

AGGREGATION-DIFFUSION ENERGIES ON CARTAN-HADAMARD MANIFOLDS OF UNBOUNDED CURVATURE

RAZVAN C. FETECU AND HANSOL PARK

ABSTRACT. We consider an aggregation-diffusion energy on Cartan-Hadamard manifolds with sectional curvatures that can grow unbounded at infinity. The energy corresponds to a macroscopic aggregation model that involves nonlocal interactions and linear diffusion. We establish necessary and sufficient conditions on the growth at infinity of the attractive interaction potential for ground states to exist. Specifically, we derive explicit conditions on the attractive potential in terms of the bounds on the sectional curvatures at infinity. To prove our results we establish a new logarithmic Hardy-Littlewood inequality for Cartan-Hadamard manifolds of unbounded curvature.

1. INTRODUCTION

In this paper we investigate the existence of ground states of the *free energy*

$$(1.1) \quad E[\rho] = \int_M \rho(x) \log \rho(x) dV(x) + \frac{1}{2} \iint_{M \times M} W(x, y) \rho(x) \rho(y) dV(x) dV(y),$$

defined for probability measures on M , where M is a general Cartan-Hadamard manifold of finite dimension, and $W : M \times M \rightarrow \mathbb{R}$ is an interaction potential – see (2.3) for the precise definition. Here, the integration is with respect to the Riemannian volume measure dV .

In terms of dynamical evolution, the energy E relates to the following nonlinear nonlocal evolution equation [2]:

$$(1.2) \quad \partial_t \rho(x) - \nabla_M \cdot (\rho(x) \nabla_M W * \rho(x)) = \Delta \rho(x),$$

where

$$W * \rho(x) = \int_M W(x, y) \rho(y) dV(y),$$

and $\nabla_M \cdot$ and ∇_M represent the Riemannian divergence and gradient, respectively. Critical points of the energy functional correspond to steady states of (1.2). In the Euclidean space $M = \mathbb{R}^n$, there exists an extensive literature on the analysis of the evolution equation (1.2) as the gradient flow of the energy E on a suitable Wasserstein space [2, 14]. In the general manifold setting, the gradient flow literature is less developed however, we note here the work in [42] for the pure interaction equation and that in [41] for the purely diffusive case.

In applications, equation (1.2) has been used to model a wide range of self-organizing phenomena in biology, physics, engineering and social sciences, such as swarming or flocking

Date: September 3, 2024.

2020 Mathematics Subject Classification. 35A30, 35B38, 53C21, 58J90.

Key words and phrases. Cartan-Hadamard manifolds, global minimizers, diffusion on manifolds, logarithmic HLS inequality, comparison theorems.

of biological organisms [11, 37], emergent behaviour in robotic swarms [27], and opinion formation [38]. When W is the Newtonian potential in \mathbb{R}^2 or \mathbb{R}^3 , (1.2) reduces to some classical models in mathematical biology and gravitational physics, namely the Patlak-Keller-Segel model of chemotaxis [35] and the Smoluchowski-Poisson equation [15].

The energy (1.1) consists of an entropy (or internal energy) and an interaction energy modelled by the potential W . In the evolution equation (1.2), these two components are reflected in the linear diffusion and the nonlocal transport term, respectively. Of particular interest is the case when W is a purely *attractive* potential, and any two points experience a pairwise attractive interaction. In such case, there is a competing effect between the nonlocal attraction and the local diffusion, as the first leads to aggregation/blow-up and the latter results in spreading. This paper investigates the subtle balance of these opposite effects that leads to ground states of the energy functional set up on general Cartan-Hadamard manifolds.

Linear diffusion on negatively-curved manifolds of unbounded curvature is a delicate subject. The Brownian motion on Cartan-Hadamard manifolds with sectional curvatures that grow too fast at infinity, is no longer stochastically complete, i.e., the process has finite lifetime [28, Section 15]. On such manifolds, the high negative curvature can sweep a Brownian particle to infinity in finite time. Stochastic incompleteness of the Brownian motion relates to the well-posedness of the Cauchy problem for the linear diffusion equation (in this case, the heat kernel no longer integrates to 1 for all $t > 0$) [28]. In the present work we do not address well-posedness issues for the evolution equation (1.2), but only consider the free energy (1.1) and investigate the existence of its ground states. In our study we place no restriction on the sectional curvatures, which are allowed to grow unbounded at infinity at any rate.

There is extensive literature on the free energy (1.1) and the aggregation-diffusion equation (1.2) posed on $M = \mathbb{R}^n$. The well-posedness and the asymptotic behaviour of solutions to equation (1.2) on \mathbb{R}^n , along with qualitative studies on its steady states, have been investigated by numerous authors, we refer to the recent review article [9] for a comprehensive discussion on such issues. In [12] the authors show that the stationary solutions of (1.2) are radially decreasing up to translation. The equilibration toward the heat kernel was investigated in [8], and the effect of the boundaries was studied in [36]. Most relevant to the present work is the work of Carrillo *et al.* [10], where the authors establish necessary and sufficient conditions on the interaction potential and diffusion, for global energy minimizers to exist. We also note that there has been separate interest in the existence and characterization of minimizers of the energy (1.1) without diffusion [3, 6, 18, 40] or with nonlinear instead of linear diffusion [4, 5, 13, 12, 19, 34].

Although the free energy (1.1) and the evolution equation (1.2) have been extensively studied in the Euclidean space, there is very little done for the manifold setup. In this paper we are exclusively interested in the set up of the aggregation-diffusion model on Cartan-Hadamard manifolds. Also, we assume that the interaction potential $W(x, y)$ depends only on the geodesic distance $d(x, y)$ between the points x and y . This assumption follows the *intrinsic* approach introduced in [26], and considered in various subsequent works on the plain interaction equation (no diffusion) on Riemannian manifolds [21, 22, 23, 25]. As examples where the manifold framework and the intrinsic approach are particularly relevant, we mention applications to robotics and biology, where limitations in environment, the topography, or mobility constraints restrict the agents to evolve on a certain configuration manifold.

The present work addresses and answers the following problem. Consider a Cartan-Hadamard manifold M and assume that the sectional curvatures at a generic point x are bounded below and above, respectively, by two negative functions $-c_m(r_x)$ and $-c_M(r_x)$, where r_x denotes the distance from x to a fixed (but arbitrary) pole on M . The functions $c_m(\cdot)$ and $c_M(\cdot)$ are allowed to grow at any rate at infinity, e.g., they can grow algebraically or exponentially. To contain the diffusion on a manifold with sectional curvatures that can possibly grow unbounded at infinity, the attractive interaction potential needs to grow correspondingly at large distances. We are then interested to establish necessary and sufficient conditions on the behaviour of the interaction potential W at infinity that prevent the spreading and result in existence of ground states of the energy functional.

The problem posed above was studied in [10] for $M = \mathbb{R}^n$ and in [24] for Cartan-Hadamard manifolds with constant curvature bounds (the case when $c_m(\cdot)$ and $c_M(\cdot)$ are constant functions). In the Euclidean case, it was found that a sharp condition for the existence of energy minimizers is that the attractive potential grows at least *logarithmically* at infinity. Similarly, for Cartan-Hadamard manifolds with sectional curvatures bounded below by a negative constant, spreading by diffusion is prevented provided W grows at least *superlinearly*. In the present work, as we allow the sectional curvatures to grow unbounded at infinity, an even stronger growth of the attractive potential is needed to contain the diffusion. We quantify precisely such conditions on W in terms of the upper and lower bounds of the sectional curvatures.

To establish the results of this paper, we derive several tools that have an interest in their own. The first is a volume comparison theorem in Riemannian geometry for manifolds with non-constant bounds of the sectional curvatures. We expect that the results of this comparison theorem are known to experts (e.g., the upper bound on the volume of geodesic balls is presented in [1, Section 1.9]). The second tool is a new logarithmic Hardy-Littlewood-Sobolev (HLS) inequality on Cartan-Hadamard manifolds of unbounded curvature. Compared to the HLS inequality on \mathbb{R}^n [10, 7], the logarithmic HLS inequality on Cartan-Hadamard manifolds includes an additional term that contains the Jacobian of the exponential map at a fixed pole. This additional term is a manifestation of the curvature of the manifold, as the Jacobian of the exponential map determines the volume growth of geodesic balls on the manifold.

We also note that there has been very recent interest on the well-posedness and long-time behaviour for PDE's on Cartan-Hadamard manifolds with either linear or nonlinear diffusion and *local* reaction terms [29, 30, 31, 32, 33]. In particular, in [31] the authors consider linear diffusion with a reaction term, and study the dichotomy global existence versus blow-up. Stochastic completeness of the manifold is a key assumption in their work, which limits the growth at infinity of the sectional curvatures to be at most quadratic. As pointed out above, we do not need to make such restrictive assumptions in our work, as our approach is entirely variational.

The summary of the paper is as follows. In Section 2 we present the assumptions, notations and some necessary background on comparison theorems in Riemannian geometry. In Section 3 we establish necessary conditions on the growth of the interaction potential for ground states to exist - the main result is Theorem 3.1. In Section 4 we present a new logarithmic HLS inequality on Cartan-Hadamard manifolds with non-constant lower bound of curvatures. Section 5 presents the proof of Theorem 5.1, which establishes sufficient conditions on the interaction potential that guarantee existence of global energy minimizers.

2. ASSUMPTIONS AND BACKGROUND

In this section we present the assumptions and set up the notations. We also present several comparison results in Riemannian geometry which will be used in our analysis.

2.1. Assumptions and notations. Throughout the paper we will make the following assumptions on the manifold M and the interaction potential W .

(M) M is an n -dimensional Cartan-Hadamard manifold, i.e., M is complete, simply connected, and has everywhere non-positive sectional curvature. We note that as a Cartan-Hadamard manifold, M has no conjugate points, it is diffeomorphic to \mathbb{R}^n and the exponential map at any point is a global diffeomorphism.

(W) The interaction potential $W : M \times M \rightarrow \mathbb{R}$ has the form

$$W(x, y) = h(d(x, y)), \quad \text{for all } x, y \in M,$$

where $h : [0, \infty) \rightarrow [0, \infty)$ is lower semi-continuous and non-decreasing. The fact that h is non-decreasing indicates that the interaction potential is purely *attractive*. Also note that h is bounded below, so it cannot have an infinite singularity at the origin.

We denote by d the intrinsic distance on M . For $x \in M$, and σ a two-dimensional subspace of $T_x M$, $\mathcal{K}(x; \sigma)$ denotes the sectional curvature of σ at x . We also denote by dV the Riemannian volume measure on M and by $\mathcal{P}_{ac}(M) \subset \mathcal{P}(M)$ the space of probability measures on M that are absolutely continuous with respect to dV . We will refer to an absolutely continuous measure directly by its density ρ , and write $\rho \in \mathcal{P}_{ac}(M)$ to mean $d\rho(x) = \rho(x)dV(x) \in \mathcal{P}_{ac}(M)$.

The precise definition of the free energy introduced in (1.1) is the following. We define $E : \mathcal{P}(M) \rightarrow [-\infty, \infty]$ by

$$(2.3) \quad E[\rho] = \begin{cases} \int_M \rho(x) \log \rho(x) dV(x) + \frac{1}{2} \iint_{M \times M} h(d(x, y)) \rho(x) \rho(y) dV(x) dV(y), & \text{if } \rho \in \mathcal{P}_{ac}(M), \\ \inf_{\substack{\rho_k \in \mathcal{P}_{ac}(M) \\ \rho_k \rightarrow \rho}} \liminf_{k \rightarrow \infty} E[\rho_k], & \text{otherwise,} \end{cases}$$

where $\rho_k \rightarrow \rho$ denotes weak convergence as measures, i.e.,

$$\int_M f(x) \rho_k(x) dV(x) \rightarrow \int_M f(x) \rho(x) dV(x), \quad \text{as } k \rightarrow \infty,$$

for all bounded and continuous functions $f : M \rightarrow \mathbb{R}$.

Remark 2.1. For interaction potentials h that are bounded from below, as considered in this paper, the energy functional can be alternatively defined as

$$E[\rho] = \begin{cases} \int_M \rho(x) \log \rho(x) dV(x) + \frac{1}{2} \iint_{M \times M} h(d(x, y)) \rho(x) \rho(y) dV(x) dV(y), & \text{if } \rho \in \mathcal{P}_{ac}(M), \\ +\infty, & \text{otherwise.} \end{cases}$$

We listed the more general definition (2.3) however, for consistency with how the energy was defined in [10] (for Euclidean spaces) and in [24] (for general manifolds).

Denote by $\mathcal{P}_1(M)$ the set of probability measures on M with finite first moment, i.e.,

$$\mathcal{P}_1(M) = \left\{ \mu \in \mathcal{P}(M) : \int_M d(x, x_0) d\mu(x) < \infty \right\},$$

for some fixed (but arbitrary) point $x_0 \in M$. For $\rho, \sigma \in \mathcal{P}(M)$, the *intrinsic* 1-Wasserstein distance is defined as

$$\mathcal{W}_1(\rho, \sigma) = \inf_{\gamma \in \Pi(\rho, \sigma)} \iint_{M \times M} d(x, y) d\gamma(x, y),$$

where $\Pi(\rho, \sigma) \subset \mathcal{P}(M \times M)$ is the set of transport plans between ρ and σ , i.e., the set of elements in $\mathcal{P}(M \times M)$ with first and second marginals ρ and σ , respectively. The space $(\mathcal{P}_1(M), \mathcal{W}_1)$ is a metric space [2].

Similar to the variational studies in Euclidean space [12, 13, 19], we fix the Riemannian centre of mass of the admissible densities. For a fixed arbitrary point $o \in M$ (referred throughout the paper as *pole*), we denote

$$\mathcal{P}_o(M) := \left\{ \rho \in \mathcal{P}_{ac}(M) \cap \mathcal{P}_1(M) : \int_M \log_o x \rho(x) dV(x) = 0 \right\},$$

where \log_o denotes the Riemannian logarithm map at o (i.e., the inverse of the Riemannian exponential map \exp_o) [20]. The integral condition above fixes the centre of mass of ρ at the pole o . This is the set of admissible densities, in which we will establish existence of global energy minimizers.

2.2. Rauch comparison theorem. A tool we use in our proofs is Rauch's comparison theorem [20, Chapter 10, Proposition 2.5]. Specifically, we use this result to compare lengths of curves on a general Cartan-Hadamard manifold with lengths of curves on spaces of constant curvature.

Theorem 2.1 (Rauch comparison theorem [17]). *Let M and \tilde{M} be Riemannian manifolds with $\dim(\tilde{M}) \geq \dim(M)$, and suppose that for all $p \in M$, $\tilde{p} \in \tilde{M}$, and $\sigma \subset T_p M$, $\tilde{\sigma} \subset T_{\tilde{p}} \tilde{M}$, the sectional curvatures \mathcal{K} and $\tilde{\mathcal{K}}$ of M and \tilde{M} , respectively, satisfy*

$$\tilde{\mathcal{K}}(\tilde{p}; \tilde{\sigma}) \geq \mathcal{K}(p; \sigma).$$

Let $p \in M$, $\tilde{p} \in \tilde{M}$ and fix a linear isometry $i : T_p M \rightarrow T_{\tilde{p}} \tilde{M}$. Let $r > 0$ be such that the restriction $\exp_p|_{B_r(0)}$ is a diffeomorphism and $\exp_{\tilde{p}}|_{B_r(0)}$ is non-singular. Let $c : [0, a] \rightarrow \exp_p(B_r(0)) \subset M$ be any differentiable curve and define $\tilde{c} : [0, a] \rightarrow \exp_{\tilde{p}}(B_r(0)) \subset \tilde{M}$ by

$$\tilde{c}(s) = \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}(c(s)), \quad s \in [0, a].$$

Then the length of c is greater or equal than the length of \tilde{c} .

2.3. Bounds on the Jacobian of the exponential map. We will follow notations and the general framework from [16]. A Jacobi field along a unit-speed geodesic γ is a differentiable vector field Y along γ that satisfies the Jacobi equation

$$(2.4) \quad \nabla_t^2 Y + \mathcal{R}(\gamma', Y)\gamma' = 0,$$

where ∇ denotes the Levi-Civita connection on M and \mathcal{R} represents the curvature tensor. Denote by \mathcal{J} the vector space of Jacobi fields along γ and by \mathcal{J}^\perp the subspace of \mathcal{J} consisting of the Jacobi fields orthogonal to γ . Jacobi fields are variation fields through geodesics, and they measure the spread and the volume growth of a geodesic spray.

Consider a point $x \in M$, a unit tangent vector $u \in T_x M$, and a unit-speed geodesic γ through x in the direction u , i.e., $\gamma(0) = x$, $\gamma'(0) = u$. Set an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x M$, with $e_n = u$, and consider the Jacobi fields $J_j(t)$, $j = 1, \dots, n-1$, along the geodesic γ that satisfy

$$J_j(0) = 0 \quad \text{and} \quad J'_j(0) = e_j, \quad j = 1, \dots, n-1.$$

We point out that $J_j(t) \in \mathcal{J}^\perp$ for all $t \geq 0$, $j = 1, \dots, n-1$. Denote by $\mathcal{A}(t; u)$ the matrix with columns $J_j(t)$, $j = 1, \dots, n-1$, i.e.,

$$\mathcal{A}(t) = [J_1(t) \ J_2(t) \ \dots \ J_{n-1}(t)].$$

Note that the matrix $\mathcal{A}(t)$ depends on the geodesic γ , in particular on the unit tangent vector u . When this dependence is important and can lead to confusion, we will use the notation $\mathcal{A}(t; u)$ instead.

By the properties of the Jacobi fields, we have

$$J_j(t) = (d \exp_x)_{te_n}(te_j), \quad j = 1, \dots, n-1,$$

and hence we can write

$$(2.5) \quad \mathcal{A}(t) = t[(d \exp_x)_{te_n}(e_1) \ (d \exp_x)_{te_n}(e_2) \ \dots \ (d \exp_x)_{te_n}(e_{n-1})].$$

In the present work we are particularly interested in the Jacobian $J(\exp_x)$ of the exponential map. Since

$$(d \exp_x)_{te_n}(e_n) = e_n,$$

by using (2.5) we can express

$$(2.6) \quad J(\exp_x)(te_n) = \det((d \exp_x)_{te_n}) = \det \begin{bmatrix} \frac{\mathcal{A}(t)}{t} & 0 \\ 0 & 1 \end{bmatrix} = \det(\mathcal{A}(t)/t) = \frac{\det \mathcal{A}(t)}{t^{n-1}}.$$

Throughout the paper we fix a generic point o in M which we will be referring to as the *pole* of the manifold. In general, a point on a manifold is called a pole if the exponential map at that point is a global diffeomorphism. On Cartan-Hadamard manifolds all points satisfy this property. We also make the notation

$$r_x := d(o, x), \quad \text{for all } x \in M.$$

The following theorem establishes estimates on the Jacobian $J(\exp_o)$. The theorem is a generalization of [16, Theorems III.4.1 and III.4.3], which consider the case when the sectional curvatures of M are bounded above and below by constants.

Theorem 2.2. (*Generalizations of [16, Theorems III.4.1 and III.4.3]*) *Suppose the sectional curvatures of M satisfy*

$$(2.7) \quad -c_m(r_x) \leq \mathcal{K}(x; \sigma) \leq -c_M(r_x) < 0,$$

for all $x \in M$ and all two-dimensional subspaces $\sigma \subset T_x M$, where $c_m(\cdot)$ and $c_M(\cdot)$ are positive continuous functions of the distance from a fixed pole o . Then, for any unit-speed geodesic γ through o , it holds that

$$\psi_M^{n-1}(t) \leq \det \mathcal{A}(t) \leq \psi_m^{n-1}(t),$$

where ψ_m and ψ_M are solutions of

$$(2.8) \quad \begin{cases} \psi_m''(\theta) = c_m(\theta)\psi_m(\theta), & \forall \theta > 0, \\ \psi_m(0) = 0, \quad \psi_m'(0) = 1, \end{cases} \quad \text{and} \quad \begin{cases} \psi_M''(\theta) = c_M(\theta)\psi_M(\theta), & \forall \theta > 0, \\ \psi_M(0) = 0, \quad \psi_M'(0) = 1. \end{cases}$$

In addition (by (2.6)), given that γ is arbitrary,

$$\left(\frac{\psi_M(\|u\|)}{\|u\|} \right)^{n-1} \leq |J(\exp_o)(u)| \leq \left(\frac{\psi_m(\|u\|)}{\|u\|} \right)^{n-1},$$

for all $u \in T_o M$.

Proof. The approach follows the proofs of [16, Theorems III.4.1 and III.4.3], with some small caveats related to the nonconstant bounds of curvature. For completeness, we will present the proof in detail in Appendix A. Note that similar bounds have been derived in the literature using Laplacian comparison techniques (see [1, Section 1.9]). \square

Remark 2.2. We note that for the upper bound on $\det \mathcal{A}(t)$, a weaker condition on curvature can be assumed. Specifically, it is sufficient that the Ricci curvatures are bounded below by $-(n-1)c_m(\cdot)$; see the proof in the Appendix, as well as [16, Theorem III.4.3] or [1, Theorem 1.3].

While in general, one cannot find explicit solutions to the IVP's in (2.8), these are linear problems and the global well-posedness of their solutions is guaranteed by standard results in ODE theory. In addition, by straightforward arguments, it can be shown that ψ_m and ψ_M satisfy

$$(2.9) \quad \psi_m(\theta) \geq 0, \psi_M(\theta) \geq 0 \quad \text{and} \quad \psi'_m(\theta) \geq 1, \psi'_M(\theta) \geq 1, \quad \text{for all } \theta \geq 0.$$

Remark 2.3. When $c_m(\cdot) \equiv c_m$ and $c_M(\cdot) \equiv c_M$ are constant functions, the IVP's in (2.8) have the explicit solutions

$$\psi_m(\theta) = \frac{\sinh(\sqrt{c_m}\theta)}{\sqrt{c_m}}, \quad \text{and} \quad \psi_M(\theta) = \frac{\sinh(\sqrt{c_M}\theta)}{\sqrt{c_M}}.$$

In this case, Theorem 2.2 reduces to the comparison results from [16, Theorems III.4.1 and III.4.3], i.e., for any unit-speed geodesic γ through o , it holds that

$$\left(\frac{\sinh(\sqrt{c_M}t)}{\sqrt{c_M}} \right)^{n-1} \leq \det \mathcal{A}(t) \leq \left(\frac{\sinh(\sqrt{c_m}t)}{\sqrt{c_m}} \right)^{n-1},$$

and

$$\left(\frac{\sinh(\sqrt{c_M}\|u\|)}{\sqrt{c_M}\|u\|} \right)^{n-1} \leq |J(\exp_o)(u)| \leq \left(\frac{\sinh(\sqrt{c_m}\|u\|)}{\sqrt{c_m}\|u\|} \right)^{n-1},$$

for all $u \in T_o M$.

Theorem 2.2 can be used to find bounds on the volume of geodesic balls in M . In the paper, \mathbb{S}^{n-1} denotes the unit sphere in \mathbb{R}^n and $\omega(n)$ the volume of the unit ball in \mathbb{R}^n . Also, $B_r(o)$ denotes the geodesic ball in M centred at o of radius r , and $|B_r(o)|$ denotes its volume.

Corollary 2.1. (*Generalizations of [16, Theorems III.4.2 and III.4.4]*) Suppose the sectional curvatures of M satisfy (2.7). Then,

$$nw(n) \int_0^r \psi_M(t)^{n-1} dt \leq |B_r(o)| \leq nw(n) \int_0^r \psi_m(t)^{n-1} dt,$$

with ψ_m and ψ_M defined as in Theorem 2.2.

Proof. We will use here the full notation $\mathcal{A}(t; u)$ to indicate the dependence of \mathcal{A} on the unit tangent vector $u = \gamma'(0)$. By using geodesic spherical coordinates centred at o , the Riemannian measure on M is given by (see [16, Theorem III.3.1])

$$(2.10) \quad dV(\exp_o(tu)) = \det \mathcal{A}(t; u) dt d\sigma(u),$$

where $d\sigma(u)$ denotes the Riemannian measure on \mathbb{S}^{n-1} . The proof of the corollary now follows immediately from Theorem 2.2 together with (2.10), by writing

$$|B_r(o)| = \int_0^r \int_{\mathbb{S}^{n-1}} \det \mathcal{A}(t; u) d\sigma(u) dt.$$

□

The volume bounds in Corollary 2.1 are known in Riemannian geometry literature. For example, the upper bound is proved in [1, Theorem 1.3] by Laplacian comparison methods.

Remark 2.4. When $c_m(\cdot) \equiv c_m$ and $c_M(\cdot) \equiv c_M$, Corollary 2.1 reduces to

$$nw(n) \int_0^r \left(\frac{\sinh(\sqrt{c_M} t)}{\sqrt{c_M}} \right)^{n-1} dt \leq |B_r(o)| \leq nw(n) \int_0^r \left(\frac{\sinh(\sqrt{c_m} t)}{\sqrt{c_m}} \right)^{n-1} dt,$$

which is the result in [16, Theorems III.4.2 and III.4.4]. The lower and upper bounds in this case represent the volumes of the ball of radius r in the hyperbolic space \mathbb{H}^n of constant curvatures $-c_M$, and $-c_m$, respectively.

2.4. Properties of solutions to the IVP (2.8). Next, we provide various properties of solutions to the IVP's in (2.8), under additional assumptions on the functions c_m and c_M . We list the results for a generic function c . Throughout the paper, $\underline{D}f$ and $\overline{D}f$ denote, respectively, the lower and the upper derivatives of a real-valued function f , defined by

$$\underline{D}f(x) = \liminf_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \quad \overline{D}f(x) = \limsup_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Lemma 2.1. *Let $c : [0, \infty) \rightarrow (0, \infty)$ be a positive and continuous function, and consider the solution ψ of the IVP:*

$$\begin{cases} \psi''(\theta) = c(\theta)\psi(\theta), & \forall \theta > 0, \\ \psi(0) = 0, \quad \psi'(0) = 1. \end{cases}$$

(Upper bound) Assume $c(\cdot)$ is non-decreasing. Then,

$$(2.11) \quad \psi(\theta) \leq \theta \exp \left(\int_0^\theta \sqrt{c(t)} dt \right), \quad \forall \theta \geq 0.$$

(Lower bound) Assume $c(\cdot)$ satisfies

$$(2.12) \quad \lim_{\theta \rightarrow \infty} \frac{\overline{D}c(\theta)}{c(\theta)^{3/2}} = 0.$$

Then, for any $0 < \epsilon < 1$, there exists $\theta_0 > 0$ such that

$$(2.13) \quad \psi(\theta_0) \exp \left((1 - \epsilon) \int_{\theta_0}^\theta \sqrt{c(t)} dt \right) \leq \psi(\theta), \quad \forall \theta \geq \theta_0.$$

Proof. See Appendix B. □

Remark 2.5. There are various examples of functions $c(\cdot)$, relevant to our work, which satisfy condition (2.12), e.g.,

$$c(\theta) = \theta^k, \quad c(\theta) = e^{\theta^k}, \quad c(\theta) = 1 - e^{-\theta^k},$$

for all $k > 0$. Specifically, the first two examples above, when used as bounds for the curvature (see (2.7)), correspond to manifolds whose negative curvature grows algebraically, respectively exponentially, at infinity.

Remark 2.6. To give some intuition on the first assumption in (2.12), consider a non-decreasing C^1 function $c(\cdot)$ which satisfies

$$\lim_{\theta \rightarrow \infty} \frac{c'(\theta)}{c(\theta)^{3/2}} > 0.$$

Then, there exists $\delta > 0$ and $\theta_0 > 0$ such that

$$\frac{c'(\theta)}{c(\theta)^{3/2}} \geq \delta, \quad \text{for all } \theta \geq \theta_0.$$

By solving the differential inequality, we get

$$-\frac{2}{c(\theta)^{1/2}} + \frac{2}{c(\theta_0)^{1/2}} \geq \delta(\theta - \theta_0),$$

which implies

$$c(\theta) \geq \left(\frac{1}{c(\theta_0)^{1/2}} - \frac{\delta(\theta - \theta_0)}{2} \right)^{-2}.$$

From here one infers that $c(\theta) \rightarrow \infty$ as $\theta \rightarrow \theta_0 + \frac{2}{\delta c(\theta_0)^{1/2}}$ and hence, $c(\theta)$ is not defined globally for all $\theta \geq 0$. Given these considerations, assumption (2.12) is natural for a globally defined function $c(\cdot)$, as used in this paper.

3. NONEXISTENCE OF A GLOBAL MINIMIZER

We will show that the energy functional (2.3) does not admit global minimizers when the attractive forces at infinity are not strong enough to contain the diffusion. We fix a pole $o \in M$ and assume that the sectional curvatures $\mathcal{K}(x; \sigma)$ are bounded above by a function $-c_M(r_x) < 0$ that can grow unbounded as $r_x \rightarrow \infty$. We denote by ψ_M the solution to the initial-value problem

$$(3.14) \quad \begin{cases} \psi_M''(\theta) = c_M(\theta)\psi_M(\theta), & \forall \theta > 0, \\ \psi_M(0) = 0, \quad \psi_M'(0) = 1. \end{cases}$$

The main result is the following.

Theorem 3.1 (Nonexistence by spreading). *Let M be an n -dimensional Cartan-Hadamard manifold and $o \in M$ a fixed pole. Assume that the sectional curvatures of M satisfy*

$$\mathcal{K}(x; \sigma) \leq -c_M(r_x),$$

for all $x \in M$ and all two-dimensional subspaces $\sigma \subset T_x M$, where $c_M(\cdot)$ is a positive and continuous function such that

$$\lim_{\theta \rightarrow \infty} \frac{\overline{D}c_M(\theta)}{c(\theta)^{3/2}} = 0.$$

Let $h : [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing lower semi-continuous function that satisfies

$$(3.15) \quad \liminf_{\theta \rightarrow \infty} \left(h(\theta) - A \int_0^{\theta-\delta} \sqrt{c_M(t/2)} dt \right) = -\infty,$$

for some $A < n-1$, and some $\delta > 0$. Then, the energy functional (2.3) is unbounded below, and hence, it admits no global minimizer in $\mathcal{P}_{ac}(M)$.

Proof. Consider the following family of densities ρ_R defined for all $R > 0$:

$$\rho_R(x) = \begin{cases} \frac{1}{|B_R(o)|}, & \text{when } x \in B_R(o), \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$(3.16) \quad \begin{aligned} E[\rho_R] &= \int_M \rho_R(x) \log \rho_R(x) dV(x) + \frac{1}{2} \iint_{M \times M} h(d(x, y)) \rho_R(x) \rho_R(y) dV(x) dV(y) \\ &\leq -\log |B_R(o)| + \frac{h(2R)}{2}, \end{aligned}$$

where for the inequality we used

$$\sup_{x, y \in \text{supp}(\rho_R)} d(x, y) = 2R.$$

We will show that $E[\rho_R] \rightarrow -\infty$ as $R \rightarrow \infty$, and hence, no global energy minimizers in $\mathcal{P}_{ac}(M)$ exist.

Since $A < n-1$, set $\epsilon \in (0, 1)$ by

$$(3.17) \quad A = (1 - \epsilon)(n - 1).$$

By the assumptions on c_M and Lemma 2.1 (see lower bound (2.13)), there exists $\theta_0 > 0$ such that

$$(3.18) \quad \psi_M(\theta) \geq \psi_M(\theta_0) \exp \left((1 - \epsilon) \int_{\theta_0}^{\theta} \sqrt{c_M(t)} dt \right), \quad \text{for all } \theta > \theta_0,$$

where ψ_M is the solution of the IVP (3.14).

Then, for any $R > \theta_0 + \frac{\delta}{2}$ (with δ as in (3.15)), combine Corollary 2.1 and (3.18) to find

$$\begin{aligned} |B_R(o)| &\geq nw(n) \int_0^R \psi_M(\theta)^{n-1} d\theta \\ &\geq nw(n) \int_{R-\frac{\delta}{2}}^R \psi_M(\theta)^{n-1} d\theta \\ &\geq nw(n) \int_{R-\frac{\delta}{2}}^R \psi_M(\theta_0)^{n-1} \exp \left((1 - \epsilon)(n - 1) \int_{\theta_0}^{\theta} \sqrt{c_M(t)} dt \right) d\theta. \end{aligned}$$

Furthermore, using that $\int_{\theta_0}^{\theta} \sqrt{c_M(t)} dt \geq \int_{\theta_0}^{R-\frac{\delta}{2}} \sqrt{c_M(t)} dt$, for all $R - \frac{\delta}{2} < \theta < R$, we get

$$\begin{aligned} |B_R(o)| &\geq nw(n)\psi_M(\theta_0)^{n-1} \int_{R-\frac{\delta}{2}}^R \exp\left((1-\epsilon)(n-1) \int_{\theta_0}^{R-\frac{\delta}{2}} \sqrt{c_M(t)} dt\right) d\theta \\ &= \frac{\delta}{2} nw(n)\psi_M(\theta_0)^{n-1} \exp\left((1-\epsilon)(n-1) \int_{\theta_0}^{R-\frac{\delta}{2}} \sqrt{c_M(t)} dt\right). \end{aligned}$$

The above yields

$$\log |B_R(o)| \geq \log\left(\frac{\delta}{2} nw(n)\psi_M(\theta_0)^{n-1}\right) + (1-\epsilon)(n-1) \int_{\theta_0}^{R-\frac{\delta}{2}} \sqrt{c_M(t)} dt.$$

We can now estimate $E[\rho_R]$ for $R > \theta_0 + \frac{\delta}{2}$ (see (3.16)) as follows:

$$\begin{aligned} E[\rho_R] &\leq -\log\left(\frac{\delta}{2} nw(n)\psi_M(\theta_0)^{n-1}\right) - (1-\epsilon)(n-1) \int_{\theta_0}^{R-\frac{\delta}{2}} \sqrt{c_M(t)} dt + \frac{h(2R)}{2} \\ &= -\log\left(\frac{\delta}{2} nw(n)\psi_M(\theta_0)^{n-1}\right) + (1-\epsilon)(n-1) \int_0^{\theta_0} \sqrt{c_M(t)} dt \\ &\quad - (1-\epsilon)(n-1) \int_0^{R-\frac{\delta}{2}} \sqrt{c_M(t)} dt + \frac{h(2R)}{2}. \end{aligned}$$

Since by the assumption (3.15) on h (see also (3.17)) and a simple change of variable, we have

$$\liminf_{R \rightarrow \infty} \left(\frac{h(2R)}{2} - (1-\epsilon)(n-1) \int_0^{R-\frac{\delta}{2}} \sqrt{c_M(t)} dt \right) = -\infty,$$

we infer $\lim_{R \rightarrow \infty} E[\rho_R] = -\infty$. \square

Example 3.1. For certain functions c_M we can compute $\int_0^{\theta-\delta} \sqrt{c_M(t/2)} dt$ and hence, find explicit expressions for the behaviour of h at infinity that lead to spreading.

(1) Constant: $c_M(\theta) \equiv c_M$. This is the case studied in [24]. Compute (see (3.15)):

$$h(\theta) - A \int_0^{\theta-\delta} \sqrt{c_M} dt = h(\theta) - A\sqrt{c_M}\theta + A\delta\sqrt{c_M}.$$

In this case, spreading occurs provided $h(\theta)$ grows slower than $A\sqrt{c_M}\theta$ at infinity (for some $A < n-1$), which recovers the result in [24, Theorem 2.1].

(2) Power law: $c_M(\theta) = \theta^k$ with $k \geq 1$. We compute

$$h(\theta) - A \int_0^{\theta-\delta} 2^{-k/2} t^{k/2} dt = h(\theta) - \frac{A2^{-k/2}}{k/2+1} (\theta-\delta)^{k/2+1},$$

which implies that spreading occurs if h grows slower than $\theta^{k/2+1}$ at infinity.

(3) Exponential growth: $c_M(\theta) = e^{\beta\theta}$ with $\beta > 0$. We find

$$h(\theta) - A \int_0^{\theta-\delta} e^{\beta t/4} dt = h(\theta) - \frac{A}{\beta/4} \left(e^{\beta(\theta-\delta)/4} - 1 \right).$$

Hence, in this case spreading cannot be prevented provided h grows slower than $e^{\beta\theta/4}$ at infinity.

4. LOGARITHMIC HLS INEQUALITY ON CARTAN-HADAMARD MANIFOLDS

This section presents a new logarithmic HLS inequality on Cartan-Hadamard manifolds, which is a key tool for our results on existence of global energy minimizers. We first recall the logarithmic HLS inequality on \mathbb{R}^n , given by the following theorem.

Theorem 4.1 (Logarithmic HLS inequality on \mathbb{R}^n [10]). *Let $\rho \in \mathcal{P}_{ac}(\mathbb{R}^n)$ satisfy $\log(1 + |\cdot|^2)\rho \in L^1(\mathbb{R}^n)$. Then there exists $C_0 \in \mathbb{R}$ depending only on n , such that*

$$-\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \log(|x-y|) \rho(x) \rho(y) dx dy \leq \frac{1}{n} \int_{\mathbb{R}^n} \rho(x) \log \rho(x) dx + C_0.$$

We generalize Theorem 4.1 to Cartan-Hadamard manifolds. The result is the following.

Theorem 4.2 (Logarithmic HLS inequality on Cartan-Hadamard manifolds). *Let M be an n -dimensional Cartan-Hadamard manifold, and $o \in M$ be a fixed (arbitrary) pole. Assume that the sectional curvatures of M satisfy, for all $x \in M$ and all sections $\sigma \subset T_x M$,*

$$(4.19) \quad -c_m(r_x) \leq \mathcal{K}(x; \sigma) \leq 0,$$

where $c_m(\cdot)$ is a continuous positive function. Let ψ_m be the solution to the following IVP:

$$(4.20) \quad \begin{cases} \psi_m''(\theta) = c_m(\theta) \psi_m(\theta), & \forall \theta > 0, \\ \psi_m(0) = 0, \quad \psi_m'(0) = 1. \end{cases}$$

Then, if $\rho \in \mathcal{P}_{ac}(M)$ and $\int_M \log(1 + |r_x|^2) \rho(x) dV(x) < \infty$, we have

$$(4.21) \quad -\int_M \int_M \log(d(x, y)) \rho(x) \rho(y) dV(x) dV(y) \leq \frac{1}{n} \int_M \rho(x) \log \rho(x) dV(x) + \frac{(n-1)}{n} \int_M \log\left(\frac{\psi_m(r_x)}{r_x}\right) \rho(x) dV(x) + C_0,$$

where $C_0 \in \mathbb{R}$ is the constant depending only on n introduced in Theorem 4.1.

Proof. Denote by $f : M \rightarrow T_o M$ the Riemannian logarithm map at o , i.e.,

$$(4.22) \quad f(x) = \log_o x, \quad \text{for all } x \in M.$$

The inverse map $f^{-1} : T_o M \rightarrow M$ is the Riemannian exponential map

$$f^{-1}(u) = \exp_o u, \quad \text{for all } u \in T_o M.$$

On Cartan-Hadamard manifolds, the exponential map is a global diffeomorphism.

Take a density $\rho \in \mathcal{P}_{ac}(M)$, and its pushforward (as measures) $f_{\#} \rho \in \mathcal{P}_{ac}(T_o M)$ by f . Then,

$$(4.23) \quad f_{\#} \rho(u) = \rho(f^{-1}(u)) |J(f^{-1})(u)|, \quad \text{for all } u \in T_o M.$$

Since M has non-positive curvature, by Rauch Comparison Theorem (see Theorem 2.1) we have

$$(4.24) \quad d(x, y) \geq |f(x) - f(y)|, \quad \text{for all } x, y \in M.$$

Inequality (4.24) can be justified as follows. Fix any two points $x, y \in M$. In the setup of Theorem 2.1, take $\tilde{M} = T_o M$, i to be the identity map, and c to be the geodesic curve joining x and y . The assumption on the curvatures \mathcal{K} and $\tilde{\mathcal{K}}$ holds, as $\mathcal{K} \leq 0 = \tilde{\mathcal{K}}$. The curve \tilde{c} constructed in Theorem 2.1 joins $\log_o x$ and $\log_o y$, and hence, its length is greater

than or equal to the length of the straight segment joining $\log_o x$ and $\log_o y$. Together with Theorem 2.1, we then infer:

$$|\log_o x - \log_o y| \leq l(\tilde{c}) \leq l(c) = d(x, y).$$

Note that (4.24) holds indeed with equal sign for $M = \mathbb{R}^n$.

By (4.24), we have

$$(4.25) \quad - \int_M \int_M \log(d(x, y)) \rho(x) \rho(y) dV(x) dV(y) \leq - \int_M \int_M \log(|f(x) - f(y)|) \rho(x) \rho(y) dV(x) dV(y),$$

since $\log(\cdot)$ is an increasing function on $(0, \infty)$. Then, using the definition of the pushforward measure and the logarithmic HLS inequality on $T_o M \simeq \mathbb{R}^n$, we get

$$(4.26) \quad - \int_M \int_M \log(|f(x) - f(y)|) \rho(x) \rho(y) dV(x) dV(y) = - \int_{T_o M} \int_{T_o M} \log(|w - z|) f_{\#} \rho(w) f_{\#} \rho(z) dw dz \\ \leq \frac{1}{n} \int_{T_o M} f_{\#} \rho(z) \log(f_{\#} \rho(z)) dz + C_0,$$

where C_0 is a constant that depends on n . Now, use again the property of the pushforward measure, to find

$$(4.27) \quad \frac{1}{n} \int_{T_o M} f_{\#} \rho(z) \log(f_{\#} \rho(z)) dz + C_0 = \frac{1}{n} \int_M \rho(x) \log(f_{\#} \rho(f(x))) dV(x) + C_0 \\ = \frac{1}{n} \int_M \rho(x) \log(\rho(x) |J(f^{-1})(f(x))|) dV(x) + C_0 \\ = \frac{1}{n} \int_M \rho(x) \log \rho(x) dV(x) + \frac{1}{n} \int_M \rho(x) \log |J(f^{-1})(f(x))| dV(x) + C_0.$$

By combining (4.25), (4.26) and (4.27), we obtain

$$(4.28) \quad - \int_M \int_M \log(d(x, y)) \rho(x) \rho(y) dV(x) dV(y) \leq \\ \frac{1}{n} \int_M \rho(x) \log \rho(x) dV(x) + \frac{1}{n} \int_M \rho(x) \log |J(f^{-1})(f(x))| dV(x) + C_0.$$

Compared to the logarithmic HLS inequality on \mathbb{R}^n , inequality (4.28) has an additional term on the right-hand-side, containing the Jacobian of the exponential map. Using Theorem 2.2 (note the lower bound (4.19) of the sectional curvatures), we can bound this Jacobian from above by

$$|J(f^{-1})(f(x))| \leq \left(\frac{\psi_m(\|f(x)\|)}{\|f(x)\|} \right)^{n-1} = \left(\frac{\psi_m(r_x)}{r_x} \right)^{n-1}.$$

The HLS inequality (4.21) now follows from the estimate above and (4.28). \square

Remark 4.1. In particular, if c_m is a constant function, $\psi_m(\theta) = \sinh(\sqrt{c_m}\theta)/\sqrt{c_m}$, and (4.21) reduces to

$$- \int_M \int_M \log(d(x, y)) \rho(x) \rho(y) dV(x) dV(y) \leq \\ \frac{1}{n} \int_M \rho(x) \log \rho(x) dV(x) + \frac{(n-1)}{n} \int_M \log \left(\frac{\sinh(\sqrt{c_m} r_x)}{\sqrt{c_m} r_x} \right) \rho(x) dV(x) + C_0.$$

This recovers the result in [24, Theorem 5.2].

5. EXISTENCE OF A GLOBAL MINIMIZER

In this section we fix a pole $o \in M$ and assume that the sectional curvatures $\mathcal{K}(x; \sigma)$ are bounded below by a function $-c_m(r_x)$ which can grow unbounded as $r_x \rightarrow \infty$. We denote by ψ_m the solution to the initial-value problem

$$(5.29) \quad \begin{cases} \psi_m''(\theta) = c_m(\theta)\psi_m(\theta), & \forall \theta > 0, \\ \psi_m(0) = 0, \quad \psi_m'(0) = 1. \end{cases}$$

The main result, given by the theorem below, establishes sufficient conditions on the growth at infinity of the attractive potential (in terms of the growth of curvatures) that guarantee existence of a global energy minimizer.

Theorem 5.1 (Existence of a global minimizer). *Let M be an n -dimensional Cartan-Hadamard manifold and $o \in M$ a fixed pole. Assume that the sectional curvatures of M satisfy*

$$(5.30) \quad -c_m(r_x) \leq \mathcal{K}(x; \sigma) \leq 0,$$

for all $x \in M$ and all two-dimensional subspaces $\sigma \subset T_x M$, where $c_m(\cdot)$ is a positive, continuous and non-decreasing function. Also assume that $h : [0, \infty) \rightarrow [0, \infty)$ is a non-decreasing lower semi-continuous function which satisfies

$$(5.31) \quad h(\theta) \geq \phi(\theta), \quad \text{and} \quad \lim_{\theta \rightarrow \infty} \frac{\phi(\theta)}{\int_0^\theta \sqrt{c_m(t)} dt} = \infty,$$

for some non-decreasing convex function ϕ . Then there exists a global minimum of $E[\rho]$ in $\mathcal{P}_o(M)$.

Remark 5.1. In the proof of Theorem 5.1 we will use in fact a weaker assumption on the growth at infinity of h , namely that

$$(5.32) \quad h(\theta) \geq \phi(\theta), \quad \text{and} \quad \lim_{\theta \rightarrow \infty} \frac{\phi(\theta)}{\log \left(\frac{\psi_m(\theta)}{\theta} \right)} = \infty,$$

for some non-decreasing convex function ϕ , where ψ_m is the solution of the IVP (5.29). Nevertheless, we chose to present Theorem 5.1 with the explicit growth at infinity given in terms of the known function $c_m(\cdot)$, rather than using the unknown solution $\psi_m(\cdot)$ of the IVP. To see that (5.32) is indeed weaker than (5.31), one can argue as following. By Lemma 2.1, it holds that

$$\psi_m(\theta) \leq \theta \exp \left(\int_0^\theta \sqrt{c_m(t)} dt \right), \quad \forall \theta \geq 0.$$

which is equivalent to

$$\log \left(\frac{\psi_m(\theta)}{\theta} \right) \leq \int_0^\theta \sqrt{c_m(t)} dt, \quad \forall \theta > 0,$$

Hence, if ϕ satisfies the asymptotic condition in (5.31), then it also satisfies (5.32).

Remark 5.2. The two conditions (3.15) and (5.31) on h , that result in nonexistence, respectively existence of minimizers, are complementary in the following sense. Suppose the sectional curvatures satisfy both a lower and an upper bound: $-c_m(r_x) \leq \mathcal{K}(x; \sigma) \leq -c_M(r_x)$,

for all points x and all sections σ , where $c_m(\cdot)$ is non-decreasing. Also assume that h satisfies (5.31). Then, for any positive A and δ , we have

$$\begin{aligned} h(\theta) - A \int_0^{\theta-\delta} \sqrt{c_M(t/2)} dt &\geq h(\theta) - A \int_0^{\theta-\delta} \sqrt{c_m(t/2)} dt \\ &\geq \phi(\theta) - A \int_0^{\theta-\delta} \sqrt{c_m(t)} dt \\ &\geq \phi(\theta) - A \int_0^\theta \sqrt{c_m(t)} dt, \end{aligned}$$

where in the first inequality we used $c_M \leq c_m$, and in the second inequality we used $h \geq \phi$ and the monotonicity of c_m . Since the second condition in (5.31) implies that $\phi(\theta) - A \int_0^\theta \sqrt{c_m(t)} dt$ tends to infinity as $\theta \rightarrow \infty$, (3.15) is always false in this case.

We present the proof of Theorem 5.1 after we establish several important preparatory results. The overall strategy is based on the direct method of the calculus of variations, where the key items are a coercivity property of the energy functional (Proposition 5.1), and the lower semi-continuity of the energy (Proposition 5.2). Another important ingredient of the proof is the property that minimizing sequences preserve, in the limit of weak convergence as measures, the centre of mass (Lemma 5.5).

Lemma 5.1 ([24]). *Let M be a Cartan-Hadamard manifold, $o \in M$ a fixed pole and $\rho \in \mathcal{P}_o(M)$. Then we have*

$$(2 - \sqrt{2})\mathcal{W}_1(\rho, \delta_o) \leq \iint_{M \times M} d(x, y) \rho(x) \rho(y) dV(x) dV(y) \leq 2\mathcal{W}_1(\rho, \delta_o).$$

Here, δ_o denotes the Dirac delta measure centred at the pole o .

Proof. This result was stated and proved in [24, Lemma 6.1]. The proof is based on Rauch comparison theorem and some simple calculations and estimates. We also note that this result does not require any information on the curvatures of M , and it applies to general Cartan-Hadamard manifolds. \square

Lemma 5.2. *Let ψ_m be the solution to (5.29), where $c_m(\cdot) \geq 0$ is a continuous non-decreasing function on $[0, \infty)$. Then,*

$$(5.33) \quad H(\theta) := \log \left(\frac{\psi_m(\theta)}{\theta} \right)$$

is a non-decreasing convex function on $(0, \infty)$.

Proof. Note first that by (2.9), $\psi_m(\theta) \geq 0$ and $\psi'_m(\theta) \geq 1$ for all $\theta \geq 0$. By direct calculation we get

$$(5.34) \quad \frac{d}{d\theta} \log \left(\frac{\psi_m(\theta)}{\theta} \right) = \frac{\psi'_m(\theta)}{\psi_m(\theta)} - \frac{1}{\theta}.$$

Set

$$f(\theta) = \frac{\psi_m(\theta)}{\psi'_m(\theta)}.$$

Then,

$$f'(\theta) = 1 - c_m(\theta) f(\theta)^2,$$

and hence,

$$f'(\theta) \leq 1.$$

This implies (also note that $f(0) = 0$) that $f(\theta) \leq \theta$ for all $\theta > 0$ and hence, by (5.34), we find $H'(\theta) \geq 0$. This shows H is non-decreasing.

To show that $H(\theta)$ is convex, we compute its second derivative:

$$(5.35) \quad \begin{aligned} \frac{d^2}{d\theta^2} \log \left(\frac{\psi_m(\theta)}{\theta} \right) &= \frac{\psi_m''(\theta)\psi_m(\theta) - \psi_m'(\theta)^2}{\psi_m(\theta)^2} + \frac{1}{\theta^2} \\ &= c_m(\theta) - \left(\frac{\psi_m'(\theta)}{\psi_m(\theta)} \right)^2 + \frac{1}{\theta^2}. \end{aligned}$$

Now note that the ODE satisfied by ψ_m in (5.29) can be written as

$$[(\psi_m'(\theta))^2]' = c_m(\theta)[(\psi_m(\theta))^2]'$$

Integrate the above and use the initial conditions on ψ_m and the monotonicity of c_m , to get

$$\begin{aligned} (\psi_m'(\theta))^2 &= 1 + \int_0^\theta c_m(t)[(\psi_m(t))^2]' dt \\ &\leq 1 + c_m(\theta)(\psi_m(\theta))^2. \end{aligned}$$

Then, by using $\psi_m(\theta) \geq \theta$, we further find

$$\begin{aligned} \left(\frac{\psi_m'(\theta)}{\psi_m(\theta)} \right)^2 &\leq \frac{1}{(\psi_m(\theta))^2} + c_m(\theta) \\ &\leq \frac{1}{\theta^2} + c_m(\theta), \end{aligned}$$

which combined with (5.35) shows the convexity of H . \square

Corollary 5.1. *Let ψ_m be the solution to (5.29), where $c_m(\cdot)$ is a positive, continuous and non-decreasing function on $[0, \infty)$. Then, we have*

$$\liminf_{\theta \rightarrow \infty} \frac{\log \left(\frac{\psi_m(\theta)}{\theta} \right)}{\theta} \geq A,$$

for some $A > 0$.

Proof. The proof follows immediately from the convexity and monotonicity of the function $H(\theta) = \log \left(\frac{\psi_m(\theta)}{\theta} \right)$ (note that $H(\theta)$ cannot be a constant if c_m is positive). \square

Remark 5.3. *Let \tilde{H} be the even extension of the function H on \mathbb{R} :*

$$\tilde{H}(\theta) = \begin{cases} H(\theta) & \text{for } \theta > 0, \\ 0 & \text{for } \theta = 0, \\ H(-\theta) & \text{for } \theta < 0. \end{cases}$$

It can be shown by direct calculations (not shown here) using L'Hôpital theorem that \tilde{H} is second order differentiable at 0, with $\tilde{H}(0) = 0$, $\tilde{H}'(0) = 0$, and $\tilde{H}''(0) = \frac{c_m(0)}{3} > 0$. Consequently, \tilde{H} is a globally convex function on \mathbb{R} .

Proposition 5.1. *Let M be an n -dimensional Cartan-Hadamard manifold whose sectional curvatures satisfy (5.30), with $o \in M$ a fixed pole, and $c_m(\cdot)$ a positive, continuous and non-decreasing function. Also assume that $h : [0, \infty) \rightarrow [0, \infty)$ is lower semi-continuous and satisfies (5.32). Then there exist constants C_0 and \mathcal{C} such that*

$$(5.36) \quad E[\rho] \geq -C_0 n + \frac{\mathcal{C}}{2} + \frac{2 - \sqrt{2}}{2} \mathcal{W}_1(\rho, \delta_o) + \frac{1}{4} \iint_{M \times M} h(d(x, y)) \rho(x) \rho(y) dV(x) dV(y),$$

for all $\rho \in \mathcal{P}_o(M)$. Specifically, C_0 is the constant from the logarithmic HLS inequality in Theorem 4.2 (which only depends on n), while \mathcal{C} depends on h , c_m and n .

Proof. By Rauch comparison theorem (see (4.24)) and the law of cosines, for any two points $x, y \in M$, we have

$$\begin{aligned} d(x, y)^2 &\geq |\log_o x - \log_o y|^2 \\ &= r_x^2 + r_y^2 - 2r_x r_y \cos \angle(xoy) \\ &\geq (r_x - r_y \cos \angle(xoy))^2. \end{aligned}$$

This yields

$$(5.37) \quad d(x, y) \geq |r_x - r_y \cos \angle(xoy)|.$$

Take any $\rho \in \mathcal{P}_o(M)$. Consider the function H defined in (5.33) and its even extension (see Remark 5.3). Since H is non-decreasing (by Lemma 5.2), along with (5.37), one can estimate

$$\begin{aligned} (5.38) \quad \iint_{M \times M} H(d(x, y)) \rho(x) \rho(y) dV(x) dV(y) &\geq \iint_{M \times M} H(|r_x - r_y \cos \angle(xoy)|) \rho(x) \rho(y) dV(x) dV(y) \\ &= \iint_{M \times M} \tilde{H}(r_x - r_y \cos \angle(xoy)) \rho(x) \rho(y) dV(x) dV(y). \end{aligned}$$

Since $\rho \in \mathcal{P}_o(M)$, we have

$$(5.39) \quad \int_M \log_o y \rho(y) dV(y) = 0.$$

In this equation, take the inner product with $\log_o x$ for some arbitrary $x \neq o$, to get

$$\int_M r_x r_y \cos \angle(xoy) \rho(y) dV(y) = 0,$$

which is equivalent to

$$(5.40) \quad \int_M r_y \cos \angle(xoy) \rho(y) dV(y) = 0.$$

Now, use the convexity of \tilde{H} and Jensen's inequality, together with (5.40), to obtain

$$\begin{aligned} \int_M \tilde{H}(r_x - r_y \cos \angle(xoy)) \rho(y) dV(y) &\geq \tilde{H} \left(\int_M (r_x - r_y \cos \angle(xoy)) \rho(y) dV(y) \right) \\ &= \tilde{H}(r_x), \end{aligned}$$

for any $x \neq o$. By integrating the inequality above with respect to $\rho(x)$ we then find

$$(5.41) \quad \iint_{M \times M} \tilde{H}(r_x - r_y \cos \angle(xoy)) \rho(x) \rho(y) dV(x) dV(y) \geq \int_M \tilde{H}(r_x) \rho(x) dV(x).$$

Finally, combine (5.38) with (5.41) and use the expression of H in (5.33), to get

$$(5.42) \quad \iint_{M \times M} \log \left(\frac{\psi_m(d(x, y))}{d(x, y)} \right) \rho(x) \rho(y) dV(x) dV(y) \geq \int_M \log \left(\frac{\psi_m(r_x)}{r_x} \right) \rho(x) dV(x).$$

Using (5.42) in Theorem 4.2 (see (4.21)) we then reach

$$(5.43) \quad \begin{aligned} - \iint_{M \times M} \log(d(x, y)) \rho(x) \rho(y) dV(x) dV(y) &\leq \frac{1}{n} \int_M \rho(x) \log \rho(x) dV(x) \\ &\quad + \frac{(n-1)}{n} \iint_{M \times M} \log \left(\frac{\psi_m(d(x, y))}{d(x, y)} \right) \rho(x) \rho(y) dV(x) dV(y) + C_0. \end{aligned}$$

Since h satisfies (5.32), we have

$$\lim_{\theta \rightarrow \infty} \left(\frac{h(\theta)}{4} - 2(n-1) \log \left(\frac{\psi_m(\theta)}{\theta} \right) \right) = \infty.$$

On the other hand, by Corollary 5.1 we infer that $\log \left(\frac{\psi_m(\theta)}{\theta} \right)$ grows at least linearly at infinity. Then, from (5.32) we conclude that ϕ (and hence h) have *superlinear* growth at infinity, which yields

$$\lim_{\theta \rightarrow \infty} \left(\frac{h(\theta)}{4} - 2n \log \theta - \theta \right) = \infty.$$

By combining the two limits above we get

$$\lim_{\theta \rightarrow \infty} \left(\frac{h(\theta)}{2} - 2(n-1) \log \left(\frac{\psi_m(\theta)}{\theta} \right) - 2n \log \theta - \theta \right) = \infty.$$

Now we note the behaviour at zero of this expression, given by

$$\lim_{\theta \rightarrow 0^+} \left(\frac{h(\theta)}{2} - 2(n-1) \log \left(\frac{\psi_m(\theta)}{\theta} \right) - 2n \log \theta - \theta \right) = \infty,$$

where we used that h is bounded from below and $\psi_m(\theta) \sim \theta$ near $\theta = 0$. By the lower semi-continuity of h , we can then conclude that there exists a constant \mathcal{C} such that

$$\frac{h(\theta)}{2} - 2(n-1) \log \left(\frac{\psi_m(\theta)}{\theta} \right) - 2n \log \theta - \theta \geq \mathcal{C}, \quad \text{for all } \theta > 0.$$

For convenience, we write the above as

$$(5.44) \quad h(\theta) \geq \frac{h(\theta)}{2} + 2(n-1) \log \left(\frac{\psi_m(\theta)}{\theta} \right) + 2n \log \theta + \theta + \mathcal{C}, \quad \text{for all } \theta > 0.$$

Finally, from (5.44) and (5.43), we get

$$\begin{aligned}
E[\rho] &= \int_M \rho(x) \log \rho(x) dV(x) + \frac{1}{2} \iint_{M \times M} h(d(x, y)) \rho(x) \rho(y) dV(x) dV(y) \\
&\geq \int_M \rho(x) \log \rho(x) dV(x) + (n-1) \iint_{M \times M} \log \left(\frac{\psi_m(d(x, y))}{d(x, y)} \right) \rho(x) \rho(y) dV(x) dV(y) \\
&\quad + n \iint_{M \times M} \log(d(x, y)) \rho(x) \rho(y) dV(x) dV(y) + \frac{1}{2} \iint_{M \times M} d(x, y) \rho(x) \rho(y) dV(x) dV(y) + \frac{\mathcal{C}}{2} \\
&\quad + \frac{1}{4} \iint_{M \times M} h(d(x, y)) \rho(x) \rho(y) dV(x) dV(y) \\
&\geq -C_0 n + \frac{\mathcal{C}}{2} + \frac{1}{2} \iint_{M \times M} d(x, y) \rho(x) \rho(y) dV(x) dV(y) + \frac{1}{4} \iint_{M \times M} h(d(x, y)) \rho(x) \rho(y) dV(x) dV(y).
\end{aligned}$$

The lower bound (5.36) on $E[\rho]$ now follows from the estimate above and Lemma 5.1. \square

A major ingredient in the proof of Theorem 5.1 is the lower semi-continuity of the energy, with respect to weak convergence of measures. Note that weak convergence of measures does not preserve absolute continuity, so we need to take this into account in the notations.

Lemma 5.3. *Assume M is a Cartan-Hadamard manifold and $h : [0, \infty) \rightarrow [0, \infty)$ is lower semi-continuous. Then, the interaction energy is lower semi-continuous with respect to weak convergence, i.e., if $\rho_k \rightharpoonup \rho_0$ weakly (as measures) as $k \rightarrow \infty$, it holds that*

$$\iint_{M \times M} h(d(x, y)) d\rho_0(x) d\rho_0(y) \leq \liminf_{k \rightarrow \infty} \iint_{M \times M} h(d(x, y)) d\rho_k(x) d\rho_k(y).$$

Proof. Using the exponential map $f^{-1} = \exp_o$ to make a change of variable, we can write the interaction energy of a probability measure ρ as

$$\iint_{M \times M} h(d(x, y)) d\rho(x) d\rho(y) = \iint_{T_o M \times T_o M} h(d(\exp_o u, \exp_o v)) df_{\#} \rho(u) df_{\#} \rho(v).$$

The right-hand-side above can be interpreted as the interaction energy of $f_{\#} \rho$ on $T_o M \simeq \mathbb{R}^n$ with the interaction potential $\tilde{h} : T_o M \times T_o M \rightarrow \mathbb{R}^n$ given by

$$\tilde{h}(u, v) = h(d(\exp_o u, \exp_o v)).$$

Note that \tilde{h} is a lower semi-continuous function, as h is lower semi-continuous and \exp_o is a diffeomorphism.

Consider now a sequence $\{\rho_k\}_{k \geq 1}$, with $\rho_k \rightharpoonup \rho_0$ weakly (as measures) as $k \rightarrow \infty$. By the change of variable formula, it is straightforward to show that $\rho_k \rightharpoonup \rho_0$ implies $f_{\#} \rho_k \rightharpoonup f_{\#} \rho_0$ as $k \rightarrow \infty$. In the Euclidean space, the interaction energy is lower semi-continuous with respect to weak convergence provided the interaction potential is lower semi-continuous [39,

Proposition 7.2]. By combining all these facts, we then find

$$\begin{aligned}
\liminf_{k \rightarrow \infty} \iint_{M \times M} h(d(x, y)) d\rho_k(x) d\rho_k(y) &= \liminf_{k \rightarrow \infty} \iint_{T_o M \times T_o M} \tilde{h}(u, v) df_{\#} \rho_k(u) df_{\#} \rho_k(v) \\
&\geq \iint_{T_o M \times T_o M} \tilde{h}(u, v) df_{\#} \rho_0(u) df_{\#} \rho_0(v) \\
&= \iint_{M \times M} h(d(x, y)) d\rho_0(x) d\rho_0(y).
\end{aligned}$$

□

Lemma 5.4. *Assume M and h satisfy the assumptions in Proposition 5.1, with o a fixed pole in M . Suppose $\{\rho_k\}_{k \geq 1} \subset \mathcal{P}_o(M)$ is a sequence of probability densities such that $E[\rho_k]$ is uniformly bounded from above, and $\rho_k \rightarrow \rho_0$ weakly (as measures) as $k \rightarrow \infty$, where ρ_0 has centre of mass at o . Then, we have*

$$\lim_{k \rightarrow \infty} \int_M \log |J(f^{-1})(f(x))| d\rho_k(x) = \int_M \log |J(f^{-1})(f(x))| d\rho_0(x),$$

where f denotes the Riemannian logarithm map at o (see (4.22)).

Proof. Since $E[\rho_k]$ is uniformly bounded from above, by Proposition 5.1 (see (5.36)), so is

$$\iint_{M \times M} h(d(x, y)) d\rho_k(x) d\rho_k(y). \text{ Denote by } U \text{ such an upper bound, i.e.,}$$

$$\iint_{M \times M} h(d(x, y)) d\rho_k(x) d\rho_k(y) \leq U, \quad \text{for all } k \geq 1.$$

Also, by Lemma 5.3, we have

$$\iint_{M \times M} h(d(x, y)) d\rho_0(x) d\rho_0(y) \leq U.$$

Now recall that ϕ in (5.31) is non-decreasing and convex, and apply the same argument used in the proof of Proposition 5.1 (see the arguments that led to (5.42)) with ϕ instead of H . Also note that the absolute continuity of the measure is not needed in the argument, only that the measure has centre of mass at o (use the more general condition $\int_M \log_o y d\rho(y) = 0$ instead of (5.39)). Hence, we can get

$$\iint_{M \times M} \phi(d(x, y)) d\rho_k(x) d\rho_k(y) \geq \int_M \phi(r_x) d\rho_k(x), \quad \text{for all } k \geq 1,$$

and

$$\iint_{M \times M} \phi(d(x, y)) d\rho_0(x) d\rho_0(y) \geq \int_M \phi(r_x) d\rho_0(x).$$

Consequently, since $h(\theta) \geq \phi(\theta)$, we infer from above that

$$(5.45) \quad \int_M \phi(r_x) d\rho_k(x) \leq U, \quad \text{for all } k \geq 1, \quad \text{and} \quad \int_M \phi(r_x) d\rho_0(x) \leq U.$$

For any $R > 0$ fixed, we have the following estimate by the triangle inequality:

$$\begin{aligned}
 (5.46) \quad & \left| \int_M \log |J(f^{-1})(f(x))| d\rho_0(x) - \int_M \log |J(f^{-1})(f(x))| d\rho_k(x) \right| \\
 & \leq \left| \int_{r_x \leq R} \log |J(f^{-1})(f(x))| d\rho_0(x) - \int_{r_x \leq R} \log |J(f^{-1})(f(x))| d\rho_k(x) \right| \\
 & \quad + \int_{r_x > R} \log |J(f^{-1})(f(x))| d\rho_0(x) + \int_{r_x > R} \log |J(f^{-1})(f(x))| d\rho_k(x),
 \end{aligned}$$

where we also used that $|J(f^{-1})(f(x))| \geq 1$ (a consequence of Theorem 2.2). Using Theorem 2.2 and (5.45), we can estimate the last two terms in the right-hand-side above as follows:

$$\begin{aligned}
 & \int_{r_x > R} \log |J(f^{-1})(f(x))| d\rho_0(x) + \int_{r_x > R} \log |J(f^{-1})(f(x))| d\rho_k(x) \\
 & \leq (n-1) \int_{r_x > R} \log \left(\frac{\psi_m(r_x)}{r_x} \right) d\rho_0(x) + (n-1) \int_{r_x > R} \log \left(\frac{\psi_m(r_x)}{r_x} \right) d\rho_k(x) \\
 & \leq (n-1) \left\| \frac{\log \left(\frac{\psi_m(\cdot)}{\cdot} \right)}{\phi(\cdot)} \right\|_{L^\infty([R, \infty))} \left(\int_{r_x > R} \phi(r_x) d\rho_0(x) + \int_{r_x > R} \phi(r_x) d\rho_k(x) \right) \\
 & \leq 2(n-1) \left\| \frac{\log \left(\frac{\psi_m(\cdot)}{\cdot} \right)}{\phi(\cdot)} \right\|_{L^\infty([R, \infty))} U.
 \end{aligned}$$

Using this result in (5.46), together with the weak convergence $\rho_k \rightharpoonup \rho_0$, we then find

$$\begin{aligned}
 & \limsup_{k \rightarrow \infty} \left| \int_M \log |J(f^{-1})(f(x))| d\rho_0(x) - \int_M \log |J(f^{-1})(f(x))| d\rho_k(x) \right| \\
 & \leq \limsup_{k \rightarrow \infty} \left| \int_{r_x \leq R} \log |J(f^{-1})(f(x))| d\rho_0(x) - \int_{r_x \leq R} \log |J(f^{-1})(f(x))| d\rho_k(x) \right| \\
 & \quad + 2(n-1) \left\| \frac{\log \left(\frac{\psi_m(\cdot)}{\cdot} \right)}{\phi(\cdot)} \right\|_{L^\infty([R, \infty))} U \\
 & = 2(n-1) \left\| \frac{\log \left(\frac{\psi_m(\cdot)}{\cdot} \right)}{\phi(\cdot)} \right\|_{L^\infty([R, \infty))} U.
 \end{aligned}$$

Finally, by assumption (5.32), we have

$$\lim_{R \rightarrow \infty} \left\| \frac{\log \left(\frac{\psi_m(\cdot)}{\cdot} \right)}{\phi(\cdot)} \right\|_{L^\infty([R, \infty))} = 0.$$

Since R is arbitrary, we then get

$$\limsup_{k \rightarrow \infty} \left| \int_M \log |J(f^{-1})(f(x))| d\rho_0(x) - \int_M \log |J(f^{-1})(f(x))| d\rho_k(x) \right| \leq 0,$$

which yields the desired conclusion. \square

We can now show that the energy functional is lower semi-continuous along minimizing sequences.

Proposition 5.2 (Lower semi-continuity of the energy). *Let $o \in M$ be a fixed pole and assume M and h satisfy the assumptions in Proposition 5.1. Suppose $\{\rho_k\}_{k \geq 1} \subset \mathcal{P}_o(M)$ such that $E[\rho_k]$ is uniformly bounded from above, and $\rho_k \rightharpoonup \rho_0$ weakly (as measures) as $k \rightarrow \infty$, where ρ_0 has centre of mass at o . Then, ρ_0 is necessarily absolutely continuous, and the energy functional is lower semi-continuous along ρ_k , i.e.,*

$$E[\rho_0] \leq \liminf_{k \rightarrow \infty} E[\rho_k].$$

Proof. The energy functional contains two components: the entropy and the interaction energy; we will consider the two parts separately. The lower semi-continuity of the interaction energy was established in Lemma 5.3. For the entropy component, we will use again the Riemannian logarithm $f = \log_o$ and map M to the tangent space $T_o M$.

We first show that given the assumptions on ρ_k , the limit ρ_0 is necessarily absolutely continuous. Indeed, assume by contradiction that it is not. Then, since $J(f^{-1}) \geq 1$, its pushforward measure $f_{\#}\rho_0$ is also singular. Following [2, Section 10.4.3], define the entropy functional $\mathcal{F} : \mathcal{P}_1(\mathbb{R}^n) \rightarrow \mathbb{R}$ by

$$\mathcal{F}[\mu] = \begin{cases} \int_{\mathbb{R}^n} \rho(x) \log(\rho(x)) dx, & \text{if } \mu = \rho dx, \\ \infty, & \text{otherwise.} \end{cases}$$

Here, dx denotes the Lebesgue measure on the Euclidean space \mathbb{R}^n . As noted in [2, Section 10.4.3], the functional \mathcal{F} is lower semi-continuous with respect to weak convergence of measures. Note that by the change of variable formula, $\rho_k \rightharpoonup \rho_0$ implies $f_{\#}\rho_k \rightharpoonup f_{\#}\rho_0$ as $k \rightarrow \infty$. Hence,

$$\liminf_{k \rightarrow \infty} \mathcal{F}[f_{\#}\rho_k] \geq \mathcal{F}[f_{\#}\rho_0],$$

and since $f_{\#}\rho_0$ is singular, this implies

$$(5.47) \quad \liminf_{k \rightarrow \infty} \mathcal{F}[f_{\#}\rho_k] = \infty.$$

Now, using the exponential map to change variables, write (see also (4.23)):

$$\int_{\mathbb{R}^n} f_{\#}\rho_k(u) \log f_{\#}\rho_k(u) du = \int_M \rho_k(x) \log \rho_k(x) dV(x) + \int_M \rho_k(x) \log |J(f^{-1})(f(x))| dV(x),$$

for all $k \geq 1$. The left-hand-side above is the entropy of $f_{\#}\rho_k$ on $T_o M \simeq \mathbb{R}^n$. From the above, together with Lemma 5.4, we infer

$$(5.48) \quad \begin{aligned} \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_{\#}\rho_k(u) \log f_{\#}\rho_k(u) du \\ = \liminf_{k \rightarrow \infty} \int_M \rho_k(x) \log \rho_k(x) dV(x) + \int_M \log |J(f^{-1})(f(x))| d\rho_0(x). \end{aligned}$$

By (5.47), the left-hand-side of (5.48) is infinite. However, both terms in the right-hand-side are bounded above, which leads to a contradiction. To argue this latter statement, note that since $h \geq 0$, the entropy of ρ_k is bounded from above by $E[\rho_k]$, which is uniformly bounded by assumption. Also, by the proof of Lemma 5.4, one can find that $\int_M \log |J(f^{-1})(f(x))| d\rho_k(x)$ is bounded above uniformly in k . The contradiction makes us conclude that ρ_0 (and $f_{\#}\rho_0$) are absolutely continuous.

The lower semi-continuity of the entropy follows now from the arguments above:

$$\begin{aligned}
& \liminf_{k \rightarrow \infty} \int_M \rho_k(x) \log \rho_k(x) dV(x) \\
&= \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_{\#} \rho_k(u) \log f_{\#} \rho_k(u) du - \lim_{k \rightarrow \infty} \int_M \rho_k(x) \log |J(f^{-1})(f(x))| dV(x) \\
&\geq \int_{\mathbb{R}^n} f_{\#} \rho_0(u) \log f_{\#} \rho_0(u) du - \int_M \rho_0(x) \log |J(f^{-1})(f(x))| dV(x) \\
&= \int_M \rho_0(x) \log \rho_0(x) dV(x),
\end{aligned}$$

where for the last line we used again the change of variable $x = \exp_o u$ and (4.23).

Finally, the lower semi-continuity of the energy can be inferred from the lower semi-continuity of its two components, as

$$\begin{aligned}
\liminf_{k \rightarrow \infty} E[\rho_k] &\geq \liminf_{k \rightarrow \infty} \int_M \rho_k(x) \log \rho_k(x) dV(x) + \liminf_{k \rightarrow \infty} \frac{1}{2} \iint_{M \times M} h(d(x, y)) \rho_k(x) \rho_k(y) dV(x) dV(y) \\
&\geq \int_M \rho_0(x) \log \rho_0(x) dV(x) + \frac{1}{2} \iint_{M \times M} h(d(x, y)) \rho_0(x) \rho_0(y) dV(x) dV(y) \\
&= E[\rho_0].
\end{aligned}$$

□

The last ingredient needed for the proof of the main theorem is the conservation of centre of mass, given by the following result established in [24]. We point out that in the next lemma, as well as in the proof of Theorem 5.1 that follows, we will use the notation $B_R(\delta_o)$ for the geodesic ball in the space $(\mathcal{P}_1(M), \mathcal{W}_1)$, of radius R and centre at δ_o . A similar notation has been used in the paper for the open ball $B_r(o)$ of radius r and centre at o , in the geodesic space (M, d) . Nevertheless, the different spaces in which these geodesic balls are considered, will be clear from the context.

Lemma 5.5 (Conservation of centre of mass [24]). *Let M be a Cartan-Hadamard manifold, $o \in M$ a fixed pole, $R > 0$ a fixed radius, and $\{\rho_k\}_{k \geq 1}$ a sequence in $\overline{B_R(\delta_o)}$ such that all ρ_k ($k \geq 1$) have centre at mass at o . Also assume that*

$$\int_M \phi(r_x) d\rho_k(x)$$

is uniformly bounded from above, where ϕ is a function with superlinear growth at infinity, i.e., $\lim_{\theta \rightarrow \infty} \frac{\phi(\theta)}{\theta} = \infty$. Then, there exists a subsequence of $\{\rho_k\}_{k \geq 1}$ which converges weakly as measures to $\rho_0 \in \overline{B_R(\delta_o)}$, where ρ_0 has centre of mass at o .

Proof. The result was stated and proved in [24, Lemma 6.7], but for completeness, we also present it in Appendix C. First, by tightness and Prokhorov's theorem, one can extract a subsequence of $\{\rho_k\}_{k \geq 1}$ which converges weakly as measures to $\rho_0 \in \overline{B_R(\delta_o)}$. Then, use the uniform boundedness of $\int_M \phi(r_x) d\rho_k(x)$ to argue that the centre of mass is preserved in the limit. See Appendix C for the details. □

We now present the proof of Theorem 5.1, one of the main results of this paper.

Proof of Theorem 5.1. The energy is bounded below on $\mathcal{P}_o(M)$ (see Proposition 5.1). In particular, $E[\rho]$ is bounded below on $\overline{B_1(\delta_o)} \cap \mathcal{P}_o(M)$, and define

$$E_0 := \inf_{\rho \in \overline{B_1(\delta_o)} \cap \mathcal{P}_o(M)} E[\rho].$$

Take a minimizing sequence $\{\rho_k\}_{k \in \mathbb{N}}$ of $E[\rho]$ on $\overline{B_1(\delta_o)} \cap \mathcal{P}_o(M)$, i.e.,

$$\{\rho_k\} \subset \overline{B_1(\delta_o)} \cap \mathcal{P}_o(M), \quad \lim_{k \rightarrow \infty} E[\rho_k] = E_0.$$

Without loss of generality, we can assume that $E[\rho_k]$ is decreasing. In particular, $E[\rho_k]$ is uniformly bounded from above and by the argument used in the proof of Lemma 5.4 (see the derivation of (5.45)), $\int_M \phi(r_x) \rho_k(x) dV(x)$ is also uniformly bounded from above. We also recall that the function ϕ used in our assumption (5.32) (or (5.31)) for h has superlinear growth at infinity, and therefore it can be used in the context of Lemma 5.5. Hence, by Lemma 5.5, there exists a subsequence $\{\rho_{k_l}\}_{l \in \mathbb{N}}$ which converges weakly to $\rho_0 \in \overline{B_1(\delta_o)}$, where ρ_0 has centre of mass at o .

By Proposition 5.2, ρ_0 is necessarily absolutely continuous and hence, $\rho_0 \in \overline{B_1(\delta_o)} \cap \mathcal{P}_o(M)$. Also, from the lower semi-continuity of the energy in Proposition 5.2, we have

$$\lim_{l \rightarrow \infty} E[\rho_{k_l}] \geq E[\rho_0].$$

Therefore,

$$E_0 = \lim_{l \rightarrow \infty} E[\rho_{k_l}] \geq E[\rho_0] \geq E_0,$$

which implies $E[\rho_0] = E_0$, i.e, ρ_0 is a minimizer of the energy on $\overline{B_1(\delta_o)} \cap \mathcal{P}_o(M)$.

Now take $R > 1$ large enough such that

$$E[\rho_0] \leq -C_0 n + \frac{\mathcal{C}}{2} + \frac{2 - \sqrt{2}}{2} R,$$

where C_0 and \mathcal{C} are the constants from Proposition 5.1 (C_0 is the constant from the HLS inequality (4.21), and depends only on the dimension n , and \mathcal{C} depends on h , c_m and n). Then, by Proposition 5.1 (note that $h \geq 0$), for any $\rho \in \mathcal{P}_o(M) \setminus \overline{B_R(\delta_o)}$ it holds that

$$\begin{aligned} E[\rho] &\geq -C_0 n + \frac{\mathcal{C}}{2} + \frac{2 - \sqrt{2}}{2} \mathcal{W}_1(\rho, \delta_o) \\ (5.49) \quad &> -C_0 n + \frac{\mathcal{C}}{2} + \frac{2 - \sqrt{2}}{2} R \\ &\geq E[\rho_0]. \end{aligned}$$

On the other hand,

$$E[\rho_0] = \inf_{\rho \in \overline{B_1(\delta_o)} \cap \mathcal{P}_o(M)} E[\rho] \geq \inf_{\rho \in \overline{B_R(\delta_o)} \cap \mathcal{P}_o(M)} E[\rho],$$

which together with (5.49), it implies

$$\inf_{\rho \in \mathcal{P}_o(M)} E[\rho] = \inf_{\rho \in \overline{B_R(\delta_o)} \cap \mathcal{P}_o(M)} E[\rho].$$

Finally, the existence of a global minimizer of $E[\rho]$ on $\overline{B_R(\delta_o)} \cap \mathcal{P}_o(M)$ can be argued exactly as we argued above the existence of the minimizer ρ_0 in $\overline{B_1(\delta_o)} \cap \mathcal{P}_o(M)$. This concludes the proof. \square

Example 5.1. We will revisit here Example 3.1 in the context of Theorem 5.1.

- (1) Constant: $c_m(\theta) \equiv c_m$. In this case, (5.31) reduces to $h \geq \phi$, with ϕ growing superlinearly at infinity, which recovers the result in [24, Theorem 2.2].
- (2) Power law: $c_m(\theta) = \theta^k$ with $k \geq 1$. Ground states exists provided $h \geq \phi$ and ϕ grows faster than $\theta^{k/2+1}$ at infinity. Note that the result is sharp in the sense that a growth of h slower than $\theta^{k/2+1}$ is not sufficient to contain the diffusion, and leads to spreading (see item (2) in Example 3.1).
- (3) Exponential growth: $c_m(\theta) = e^{\beta\theta}$ with $\beta > 0$. For this case, spreading is prevented provided $h(\theta)$ grows at infinity faster than $e^{\beta\theta/2}$. Hence, an exponential growth of the attractive interactions is needed to contain diffusion on a manifold with exponentially growing curvatures.

We conclude the paper with a discussion on the assumption of the monotonicity of c_m . This assumption can be dropped by requiring a stronger assumption on the behaviour at infinity of h . Indeed, assume that the sectional curvatures satisfy (5.30), where c_m is not necessarily non-decreasing. Define

$$\tilde{c}_m(\theta) := \max_{0 \leq t \leq \theta} c_m(t).$$

Then, \tilde{c}_m is continuous and non-decreasing and by (5.30), the curvatures satisfy

$$-\tilde{c}_m(r_x) \leq \mathcal{K}(x; \sigma) \leq 0,$$

for all $x \in M$ and all two-dimensional subspaces $\sigma \subset T_x M$. Hence, we can apply Theorem 5.1 with \tilde{c}_m instead of c_m , and obtain the following corollary.

Corollary 5.2. *Let M be an n -dimensional Cartan-Hadamard manifold and $o \in M$ a fixed pole. Assume that the sectional curvatures of M satisfy (5.30), where $c_m(\cdot)$ is a positive, continuous function. Also assume that $h : [0, \infty) \rightarrow [0, \infty)$ is a non-decreasing lower semi-continuous function which satisfies*

$$h(\theta) \geq \phi(\theta), \quad \text{and} \quad \lim_{\theta \rightarrow \infty} \frac{\phi(\theta)}{\int_0^\theta \sqrt{\max_{0 \leq s \leq t} c_m(s)} dt} = \infty,$$

for some convex function ϕ . Then there exists a global minimum of $E[\rho]$ in $\mathcal{P}_o(M)$.

APPENDIX A. PROOF OF THEOREM 2.2

Part I: Left inequality. This part follows exactly the steps of the proof of [16, Theorem III.4.1] (constant bounds of \mathcal{K}), but we will reproduce it here for completeness. Following the proof of [16, Theorem II.6.4], for $Y \in \mathcal{J}^\perp$, one has

$$|Y'| = \langle Y, \nabla_t Y \rangle |Y|^{-1},$$

and then,

$$\begin{aligned} |Y|'' &= |Y|^{-1} (|\nabla_t Y|^2 - \langle Y, \mathcal{R}(\gamma', Y) \gamma' \rangle) - |Y|^{-3} \langle Y, \nabla_t Y \rangle^2 \\ &\geq c_M |Y| + |Y|^{-3} (|\nabla_t Y|^2 |Y|^2 - \langle Y, \nabla_t Y \rangle^2) \\ \text{(A.50)} \quad &\geq c_M |Y|, \end{aligned}$$

where in the first line we used the Jacobi equation (2.4), for the second line we used the upper bound on the sectional curvatures, and the third line follows from Cauchy-Schwarz inequality.

We set

$$|Y|(0) = 0, \quad |Y|'(0) = 1.$$

It then follows from (A.50) and (2.8) that $F := |Y|'\psi_M - |Y|\psi'_M$ is non-decreasing, and hence, $F \geq 0$ (as $F(0) = 0$). We can write this as

$$(A.51) \quad \frac{|Y|'}{|Y|} \geq \frac{\psi'_M}{\psi_M}.$$

Instead of calculating the determinant of \mathcal{A} , we work with the self-adjoint matrix \mathcal{B} defined as

$$\mathcal{B} := \mathcal{A}^* \mathcal{A}.$$

From this definition, we have

$$(A.52) \quad \frac{(\det \mathcal{A})'}{\det \mathcal{A}} = \frac{1}{2} \frac{(\det \mathcal{B})'}{\det \mathcal{B}} = \frac{1}{2} \text{tr}(\mathcal{B}' \mathcal{B}^{-1}),$$

where the second equal sign comes from Jacobi's formula for the derivative of a determinant.

Fix an arbitrary $\tau > 0$. Since any self-adjoint matrix is diagonalizable, we can choose an orthonormal basis $\{e_1, \dots, e_{n-1}\}$ of $\gamma'(0)^\perp$ consisting of eigenvectors of $\mathcal{B}(\tau)$. Then consider the solution $\{\eta_1, \dots, \eta_{n-1}\}$ of the Jacobi equation in $\gamma'(0)^\perp$, given by

$$\eta_i(t) = \mathcal{A}(t)e_i, \quad \text{for all } 1 \leq i \leq n-1.$$

It holds immediately that for all t ,

$$\langle \eta_i(t), \eta_j(t) \rangle = \langle \mathcal{A}(t)e_i, \mathcal{A}(t)e_j \rangle = \langle \mathcal{B}(t)e_i, e_j \rangle, \quad 1 \leq i, j \leq n-1.$$

By differentiating the above, we compute

$$\begin{aligned} \text{tr}(\mathcal{B}' \mathcal{B}^{-1})(t) &= \sum_{i,j=1}^{n-1} \langle \mathcal{B}'(t)e_i, e_j \rangle \cdot \langle e_j, \mathcal{B}^{-1}(t)e_i \rangle \\ &= \sum_{i,j=1}^{n-1} (\langle \eta'_i(t), \eta_j(t) \rangle + \langle \eta_i(t), \eta'_j(t) \rangle) \langle \mathcal{B}^{-1}(t)e_i, e_j \rangle. \end{aligned}$$

Using the assumption on the basis $\{e_1, \dots, e_{n-1}\}$, $\langle \mathcal{B}(\tau)e_i, e_j \rangle = \lambda_i \delta_{ij}$ for some eigenvalues λ_i , $i = 1, \dots, n-1$. Also, $\langle \mathcal{B}(\tau)^{-1}e_i, e_j \rangle = \lambda_i^{-1} \delta_{ij}$. Then, by evaluating at $t = \tau$, we find

$$\begin{aligned} \text{tr}(\mathcal{B}' \mathcal{B}^{-1})(\tau) &= \sum_{i,j=1}^{n-1} (\langle \eta'_i(\tau), \eta_j(\tau) \rangle + \langle \eta_i(\tau), \eta'_j(\tau) \rangle) \lambda_i^{-1} \delta_{ij} \\ &= 2 \sum_{i=1}^{n-1} \frac{\langle \eta'_i(\tau), \eta_i(\tau) \rangle}{\lambda_i}. \end{aligned}$$

Finally, we use

$$\langle \eta_i(\tau), \eta_i(\tau) \rangle = \langle \mathcal{B}(\tau)e_i, e_i \rangle = \lambda_i, \quad i = 1, \dots, n-1,$$

to conclude that

$$\frac{1}{2} \text{tr}(\mathcal{B}' \mathcal{B}^{-1})(\tau) = \sum_{i=1}^{n-1} \frac{\langle \eta'_i(\tau), \eta_i(\tau) \rangle}{\langle \eta_i(\tau), \eta_i(\tau) \rangle} = \sum_{i=1}^{n-1} \frac{|\eta'_i(\tau)|}{|\eta_i(\tau)|} \geq (n-1) \frac{\psi'_M(\tau)}{\psi_M(\tau)},$$

where for the last inequality we used (A.51).

Since τ was arbitrary, combining the above inequality with (A.52) we find

$$(A.53) \quad (n-1) \frac{\psi'_M}{\psi_M} \leq \frac{(\det \mathcal{A})'}{\det \mathcal{A}}.$$

Inequality (A.53) can be integrated to get the desired conclusion. We explain this last step, together with the right inequality, in Part II.

Part II: Right inequality. This part also follows closely the proof of the constant bounds case (see [16, Theorem III.4.3]), but with some small extra technicalities. We present a lemma first.

Lemma A.1. *Let Ψ satisfy the following ODE for some $\epsilon > 0$:*

$$\Psi'(t) = (n-1)c(t) - \frac{\Psi^2(t)}{n-1}, \quad \text{for all } t \geq \epsilon, \quad \Psi(\epsilon) > 0,$$

where $c(t) \geq 0$ for all $t \geq \epsilon$. Then,

$$\Psi(t) > 0, \quad \text{for all } t > \epsilon.$$

Proof. We have

$$\Psi'(t) \geq -\frac{\Psi^2(t)}{n-1}, \quad \text{for } t \geq \epsilon,$$

which yields

$$\frac{d}{dt} \left(\frac{1}{\Psi(t)} \right) \leq \frac{1}{n-1}.$$

We can solve this differential inequality to get

$$\frac{1}{\Psi(t)} - \frac{1}{\Psi(\epsilon)} \leq \frac{t-\epsilon}{n-1}, \quad \text{for } t \geq \epsilon.$$

Hence, we have

$$\frac{1}{\Psi(t)} \leq \frac{1}{\Psi(\epsilon)} + \frac{t-\epsilon}{n-1}, \quad \text{for all } t \geq \epsilon,$$

which then implies the conclusion of the lemma. \square

We define

$$\Psi_m := (n-1) \frac{\psi'_m}{\psi_m}, \quad \Psi_M := (n-1) \frac{\psi'_M}{\psi_M}, \quad \Phi := \frac{(\det \mathcal{A})'}{\det \mathcal{A}}.$$

Using (2.8) we find that Ψ_m and Ψ_M satisfy the following equations:

$$\Psi'_m = (n-1)c_m - \frac{\Psi_m^2}{n-1}, \quad \text{and} \quad \Psi'_M = (n-1)c_M - \frac{\Psi_M^2}{n-1}.$$

Also, by the initial data in (2.8), we have

$$\Psi_m(t), \Psi_M(t) \sim \frac{n-1}{t}, \quad \text{as } t \rightarrow 0^+.$$

From Lemma A.1, we know that both Ψ_m and Ψ_M are positive. From the left inequality (Part I), we have

$$\Psi_M \leq \Phi,$$

and this yields that Φ is also positive.

By Jacobi's formula, we have

$$\Phi = \text{tr}(\mathcal{A}'\mathcal{A}^{-1}).$$

We denote $\mathcal{U} := \mathcal{A}'\mathcal{A}^{-1}$. From the argument in the proof of [16, Theorem III.4.3], it holds that

$$(\mathrm{tr} \mathcal{U})' + \mathrm{tr} \mathcal{U}^2 + \mathrm{tr} \mathcal{R} = 0.$$

Since $\mathcal{K}(x) \geq -c_m(r_x)$ and

$$\mathrm{tr} \mathcal{U}^2 \geq \frac{(\mathrm{tr} \mathcal{U})^2}{n-1},$$

we get¹

$$(A.54) \quad \Phi' \leq (n-1)c_m - \frac{\Phi^2}{n-1}.$$

We calculate now the time evolution of the functional $F := 1 - \frac{\Phi}{\Psi_m}$, and estimate it using (A.54):

$$\begin{aligned} -F' &= \frac{\Phi' \Psi_m - \Phi \Psi_m'}{\Psi_m^2} \\ &\leq \frac{(n-1)c_m(\Psi_m - \Phi) - \frac{\Phi \Psi_m}{n-1}(\Phi - \Psi_m)}{\Psi_m^2} \\ &= \frac{(n-1)c_m}{\Psi_m} \left(1 - \frac{\Phi}{\Psi_m}\right) + \frac{\Phi}{n-1} \left(1 - \frac{\Phi}{\Psi_m}\right) \\ &= \left(\frac{(n-1)c_m}{\Psi_m} + \frac{\Phi}{n-1}\right) F. \end{aligned}$$

This yields

$$(A.55) \quad F' \geq -\left(\frac{(n-1)c_m}{\Psi_m} + \frac{\Phi}{n-1}\right) F.$$

We also find

$$\lim_{t \rightarrow 0+} \frac{\Phi(t)}{\Psi_m(t)} = \lim_{t \rightarrow 0+} \frac{(\det \mathcal{A})'(t) \psi_m(t)}{(n-1) \det \mathcal{A}(t) \psi_m'(t)} = \lim_{t \rightarrow 0+} \left(\frac{t(\det \mathcal{A})'(t)}{(n-1) \det \mathcal{A}(t)} \right) \left(\frac{\psi_m(t)}{\psi_m'(t)t} \right) = 1.$$

Hence, we have $\lim_{t \rightarrow 0+} F(t) = 0$, and we can define $F(0) = 0$. Let assume that for some $T > 0$, $F(T) < 0$. Then, we can define $0 \leq T_0 < T$ which satisfies

$$T_0 = \sup\{t < T : F(t) = 0\}.$$

Then,

$$F(t) \leq 0, \quad \forall T_0 \leq t \leq T,$$

which by (B.61), it yields

$$F' \geq 0, \quad \forall T_0 \leq t \leq T.$$

Finally, we get

$$F(T) - F(T_0) = \int_{T_0}^T F'(t) dt \geq 0.$$

However, we assumed $F(T_0) = 0$ and $F(T) < 0$, so we have a contradiction. We conclude that $F(t) \geq 0$ for all $t \geq 0$, or equivalently,

$$\Phi(t) \leq \Psi_m(t), \quad \forall t > 0.$$

¹To obtain inequality (A.54) one needs the weaker assumption $\mathrm{tr} \mathcal{R} \geq -(n-1)c_m$ on curvatures – see also Remark 2.2.

Using the expressions of Φ and Ψ , we then have

$$(A.56) \quad \frac{(\det \mathcal{A})'}{\det \mathcal{A}} \leq (n-1) \frac{\psi'_m}{\psi_m}.$$

Now, by (A.53) and (A.56), we have both inequalities:

$$(n-1) \frac{\psi'_M}{\psi_M} \leq \frac{(\det \mathcal{A})'}{\det \mathcal{A}} \leq (n-1) \frac{\psi'_m}{\psi_m}.$$

We integrate this double inequality from $\epsilon > 0$ to θ to get

$$(A.57) \quad \psi_M(\theta)^{n-1} \left(\frac{\det \mathcal{A}(\epsilon)}{\psi_M(\epsilon)^{n-1}} \right) \leq \det \mathcal{A}(\theta) \leq \psi_m(\theta)^{n-1} \left(\frac{\det \mathcal{A}(\epsilon)}{\psi_m(\epsilon)^{n-1}} \right).$$

Since $\mathcal{A}'(0) = I$ and $\mathcal{A}(0) = 0$, we have

$$\lim_{\epsilon \rightarrow 0^+} \frac{\det \mathcal{A}(\epsilon)}{\epsilon^{n-1}} = 1.$$

Together with the initial conditions for ψ_m and ψ_M (see (2.8)), this further yields

$$\lim_{\epsilon \rightarrow 0^+} \frac{\det \mathcal{A}(\epsilon)}{\psi_m(\epsilon)^{n-1}} = \lim_{\epsilon \rightarrow 0^+} \frac{\det \mathcal{A}(\epsilon)}{\psi_M(\epsilon)^{n-1}} = 1.$$

Finally, we let $\epsilon \rightarrow 0^+$ in (A.57) to get

$$\psi_M(\theta)^{n-1} \leq \det \mathcal{A}(\theta) \leq \psi_m(\theta)^{n-1}.$$

This concludes the proof of Theorem 2.2.

APPENDIX B. PROOF OF LEMMA 2.1

In this appendix, we prove each part of Lemma 2.1.

Proof of upper bound. In this part, we assume that c is positive, continuous, and non-decreasing, and show (2.11). From $\psi''(\theta) = c(\theta)\psi(\theta)$, we have

$$\frac{d}{dt} \psi'(t)^2 = 2\psi''(t)\psi'(t) = 2c(t)\psi(t)\psi'(t) = c(t) \frac{d}{dt} \psi(t)^2.$$

By integrating the above equality on $0 \leq t \leq \theta$, we then obtain

$$\psi'(\theta)^2 - \psi'(0)^2 = \int_0^\theta \frac{d}{dt} \psi'(t)^2 dt = \int_0^\theta c(t) \frac{d}{dt} \psi(t)^2 dt,$$

and since c is non-decreasing, we further get

$$\psi'(\theta)^2 - \psi'(0)^2 \leq c(\theta) \int_0^\theta \frac{d}{dt} \psi(t)^2 dt = c(\theta) (\psi(\theta)^2 - \psi(0)^2).$$

Now substitute the initial conditions $\psi(0) = 0$ and $\psi'(0) = 1$ in the above, to estimate

$$(B.58) \quad \begin{aligned} \psi'(\theta) &\leq \sqrt{1 + c(\theta)\psi(\theta)^2} \\ &\leq 1 + \sqrt{c(\theta)}\psi(\theta). \end{aligned}$$

The differential inequality (B.58) can be integrated, with the explicit solution

$$\psi(\theta) \leq \exp \left(\int_0^\theta \sqrt{c(t)} dt \right) \left(\int_0^\theta \left(\exp \left(- \int_0^t \sqrt{c(s)} ds \right) \right) dt \right).$$

Then, by

$$\int_0^\theta \left(\exp \left(- \int_0^t \sqrt{c(s)} ds \right) \right) dt \leq \int_0^\theta 1 dt = \theta,$$

(2.11) follows.

Proof of lower bound. For $0 < \epsilon < 1$ arbitrary fixed, define $F_\epsilon(\theta)$ by

$$F_\epsilon(\theta) = \frac{1}{(1-\epsilon)\sqrt{c(\theta)}}.$$

The lower derivative of F_ϵ can be obtained as

$$(B.59) \quad \underline{D}F_\epsilon(\theta) = -\frac{1}{2(1-\epsilon)} \times \frac{\overline{D}c(\theta)}{c(\theta)^{3/2}},$$

where we used

$$\begin{aligned} \underline{D}\left(\frac{1}{\sqrt{c(\theta)}}\right) &= \liminf_{h \rightarrow 0} \frac{\frac{1}{\sqrt{c(\theta+h)}} - \frac{1}{\sqrt{c(\theta)}}}{h} \\ &= \liminf_{h \rightarrow 0} \left(-\frac{c(\theta+h) - c(\theta)}{h} \cdot \frac{1}{\sqrt{c(\theta)}\sqrt{c(\theta+h)}(\sqrt{c(\theta)} + \sqrt{c(\theta+h)})} \right) \\ &= -\limsup_{h \rightarrow 0} \left(\frac{c(\theta+h) - c(\theta)}{h} \right) \frac{1}{2c(\theta)\sqrt{c(\theta)}} \\ &= -\frac{1}{2c(\theta)^{3/2}} \overline{D}c(\theta). \end{aligned}$$

Compute

$$(B.60) \quad 1 - c(\theta)F_\epsilon(\theta)^2 = 1 - \frac{1}{(1-\epsilon)^2} = \frac{-2\epsilon + \epsilon^2}{(1-\epsilon)^2}.$$

Then, since $\lim_{\theta \rightarrow \infty} \frac{\overline{D}c(\theta)}{c(\theta)^{3/2}} = 0$, there exists $\theta_\epsilon > 0$ such that

$$\frac{\overline{D}c(\theta)}{c(\theta)^{3/2}} \leq \frac{2(2\epsilon - \epsilon^2)}{1-\epsilon}, \quad \forall \theta \geq \theta_\epsilon,$$

which combined with (B.59) and (B.60), yields

$$(B.61) \quad 1 - c(\theta)F_\epsilon(\theta)^2 \leq \underline{D}F_\epsilon(\theta), \quad \forall \theta \geq \theta_\epsilon.$$

Denote

$$G(\theta) = \frac{\psi(\theta)}{\psi'(\theta)}, \quad \text{for } \theta \geq 0.$$

Compute

$$\begin{aligned} G'(\theta) &= \frac{\psi'(\theta)\psi'(\theta) - \psi(\theta)\psi''(\theta)}{\psi'(\theta)^2} \\ (B.62) \quad &= 1 - c(\theta)G(\theta)^2. \end{aligned}$$

Note that $G(0) = \frac{\psi(0)}{\psi'(0)} = 0$ and $G'(0) = 1 - c(0)G(0)^2 = 1$.

We will show that

$$(B.63) \quad \liminf_{\theta \rightarrow \infty} \sqrt{c(\theta)} G(\theta) \leq 1.$$

Indeed, if we assume $\liminf_{\theta \rightarrow \infty} \sqrt{c(\theta)} G(\theta) > 1$, then there exists some positive $\delta > 0$ and $\bar{\theta} > 0$ such that

$$\sqrt{c(\theta)} G(\theta) \geq \sqrt{1 + \delta}, \quad \forall \theta \geq \bar{\theta}.$$

Hence,

$$G'(\theta) = 1 - c(\theta) G(\theta)^2 < 1 - (1 + \delta) = -\delta, \quad \forall \theta \geq \bar{\theta},$$

which implies

$$G(\theta) < G(\bar{\theta}) - \delta(\theta - \bar{\theta}), \quad \forall \theta \geq \bar{\theta}.$$

However, by choosing $\theta > \bar{\theta} + \frac{G(\bar{\theta})}{\delta}$, we find $G(\theta) < 0$, which is not possible. Therefore, (B.63) holds.

By (B.63), there exists $\theta_0 > \theta_\epsilon$ such that

$$\sqrt{c(\theta_0)} G(\theta_0) \leq \frac{1}{1 - \epsilon},$$

which yields

$$(B.64) \quad G(\theta_0) \leq \frac{1}{(1 - \epsilon)\sqrt{c(\theta_0)}} = F_\epsilon(\theta_0).$$

Now note that F_ϵ is a supersolution to the ODE solved by G (see (B.61) and (B.62)) for all $\theta \geq \theta_0$. Hence, together with (B.64), we infer

$$F_\epsilon(\theta) \geq G(\theta), \quad \forall \theta \geq \theta_0.$$

Finally, we use the definitions of F_ϵ and G to get

$$\frac{1}{(1 - \epsilon)\sqrt{c(\theta)}} \geq \frac{\psi(\theta)}{\psi'(\theta)}, \quad \forall \theta \geq \theta_0,$$

which is equivalent to

$$\psi'(\theta) \geq (1 - \epsilon)\sqrt{c(\theta)}\psi(\theta), \quad \forall \theta \geq \theta_0.$$

By integrating the above differential inequality, one then finds (2.13).

APPENDIX C. PROOF OF LEMMA 5.5

In this appendix, we prove Lemma 5.5, following [24, Lemma 6.7].

Consider the open ball $B_R(\delta_o)$ in $(\mathcal{P}_1(M), \mathcal{W}_1)$ centred at δ_o and of radius $R > 0$. The ball $B_R(\delta_o)$ (and its closure in \mathcal{W}_1) are tight. Indeed, for any $\rho \in B_R(\delta_o)$,

$$\int_M r_x d\rho(x) = \mathcal{W}_1(\rho, \delta_o) < R,$$

and on the other hand,

$$\int_M r_x d\rho(x) \geq \int_{r_x > L} r_x d\rho(x) \geq L \int_{r_x > L} d\rho(x),$$

for any $L > 0$. By combining the inequalities we then find

$$\int_{r_x > L} d\rho(x) < \frac{R}{L}, \quad \text{for all } L > 0.$$

Therefore, for any $\epsilon > 0$, choose $L = \frac{R}{\epsilon}$ and get

$$\int_{r_x > \frac{R}{\epsilon}} d\rho(x) < \epsilon,$$

which shows tightness of $B_R(\delta_0)$.

Since $\overline{B_R(\delta_o)}$ is a tight and closed set in $\mathcal{P}_1(M)$, by Prokhorov's theorem, the sequence $\{\rho_k\}_{k \in \mathbb{N}}$ has a subsequence that converges weakly to $\rho_0 \in \overline{B_R(\delta_o)}$. It remains to show that ρ_0 has centre of mass at o .

Fix an arbitrary unit tangent vector $v \in T_o M$. As $\rho_0 \in \overline{B_R(\delta_o)}$, we have

$$\left| \int_M \log_o x \cdot v d\rho_0(x) \right| \leq \|v\| \int_M r_x d\rho_0(x) \leq R.$$

Fix $\epsilon > 0$ arbitrary small. Since the integral $\int_M \log_o x \cdot v d\rho_0(x)$ is convergent, there exists $r_1 > 0$ such that

$$(C.65) \quad \left| \int_{r_x > r} \log_o x \cdot v d\rho_0(x) \right| < \epsilon, \quad \text{for any } r > r_1.$$

On the other hand, since ϕ grows superlinearly at infinity, there exists $r_2 > 0$ such that

$$0 < \frac{\theta}{\phi(\theta)} < \epsilon, \quad \text{for any } r > r_2.$$

Then, for any $r > r_2$,

$$(C.66) \quad \begin{aligned} \left| \int_{r_x > r} \log_o x \cdot v d\rho_k(x) \right| &\leq \int_{r_x > r} r_x d\rho_k(x) \\ &= \int_{r_x > r} \phi(r_x) \left(\frac{r_x}{\phi(r_x)} \right) d\rho_k(x) \\ &\leq \left\| \frac{\theta}{\phi(\theta)} \right\|_{L^\infty((r, \infty))} \times \int_M \phi(r_x) d\rho_k(x) \\ &\leq U\epsilon, \end{aligned}$$

where U denotes a uniform upper bound of $\int_M \phi(r_x) d\rho_k(x)$.

By combining (C.65) and (C.66), for any $r > \max(r_1, r_2)$, we get

$$(C.67) \quad \begin{aligned} \left| \int_M \log_o x \cdot v d\rho_k(x) - \int_M \log_o x \cdot v d\rho_0(x) \right| &\leq \left| \int_{r_x \leq r} \log_o x \cdot v d\rho_k(x) - \int_{r_x \leq r} \log_o x \cdot v d\rho_0(x) \right| \\ &\quad + \left| \int_{r_x > r} \log_o x \cdot v d\rho_k(x) - \int_{r_x > r} \log_o x \cdot v d\rho_0(x) \right| \\ &\leq \left| \int_{r_x \leq r} \log_o x \cdot v d\rho_k(x) - \int_{r_x \leq r} \log_o x \cdot v d\rho_0(x) \right| + (U + 1)\epsilon. \end{aligned}$$

Since $\{x : r_x \leq r\}$ is a bounded set, by continuity of the Riemannian logarithm we have

$$\lim_{k \rightarrow \infty} \int_{r_x \leq r} \log_o x \cdot v d\rho_k(x) = \int_{r_x \leq r} \log_o x \cdot v d\rho_0(x).$$

Finally, letting $k \rightarrow \infty$ in (C.67), we find

$$\limsup_{k \rightarrow \infty} \left| \int_M \log_o x \cdot v \, d\rho_k(x) - \int_M \log_o x \cdot v \, d\rho_0(x) \right| \leq (U + 1)\epsilon,$$

for any $\epsilon > 0$. From here we infer that

$$\lim_{k \rightarrow \infty} \left| \int_M \log_o x \cdot v \, d\rho_k(x) - \int_M \log_o x \cdot v \, d\rho_0(x) \right| = 0,$$

and hence,

$$\lim_{k \rightarrow \infty} \int_M \log_o x \cdot v \, d\rho_k(x) = \int_M \log_o x \cdot v \, d\rho_0(x).$$

Since ρ_k has centre at mass at o , we have $\int_M \log_o x \cdot v \, d\rho_k(x) = 0$ for all $k \geq 1$, and consequently,

$$\int_M \log_o x \cdot v \, d\rho_0(x) = 0.$$

Since the constant unit vector v is arbitrary, we infer the conservation of the centre of mass property.

REFERENCES

- [1] L. Alías, P. Mastrolia, and M. Rigoli. *Maximum Principles and Geometric Applications*. Springer Monographs in Mathematics. Springer International Publishing, 2019.
- [2] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.
- [3] D. Balagué, J. A. Carrillo, T. Laurent, and G. Raoul. Dimensionality of local minimizers of the interaction energy. *Arch. Ration. Mech. Anal.*, 209(3):1055–1088, 2013.
- [4] M. Burger, M. Di Francesco, and M. Franek. Stationary states of quadratic diffusion equations with long-range attraction. *Comm. Math. Sci.*, 11(3):709–738, 2013.
- [5] M. Burger, R. C. Fetecau, and Y. Huang. Stationary states and asymptotic behavior of aggregation models with nonlinear local repulsion. *SIAM J. Appl. Dyn. Syst.*, 13(1):397–424, 2014.
- [6] J. A. Cañizo, J. A. Carrillo, and F. S. Patacchini. Existence of compactly supported global minimisers for the interaction energy. *Arch. Ration. Mech. Anal.*, 217(3):1197–1217, 2015.
- [7] E. Carlen and M. Loss. Competing symmetries, the logarithmic HLS inequality and Onofri’s inequality on \mathbb{S}^n . *Geom. Funct. Anal.*, 2:90–104, 1992.
- [8] J. Carrillo, D. Gómez-Castro, Y. Yao, and C. Zeng. Asymptotic simplification of aggregation-diffusion equations towards the heat kernel. *Arch. Rational Mech. Anal.*, 247(11), 2023.
- [9] J. A. Carrillo, K. Craig, and Y. Yao. Aggregation-diffusion equations: Dynamics, asymptotics, and singular limits. In N. Bellomo, P. Degond, and E. Tadmor, editors, *Active Particles, Volume 2: Advances in Theory, Models, and Applications*, Modeling and Simulation in Science, Engineering and Technology, pages 65–108. Birkhäuser, 2019.
- [10] J. A. Carrillo, M. Delgadino, and F. S. Patacchini. Existence of ground states for aggregation-diffusion equations. *Analysis and Applications*, 17(3):393–423, 2019.
- [11] J. A. Carrillo, M. Fornasier, G. Toscani, and F. Vecil. Particle, kinetic, and hydrodynamic models of swarming. In *Mathematical modeling of collective behavior in socio-economic and life sciences*, Model. Simul. Sci. Eng. Technol., pages 297–336. Birkhäuser Boston, Inc., Boston, MA, 2010.
- [12] J. A. Carrillo, S. Hittmeir, B. Volzone, and Y. Yao. Nonlinear aggregation-diffusion equations: radial symmetry and long time asymptotics. *Invent. Math.*, 218:889–977, 2019.
- [13] J. A. Carrillo, F. Hoffmann, E. Mainini, and B. Volzone. Ground states in the diffusion-dominated regime. *Calc. Var. Partial Differ. Equ.*, 57:127, 2018.
- [14] J. A. Carrillo, R. J. McCann, and C. Villani. Contractions in the 2-Wasserstein length space and thermalization of granular media. *Arch. Ration. Mech. Anal.*, 179(2):217–263, 2006.
- [15] P.-H. Chavanis, P. Laurençot, and M. Lemou. Chapman–Enskog derivation of the generalized Smoluchowski equation. *Physica A*, 341:145–164, 2004.

- [16] I. Chavel. *Riemannian Geometry : A Modern Introduction*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, second edition, 2006.
- [17] J. Cheeger and D. G. Ebin. *Comparison theorems in Riemannian geometry*, volume 9 of *North-Holland Mathematical Library*. North-Holland Publishing Company, Amsterdam, 1975.
- [18] R. Choksi, R. C. Fetecau, and I. Topaloglu. On minimizers of interaction functionals with competing attractive and repulsive potentials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(6):1283–1305, 2015.
- [19] M. G. Delgadino, X. Yan, and Y. Yao. Uniqueness and nonuniqueness of steady states of aggregation-diffusion equations. *Comm. Pure Appl. Math.*, 75(1):3–59, 2022.
- [20] M. P. do Carmo. *Riemannian Geometry*. Mathematics: Theory and Applications. Birkhäuser, Boston, second edition, 1992.
- [21] R. C. Fetecau, S.-Y. Ha, and H. Park. An intrinsic aggregation model on the special orthogonal group $SO(3)$: well-posedness and collective behaviours. *J. Nonlinear Sci.*, 31(5):74, 2021.
- [22] R. C. Fetecau and H. Park. Equilibria and energy minimizers for an interaction model on the hyperbolic space. *Physica D*, 446:133670, 2023.
- [23] R. C. Fetecau and H. Park. Long-time behaviour of interaction models on Riemannian manifolds with bounded curvature. *J. Geom. Anal.*, 33(7):218, 2023.
- [24] R. C. Fetecau and H. Park. Ground states for aggregation-diffusion models on Cartan-Hadamard manifolds. *arXiv preprint <http://arxiv.org/abs/2306.04856>*, 2024. submitted.
- [25] R. C. Fetecau and F. S. Patacchini. Well-posedness of an interaction model on Riemannian manifolds. *Comm. Pure Appl. Anal.*, 21(11):3559–3585, 2021.
- [26] R. C. Fetecau and B. Zhang. Self-organization on Riemannian manifolds. *J. Geom. Mech.*, 11(3):397–426, 2019.
- [27] V. Gazi and K. M. Passino. Stability analysis of swarms. In *Proc. American Control Conf.*, pages 8–10, Anchorage, AK, 2002.
- [28] A. Grigor'yan. Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. *Bull. Amer. Math. Soc.*, 36(2):135–249, 1999.
- [29] G. Grillo, G. Meglioli, and F. Punzo. Global existence of solutions and smoothing effects for classes of reaction-diffusion equations on manifolds. *J. Evol. Equ.*, 21(2):2339–2375, 2021.
- [30] G. Grillo, G. Meglioli, and F. Punzo. Smoothing effects and infinite time blowup for reaction-diffusion equations: An approach via Sobolev and Poincaré inequalities. *J. Math. Pures Appl.*, 151:99–131, 2021.
- [31] G. Grillo, G. Meglioli, and F. Punzo. Blow-up versus global existence of solutions for reaction-diffusion equations on classes of Riemannian manifolds. *Annali di Matematica Pura ed Applicata*, 202(3):1255–1270, 2023.
- [32] G. Grillo, M. Muratori, and J. L. Vázquez. The porous medium equation on Riemannian manifolds with negative curvature. The large-time behaviour. *Advances in Mathematics*, 314:328–377, 2017.
- [33] G. Grillo, M. Muratori, and J. L. Vázquez. The porous medium equation on Riemannian manifolds with negative curvature: the superquadratic case. *Mathematische Annalen*, 373:119–153, 2019.
- [34] G. Kaib. Stationary states of an aggregation equation with degenerate diffusion and bounded attractive potential. *SIAM J. Math. Anal.*, 49(1):272–296, 2017.
- [35] E. F. Keller and L. A. Segel. Model for chemotaxis. *J. Theor. Biol.*, 30(2):225–234, 1971.
- [36] D. Messenger and R. C. Fetecau. Equilibria of an aggregation model with linear diffusion in domains with boundaries. *Math. Models Methods Appl. Sci.*, 30(4):805–845, 2020.
- [37] A. Mogilner and L. Edelstein-Keshet. A non-local model for a swarm. *J. Math. Biol.*, 38:534–570, 1999.
- [38] S. Motsch and E. Tadmor. Heterophilous dynamics enhances consensus. *SIAM Review*, 56:577–621, 2014.
- [39] F. Santambrogio. *Optimal Transport for Applied Mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and Their Applications*. Birkhäuser, 2015.
- [40] R. Simione, D. Slepčev, and I. Topaloglu. Existence of ground states of nonlocal-interaction energies. *J. Stat. Phys.*, 159(4):972–986, 2015.
- [41] K.-T. Sturm. Convex functionals of probability measures and nonlinear diffusions on manifolds. *J. Math. Pures Appl.*, 84(2):149–168, 2005.
- [42] L. Wu and D. Slepčev. Nonlocal interaction equations in environments with heterogeneities and boundaries. *Comm. Partial Differential Equations*, 40(7):1241–1281, 2015.

(Razvan C. Fetecau)

DEPARTMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, 8888 UNIVERSITY DR., BURNABY, BC V5A
1S6, CANADA

Email address: `van@math.sfu.ca`

(Hansol Park)

DEPARTMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, 8888 UNIVERSITY DR., BURNABY, BC V5A
1S6, CANADA

Email address: `hansol_park@sfu.ca`