Research Notes brings mathematical research ideas forth to the CMS readership in a generally accessible manner that promotes discussion of relevant topics including research (both pure and applied), activities, and noteworthy news items. Comments, suggestions, and submissions are welcome.

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Les articles de recherche présente des sujets mathématiques aux lecteurs de la SMC dans un format généralement accessible qui favorise les discussions sur divers sujets pertinents, dont la recherche (pure et appliquée), les activités et des nouvelles dignes de mention. Vos commentaires, suggestions et propositions sont le bienvenue.

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Self-Collective Behaviour In Biological Aggregations

Razvan Fetecau, *Self-collective behaviour in biological aggregations*

Swarming and self-organization have been buzz words in certain applied mathematics communities lately. The interest started from the spectacular formations of animal groups that we often see in nature (e.g., flocks of birds, schools of fish, swarms of insects). But research in self-collective behaviour quickly reached far beyond biology, with applications in robotics, opinion formation, social networks, pedestrian flow.

This short note regards a certain differential equation model for aggregation that has attracted a lot of interest due to its simplicity and its rich behaviour of solutions. The model considers an interaction potential K which incorporates inter-individual social interactions such as long-range attraction and short-range repulsion.

Denote by ρ the macroscopic density of a group of individuals in $\mathbb{R}^N$. The model consists in the following active transport equation for ρ:

$$\rho_t + \nabla \cdot (\rho v) = 0, \tag{1a}$$

$$v = -\nabla K * \rho, \tag{1b}$$

where the asterisk $*$ denotes convolution.

Despite its simplicity, the model can capture a wide variety of "swarm" behaviour. A provoking gallery of solutions that can be obtained with model (1) is presented for instance in [4]. It contains aggregations on disks, annuli, rings, soccer balls, and others.

The choice of the aggregation potential K is essential for both analysis and applications of model (1). We focus here on a simple potential in the form of a power-law:

$$K(x) = \frac{1}{q}|x|^q - \frac{1}{p}|x|^p, \qquad \text{for } -N < p < q, \tag{2}$$

where the exponents p and q correspond to repulsion and attraction, respectively.

Figure 1 shows some equilibria in two dimensions for model (1) with K in power-law form. We observe a variety of possible steady states: (a) constant density on a disk, (b) non-uniform density on a disk with higher concentration toward the boundary, (c) aggregation on a circle, and (d) non-symmetric aggregation on three arcs.

In [2] and [3] the authors investigate the case when $p = 2 - N$, i.e., when the repulsion is Newtonian. The equilibria in Figures 1(a) and 1(b) correspond to Newtonian repulsion, with attraction exponents $q = 2$ and $q = 10$, respectively. Note that in two dimensions the Newtonian repulsion is given by $-\log|x|$ instead. The focus in [2], [3] is the intricate balance between the power-law attraction and the singular repulsion, which yields a very interesting and at the same time biologically relevant set of equilibria for model (1).

It is shown in [2], [3] that for $p = 2 - N$ and for all values of $q > 2 - N$, the aggregation model has a unique steady state supported on a ball. This steady state is radial and monotone in the radial coordinate, with an interesting demarcation at $q = 2$. More specifically, the equilibria are decreasing about the origin for $2 - N < q < 2$ and increasing for $q > 2$ (Figure 1(b)), while $q = 2$ corresponds to a constant equilibrium density (Figure 1(a)). Numerical simulations suggest that all these equilibria are global attractors for the dynamics.

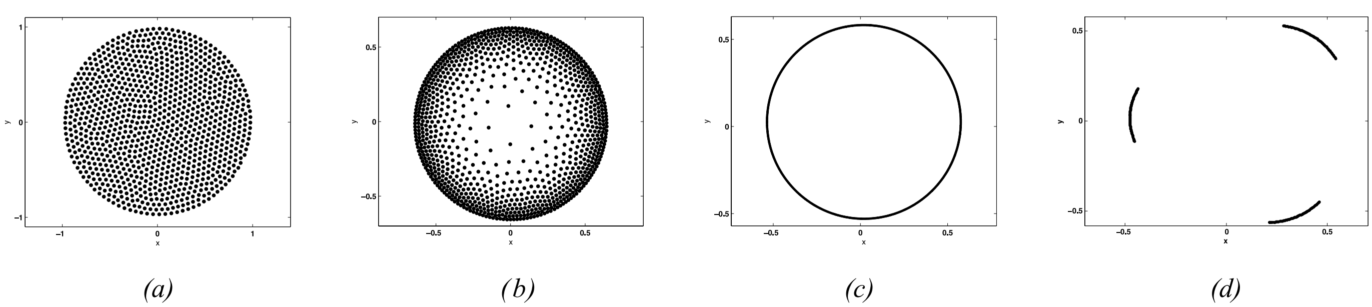

Figure 1. Various equilibria for aggregation model (1) in two dimensions, with K in power-law form (2).

The limits $q \to \infty$ and $q \searrow 2-N$, i.e., when attraction becomes infinitely strong or as singular as the (Newtonian) repulsion, are investigated in [2]. As $q \to \infty$, the radii of the equilibria approach a constant, but the qualitative features change dramatically, as mass aggregates toward the edge of the swarm, leaving an increasingly void region in the centre --- the onset of this effect can be observed for $q = 10$ in Figure 1(b). As $q \searrow 2-N$, the radii of equilibria approach 0 and mass concentrates at the origin.

Alternatively, the steady states of (1) can be investigated by variational methods. Indeed, PDE (1) is the gradient flow of the energy

$$E[\rho] := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x-y)\rho(x)\rho(y)\,dxdy \qquad (3)$$

with respect to the 2-Wasserstein metric.

In [1] the authors establish, via Lions concentration compactness principle, the existence of global minimizers of $E[\rho]$ for potentials K in the power-law form (2). Minimizers are sought in various minimization classes, depending on the sign of the repulsion exponent p. A major distinction is that for $p < 0$ the minimizers are sought among bounded and integrable density *functions*, while for $p > 0$ the minimization class consists of probability *measures*. These choices of minimization classes are supported by numerical simulations. Indeed, Figures 1(a) and 1(b) correspond to $p = 2 - N < 0$, while the measure accumulations in Figures 1(c) and 1(d) correspond to $p > 0$ ($p = 1$, $q = 10$ for (c) and $p = 1.5$, $q = 7$ for (d)).

In the last decade various other swarming models have been proposed and investigated. Lots of qualitative features of biological aggregations have been captured by these models, but solid quantitative studies are still lacking. We end this glimpse into the subject by noting that understanding collective behaviour in nature or otherwise surely has a long and exciting road ahead.

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References

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[3] R.C. Fetecau, Y. Huang, and T. Kolokolnikov. Swarm dynamics and equilibria for a nonlocal aggregation model. *Nonlinearity*, 24(10):2681--2716, 2011.

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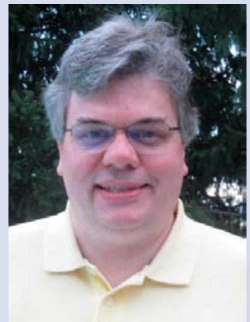
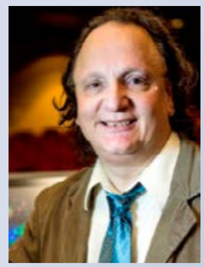
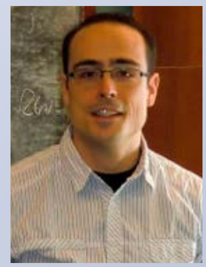
2014 CMS Winter Meeting Recap, continued from page 11

The following individuals were recognized at the Winter Meeting.
Les personnes suivantes ont été reconnus à la Réunion d'hiver.



Graham Wright Award for Distinguished Service / Prix Graham Wright pour service méritoire - Shawn Godin





2014 G. de B. Robinson Award / lauréats du Prix G. de B. Robinson de 2014 - Jonathan M. Borwein (Newcastle, NSW), Armin Straub (Illinois), James Wan (Newcastle, NSW), Wadim Zudilin (Newcastle, NSW) and/et Jan Nekovář (Université Pierre et Marie Curie)

Jeffrey Rosenthal's public lecture
L'exposé public de Jeffrey Rosenthal