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A NOTE ON A NON-LOCAL KURAMOTO-SIVASHINSKY EQUATION

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Abstract. In this note we outline some improvements to a result of Hilhorst, Peletier, Rotariu and Sivashinsky [5] on the L_2 boundedness of solutions to a non-local variant of the Kuramoto-Sivashinsky equation with additional stabilizing and destabilizing terms. We are able to make the following improvements: in the case of odd data we reduce the exponent in the estimate $\limsup_{t\to\infty} \|u\| \le CL^\nu$ from $\nu = \frac{11}{5}$ to $\nu = \frac{3}{2}$, and for the case of general initial data we establish an estimate of the above form with $\nu = \frac{13}{6}$. We also remove the restrictions on the magnitudes of the parameters in the model and track the dependence of our estimates on these parameters, assuming they are at least $O(1)$.

1. **Introduction.**

Background. We consider the following initial-value problem for the non-local variant of the Kuramoto-Sivashinsky (KS) equation analyzed by Hilhorst, Peletier, Rotariu and Sivashinsky [5]:

$$u_t = -u_{xxxx} - u_{xx} - uu_x + 2\pi\kappa I(u) - \alpha(xu_x + 2u), \tag{1a}$$

$$u(\pm L) = 0, \quad u_{xx}(\pm L) = 0, \tag{1b}$$

$$u(x,0) = u_0(x), \tag{1c}$$

where $I(u)$ is the following non-local operator:

$$I(u) = \sum_n \frac{n}{L^2}\sin(\frac{n\pi x}{L})\int_{-L}^{L}\sin(\frac{n\pi y}{L})u(y)dy + \sum_n \frac{(2n+1)}{L^2}\cos(\frac{(2n+1)\pi x}{L})\int_{-L}^{L}\cos(\frac{(2n+1)\pi y}{L})u(y)dy. \tag{2}$$

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The non-local term $I(u)$ represents an additional instability due to thermal expansion effects (see [7]), while the term $\alpha(xu_x + 2u)$ represents the stabilizing effect of fluid stretching (see [6]).

In [5], using a modification of the prior construction of Collet, Eckmann, Epstein and Stubbe [2], Hilhorst et al. were able to show that the above equation satisfies an L_2 bound of the form $\|u\|_2 \leq CL^{\frac{11}{5}}$ under the assumptions that the coefficients satisfy $\kappa < \frac{1}{3^{7/4}}(1+3\alpha)^{3/4}$ and that the initial condition is odd (which is preserved under the flow). In this paper we show how to use a result of two of the authors [1] to improve the exponent as well as to remove the restriction on κ and the restriction to odd solutions. We get an estimate $\|u\|_2 \leq C(\kappa, \alpha)L^{\frac{3}{2}}$ for odd solutions and $\|u\|_2 \leq C(\kappa, \alpha)L^{\frac{13}{6}}$ for the general case.

It should be noted that the method of Giacomelli and Otto [3] gives a better estimate for the standard Kuramoto-Sivashinsky equation. In this problem, however, the method appears to be difficult to apply due to the effect of the fluid stretching term at small wavenumbers.

Notation. In this paper we choose to work with the parameter κ, whereas in [5] the authors work with $\gamma = \frac{2\pi\kappa}{L}$. We track the dependence of our estimates on κ and α, assuming that these parameters are at least $O(1)$. We do not assume, however, any relationship between κ, α and L.

Throughout this paper $\|\cdot\|_2$ will denote the usual L_2 norm: $\|\phi\|_2^2 = \int_{-L}^{L} \phi^2(x)dx$.

2. **Main results.**

This paper follows the Lyapunov function construction detailed in numerous previous papers [8, 2, 4]. The main idea is to establish that the L_2 norm of $u(x,t) - \phi(x)$ is a Lyapunov functional for some suitably chosen function ϕ.

Lemma 1. *Suppose $u(x,t)$ solves (1) and $\phi(x)$ satisfies the boundary condition (1b). The rate of change of the Lyapunov functional $\|u - \phi\|_2$ satisfies the estimate*

$$\begin{aligned} \frac{d}{dt}\|u-\phi\|_2^2 &\leq \int_{-L}^{L} \left[-u_{xx}^2 + \left(3 + \frac{4\kappa}{\pi}\right) u_x^2 + (2\pi\kappa - \alpha - \phi_x)u^2 \right] dx + R(\phi,\phi) \\ &\leq \int_{-L}^{L} \left[-\frac{1}{2}u_{xx}^2 - (\phi_x - \lambda)u^2 \right] dx + R(\phi,\phi), \end{aligned} \tag{3}$$

where the bilinear remainder term $R(\phi,\phi)$ is given by

$$R(\phi,\phi) = \int_{-L}^{L} \left[\phi_{xx}^2 + \phi_x^2 + (2\pi\kappa + \alpha)\phi^2 + \alpha x^2 \phi_x^2 \right] dx \tag{4}$$

and the constant λ by

$$\lambda = \frac{1}{2}\left(3 + \frac{4\kappa}{\pi}\right)^2 + 2\pi\kappa - \alpha. \tag{5}$$

Proof. Start from

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{-L}^{L}(u(x,t)-\phi(x))^2 dx &= \int_{-L}^{L}(u-\phi)u_t\, dx \\ &= \int_{-L}^{L}(u-\phi)(-u_{xxxx} - u_{xx} - uu_x + 2\pi\kappa I(u) - \alpha(xu_x + 2u))\, dx. \end{aligned}$$

After integrating by parts and using the homogeneous boundary conditions (1b), we get

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = \int_{-L}^{L}\Big[-u_{xx}^2 + u_x^2 + 2\pi\kappa u I(u) - \frac{3}{2}\alpha u^2 + \phi_{xx}u_{xx} - \phi_x u_x - \frac{1}{2}\phi_x u^2 - 2\pi\kappa\phi I(u) + \alpha(\phi - x\phi_x)u\Big]dx. \quad (6)$$

The Cauchy-Schwartz inequality implies $|\int fg dx| \le \frac{1}{2}\int f^2 dx + \frac{1}{2}\int g^2 dx$, for any two functions f and g in L^2. We apply this inequality to estimate $\int uI(u)$, $\int \phi_{xx}u_{xx}$, $-\int \phi_x u_x$, $-\int \phi I(u)$, $\int \phi u$ and $-\int x\phi_x u$. Hence, from (6) we have

$$\frac{d}{dt}\|u-\phi\|_2^2 \le \int_{-L}^{L}\left[-u_{xx}^2 + 3u_x^2 + (2\pi\kappa - \alpha)u^2 + 4\pi\kappa I(u)^2 - \phi_x u^2\right]dx + R(\phi,\phi),$$

where

$$R(\phi,\phi) = \int_{-L}^{L}\left[\phi_{xx}^2 + \phi_x^2 + (2\pi\kappa + \alpha)\phi^2 + \alpha x^2\phi_x^2\right]dx.$$

A straightforward computation yields

$$\|I(u)\|_2^2 = \frac{1}{\pi^2}\|u_x\|_2^2.$$

Applying Cauchy-Schwartz again for the term containing u_x^2,

$$\left(3 + \frac{4\kappa}{\pi}\right)\int_{-L}^{L} u_x^2 dx \le \frac{1}{2}\int_{-L}^{L} u_{xx}^2 + \frac{1}{2}\left(3 + \frac{4\kappa}{\pi}\right)^2\int_{-L}^{L} u^2 dx,$$

the result follows. □

Thus, if the potential ϕ_x can be chosen so that the quadratic form satisfies the coercivity estimate $\int_{-L}^{L}\frac{1}{2}u_{xx}^2 + (\phi_x - \lambda)u^2 \ge \|u\|_2^2$, then the rate of change of the Lyapunov functional is negative for $\|u\|$ larger than $R(\phi,\phi)$, and a ball in L_2 centered about ϕ of sufficiently large radius is exponentially attracting and invariant in forward time.

The main tool for constructing such a function ϕ is provided by a result derived in [1], which guarantees the existence of a $2L$-periodic, mean-zero function ϕ_x such that

$$\int_{-L}^{L}\frac{1}{4}u_{xx}^2 + \phi_x u^2 \ge (1+\lambda)\|u\|_2^2, \quad (7)$$

for any fixed constants $\lambda > 0$ and for all $u \in H^2$ satisfying a Dirichlet boundary condition at the origin (this condition is satisfied, for instance, if we assume the solution u is odd).

The potential ϕ_x constructed in [1] is of the form

$$\phi_x = \mu + L^{\frac{4}{3}}q(L^{\frac{1}{3}}x), \quad (8)$$

where $q(y)$ is a compactly supported function with $\int q(y)dy = -2\mu$, so that ϕ_x is mean zero. More specifically,

$$q(y) = \frac{Q(y)}{y^2}, \quad (9)$$

where Q is chosen to be

$$Q(y) = \begin{cases} -q_0 f(\frac{y}{\delta}) & y \in (0, \delta) \\ -q_0 & y \in (\delta, \frac{a}{2} - \delta) \\ -q_0 + (q_0 + q_1) f(\frac{y - \frac{a}{2} + \delta}{\delta}) & y \in (\frac{a}{2} - \delta, \frac{a}{2}) \\ q_1 & y \in (\frac{a}{2}, a) \\ q_1 f(1 + \frac{a-y}{\delta}) & y \in (a, a+\delta). \end{cases} \tag{10}$$

Here, $f(y)$ is a C^∞ function satisfying

$$\begin{aligned} &\lim_{y\to 0} f(y) = 0, \\ &\lim_{y\to 0} f^{(k)}(y) = 0 \quad \text{for } k \geq 1, \\ &\lim_{y\to 1} f(y) = 1, \\ &\lim_{y\to 1} f^{(k)}(y) = 0 \quad \text{for } k \geq 1, \end{aligned}$$

and q_0, q_1, a and δ are positive constants.

Provided q_0, q_1 and a satisfy

$$\begin{aligned} q_0 a^2 &< 1, \\ q_1 &> \frac{q_0}{1 - a^2 q_0}, \end{aligned}$$

the above potential has the property (see [1]) that

$$\int_{-L}^{L} \frac{1}{4} u_{xx}^2 + \phi_x u^2 \geq \mu \|u\|_2^2, \tag{11}$$

for all $u \in H^2$ with $u(0) = 0$.

Hence, if we are to satisfy (7), we must choose $\mu = O(\lambda)$. It is easy to see from (9) and (10) that for δ small, $\int q(y)dy = -\frac{C}{\delta} + O(1)$. To satisfy the mean-zero condition on ϕ_x, we must choose $\delta = O(\mu^{-1})$, and thus we have $\delta = O(\lambda^{-1})$.

The construction outlined above yields the following estimate:

Lemma 2. *The construction (8)-(10) with $\mu = O(\lambda)$ and $\delta = O(\lambda^{-1})$ gives a potential ϕ_x satisfying*

$$\|\phi\|_2 \leq C_1 \lambda L^{\frac{3}{2}}, \tag{12}$$

$$\|\phi_x\|_2 \leq C_2 \lambda^{\frac{3}{2}} L^{\frac{7}{6}}, \tag{13}$$

$$\|\phi_{xx}\|_2 \leq C_3 \lambda^{\frac{5}{2}} L^{\frac{3}{2}}, \tag{14}$$

for $\lambda, L \geq O(1)$.

Here and elsewhere in this paper, C_i denotes a generic constant, independent of L.

Proof. A short calculation shows

$$\begin{aligned} \|\phi_x\|_2^2 &= \int_{-L}^{L} (\mu + L^{\frac{4}{3}} q(xL^{\frac{1}{3}}))^2 dx \\ &\leq 2 \int_{-L}^{L} \left(\mu^2 + L^{\frac{8}{3}} q^2(xL^{\frac{1}{3}})\right) dx \\ &= 4\mu^2 L + 2L^{\frac{7}{3}} \int_{-L^{4/3}}^{L^{4/3}} q^2(y) dy. \end{aligned}$$

From (9) and (10) it is easy to see that $\int q^2(y)dy = O(\delta^{-3}) = O(\lambda^3)$, so the second term dominates the first for $\lambda, L \geq O(1)$. The estimates of the other two norms follow analogously. □

Remark 1. Using a similar approach, Wittenberg [9] showed that the construction of Collet, Eckmann, Epstein and Stubbe [2] can be made to give an arbitrarily large lowest eigenvalue, at the cost of increasing the norms of ϕ, ϕ_x, ϕ_{xx}. It appears to us that this, together with the prior results of Hilhorst et al., suffice to remove the restrictions on the coefficients κ, α in [5], though keeping the same exponent, $\nu = \frac{11}{5}$.

The construction presented above can be trivially modified to apply to u satisfying a Dirichlet boundary condition at $\pm L$ rather than at the origin (as assumed in (1b)). We will denote this new potential function $\tilde{\phi}_x$. As in (8), the potential function is given by a constant plus a compactly supported piece (see [1]):

$$\tilde{\phi}_x = \mu + L^{\frac{4}{3}} \left(q((x-L)L^{\frac{1}{3}}) + q((x+L)L^{\frac{1}{3}}) \right). \tag{15}$$

Note that in (15) the support of the non-constant piece of $\tilde{\phi}_x$ is localized near the point(s) where a Dirichlet boundary condition is satisfied. In this case we have Dirichlet boundary conditions at both $-L$ and L, and for the sake of symmetry we chose to use both in the expression (15). Alternatively, one can just use a compactly supported function localized near one or the other. From translation invariance it is clear that $\tilde{\phi}_x$ satisfies (12)-(14).

The main advantage of the real-space construction of ϕ_x presented here is the improved estimate of $\|x\phi_x\|_2$. In the paper by Hilhorst et al. [5], the fact that ϕ_x is constructed in Fourier space means that there are no good estimates available for such a quantity. The only obvious estimate is $\|x\phi_x\|_2 \leq L\|\phi_x\|_2$. This estimate is not very tight, since the Fourier space construction makes it difficult to take advantage of the fact that the ϕ_x constructed there has most of its mass concentrated near the origin in real space. The real space construction used in [1] makes a more refined estimate of this term trivial.

Hilhorst et al. [5] consider only the case of odd initial data and obtain an attracting ball of size $L^{\frac{11}{5}}$. It is for this data that we see the biggest improvement, obtaining an attracting ball of radius $C(\kappa, \alpha)L^{\frac{3}{2}}$. On the other hand, for the potential function $\tilde{\phi}_x$ constructed for general initial data the estimate $\|x\phi_x\|_2 \leq L\|\tilde{\phi}_x\|_2$ scales correctly with L. In this case the only improvement is due to a slightly better construction of the potential function.

Lemma 3. *In the above constructions (8) and (15) with $\mu = O(\lambda)$ and $\delta = O(\lambda^{-1})$, the potentials ϕ_x, $\tilde{\phi}_x$ satisfy the estimates*

$$\|x\phi_x\|_2 \leq C\lambda L^{\frac{3}{2}}, \tag{16}$$

$$\|x\tilde{\phi}_x\|_2 \leq C\lambda^{\frac{3}{2}} L^{\frac{13}{6}}. \tag{17}$$

Proof. The proof is elementary. Compute

$$\begin{aligned} \|x\phi_x\|_2^2 &= \int_{-L}^{L} x^2(\mu + L^{\frac{4}{3}} q(xL^{\frac{1}{3}}))^2 dx \leq 2\mu^2 \int_{-L}^{L} x^2 dx + 2\int_{-L}^{L} x^2 L^{\frac{8}{3}} q^2(xL^{\frac{1}{3}}) dx \\ &= \frac{4}{3}\mu^2 L^3 + L^{\frac{5}{3}} \int_{-L^{4/3}}^{L^{4/3}} y^2 q^2(y) dy. \end{aligned}$$

From (9) and (10) we get that $\int y^2 q^2(y)dy = O(\delta^{-1}) = O(\lambda)$. Since $\mu^2 = O(\lambda^2)$, the first term dominates the second, and (16) follows.

The estimate (17) is derived similarly. For simplicity, we show it by using only the component of $\tilde{\phi}_x$ localized around $x = L$:

$$\begin{aligned}\|x\tilde{\phi}_x\|_2^2 &= \int_{-L}^{L} x^2(\mu + L^{\frac{4}{3}} q((x-L)L^{\frac{1}{3}}))^2 dx \\ &= \int_{-2L^{4/3}}^{0} (L + L^{-\frac{1}{3}} y)^2 (\mu + L^{\frac{4}{3}} q(y))^2 L^{-\frac{1}{3}} dy \\ &= \int_{-2L^{4/3}}^{0} (L^2 + 2L^{\frac{2}{3}} y + +L^{-\frac{2}{3}} y^2)(\mu^2 + 2\mu L^{\frac{4}{3}} q(y) + L^{\frac{8}{3}} q^2(y)) L^{-\frac{1}{3}} dy.\end{aligned}$$

Take the product of the two brackets and note that the leading order term is

$$\int_{-2L^{4/3}}^{0} L^2 L^{\frac{8}{3}} q^2(y)) L^{-\frac{1}{3}} dy = O(\lambda^3 L^{\frac{13}{3}}).$$

Hence (17) follows. It is interesting to note that in one case the dominant contribution to the integral is given by the constant term, while in the second case the dominant contribution is due to the localized term. □

By using (12)-(14), (16) and (17), one can now examine the size of $R(\phi, \phi)$ given in (4) and derive the following main result of this note.

Theorem 1. *Consider the initial-boundary value problem (1).*

1. *If the initial data $u_0 \in H^2[-L, L]$ is odd, then there exists a function ϕ constructed as in (8) and satisfying (12)-(14) such that*
$$\text{limsup}_{t\to\infty} \|u - \phi\|_2 \le C_1 \kappa^5 L^{\frac{3}{2}} + C_2 \alpha^{\frac{1}{2}} \kappa^2 L^{\frac{3}{2}}. \tag{18}$$
2. *For arbitrary initial data $u_0 \in H^2[-L, L]$, there exists a function $\tilde{\phi}$ constructed as in (15) such that*
$$\text{limsup}_{t\to\infty} \|u - \tilde{\phi}\|_2 \le C_1 \alpha^{\frac{1}{2}} \kappa^3 L^{\frac{13}{6}} + C_2 \kappa^5 L^{\frac{3}{2}}. \tag{19}$$

Proof. The attracting L^2-ball has size of order $O(\sqrt{R(\phi, \phi)})$. By examining (4), we note that the competing terms are $\|\phi_{xx}\|_2$, $(2\pi\kappa + \alpha)^{\frac{1}{2}} \|\phi\|_2$ and $\alpha^{\frac{1}{2}} \|x\phi_x\|_2$. For odd initial data, use (12), (14), (16) and the fact that $\lambda \le C\kappa^2$ (see (5)) to get

$$\begin{aligned}&\|\phi_{xx}\|_2 \le C_1 \kappa^5 L^{\frac{3}{2}}, \\ &(2\pi\kappa + \alpha)^{\frac{1}{2}} \|\phi\|_2 \le C_2 (2\pi\kappa + \alpha)^{\frac{1}{2}} \kappa^2 L^{\frac{3}{2}}, \\ &\alpha^{\frac{1}{2}} \|x\phi_x\|_2 \le C_3 \alpha^{\frac{1}{2}} \kappa^2 L^{\frac{3}{2}}.\end{aligned}$$

The estimate (18) follows now easily. For arbitrary initial data we use $\tilde{\phi}$, constructed as in (15). Due to translation invariance, the estimates for $\|\tilde{\phi}_{xx}\|_2$ and $(2\pi\kappa + \alpha)^{\frac{1}{2}} \|\tilde{\phi}\|_2$ are the same as those for ϕ. In addition, from (17),

$$\alpha^{\frac{1}{2}} \|x\tilde{\phi}_x\|_2 \le C_3 \alpha^{\frac{1}{2}} \kappa^3 L^{\frac{13}{6}}.$$

Hence, regardless of the relative sizes of κ and α, the term $(2\pi\kappa + \alpha)^{\frac{1}{2}} \|\tilde{\phi}\|_2$ is subdominant and (19) follows. □

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