| \int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X 1+f(Z)\,dZ\right)u_0(Y)\big(1+f(Y)\big)\,dY \right.\\
&\qquad \left. - \int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X 1+g(Z)\,dZ\right)u_0(Y)\big(1+g(Y)\big)\,dY\right\|_\infty. \qquad (43)
\end{aligned}$$

The first piece $\|1+g\|_\infty$ of this product of norms can be estimated above by 2. To estimate the second piece of the product we write

$$\begin{aligned}
&\int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X 1+f(Z)\,dZ\right)u_0(Y)\big(1+f(Y)\big)\,dY - \int_{\mathbb{R}} \psi^{\alpha\prime}\left(\int_Y^X 1+g(Z)\,dZ\right)u_0(Y)\big(1+g(Y)\big)\,dY \\
&\quad = \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(X)-\eta(Y)\big)u_0(Y)\eta_Y(Y)\,dY - \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big)u_0(Y)\xi_Y(Y)\,dY. \qquad (44)
\end{aligned}$$

Recall the decomposition $u_0 = v_0 + w_0$, where v_0 is Lipschitz and $w_0 \in L^1$. We plug this decomposition of u_0 into the right-hand side of (44) and estimate the v_0 and w_0 terms separately.

We keep our old definition of η as in (19). We also define ξ by

$$\xi(X) = X + \int_{-\infty}^X g(Z)\,dZ, \qquad (45)$$

and note that since $g \in U$, we can show just as before that ξ is a diffeomorphism of $\mathbb{R}$.

The difference of the w_0 terms on the right-hand side of (44) can be estimated as follows:

$$\begin{aligned}
&\int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(X)-\eta(Y)\big)w_0(Y)\eta_Y(Y)\,dY - \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big)w_0(Y)\xi_Y(Y)\,dY \\
&\quad = \int_{\mathbb{R}} \big[\psi^{\alpha\prime}\big(\eta(X)-\eta(Y)\big) - \psi^{\alpha\prime}\big(\xi(X)-\eta(Y)\big)\big]w_0(Y)\eta_Y(Y)\,dY \\
&\qquad + \int_{\mathbb{R}} \big[\psi^{\alpha\prime}\big(\xi(X)-\eta(Y)\big) - \psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big)\big]w_0(Y)\eta_Y(Y)\,dY \\
&\qquad + \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big)w_0(Y)\big(\eta_Y(Y)-\xi_Y(Y)\big)\,dY. \qquad (46)
\end{aligned}$$

Let T.V. denote total variation. Then the first integral on the right-hand side can be bounded:

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \left[\psi^{\alpha\prime}\big(\eta(X)-\eta(Y)\big)-\psi^{\alpha\prime}\big(\xi(X)-\eta(Y)\big)\right] w_0(Y)\eta_Y(Y)\,dY\right| \\
&\quad\leqslant \|w_0\|_\infty \int_{\mathbb{R}} \left|\psi^{\alpha\prime}\big(\eta(X)-y\big)-\psi^{\alpha\prime}\big(\xi(X)-y\big)\right| dy \\
&\quad\leqslant \|w_0\|_\infty \left|\eta(X)-\xi(X)\right| \text{T.V.}\,\psi^{\alpha\prime} \\
&\quad\leqslant \|w_0\|_\infty \left|\int_{-\infty}^{X} \big(f(Y)-g(Y)\big)\,dY\right| \text{T.V.}\,\psi^{\alpha\prime} \\
&\quad\leqslant \|w_0\|_\infty \|f-g\|_{\mathbf{E}}\, \text{T.V.}\,\psi^{\alpha\prime}.
\end{aligned}$$

In the above derivation, we (i) changed variables via $y=\eta(Y)$, (ii) used $\psi^{\alpha\prime\prime}\in L^1(\mathbb{R})$, which implies that T.V. $\psi^{\alpha\prime}$ is finite, and (iii) used definitions (19) and (45) for η and ξ, respectively.

For the second integral on the right-hand side of (46), we use the mean value theorem for $\psi^{\alpha\prime}$:

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \left[\psi^{\alpha\prime}\big(\xi(X)-\eta(Y)\big)-\psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big)\right] w_0(Y)\eta_Y(Y)\,dY\right| \\
&\quad\leqslant \|\eta_Y\|_\infty \left\|\psi^{\alpha\prime\prime}\right\|_\infty \|\eta-\xi\|_\infty \|w_0\|_{L^1} \\
&\quad\leqslant 2\left\|\psi^{\alpha\prime\prime}\right\|_\infty \|f-g\|_{\mathbf{E}} \|w_0\|_{L^1}.
\end{aligned}$$

Finally, the third integral on the right-hand side of (46) can be estimated as

$$\begin{aligned}
\left|\int_{\mathbb{R}} \psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big) w_0(Y)\big(\eta_Y(Y)-\xi_Y(Y)\big)\,dY\right| &\leqslant \|w_0\|_\infty \|\eta_X-\xi_X\|_\infty \int_{\mathbb{R}} \left|\psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big)\right| dY \\
&= \|w_0\|_\infty \|f-g\|_\infty \int_{\mathbb{R}} \left|\psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big)\right| \frac{\xi_Y(Y)}{\xi_Y(Y)}\,dY \\
&\leqslant \frac{1}{\gamma}\|w_0\|_\infty \|f-g\|_\infty \left\|\psi^{\alpha\prime}\right\|_{L^1}.
\end{aligned}$$

For the last inequality we used the fact that $\xi_Y(Y)=1+g(Y)>\gamma$ to bound $1/\xi_Y(Y)$. The ξ_Y from the numerator was used to produce the L^1-norm of $\psi^{\alpha\prime}$.

We now use these results back in (46). Define the constant C_1 by

$$C_1=\|w_0\|_\infty \,\text{T.V.}\,\psi^{\alpha\prime}+2\left\|\psi^{\alpha\prime\prime}\right\|_\infty \|w_0\|_{L^1}+\frac{1}{\gamma}\|w_0\|_\infty \left\|\psi^{\alpha\prime}\right\|_{L^1}.$$

Therefore,

$$\left|\int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(X)-\eta(Y)\big) w_0(Y)\eta_Y(Y)\,dY-\int_{\mathbb{R}} \psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big) w_0(Y)\xi_Y(Y)\,dY\right| \leqslant C_1\|f-g\|_{\mathbf{E}}, \tag{47}$$

where the constant C_1 depends on the initial condition u_0, the kernel ψ and the smoothing parameter α. C_1 also depends on γ, a fixed number used in the definition of the set U.

Now we aim to derive a similar estimate for the difference of the v_0 terms on the right-hand side of (44). We have

$$\begin{aligned}
&\int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(X)-\eta(Y)\big)v_0(Y)\eta_Y(Y)\,dY - \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big)v_0(Y)\xi_Y(Y)\,dY \\
&\quad= \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(X)-y\big)v_0\big(\eta^{-1}(y)\big)\,dy - \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\xi(X)-y\big)v_0\big(\xi^{-1}(y)\big)\,dy \\
&\quad= \int_{\mathbb{R}} \big[\psi^{\alpha\prime}\big(\eta(X)-y\big)-\psi^{\alpha\prime}\big(\xi(X)-y\big)\big]v_0\big(\eta^{-1}(y)\big)\,dy \\
&\qquad+ \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\xi(X)-y\big)\big(v_0\big(\eta^{-1}(y)\big)-v_0\big(\xi^{-1}(y)\big)\big)\,dy.
\end{aligned}$$

We have made the substitution $y=\eta(Y)$ in the first integral and $y=\xi(Y)$ in the second integral. Now take the absolute value of both sides of the previous equation. The right-hand side may be estimated as follows:

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(X)-\eta(Y)\big)v_0(Y)\eta_Y(Y)\,dY - \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big)v_0(Y)\xi_Y(Y)\,dY\right| \\
&\quad\leqslant \|v_0\|_\infty \int_{\mathbb{R}} \big|\psi^{\alpha\prime}\big(\eta(X)-y\big)-\psi^{\alpha\prime}\big(\xi(X)-y\big)\big|\,dy + \big\|\psi^{\alpha\prime}\big\|_{L^1}\big\|v_0\big(\eta^{-1}(y)\big)-v_0\big(\xi^{-1}(y)\big)\big\|_\infty \\
&\quad\leqslant \|v_0\|_\infty\big|\eta(X)-\xi(X)\big|\,\mathrm{T.V.}\,\psi^{\alpha\prime} + K\big\|\psi^{\alpha\prime}\big\|_{L^1}\big\|\eta^{-1}-\xi^{-1}\big\|_\infty. \qquad (48)
\end{aligned}$$

Here we used the Lipschitz continuity of v_0. Using definitions (19) and (45) for η and ξ, respectively, the first piece from (48) can be bounded:

$$\|v_0\|_\infty\big|\eta(X)-\xi(X)\big|\,\mathrm{T.V.}\,\psi^{\alpha\prime} \leqslant \|v_0\|_\infty\|f-g\|_{\mathbf{E}}\,\mathrm{T.V.}\,\psi^{\alpha\prime}.$$

For the second piece from (48), we fix x, then plug $X_1=\eta^{-1}(x)$ into (19) and $X_2=\xi^{-1}(x)$ into (45), and subtract. This gives

$$\begin{aligned}
\eta^{-1}(x)-\xi^{-1}(x) &= -\int_{-\infty}^{\eta^{-1}(x)} f(z)\,dz + \int_{-\infty}^{\xi^{-1}(x)} g(z)\,dz \\
&= -\int_{-\infty}^{\eta^{-1}(x)} f(z)\,dz + \int_{-\infty}^{\eta^{-1}(x)} g(z)\,dz - \int_{-\infty}^{\eta^{-1}(x)} g(z)\,dz + \int_{-\infty}^{\xi^{-1}(x)} g(z)\,dz \\
&= -\int_{-\infty}^{\eta^{-1}(x)} \big(f(z)-g(z)\big)\,dz + \int_{\eta^{-1}(x)}^{\xi^{-1}(x)} g(z)\,dz.
\end{aligned}$$

Taking absolute values, we obtain

$$\big|\eta^{-1}(x)-\xi^{-1}(x)\big| \leqslant \left|\int_{-\infty}^{\eta^{-1}(x)} \big(f(z)-g(z)\big)\,dz\right| + \big|\eta^{-1}(x)-\xi^{-1}(x)\big|\|g\|_\infty.$$

Combining the left-hand side with the second term on the right-hand side, we get

$$\big(1-\|g\|_\infty\big)\big|\eta^{-1}(x)-\xi^{-1}(x)\big| \leqslant \left| \int_{-\infty}^{\eta^{-1}(x)} \big(f(z)-g(z)\big)\,dz \right|.$$

Taking the supremum over all x and using $\|g\|_\infty < 1-\gamma$ together with the fact that η^{-1} is a diffeomorphism, we get

$$\big\|\eta^{-1}-\xi^{-1}\big\|_\infty \leqslant \frac{1}{\gamma}\|f-g\|_{\mathbf{E}}. \tag{49}$$

Thus the total estimate for (48) is

$$\left| \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\eta(X)-\eta(Y)\big)v_0(Y)\eta_Y(Y)\,dY - \int_{\mathbb{R}} \psi^{\alpha\prime}\big(\xi(X)-\xi(Y)\big)v_0(Y)\xi_Y(Y)\,dY \right| \leqslant C_2\|f-g\|_{\mathbf{E}}, \tag{50}$$

where

$$C_2 = \|v_0\|_\infty \,\mathrm{T.V.}\,\psi^{\alpha\prime} + \frac{K}{\gamma}\big\|\psi^{\alpha\prime}\big\|_{L^1}$$

depends on the initial condition, ψ, α and γ.

This bound for (48) combined with (47) gives an overall bound of $(C_1+C_2)\|f-g\|_{\mathbf{E}}$ for (44). Now, we can go all the way back to (40); the first term on its right-hand side is estimated with (42), while the second is bounded by $\|1+g\|_\infty(C_1+C_2)\|f-g\|_{\mathbf{E}}$, according to (43) and the estimate for (44).

Therefore,

$$\begin{aligned}\big\|V[f]-V[g]\big\|_\infty &\leqslant \|f-g\|_\infty\big\|\psi^{\alpha\prime}\big\|_{L^1}\|u_0\|_\infty + \|1+g\|_\infty(C_1+C_2)\|f-g\|_{\mathbf{E}}\\ &\leqslant C_3\|f-g\|_{\mathbf{E}},\end{aligned} \tag{51}$$

where

$$C_3 = \big\|\psi^{\alpha\prime}\big\|_{L^1}\|u_0\|_\infty + 2(C_1+C_2).$$

Part II: Estimate

$$\sup_{X\in\mathbb{R}}\left| \int_{-\infty}^{X} \big(V[f](Z)-V[g](Z)\big)\,dZ \right|.$$

Note that for any $h\in U$, we have

$$V[h](X) = \frac{d}{dX}\int_{\mathbb{R}} \psi^\alpha\left(\int_Y^X \big(1+h(Z)\big)\,dZ\right)u_0(Y)\big(1+h(Y)\big)\,dY.$$

Again, we argue that $h \in U$ implies that $1 + h > \gamma > 0$, so that $\int_Y^X (1 + h(Z))\,dZ$ is infinite when $X \to -\infty$. Since ψ^α vanishes at $\pm\infty$, the fundamental theorem of calculus gives

$$\int_{-\infty}^{X} V[h](Z)\,dZ = \int_{\mathbb{R}} \psi^\alpha \left(\int_Y^X \bigl(1 + h(Z)\bigr)\,dZ \right) u_0(Y)\bigl(1 + h(Y)\bigr)\,dY.$$

This implies

$$\begin{aligned}
&\int_{-\infty}^{X} \bigl(V[f](Z) - V[g](Z)\bigr)\,dZ \\
&\quad = \int_{\mathbb{R}} \psi^\alpha \left(\int_Y^X \bigl(1 + f(Z)\bigr)\,dZ \right) u_0(Y)\bigl(1 + f(Y)\bigr)\,dY \\
&\qquad - \int_{\mathbb{R}} \psi^\alpha \left(\int_Y^X \bigl(1 + g(Z)\bigr)\,dZ \right) u_0(Y)\bigl(1 + g(Y)\bigr)\,dY \\
&\quad = \int_{\mathbb{R}} \psi^\alpha \bigl(\eta(X) - \eta(Y)\bigr) u_0(Y)\eta_Y(Y)\,dY - \int_{\mathbb{R}} \psi^\alpha \bigl(\xi(X) - \xi(Y)\bigr) u_0(Y)\xi_Y(Y)\,dY. \qquad (52)
\end{aligned}$$

Note that (52) has the same form as the right-hand side of (44), but with ψ^α in place of $\psi^{\alpha\prime}$. Using the decomposition of u_0 into the Lipschitz continuous part v_0 and the integrable component w_0, we can derive in a completely analogous way estimates similar to (47) and (50). The constants C_1', C_2' corresponding to C_1 and C_2, have the same form as the latter, except that $\psi^{\alpha\prime}$ must be replaced by ψ^α. In this way, we derive

$$\sup_{X \in \mathbb{R}} \left| \int_{-\infty}^{X} \bigl(V[f](Z) - V[g](Z)\bigr)\,dZ \right| \leqslant \bigl(C_1' + C_2'\bigr)\|f - g\|_{\mathbf{E}}, \qquad (53)$$

where C_1' and C_2' depend only on u_0, ψ, α and γ.

Finally, let $L = C_3 + C_1' + C_2'$. Adding (51) and (53), we obtain $\|V[f] - V[g]\|_{\mathbf{E}} \leqslant L\|f - g\|_{\mathbf{E}}$, as desired.

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