Calculus I

MATH 150/151 Course Notes

Department. of Mathematics, SFU

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Contents

Preface

This booklet contains the note templates for courses *Math 150/151 - Calculus I* at Simon Fraser University. Students are expected to use this booklet during each lecture by following along with the instructor, filling in the details in the blanks provided.

Definitions and theorems appear in highlighted boxes.

Next to some examples you'll see [link to applet]. The link will take you to an online interactive applet to accompany the example - just like the ones used by your instructor in the lecture. The link above will take you to the following url [Mul22] containing all the applets:

`http://www.sfu.ca/~jtmulhol/calculus-applets/html/appletsforcalculus.html`

Try it now.

No project such as this can be free from errors and incompleteness. We will be grateful to everyone who points out any typos, incorrect statements, or sends any other suggestion on how to improve this manuscript.

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Greek Alphabet

lower case	capital	name	pronunciation	lower case	capital	name	pronunciation
α	A	alpha	(al-fah)	ν	N	nu	(new)
β	B	beta	(bay-tah)	ξ	Ξ	xi	(zie)
γ	Γ	gamma	(gam-ah)	o	O	omicron	(om-e-cron)
δ	Δ	delta	(del-ta)	π	Π	pi	(pie)
ε	E	epsilon	(ep-si-lon)	ρ	P	rho	(roe)
ζ	Z	zeta	(zay-tah)	σ	Σ	sigma	(sig-mah)
η	H	eta	(ay-tah)	τ	T	tau	(taw)
θ	Θ	theta	(thay-tah)	υ	Υ	upsilon	(up-si-lon)
ι	I	iota	(eye-o-tah)	ϕ	Φ	phi	(fie)
κ	K	kappa	(cap-pah)	χ	X	chi	(kie)
λ	Λ	lambda	(lamb-dah)	ψ	Ψ	psi	(si)
μ	M	mu	(mew)	ω	Ω	omega	(oh-may-gah)

Part One: Review of Functions and Models

1. Functions and Models

In this chapter we lay down the foundations for this course. We introduce functions, how to represent them, and how to work with them. Since our functions will take real numbers as input we start with a brief review of types of numbers.

1.1 Review: Functions

1.1.1 Basic Sets of Numbers

- **natural numbers**: the set of counting numbers
$$\mathbb{N} = \{1,2,3,4,5,\ldots\}$$
(Some authors include 0 in this set.)
- **integers**: the set of natural numbers with their negatives
$$\mathbb{Z} = \{\ldots,-5,-4,-3,-2,-1,0,1,2,3,4,5,\ldots\}$$
- **rational numbers**: the set of ratios of integers
$$\mathbb{Q} = \left\{ \frac{a}{b} \ : \ a,b \in \mathbb{Z}, b \neq 0\right\}$$
- **real numbers** $\mathbb{R}$:
These are more difficult to define, but we already have an intuitive idea of what they are.

  They include all the rational numbers $\mathbb{Q}$ and all the numbers which fill in all the gaps between the rational numbers.

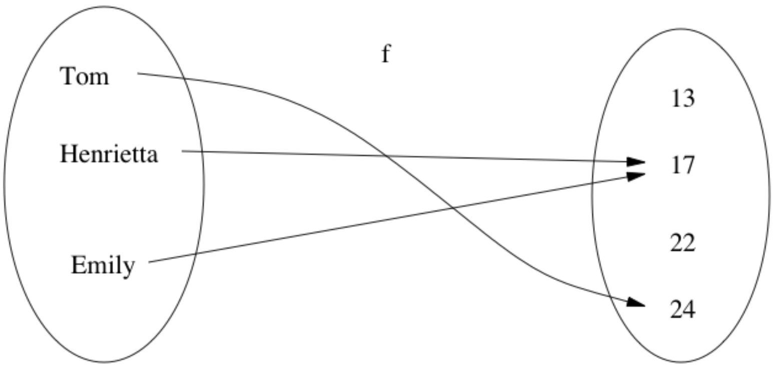
1.1.2 Four Ways to Define a Function

Definition 1.1.1 A **function** (or map) is a rule or correspondence that associates each element of a set X, called the *domain*, with a unique element of a set Y, called the *codomain*.

The **range** of f is the set of all elements in Y which correspond to an element of X:

$$\text{range } f = \{f(x) : x \in X\}.$$

Example 1.1 The following function maps each person to their age.



domain =

codomain =

range =

Reminder In calculus we will only consider functions whose domain and codomain consist of real numbers. Functions can then be described in various ways:

(a) **verbally** (word description)

ex. The area of a circle is π times the radius squared.

(b) **algebraically** (by a formula)

ex. $A(r) = \pi r^2$

(c) **numerically** (by a table of values)

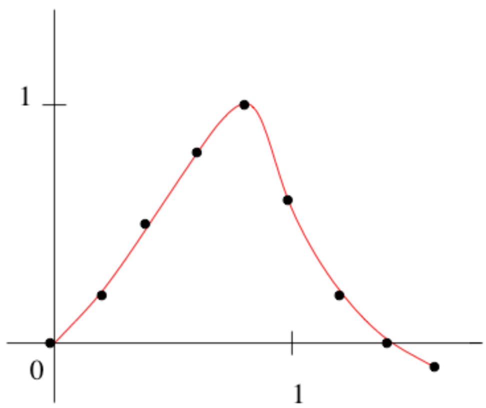
ex.

time (s)	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
velocity (m/s)	0	0.2	0.5	0.8	1.0	0.6	0.2	0	-0.1

or by a set of ordered pairs

$$\{(0,0),(0.2,0.2),(0.4,0.5),(0.6,0.8),(0.8,1.0),(1.0,0.6),(1.2,0.2),(1.4,0),(1.6,-0.1)\}$$

(d) **visually** (by a graph)



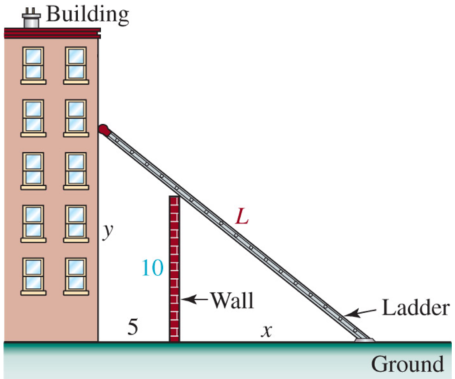
Example 1.2 Let $f(x) = x^2$.

(a) Find the following values: $f(2), f(-1), f(0), f(2/3), f(\sqrt{2}), f(\pi), f(a+h)$.

(b) Sketch the graph of f.

Example 1.3 A 10-ft wall stands 5 ft from a building and a ladder of variable length L, supported by the wall, is placed so it reaches from the ground to the building. Let y denote the vertical distance from the ground to where the tip of the ladder touches the building, and let x denote the horizontal distance from the wall to the base of the ladder.

(a) Find an expression for the height y as a function of x.
(b) Find an expression for the length L as a function of x.
(c) Determine the domain and range of the function $L(x)$ found in part (b).



Reminder If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

Example 1.4 (a) Find the domain of the function

$$g(x) = \frac{1}{x^2 - x}.$$

(b) Find the domain of the function

$$h(t) = \sqrt{16 - t^2}.$$

What is the range?

Reminder The **graph** of a function f is defined to be the set of all points (x, y) in the Cartesian plane satisfying the equation

$$y = f(x).$$

Example 1.5 Sketch the graphs of the following functions.

(a) $f(x) = x + 1$

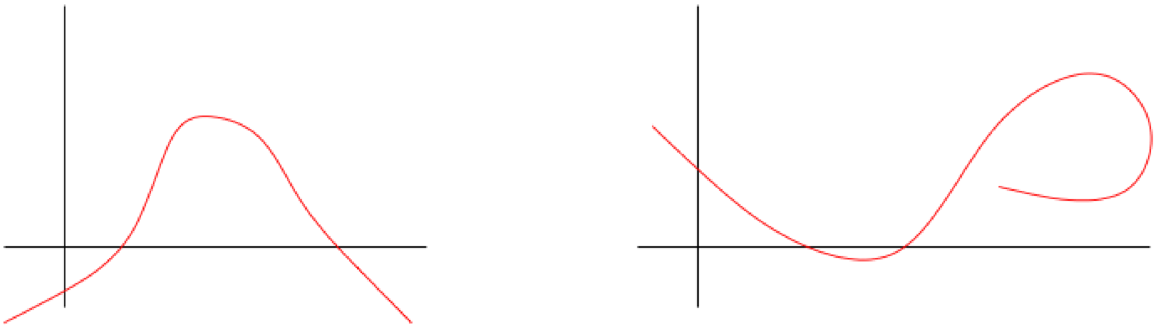
(b) $g(t) = t^2 + 1$

(c) $h(x) = \begin{cases} 2x + 3 & \text{if } x \leq 0 \\ x^2 + 3 & \text{if } x > 0 \end{cases}$

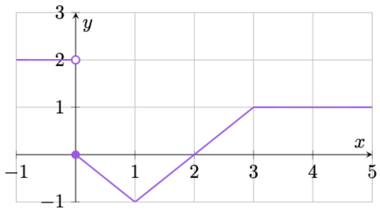
(d) $f(x) = |x|$

Reminder **Vertical line test** for testing whether a curve is the graph of a function:
If every vertical line intersects the curve at most once then the curve is a graph of a function.

Example 1.6 Which curve is the graph of a function?



Exercise 1.1 Find a formula for the function f graphed in the figure:



Exercise 1.2 Sketch the graph of $h(x) = \begin{cases} 2x+3 & \text{if } x \leq 0 \\ x^2+3 & \text{if } x > 0 \end{cases}$

Exercise 1.3 (a) Sketch the graph of $f(x) = |x|$ and write it as a piecewise defined function.
(b) Consider $g(x) = \sqrt{x^2}$. Is it true that $g(x) = x$?
(c) What is the relationship between $g(x) = \sqrt{x^2}$ and $f(x) = |x|$?

Exercise 1.4 Sketch the graph of

$$p(x) = |x| + |x+1|$$

1.2 Mathematical Models: A Catalog of Essential Functions

1.2.1 Linear Functions

Lines (linear function):

A line is determined by two bits of information: ________________ and ________________.

Or, equivalently, by ____.

Example 1.7 Find the equation of the line in each of the following cases.

(a) slope = 2, containing $P = (1,3)$.

(b) containing the points $(1,3)$ and $(-2,7)$.

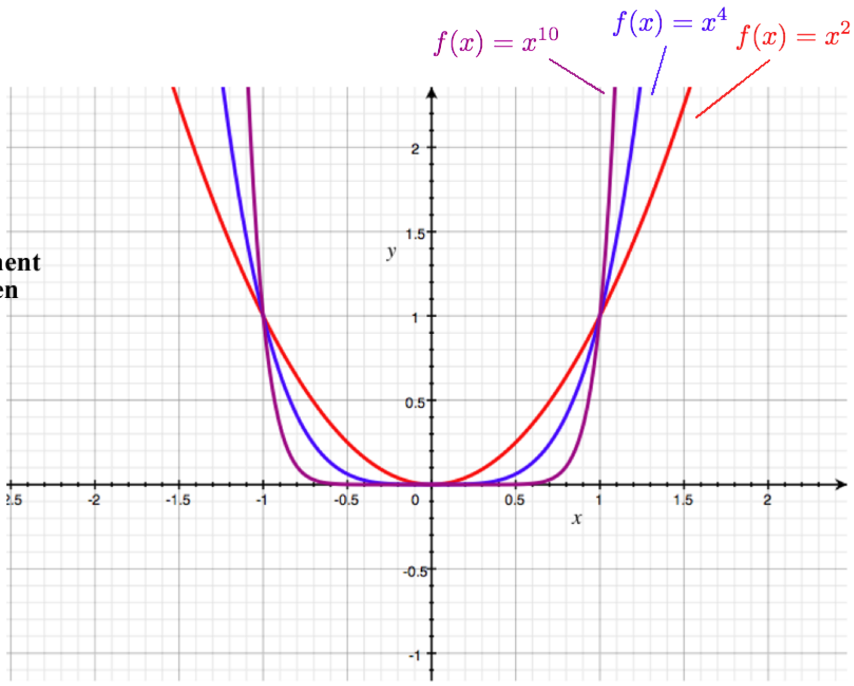
1.2.2 Some Common Functions Power, Polynomial, Rational

(a) **Power Functions:** A power function is a function of the form

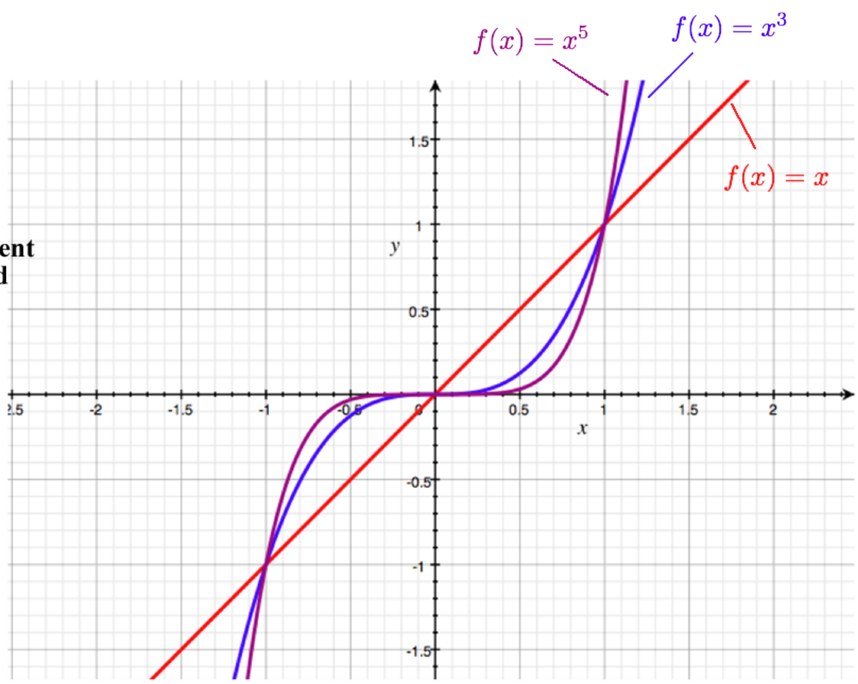
$$f(x) = x^a$$

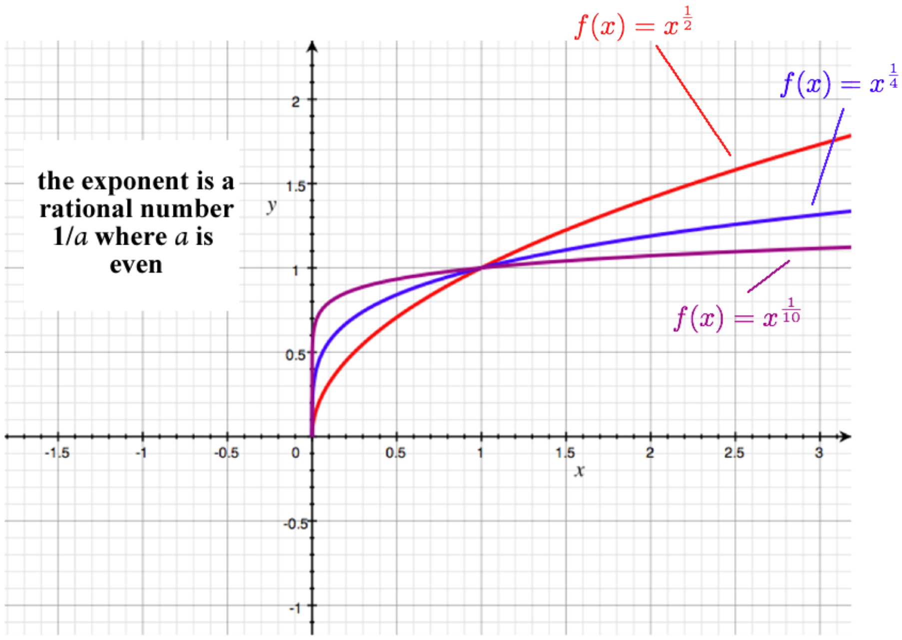
where a is a fixed real number.

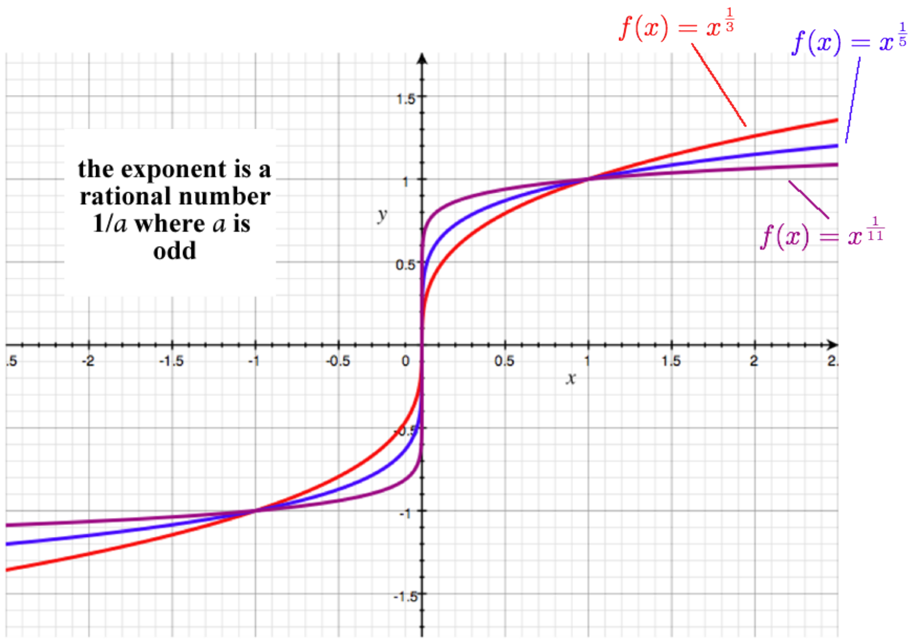
the exponent a is even



the exponent a is odd



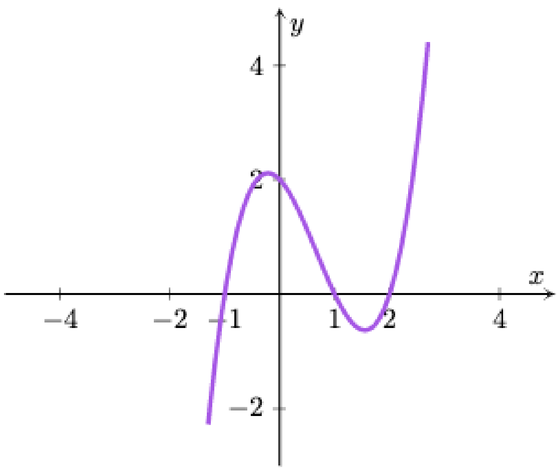

$f(x) = x^{\frac{1}{2}}$
$f(x) = x^{\frac{1}{4}}$
$f(x) = x^{\frac{1}{10}}$
the exponent is a rational number 1/*a* where *a* is even
y
x


$f(x) = x^{\frac{1}{3}}$
$f(x) = x^{\frac{1}{5}}$
$f(x) = x^{\frac{1}{11}}$
the exponent is a rational number 1/*a* where *a* is odd
y
x

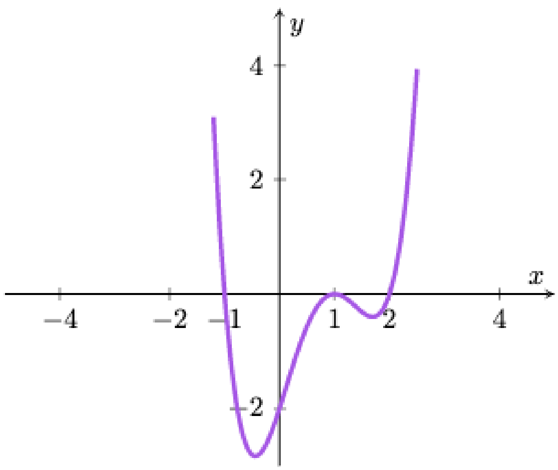
(b) **Polynomials:** A polynomial is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where n is an integer and the a_i are fixed real numbers, which are called the coefficients of f.



$f(x) = x^3 - 2x^2 - x + 2$
$= (x-1)(x-2)(x+1)$



$f(x) = x^4 - 3x^3 + x^2 + 3x - 2$
$= (x-1)^2(x-2)(x+1)$

Exercise 1.5 A polynomial of degree 0 is of the form: ■

Exercise 1.6 A polynomial of degree 1 is of the form: ■

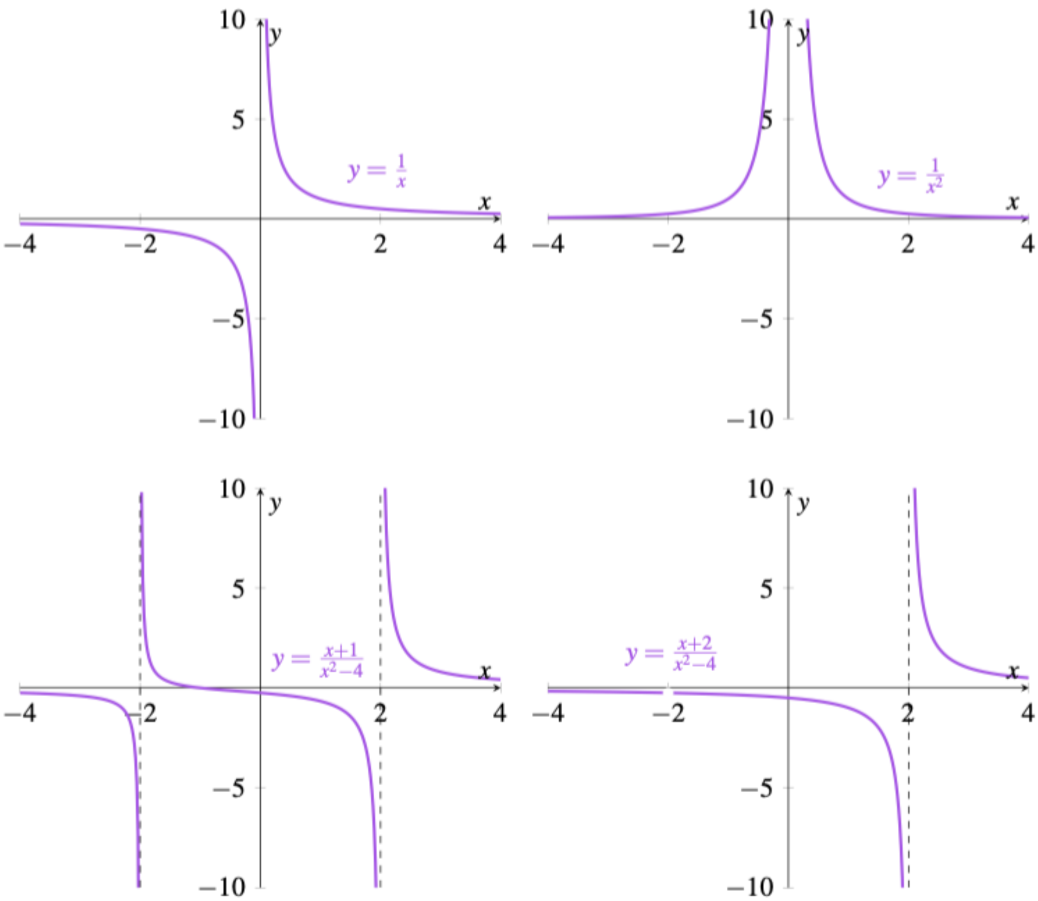
Exercise 1.7 A polynomial of degree 2 is of the form: ■

Exercise 1.8 A polynomial of degree 3 is of the form: ■

(c) **Rational Functions:** A rational function is the ratio of two polynomials:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

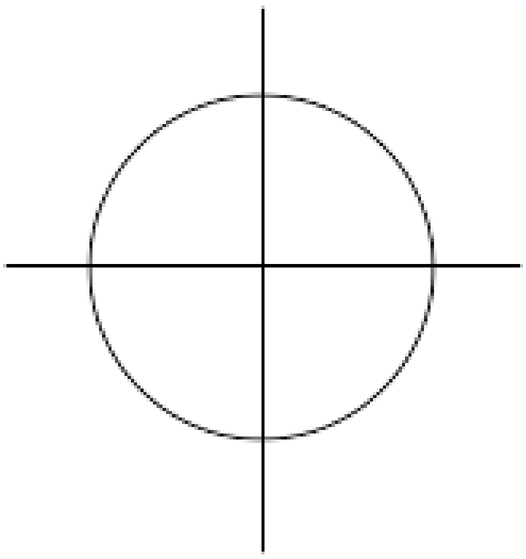


1.2.3 Trigonometric Functions

Let us recall the trigonometric functions sine, cosine, and tangent.

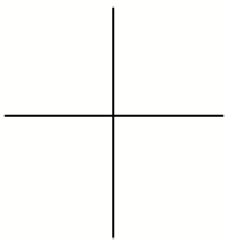
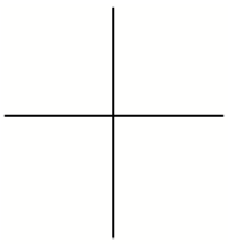
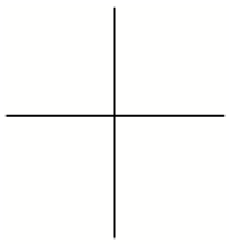
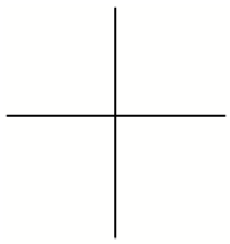
Why are there 360° in a full rotation? (° is read "degrees")

Radian measure of an angle:



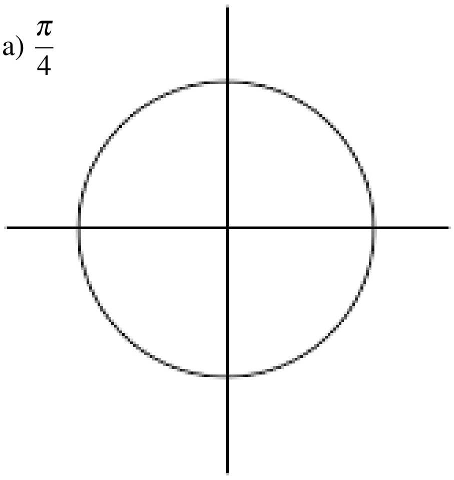
Sketch the following angles (radians) in standard position and give the measure of the angle in degrees:

a) $\frac{\pi}{2}$ b) $\frac{\pi}{4}$ c) $-\frac{5\pi}{6}$ d) $\frac{13\pi}{3}$

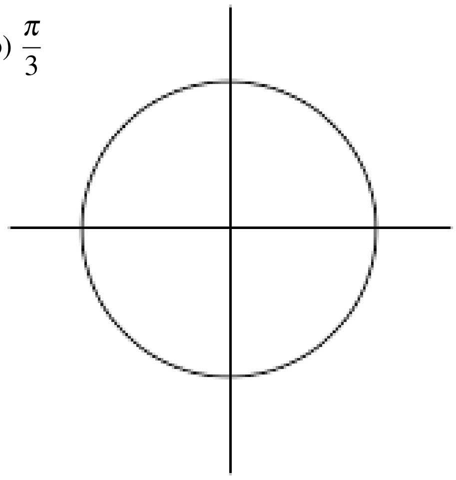





Determine the coordinates of the point where the terminal side of the angle intersects the unit circle.

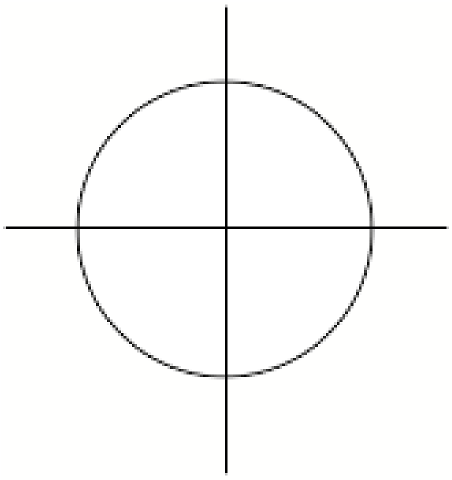
a) $\frac{\pi}{4}$



b) $\frac{\pi}{3}$



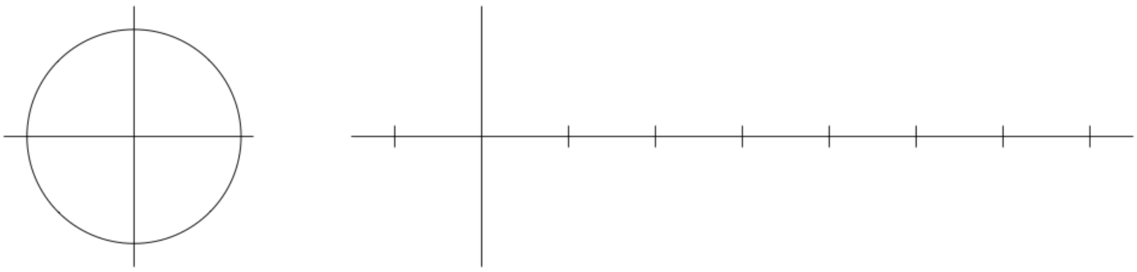
Definition: The **sin** and **cos** of an angle:

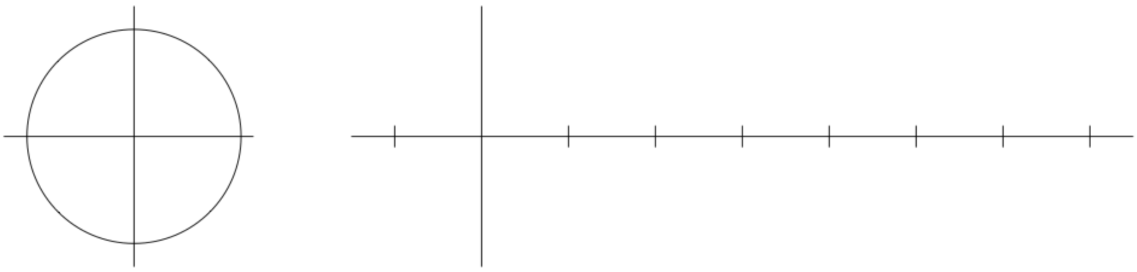


We can fill out the following table

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	–

Sketching the graphs of sin and cos.





1.3 New Functions from Old Functions

Big idea: take well understood functions (eg: $f(x) = x^2$, $g(x) = \sin(x)$....) and perform well understood transformations (eg: shift to the right, reflection...) to create and get graphs of new, related functions.

Example 1.8 Sketch the graphs of $y = x^2 + 2$, $y = (x-1)^2$, and $y = (x-1)^2 + 2$.

Example 1.9 Sketch the graph of $y = 3x^2 - 6x + 1$.

Vertical & Horizontal Shifts:

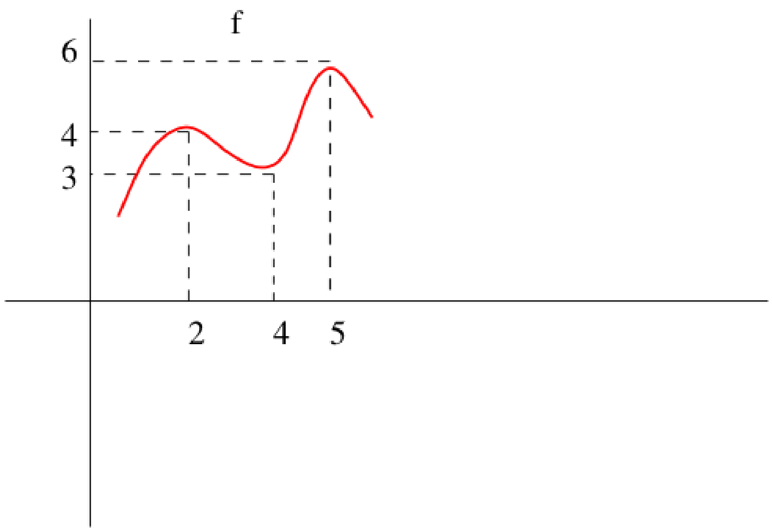
Suppose $c > 0$. To obtain the graph of
$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Example 1.10 Sketch the graph of $y = \sin(2x)$.

Vertical and Horizontal Stretching and Reflecting

Suppose $c > 1$. To obtain the graph of
$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = \frac{1}{c}f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Example 1.11 Given the graph of f sketch the graph of $\frac{1}{2}(f(x-4)-8)$.



Exercise 1.9 Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{7-3x}$ ■

Exercise 1.10 — Absolute value function.

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Sketch the graph of the $y = |x^2 - 1|$. ■

1.3.1 Algebra of Functions

We can combine functions in different ways to create new functions.

Definition 1.3.1 — Sum, Difference, Product, Quotient. Let f and g be two functions. The **sum** $f+g$, the **difference** $f-g$, the **product** fg, and the **quotient** $\frac{f}{g}$ are defined as follows:

$$(f+g)(x) = f(x)+g(x) \quad \text{domain} = A\cap B$$
$$(f-g)(x) = f(x)-g(x) \quad \text{domain} = A\cap B$$
$$(fg)(x) = f(x)\cdot g(x) \quad \text{domain} = A\cap B$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{domain} = \{x\in A\cap B \mid g(x)\neq 0\}$$

Example 1.12 If $f(-1)=1$, $f(2)=3$, $g(-1)=-5$ and $g(2)=17$ find $(f+g)(-1)$, $(fg)(-1)$ and $(f/g)(2)$.

1.3.2 Composition of Functions

Another way we can define new functions from old ones is by *composition*.

Definition 1.3.2 — Composition. If f and g are two functions we write

$$(f\circ g)(x) = f(g(x))$$

for the function obtained by applying f to the output of g. The function $f\circ g$ is called the **composition** of f with g.

The **domain of the composite function** $f\circ g$ is the set of all x such that

(a) x is in the domain of g,

(b) $g(x)$ is in the domain of f.

Example 1.13 If $f(x) = x^2$ and $g(x) = 2x+1$ find $f \circ g$ and $g \circ f$.

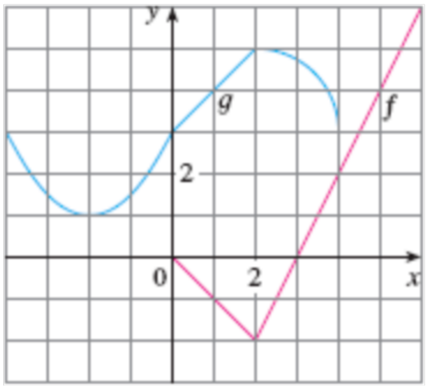
Example 1.14 If $f(x) = 2^x$, $g(t) = 3\sqrt{t}$ and $h(\theta) = \sin\theta$ find $h \circ f \circ g$. What is the domain of $h \circ f \circ g$?

Example 1.15 Given $F(x) = (1+2\sin x)^3$ find functions f, g and h such that $F = f \circ g \circ h$.

Exercise 1.11 If $f(x) = \dfrac{x+1}{x+2}$, find $f \circ f$ and its domain. ■

Exercise 1.12 Draw a graph of $f(x) = \sin|x|$. ■

Exercise 1.13 Use the given graph of $f(x)$ and $g(x)$ to evaluate each of the following, or explain why it is undefined:



(a) $(f \circ g)(1)$

(b) $(g \circ g)(-2)$ ■

(c) $(g \circ f)(6)$

1.4 Exponential Functions & Inverse Functions and Logarithms

1.4.1 Exponential Functions

Reminder For all $a \in (0,1) \cup (1,\infty)$ and all $x, y \in \mathbb{R}$:

(a) $a^{x+y} = a^x \cdot a^y$

(b) $a^{x-y} = \dfrac{a^x}{a^y}$

(c) $(a^x)^y = a^{xy}$

(d) $(ab)^x = a^x \cdot b^x$

Reminder Sketch the graphs of the functions $f(x) = 2^x$ and $g(x) = 3^x$.

Reminder Sketch the graph of the function $F(x) = \left(\frac{1}{2}\right)^x$.

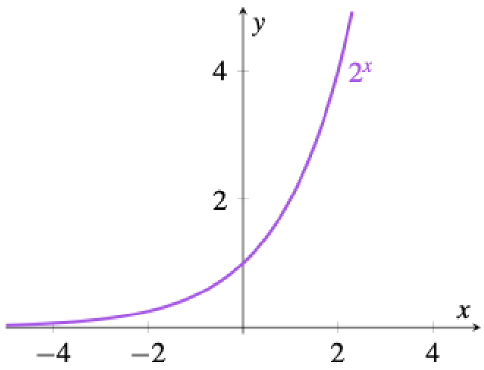
Reminder Evaluate $f(2)$, $f(-2)$, $f(\frac{1}{2})$ and $f(\frac{3}{2})$ if $f(x) = 4^x$.

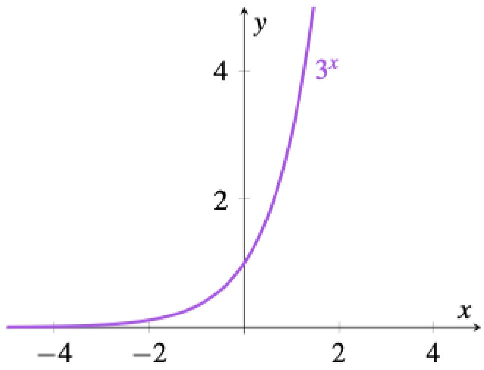
BIG Question. What is $4^{\sqrt{2}}$?

Reminder Napier's constant:

$$e \approx 2.718281828459045235360287471352$$

(John Napier, 1550-1617)





1.4.2 Inverse Functions

Definition 1.4.1 A function f is called a **one-to-one function** if it never takes on the same value twice; that is

$$\text{if } x_1 \neq x_2 \text{ then } f(x_1) \neq f(x_2)\ .$$

Example 1.16 Which of the following functions are one-to-one?

(a) $f(x) = x^2$
(b) $g(x) = x^3$
(c) $h(x) = e^x$
(d) $i(x) = \sin x$
(e) $j(x) = \sin x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Example 1.17 Horizontal Line Test. A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Definition 1.4.2 Let f be one-to-one function with domain A and range B. Then its **inverse function** has domain B and range A and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any $y \in B$.

Note:

- f^{-1} reverses the effect of f.
- If f is not one-to-one, then f^{-1} would not be uniquely defined.
- **Caution!** Do not mistake the "-1" in f^{-1} for an exponent, i.e. $f^{-1}(x)$ **does not mean** $\dfrac{1}{f(x)}$.

  (How would you write this then?)

Example 1.18 Find a formula for the inverse of $f(x) = \dfrac{x}{3x+1}$.

1.4.3 Logarithmic Function

Definition 1.4.3 — Logarithmic Function. The inverse function of the exponential function $f(x) = a^x$ is called the **logarithmic function with base a**.

All You Need To Know. For any $a > 0$, $a \neq 1$, any $x > 0$, and any $y \in \mathbb{R}$

$$\log_a x = y \Leftrightarrow a^y = x$$

Example 1.19 Determine $\log_2(16)$, $\log_2\left(\frac{1}{8}\right)$ and $\log_2(1)$.

Example 1.20 Can you find $\log_2(-32)$?

Reminder For all $a \in (0,1) \cup (1,\infty)$ and any positive x and y:

(a) $\log_a(xy) = \log_a x + \log_a y$

(b) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

(c) $\log_a(x^r) = r\log_a x$ (r is a real number)

Notation:

$$\log_{10} x = \log x$$

$$\log_e x = \ln x$$

Reminder Sketch the graph of the function $y = \ln x$

Exercise 1.14 Solve the equation $e^{x^3-3} - 9 = 0$ for x. ■

1.4.4 Inverse Trig Functions

Here we will limit our discussion to sin.

Definition 1.4.4 The inverse function of the sine function $f(x) = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, is called **arcsine** and is denoted by either $\sin^{-1}$ or arcsin.

All You Need To Know. For any $-1 \le x \le 1$, and any $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\sin^{-1} x = y \iff \sin y = x$$

Example 1.21 Determine the following.

(a) $\sin^{-1}(\frac{\sqrt{3}}{2})$

(b) $\sin(\sin^{-1}(\frac{1}{3}))$

(c) $\sin^{-1}(\sin(\frac{3\pi}{4}))$

Exercise 1.15 Find a formula for the inverse of $f(x) = \frac{x}{4x+3}$. ■

Exercise 1.16 If $f(x)$ is a one-to-one function and $f(1) = 4$, $f(3) = 5$ and $f(8) = -9$, then find $f^{-1}(-9)$, $f^{-1}(5)$ and $f^{-1}(4)$.

■

Exercise 1.17 Find the inverse function of $f(x) = 3 - 2x^3$. ■

Exercise 1.18 Sketch the graphs of $f(x) = \sqrt{-1-x}$ and its inverse function using the same coordinate axes.
Recall: The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$. ■

Exercise 1.19 Express $\frac{1}{3}\ln a - \frac{1}{4}\ln b$ as a single logarithm. ■

1.5 Summary

In this chapter we introduced the fundamental concepts which form the foundation for calculus. Calculus is all about how the output of a function changes when its input changes. Therefore, a solid understanding of functions and function notation is essential.

Describing a Function: The four ways to define a function are: verbally, algebraically, numerically, visually. Be sure you are familiar with all four, and are able to translate a function given in terms of one description into another description. I.e. given a function in an algebraic form, you should be able to produce a table of values (numerical form) and graph it (visual form).

Lines: A line is a graph of a linear function $f(x) = mx + b$, where m is the slope of the line, and b is the the y-intercept. Two points on the line is all that is needed to determine a line. If (x_1, y_1) and (x_2, y_2) are two points on a line, then the slope is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. If m is the slope of the line and (x_1, y_1) is any point on the line then the the equation of the line is

$$y = y_1 + m(x - x_1)$$

Therefore, if (x_1, y_1) and (x_2, y_2) are two points on a line, then

$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

Vocabulary

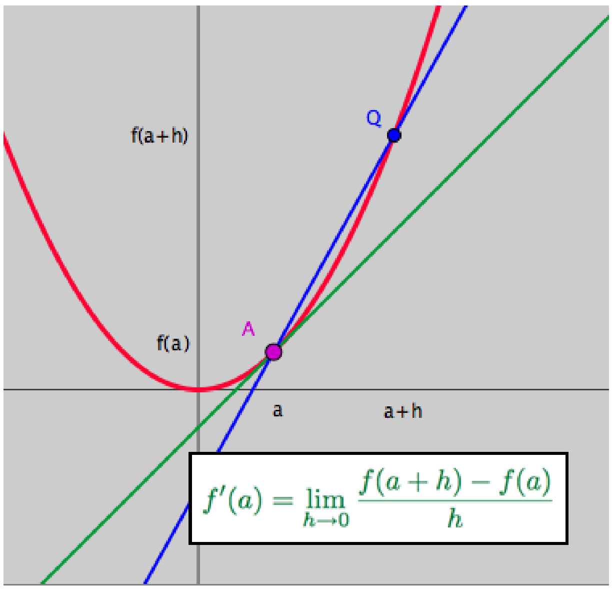
- integer, rational number, real number,
- function, domain, codomain, range
- one-to-one, onto, inverse,
- graph,
- liner function, power function, polynomial, rational function,
- trigonometric function,
- exponential, logarithm,
- transformation (vertical/horizontal shifting/stretching/compressing)
- composition of functions

Skills to acquire

- know the graphs of the elementary functions
- determine the domain and range of a function given in algebraic form
- determine the composition of two of more functions
- work with point/slope, slope/intercept, and standard, forms of the equation of a line
- solve equation involving polynomials, trigonometric, exponential and logarithms.

Part Two: Differentiation

2. Limits and Differentiation



We start with the two motivating problems for calculus: tangent and velocity problems. Although they have different origins they can be solved using the same ideas. This leads us to study *limits* and then the main topic of the course *derivatives*.
Topics we will cover are:

- the limit of a function, continuity
- the derivative of a function at a point,
- the derivative function.

2.1 The Tangent and Velocity Problems

If I were again beginning my studies, I would follow the advice of Plato and start with mathematics.

Galileo Galilei, Italian philosopher and astronomer, 1564-1642.

Motivating Problem **The Tangent Problem.** Find an equation of the tangent line ℓ to a curve with equation $y = f(x)$ at a given point P.

Three Questions.

(a) **What is** the tangent line ℓ to a curve with equation $y = f(x)$ at a given point P?

(b) If a curve with equation $y = f(x)$ and a point P on the curve are given, **does the tangent** ℓ **exist**?

(c) If a curve with equation $y = f(x)$ and a point $P = (x_0, f(x_0))$ are given and if the tangent line ℓ exists then an equation of ℓ is given by

$$y - f(x_0) = m(x - x_0) \ .$$

How do we calculate the slope m?

Hint. Find the slopes of the secant lines to the parabola $y = x^2$ through the points $(1,1)$ and:

(a) $(2,4)$

(b) $(1.5, 1.5^2)$

(c) $(1.1, 1.1^2)$

(d) $(1.001, 1.001^2)$

BIG Question. What if the second point is **VERY**, **VERY** close to the point $(1,1)$?

Motivating Problem **Velocity Problem.** By definition

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{time elapsed}}$$

What if the period of time elapsed is **very small**?

Example 2.1 The position of the car is given by the values in the table.

t	0	1	2	3	4	5
s	0	10	32	70	119	178

where t is in seconds and s is in feet.
Find the average velocity for the time beginning when $t = 2$ and lasting

(a) 3 seconds

(b) 2 seconds

(c) 1 second

Question. What is the meaning of the number that we see on the car speedometer as we travel in city traffic?

Answer. The number represents the **instantaneous velocity**.

2.2 The Limit of a Function

Black holes are where God divided by zero.

Steven Wright, American comedian, 1955-

Motivating Problem Let $f(x) = \dfrac{x^2 - x - 2}{x - 2}$.

(a) Determine the domain of f.

(b) Complete the table

x	$f(x)$	x	$f(x)$
1		3	
1.9		2.1	
1.99		2.01	
1.999		2.001	
1.9999		2.0001	

Definition 2.2.1 — Intuitive Definition of Limit. We write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit of f(x), as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Example 2.2 Guess the value of

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Example 2.3 What can we say about

$$\lim_{x \to 0} \frac{|x|}{x} ?$$

Definition 2.2.2 We write

$$\lim_{x \to a^+} f(x) = L$$

and say

"the right-hand limit of f(x), as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a and x **greater than** a.

Example 2.4 Sketch the graph of the function

$$f(x) = \begin{cases} x+1 & \text{if} \quad x \le -1 \\ x^2 & \text{if} \quad x \in (-1,0) \\ 1 & \text{if} \quad x = 0 \\ x^2 & \text{if} \quad x \in (0,1] \\ x+1 & \text{if} \quad x > 1 \end{cases}$$

Find

(a) $\lim\limits_{x \to -1^-} f(x)$

(b) $\lim\limits_{x \to -1^+} f(x)$

(c) $\lim\limits_{x \to 0^-} f(x)$

(d) $\lim\limits_{x \to 0^+} f(x)$

(e) $\lim\limits_{x \to 0} f(x)$

Fact.

$$\lim_{x \to a} f(x) = L \iff \left(\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L \right)$$

Example 2.5 Sketch the graph of $f(x) = \dfrac{1}{(x+1)^2}$.

Definition 2.2.3 Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

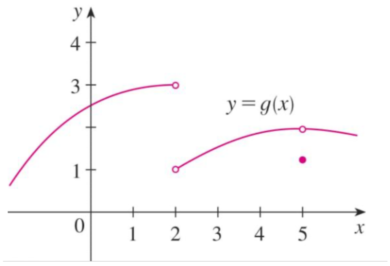
Example 2.6 Sketch the graph of the following function. $g(x) = \dfrac{x+3}{x-1}$

Exercise 2.1 Read Example 10 in text regarding $f(x) = \tan(x)$. ■

Definition 2.2.4 The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Exercise 2.2 For the function g whose graph is shown, find the following limits.



(a) $\lim_{x\to 2^-} g(x)$
(b) $\lim_{x\to 2^+} g(x)$
(c) $\lim_{x\to 2} g(x)$
(d) $\lim_{x\to 5^+} g(x)$
(e) $\lim_{x\to 5^-} g(x)$
(f) $\lim_{x\to 5} g(x)$

■

Exercise 2.3 Given

$$f(x) = \begin{cases} -2x, & \text{if } x \le 3 \\ \frac{4}{x-5} & \text{if } x > 3 \end{cases}$$

find the following limits, if they exist.

(a) $\lim_{x\to 0} f(x) =$

(b) $\lim_{x\to 3^-} f(x) =$

(c) $\lim_{x\to 3^+} f(x) =$

(d) $\lim_{x\to 3} f(x) =$

(e) $\lim_{x\to 5^+} f(x) =$

(f) $\lim_{x\to 5^-} f(x) =$

(g) $\lim_{x\to 5} f(x) =$

■

Exercise 2.4 Sketch the graph of an example of a function f that satisfies **all** of the given conditions:

$$\lim_{x\to 0^-} f(x) = 2 \qquad \lim_{x\to 0^+} f(x) = 1 \qquad \lim_{x\to 4^-} f(x) = 5$$
$$\lim_{x\to 4^+} f(x) = 1 \qquad f(0) = 2 \qquad f(4) = 3$$

■

2.3 Calculating Limits Using the Limit Laws

Laws are like sausages. It's better not to see them being made.
Otto von Bismarck , German statesman, 1815 - 1898

Example 2.7 Guess the value of $\lim_{t\to 0}\frac{\sqrt{t+9}-3}{t}$.

We've computed a few values of the function for t near 0.

t	$\frac{\sqrt{t+9}-3}{t}$	t	$\frac{\sqrt{t+9}-3}{t}$
1	0.16228...	-0.1	0.16713...
0.5	0.16214...	-0.01	0.16671...
0.1	0 16621...	-0.001	0.16667...
0.01	0.16662...		
0.001	0.16667...		

Limit Laws. Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

exist. Then

(a) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
(b) $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
(c) $\lim_{x\to a}(c\cdot f(x)) = c\cdot \lim_{x\to a} f(x)$
(d) $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
(e) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$.
(f) $\lim_{x\to a}[f(x)]^{p/q} = [\lim_{x\to a} f(x)]^{p/q}$

Two Special Limit Laws.

(a) $\lim_{x\to a} c = c$

(b) $\lim_{x\to a} x = a$

Example 2.8 Evaluate $\lim_{x\to 2}(x^3+3x^2-4x+5)$.

Theorem 2.3.1 — Direct Substitution Property. If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)\ .$$

Key Fact: If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limits exist.

Example 2.9 Find the following limits.

(a) $\lim_{x\to -1} \dfrac{x+1}{x^3+1}$

(b) $\lim_{t\to 0} \dfrac{\sqrt{t+9}-3}{t}$

(c) $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ if $f(x) = x^2$

Example 2.10 Find $\lim_{x\to 0} \frac{x^2}{|x|}$

Reminder.

$$\lim_{x\to a} f(x) = L \iff (\lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L)$$

Theorem 2.3.2 If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) .$$

Theorem 2.3.3 — Squeeze Theorem. If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

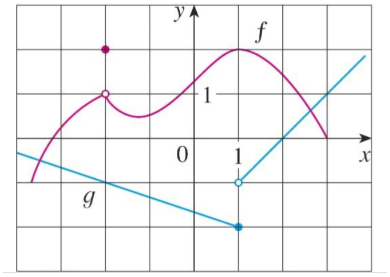
then

$$\lim_{x\to a} g(x) = L .$$

Example 2.11 Show that

$$\lim_{x\to 0} \left[x^2 \left(\sin\frac{1}{x} + \cos\frac{1}{x} \right) \right] = 0 .$$

Exercise 2.5 Evaluate the following limits, if they exist.



(a) $\lim_{x\to -2}[f(x)+5g(x)]$

(b) $\lim_{x\to 1}[f(x)g(x)]$

(c) $\lim_{x\to 2}\frac{f(x)}{g(x)}$

■

Exercise 2.6 Evaluate $\lim_{x\to 1}\frac{2x^3+3x^2-4}{2-3x}$. ■

Exercise 2.7 Find the following limits.

a) $\lim_{x\to 3}\frac{x^2-4}{2-x}$

b) $\lim_{x\to 2}\frac{x^2-4}{2-x}$

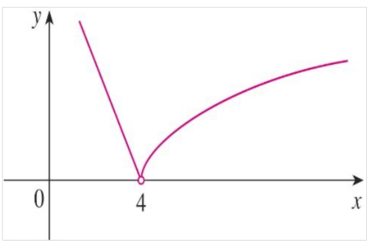
■

Exercise 2.8 Find $\lim_{x\to 1} g(x)$ if $g(x)=\begin{cases} x+1 & \text{if } x\neq 1 \\ \pi & \text{if } x=1 \end{cases}$ ■

Exercise 2.9 Find $\lim_{x\to -3}\frac{|x+3|}{x^2+x-6}$. ■

Exercise 2.10 Let $f(x)=\begin{cases} \sqrt{x-4} & \text{if } x>4 \\ 8-2x & \text{if } x<4 \end{cases}$.

Determine if $\lim_{x\to 4} f(x)$ exists.



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Exercise 2.11 Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \sqrt{x^3+x^2}\sin\left(\frac{\pi}{x}\right) = 0$$

■

Exercise 2.12 If $\lim_{x\to 1} \frac{f(x)-4}{x-1} = 8$, find $\lim_{x\to 1} f(x)$. ■

Exercise 2.13 If $\lim_{x\to 0} \frac{f(x)}{x^3} = 4$, find $\lim_{x\to 0} \frac{f(x)}{x^2}$. ■

Exercise 2.14 Show, by means of an example, that $\lim_{x\to a}[f(x)+g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists. ■

Exercise 2.15 Find the following limits, if they exist:

(a) $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right)$

(b) $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$

(c) $\lim_{x\to 25} \frac{5-\sqrt{x}}{25x-x^2}$

(d) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{|x|}\right)$

(e) $\lim_{t\to 1} \frac{t^4-1}{t^3-1}$

(f) $\lim_{x\to -6} \frac{2x+12}{|x+6|}$

(g) $\lim_{h\to 0} \frac{\frac{1}{3+h}-\frac{1}{3}}{h}$

(h) $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)}$

■

2.4 The Precise Definition of Limit (omitted)

There's a delta for every epsilon,
It's a fact that you can always count upon.
There's a delta for every epsilon
And now and again,
There's also an N.
Tom Lehrer, American singer-songwriter, satirist, pianist, and mathematician, 1928 - .

Motivating Problem **The ε, δ Game.**
Consider the function $f(x) = 3x - 1$ and the point $x = 1$. There are two players in this game: Player A and Player B. The game is played as follows. Player A chooses a number, say ε. The object of Player B is to find a number δ so that all values in the interval $(1-\delta, 1+\delta)$ have image in the interval $(f(1)-\varepsilon, f(1)+\varepsilon)$. The winner is determined as follows:

1) If Player A can pick a number ε such that Player B cannot find such a δ then Player A wins.
2) If Player B can find a δ for any ε given by Player A then Player B wins.

Who wins Player A or Player B?

Definition 2.4.1 Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of f(x) as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta \,.$$

[link to applet]

Example 2.12 Prove the statement using the ε, δ definition.

$$\lim_{x \to 3}(2 - 5x) = -13$$

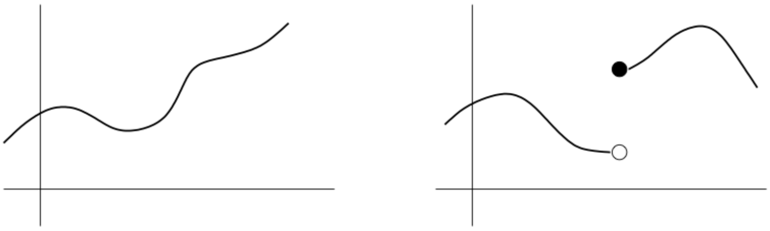
To Be Continued ... ... in Math 242.

2.5 Continuity

If I were asked to name, in one word, the pole star round which the mathematical firmament revolves, the central idea which pervades the whole corpus of mathematical doctrine, I should point to ***Continuity*** *as contained in our notions of space, and say, it is this, it is this!*

JJ Sylvester, English mathematician, 1814-1897

Example 2.13 What is the difference between the two graphs?



Definition 2.5.1 A function f is **continuous at a number a** if

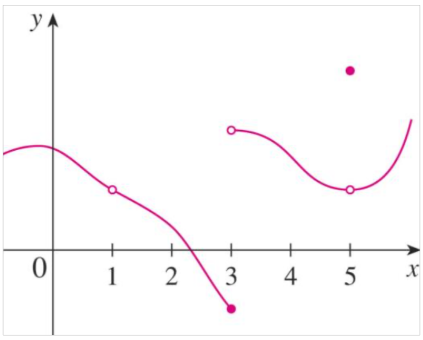
$$\lim_{x\to a} f(x) = f(a) .$$

Procedure to check:

(a) If $f(a)$ is defined - that is, is a in the domain of f?
(b) Does $\lim_{x\to a} f(x)$ exist? (i.e. one-sided limits exist and are equal)
(c) Does $\lim_{x\to a} f(x) = f(a)$?

If any of these conditions fail, the function is not continuous at a.

Example 2.14 The figure below shows the graph of a function f. At which numbers is f not continuous? Why?



Definition 2.5.2 If

(1) f is defined on an open interval containing a, except perhaps at a, and

(2) f is **not** continuous at a

we say that f is **discontinuous** at a.

Example 2.15 Where are each of the following functions discontinuous?

(a)

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if} \quad x \neq 2 \\ 5 & \text{if} \quad x = 2 \end{cases}$$

(b)

$$g(x) = \begin{cases} \dfrac{1}{x-2} & \text{if} \quad x \neq 2 \\ 5 & \text{if} \quad x = 2 \end{cases}$$

(c)

$$h(x) = \begin{cases} 1 & \text{if} \quad x \in [1,2) \\ 2 & \text{if} \quad x \in [2,3) \end{cases}$$

Definition 2.5.3 A function f is **continuous from the right at a number a** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a) \, .$$

Definition 2.5.4 A function f is **continuous on an interval** if it is continuous at every number in that interval. We understand *continuous at the endpoint* to mean *continuous from the right* or *continuous from the left*.

Example 2.16 Find the number c that makes $f(x)$ continuous for every x.

$$f(x) = \begin{cases} \dfrac{x^4 - 1}{x^3 - 1} & \text{if } x \neq 1 \\ c & \text{if } x = 1 \end{cases}$$

Fact. The following types of functions are continuous on their domains:

(a) polynomials
(b) rational functions
(c) root functions
(d) trigonometric functions
(e) inverse trigonometric functions
(f) exponential functions
(g) logarithmic functions

More Facts. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

$$f+g,\ f-g,\ cf,\ fg,\ \frac{f}{g} \text{ if } g(a) \neq 0\ .$$

Example 2.17 For which $a,b \in \mathbb{R}$ is the function

$$f(x) = \begin{cases} \dfrac{\sqrt{1-x}-1}{ax} & \text{if} \quad x \in (0,1] \\ 1 & \text{if} \quad x = 0 \\ \dfrac{bx^4+bx}{x^2+x} & \text{if} \quad x \in (-1,0) \end{cases}$$

continuous on $(-1,1]$?

Theorem 2.5.1 If f is continuous at b and $\lim_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b) \, .$$

Example 2.18 Evaluate

$$\lim_{x\to 0} e^{\frac{\sqrt{1-x}-1}{x}}$$

Theorem 2.5.2 If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Theorem 2.5.3 — Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$.

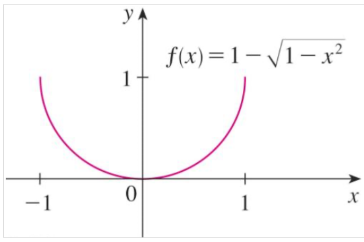
Example 2.19 Use the Intermediate Value Theorem to prove that $\sqrt{2}$ exists.
(Hint: Prove that there is $c \in \mathbb{R}$ such that $c^2 = 2$.)

Example 2.20 Use the Intermediate Value Theorem to show that the equation

$$e^x = 2 - x$$

has at least one real solution.

Exercise 2.16 Show that the function $f(x) = 1 - \sqrt{1-x^2}$ is continuous on $[-1, 1]$.



■

Exercise 2.17 Find $\lim_{x \to \pi} \frac{e^x}{2 + \cos x}$. ■

Exercise 2.18 Where is each of the following functions continuous?

(a) $h(x) = \sin(x^2)$

(b) $u(x) = \ln(1 + \cos x)$

(c) $f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$

■

Exercise 2.19 Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2. ■

Exercise 2.20 Find a constant c such that the following function is continuous everywhere.

$$h(x) = \begin{cases} x^2 + 2 & \text{if } x \le 0 \\ x + c, & \text{if } x > 0 \end{cases}$$

■

Exercise 2.21 Graph the following function and find all values of x where the function is discontinuous. Determine the type of each discontinuity.

$$f(x) = \begin{cases} |x+3| & \text{if } x < -1 \\ -(x-1)^2 + 4, & \text{if } -1 \le x \le 2 \\ 3 & \text{if } x > 2 \end{cases}$$

■

Exercise 2.22 Are there values of a for which the following function is continuous everywhere?

$$f(x) = \begin{cases} a^3 - x^3 & \text{if } x \le \pi \\ a \sin x & \text{if } x > \pi \end{cases}$$

■

2.6 Limits at Infinity: Horizontal Asymptotes

Infinity is a floorless room without walls or ceiling. - Anonymous

Definition 2.6.1 Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Motivating Problem Sketch the graphs of the following functions

(a) $f(x) = \dfrac{1}{x}$

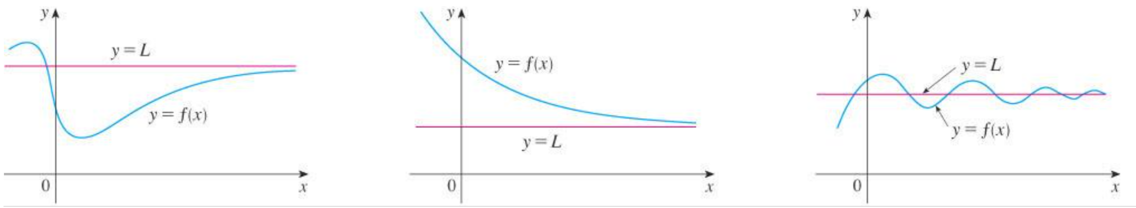
(b) $g(x) = e^x$

(c) $h(x) = \tan^{-1} x$

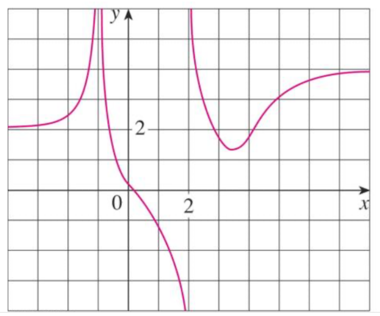
(d) $i(x) = \dfrac{1}{1+x^2}$

Definition 2.6.2 The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L\,.$$



Example 2.21 Find all asymptotes of the curve:



Fact. If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0\,.$$

If $r > 0$ is a rational number such that x^r is defined for all x then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0\,.$$

Example 2.22 Evaluate

(a)

$$\lim_{x\to\infty} \frac{3x^3 - 4x^2 - 1}{6x^3 + x + 2}$$

(b)

$$\lim_{x \to -\infty} \frac{\sqrt{3x^2 - 5} - 1}{2x + 5}$$

(c)

$$\lim_{x \to -\infty} (\sqrt{x^2 + ax} - \sqrt{x^2 + bx})$$

Exercise 2.23 Find the following limits.

(a) $\lim_{x\to\infty} x^2$

(b) $\lim_{x\to\infty} \dfrac{x^2+2x-1}{x^3+3}$

(c) $\lim_{x\to\infty} \dfrac{x^4+5x^3-1}{x^2+x+1}$

(d) $\lim_{x\to\infty} e^x$

(e) $\lim_{x\to 0^-} e^{1/x}$

(f) $\lim_{x\to 4^+} \tan^{-1}\left(\dfrac{1}{4-x}\right)$

(g) $\lim_{x\to\infty} \cos x$

(h) $\lim_{x\to\infty} (x^2-x)$

(i) $\lim_{x\to\infty} \dfrac{e^x}{x^2}$

■

Exercise 2.24 Draw a rough sketch of the graph of $y=(x+4)^5(x+1)^2(1-x)$. ■

2.7 Derivatives and Rates of Change

The real voyage of discovery consists not in seeking new landscapes, but in having new eyes.

Marcel Proust, French author, 1871- 1922

2.7.1 Derivative

Definition 2.7.1 The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Note. If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 2.23 (a) Find the slope of the tangent line to the graph of $f(x) = x^3$ at the point
(i) $x = 1$
(ii) $x = 2$

(b) Find the equation of the tangent line at each of the points above.

Example 2.24 (a) Find the slope of the tangent to the curve

$$y = \frac{1}{\sqrt{x}}$$

at the point where $x = a$.

(b) Find the equation of the tangent line at the point $(1, 1)$.

The Most Important Definition in this Course:

Definition 2.7.2 — Definition of Derivative. The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Note. If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists then $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Example 2.25 Find the derivative of the function

$$y = \frac{1}{x-1}$$

at the point where $x = 3$.

Example 2.26 The following limit represents the derivative of some function f at some number a. State f and a.

$$\lim_{h \to 0} \frac{2^{h+3} - 8}{h}$$

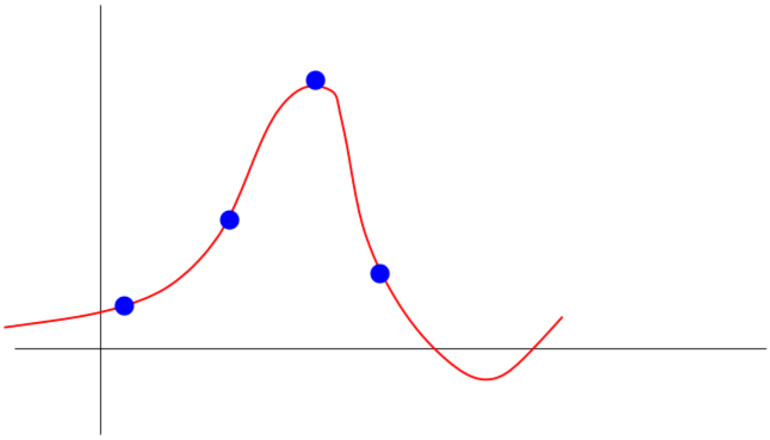
Example 2.27 Let $f(x) = |x|$. Does $f'(0)$ exist?

Must Know! An equation of the tangent line to $y = f(x)$ at $(a, f(a))$ is given by

$$y - f(a) = f'(a)(x - a) \ .$$

Example 2.28 Find the equation of the tangent line to $f(x) = \dfrac{1}{x-1}$ at the point where $x = 3$.

Example 2.29 Compare the derivatives at each of the points on the graph.



Reminder By definition

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}}$$

More Precisely... Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the **displacement** of the object from the origin at **time** t.

The average velocity of the object in the time interval from $t = a$ to $t = a+h$ is given by

$$\text{average velocity} = \frac{f(a+h) - f(a)}{h} .$$

BIG Question. What if h is small?

Definition 2.7.3 We define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ as

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} .$$

Example 2.30 If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after t seconds is given by

$$H = 58t - 0.83t^2 .$$

(a) Find the velocity of the arrow when $t = a$.

(b) When will the arrow hit the moon?

(c) With what velocity will the arrow hit the moon?

2.7.2 Rates of Change

Rates of Change. Let f be a function defined on an interval I and let $x_1, x_2 \in I$. Then the **increment** of x is defined as

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1) .$$

The **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ is defined as

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} .$$

Must Know! The **instantaneous rate of change of y with respect to x** is defined as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} .$$

Example 2.31 If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank in gallons after t minutes as

$$V = 100,000\left(1 - \frac{t}{60}\right)^2 \quad 0 \leq t \leq 60\ .$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t. What are the units?

Example 2.32 The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.

(a) What is the meaning of the derivative $f'(8)$? What are the units?

(b) Is $f'(8)$ positive or negative? Explain.

Exercise 2.25 Each limit represents the derivative of some function f at some number a. State f and a.

(a) $\lim\limits_{h\to 0} \dfrac{\sqrt[4]{16+h}-2}{h}$

(b) $\lim\limits_{x\to 5} \dfrac{2^x-32}{x-5}$

(c) $\lim\limits_{x\to \pi/4} \dfrac{\tan x-1}{x-\pi/4}$

2.8 The Derivative as a Function

I turn away with fear and horror from this lamentable sore of continuous functions without derivatives.

Charles Hermite, French mathematician, 1822-1901

Reminder The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

if this limit exists.

Example 2.33 Find the derivative of the function $f(x) = x^2$ at

(a) $x = 0$,
(b) $x = 1$,
(c) $x = 2$,
(d) $x = 10$.

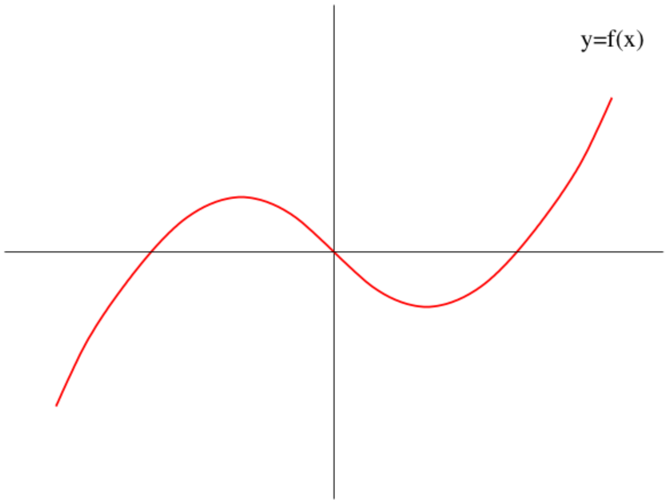
Motivating Problem Given any number x for which this limit exists, we assign to x the number $f'(x)$

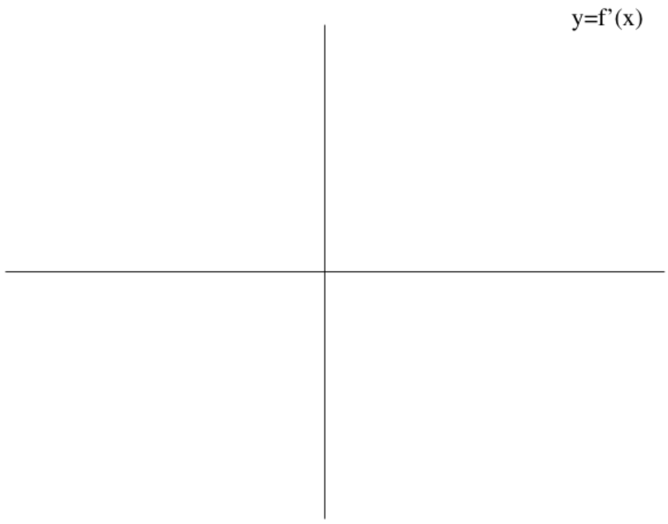
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- We regard $f'(x)$ as a *new function* - called the **derivative of** f
- How is $f'(x)$ interpreted geometrically?

- What is the **domain** of f'?

Example 2.34 The graph of f is given. Sketch the graph of f'.





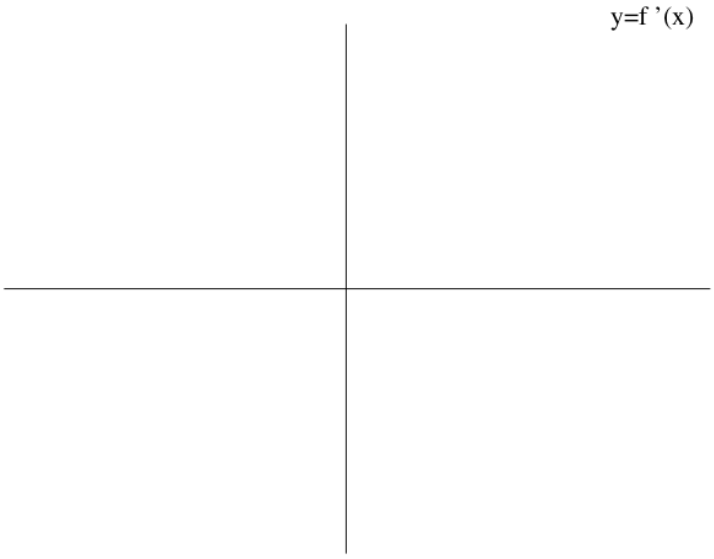
Example 2.35 Let

$$f(x) = x^{\frac{2}{3}} = \sqrt[3]{x^2}\ .$$

(a) Determine the domain of f.

(b) Determine the formula for $f'(x)$. What is the domain of f'?

(c) Sketch graphs of f and f'.



Notation. For $y = f(x)$ it is common to write:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Also,

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right]_{x=a}.$$

Definition 2.8.1 A function is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a,b) [or (a,∞) or $(-\infty,a)$ or $(-\infty,\infty)$] if it is differentiable at every number in the interval.

Two Questions.

(a) Is every continuous function differentiable?

(b) Is every differentiable function continuous?

Three Cases. A function f is not differentiable at a number a from its domain if:

(a) The graph of f has a **corner** at the point $(a, f(a))$;

(b) f is not continuous at a;

(c) The graph of f has a **vertical tangent line** when $x = a$.

Higher Derivatives. Suppose that f is a differentiable function. The **second derivative** of f is the derivative of f'.

Notation.

$$(f')' = f''$$

$$(y')' = y''$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

- The **third** derivative f''' is the derivative of the second derivative: $f''' = (f'')'$.
- The **fourth** derivative, f'''' is usually denoted by $f^{(4)}$.
- In general, the n-th derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times.

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

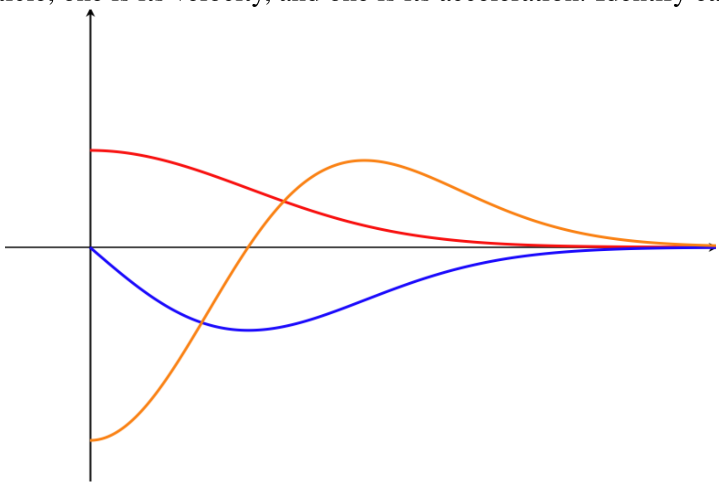
Example 2.36 Find $f''(x)$ if $f(x) = x^2$.

Example 2.37 If $f(x) = x^3$, find $f^{(4)}(x)$.

Acceleration. The instantaneous rate of change of velocity with respect to time is called the **acceleration** of the object.

$$a(t) = v'(t) = s''(t).$$

Example 2.38 The figure shows the graphs of three functions. One is the position function of a particle, one is its velocity, and one is its acceleration. Identify each curve.



2.9 Summary

In this chapter we were introduced to *limits*, and used this to build the definition of *derivative*.

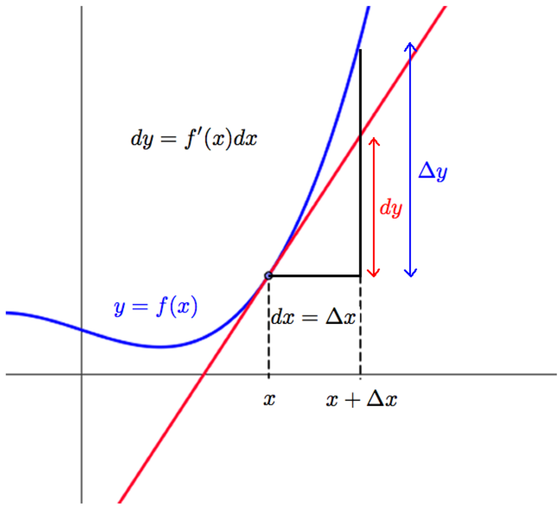
Vocabulary

- limit, limit laws,
- infinite limit, vertical asymptote,
- continuous at a point, continuous on a domain,
- limit at infinity, horizontal asymptote,
- average rate of change
- derivative (instantaneous rate of change)

Skills to acquire

- calculate the limit of a function at a given point
- determine where a function is continuous and where it is discontinuous
- compute the derivative of a function using the definition of derivative

3. Differentiation Rules



In this chapter we develop quicker methods for computing derivatives. These are known as the *differentiation rules* and allow us to avoid having to use a limit to compute a derivative. Once we have these quick methods to compute derivatives we looks at some applications.

Topics we will cover are:

- derivatives of elementary functions,
- product rule, quotient rule, chain rule,
- implicit differentiation,
- related rates,
- linear approximation and differentials
- applications to the natural and social sciences.

3.1 Derivatives of Polynomials and Exponential Functions

Young man, in mathematics you don't understand things, you just get used to them.
John von Neumann, Hungarian mathematician and polymath, 1903-1957

Reminder The **derivative of a function f** is the function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for all x for which this limit exists. Recall that we also use the notation $\frac{d}{dx}(f(x)) = f'(x)$ for the derivative.

Must Know!

(a) **Derivative of a Constant.**

$$\frac{d}{dx}(c) = 0$$

(b) We have already seen that the following are true:

$$\frac{d}{dx}(x) = 1, \quad \frac{d}{dx}(x^2) = 2x, \quad \frac{d}{dx}(x^3) = 3x^2.$$

You may be able to see a pattern. In fact, we have the following rule.

The Power Rule. If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

(c) **Constant Multiple Rule.** If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}(cf(x)) = c \cdot \frac{d}{dx}f(x)$$

(d) **Sum Rule.** If f and g are differentiable functions, then

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

(e) **The Derivative of a Polynomial.** If

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$ then

$$p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1 \ .$$

Example 3.1 Find an equation of the tangent line to the curve $y = 2x^3 - 7x^2 + 3x + 4$ at the point $(1,2)$.

Example 3.2 Find an equation for the straight line that passes through the point $(0,2)$ and it is tangent to the curve $y = x^3$.
Comment: We don't know where this line touches x^3, be we do know what a *generic* point looks like on this curve:

Fact. If $f(x) = a^x$, $a > 0$, $a \neq 1$, is an exponential function then

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}$$

exists.
Fact It is straightforward to show that if $f(x) = a^x$ then

$$f'(x) = f'(0) \cdot a^x \ .$$

Must Know! e is is the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1 \ .$$

$e \approx 2.71828$

Derivative of the Natural Exponential Function. If $f(x) = e^x$ is the natural exponential function then

$$f'(x) = f(x) \ .$$

Thus

$$\frac{d}{dx}(e^x) = e^x \ .$$

Example 3.3 Differentiate the function

$$f(x) = 2x^3 + 3x^{\frac{2}{3}} - e^{x+2} \ .$$

Example 3.4 At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

Exercise 3.1 Find the points on the curve $y = x^4 - 4x^3 + 4x^2$ where the tangent line is horizontal. ■

Exercise 3.2 Find equations of the tangent line and *normal line* to the curve $y = x^4 + 2e^x$ at the point $(0,2)$. ■

Exercise 3.3 Where does the normal line to the parabola $y = x^2 - 1$ at the point $(-1,0)$ intersect the parabola a second time? ■

Exercise 3.4 Find the coordinates of the points where the tangent lines to the parabola $y = x^2$ that pass through the point $(0,-4)$ intersect the parabola. ■

Exercise 3.5 Find the n-th derivative of the function $y = 1/x$ by calculating the first few derivatives and observing the pattern that occurs. ■

Exercise 3.6 Find a parabola with equation $y = ax^2 + bx + c$ that has a tangent line with a slope of 4 at $x = 1$, slope of -8 at $x = -1$ and passes through the point $(2,15)$. ■

Exercise 3.7 Is the function

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$$

differentiable at $x = 1$? ■

3.2 The Product and Quotient Rules

Five out of four people have trouble with fractions.

Steven Wright, American comedian, 1955-

Motivating Problem Suppose we have two functions $f(x) = \sqrt[3]{x^2}$ and $g(x) = e^x$ and we want to compute the derivative of their product

$$\frac{d}{dx}(\sqrt[3]{x^2}e^x).$$

How do we do this?

Product Rule. If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \ .$$

In Lagrange's notation this is written as $(fg)' = f \cdot g' + g \cdot f'$.

Example 3.5 (a) Differentiate $f(x) = \sqrt[3]{x^2} \cdot e^x$.

(b) Differentiate $g(x) = (x+1)(2x^2 - x + 1)$.

Quotient Rule. If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

In Lagrange's notation this is written as $\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$.

Example 3.6 (a) Differentiate $y = \frac{2t^2 - 1}{t^3 + 1}$.

(b) Differentiate $f(x) = e^{-x}$.

Example 3.7 If $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$, and $g'(3) = 5$, find the following values.

(a) $(f+g)'(3)$

(b) $(fg)'(3)$

(c) $\left(\frac{f}{g}\right)'(3)$

(d) $\left(\frac{f}{f-g}\right)'(3)$

Exercise 3.8 Differentiate $g(x) = \dfrac{3x^2 + 2\sqrt{x}}{x}$.

(Hint: You don't *need* to use the Quotient Rule every time you see a quotient... sometimes it's easier to simplify the function first.) ■

Exercise 3.9 Find the derivatives of the functions:

(a) $y(t) = (3t - e^t)\left(\frac{1}{\sqrt{t}} + \sqrt{t}\right)$

(b) $f(x) = \dfrac{x^3 - 1}{x^2 + 7x + 3}$

(c) $y(t) = \dfrac{2}{at}$

(d) $u(s) = \dfrac{s + \sqrt{s}}{s^{1/4}}$

■

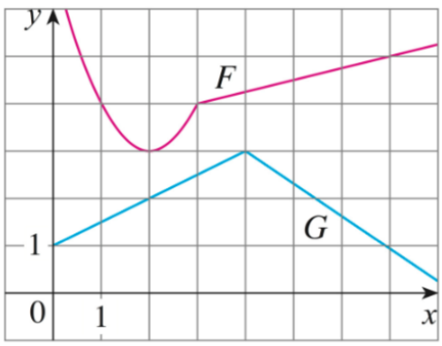
Exercise 3.10 If f is a differentiable function, find y' if

$$y(t) = \frac{1 + tf(t)}{\sqrt{t}}.$$

■

Exercise 3.11 Find an equation of the tangent line to the curve $y = \dfrac{e^x}{1+x^2}$ at the point $(1, \frac{e}{2})$. ■

Exercise 3.12 Let $P(x) = F(x)G(x)$ and $Q(x) = \frac{F(x)}{G(x)}$ where F and G are functions whose graphs are shown.



Find

(a) $P'(2)$

(b) $Q'(4)$

(c) $P'(5)$

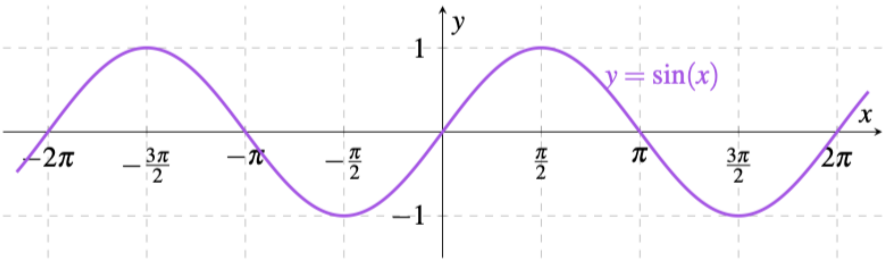
(d) $Q'(7)$

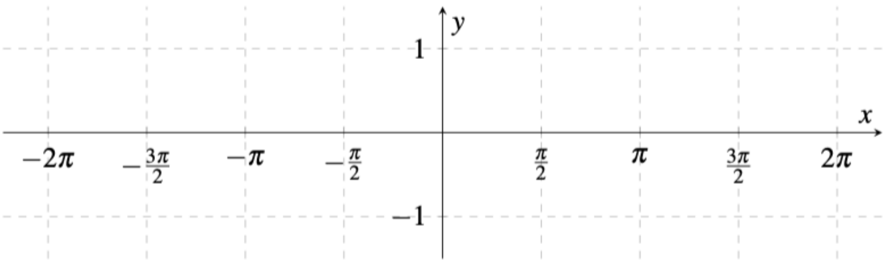
■

3.3 Derivatives of Trigonometric Functions

Trigonometry is the mathematics of sound and music.
Frank Wattenberg, American mathematician, 1952-

Motivating Problem What is the derivative of $\sin x$?





Must Know!

(a)
$$\frac{d}{dx}(\sin x) = \cos x$$

(b)
$$\frac{d}{dx}(\cos x) = -\sin x$$

(c)
$$\frac{d}{dx}(\tan x) = \sec^2 x$$

(d)
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

(e)
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

(f)
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Motivating Problem Prove that

$$\frac{d}{dx}(\sin x) = \cos x \ .$$

Trigonometric Limits. Above we used the very important results

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0.$$

We now prove these results.

Example 3.8 (a) Differentiate $y = \dfrac{1+\tan x}{x-\cot x}$.

(b) Find the points on the curve

$$y = \frac{\cos x}{2+\sin x}$$

at which the tangent is horizontal.

Example 3.9 A ladder 10 ft long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

Example 3.10 Evaluate

(a) $\lim_{x \to 0} \frac{\sin 2x}{x}$

(b) $\lim_{\theta \to 0} \frac{\sin 2\theta}{\cos \theta - 1}$

Exercise 3.13 Let $f(x) = \cos x$. Find $f^{(27)}(x)$. ■

Exercise 3.14 Find $\lim\limits_{x \to 0} \dfrac{\sin 7x}{\sin \pi x}$. ■

Exercise 3.15 Calculate $\lim\limits_{x \to 0} x \cot x$. ■

Exercise 3.16 Differentiate $f(x) = \dfrac{\cot x}{1 + \csc x}$. For what values of x does the graph of f have a horizontal tangent line? ■

3.4 Chain Rule

Puzzle: A duck before two ducks, a duck behind two ducks, and a duck in the middle. How many ducks are there?

Reminder The **composition** of the functions f and g is defined by

$$(f \circ g)(x) = f(g(x)) \ .$$

Example 3.11 Let $f(u) = \sin u$ and $g(x) = 1 + x^2$. Find $F = f \circ g$.

Motivating Problem What is $F'(x)$?

Theorem 3.4.1 — Chain Rule. If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by

$$F'(x) = f'(g(x)) \cdot g'(x) \ .$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \ .$$

Example 3.12

(a) Let $f(u) = \sin u$ and $g(x) = 1 + x^2$ and let $F = f \circ g$. Find the derivative of F.

(b) Find $y' = \dfrac{dy}{dx}$, if

(i) $y = (2-5x)^3$

(i) $y = (x+\sin x)^5(1+e^x)^2$

(c) Express the derivative dy/dx in terms of x if

$$y = u^5 \text{ and } u = \frac{(4x-1)^2}{x} .$$

Example 3.13 Find f'.

(a)

$$f(x) = \sqrt{2+5x^2}$$

(b)

$$f(x) = (\tan(x^2))^3$$

(c)

$$f(x) = e^{\cos x}$$

Notice: We can use the Chain Rule to differentiate an exponential function with any base $a > 0$:

- Recall: $a = e^{\ln a}$
- So: $a^x = (e^{\ln a})^x = e^{(\ln a)x}$
- Thus:

$$\frac{d}{dx}(a^x) =$$

Example 3.14

(a) A pebble dropped into a lake creates an expanding circular ripple. Suppose that the radius of the circle is increasing at the rate of 2 in/s. At what rate is its area increasing when its radius is 10 in?

(b) Suppose that $f(0) = 0$ and $f'(0) = 1$. Calculate the derivative of $f(f(f(x)))$ at $x = 0$.

(c) Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants.

(i) Find $\lim_{t\to\infty} p(t)$.

(ii) Find the rate of spread of the rumor.

Exercise 3.17 Find an equation of the tangent line to the curve $y = \sin x + \sin^2 x$ at $(0,0)$. ■

Exercise 3.18 Differentiate $f(x) = \sin(\cos(\tan(x)))$. ■

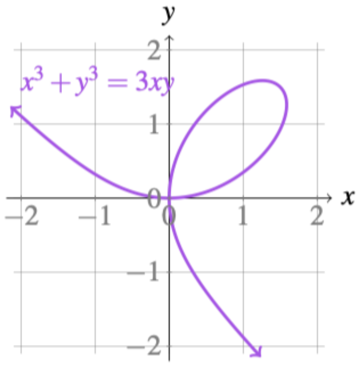
3.5 Implicit Differentiation

Dictionary. implicit
adjective
Definition:
1. implied: not stated, but understood in what is expressed
Asking us when we would like to start was an implicit acceptance of our terms.
2. absolute: not affected by any doubt or uncertainty
implicit trust
3. contained: present as a necessary part of something
Confidentiality is implicit in the relationship between doctor and patient.

Motivating Problem The curve

$$x^3 + y^3 = 3xy$$

is called the **folium of Descartes**. Find the equation of the tangent line at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.



Implicitly Defined Function. An equation in two variables x and y may have one or more solutions for y in terms of x or for x in terms of y. These solutions are functions that are said to be **implicitly defined** by the equation.

Example 3.15

(a) $x^2 + y^2 = 1$

(b) $x^3 + y^3 = 3xy$

Algorithm — Implicit Differentiation.

(a) Use the chain rule to differentiate both sides of the given equation, thinking of x as the independent variable.

(b) Solve the resulting equation for $\frac{dy}{dx}$.

Example 3.16 The curve

$$x^3 + y^3 = 3xy$$

is called the **folium of Descartes**. Find the equation of the tangent line at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Example 3.17 Suppose that water is being emptied from a spherical tank of radius 10 ft. If the depth of water in the tank is 5 ft and is decreasing at the rate of 3 ft/sec, at what rate is the radius r of the top surface of the water decreasing?

Differentiation of an Inverse Function. Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

Inverse Trig Derivatives - Must Know:	
$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1}(x)) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1}(x)) = -\frac{1}{x\sqrt{x^2-1}}$

Proof of $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$:

Example 3.18 Determine the points on the circle

$$(x-1)^2+(y-2)^2=4$$

where the tangent line is horizontal or vertical.

Example 3.19 For the curve

$$x^2+y^2=5$$

find y'' by implicit differentiation.

Exercise 3.19 Find y' if $y = x\tan^{-1}\sqrt{x}$. ■

Exercise 3.20 Find y' if $\cos(y^2) + x = e^y$. ■

3.6 Derivatives of Logarithmic Functions

One real estate development company advertised that an investment with it would grow logarithmically.

From Ed Barbeau's column, *Fallacies, Flaws, and Flimflam*, in College Math. Journal 36 (2005), 394-396.

Must Know!

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

When $a = e$ this becomes

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

In general, using the Chain Rule, we have

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \qquad \text{or} \qquad \frac{d}{dx}[\ln(g(x))] = \frac{g'(x)}{g(x)}$$

Example 3.20 Differentiate

(a) $y = \log_2(3x^2 + e^x)$

(b) $y = \ln(x + \sqrt{x^2 - 1})$

(c) $y = \sqrt{\ln x}$

(d) $y = \ln \sqrt{x}$

(e) $y = \ln\left(\dfrac{x^2}{(x+3)^4}\right)$

Notice the differences in the following statements. Let a and b be constants. Then:

(a) $\frac{d}{dx}(a^b) = 0$ (constant rule)
(b) $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1} f'(x)$ (power rule & chain rule)
(c) $\frac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$ (exponential rule & chain rule)
(d) $\frac{d}{dx}[f(x)]^{g(x)} =$

Algorithm — Logarithmic Differentiation.

(a) Take the natural logarithms of both sides of an equation $y = f(x)$.
(b) Use Laws of Logarithms to rewrite (as a simpler operation).
(c) Differentiate implicitly with respect to x.
(d) Solve the resulting equation for y'.
(e) Can you write in terms of x only?

Example 3.21 Differentiate

(a)

$$y = \frac{\sqrt[4]{x^3}\sqrt[5]{x^3+1}}{(2x+1)^3}$$

(b)

$$y = x^{x^2}$$

(c)

$$y = \ln|x|$$

Exercise 3.21 Find y' if $x^y = y^x$. ■

Special Limit - Must Know!

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

Letting $n = \frac{1}{x}$, then as $x \to 0^+$, $n \to \infty$:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Exercise 3.22 Show that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

for any $x > 0$. ■

3.7 Rates of Change in the Natural and Social Sciences

If you want to see practical applied mathematics, read chemical engineering; if you want to see theoretical applied mathematics, read electrical engineering.
And if you want to read pure math, read economics.

(Unknown blogger.)

Reminder The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the average rate of change of y with respect to x over the interval $[x_1, x_2]$.

- It can be interpreted as the slope of the ___________________.
- Its limit as $\Delta x \to 0$ is _____________________________.

Interpretation?

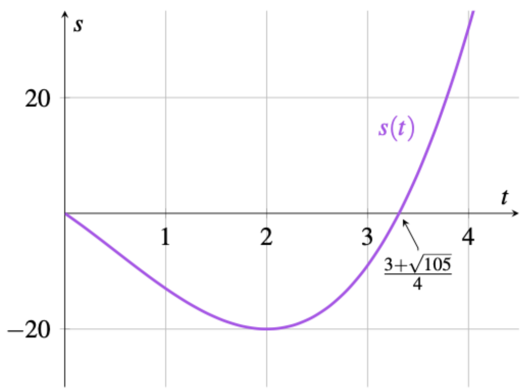
Whenever $y = f(x)$ has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change.

Example 3.22 — Physics. The equation of motion for a particle is given by

$$s = 2t^3 - 3t^2 - 12t,\ t \geq 0$$

where s is in meters and t is in seconds.

(a) Find the velocity and acceleration as functions of t.
(b) The graph of $s = s(t)$ is shown. Sketch the graphs of the velocity and acceleration functions for $0 \leq t \leq 4$.
(c) When is the particle speeding up? Slowing down?
(d) What does the expression $s'''(t) = a'(t)$ represent?



Exercise 3.23 Let $v(t)$ be a function which gives the velocity of a particle at time t. Consider the **speed function** $w(t) = |v(t)|$.

(a) The particle is *speeding up* when $w'(t) > 0$. Show that this is equivalent to the condition that $v(t)$ and $a(t)$ have the same sign.

(b) Similarly, the particle is *slowing down* when $w'(t) < 0$. Show that this is equivalent to the condition that $v(t)$ and $a(t)$ have opposite signs.

[**Hint**: Remove the absolute value signs by writing $w(t)$ as a piecewise defined function. Then differentiate the piecewise function, paying careful attention to the conditions which define each piece.] ■

Example 3.23 — Chemistry. If one molecule C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentrations of A and B have a common value $[A] = [B] = a$ moles/L then

$$[C] = \frac{a^2kt}{akt+1}$$

where k is a constant.

(a) Find the rate of reaction at time t.

(b) Show that if $x = [C]$, then

$$\frac{dx}{dt} = k(a-x)^2$$

(c) What happens to the concentration as $t \to \infty$?

(d) What happens to the rate of reaction as $t \to \infty$?

Example 3.24 — Economics. Suppose that the cost (in dollars) for a company to produce x pairs of a new line of jeans is

$$C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3$$

(a) Find the marginal cost function.
(b) Find $C'(100)$ and explain its meaning. What does it predict?
(c) Compare $C'(100)$ with the cost of manufacturing the 101st pair of jeans.

Example 3.25 — Geometry. The height of a certain cylinder is always twice its radius r. If its radius is changing, show that the rate of change of its volume with respect to r is equal to its surface area.

3.8 Exponential Growth and Decay

It's the whole issue with exponential growth, it's very slow in the beginning but over the long term it gets ridiculous.

Drew Curtis, Founder and chief administrator of Fark.com, 1973 -

Exponential Growth and Decay: A quantity q is said to be growing (or decaying) exponentially if

$$q = Ae^{kt}$$

where A and k are constants.

Natural Growth Equation. The solution of the initial-value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

is

$$y(t) = y_0 e^{kt} \; .$$

Example 3.26 Calculopolis had a population of 25000 in 1980 and the population of 30000 in 1990. What population can the Calculopolis planers expect in the year 2020 if the population grows at a rate proportional to its size?

Radioactive Decay. Radioactive material is known to decay at a rate proportional to the amount present. This means, if N is the amount (mass) of radioactive material at time t then it must satisfy the model:

$$\frac{dN}{dt} = -kN, \ k > 0.$$

Example 3.27 It takes 8 days for 20% of a particular radioactive material to decay. How long does it take for 100 grams of material to decay to 50 grams? 40 grams? 0 grams?

R Usually k is specified in terms of the **half-life** of the isotope

$$\tau = \frac{\ln 2}{k}.$$

This is the time required for half of any given quantity to decay.

Newton's Law of Cooling and Heating. If a warm object is put in cooler surroundings its temperature will steadily decrease. A law of physics known as **Newtons law of cooling** says that the rate at which the object cools is proportional to the difference between its temperature and the surrounding temperature. This law is modeled by the differential equation:

$$\frac{dT}{dt} = k(T - M)$$

where

- $T(t)$ is the temperature of the object at time t
- M is the temperature of the surroundings (ambient temperature - which is constant)
- k a constant (called the *cooling constant*)

Example 3.28 When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room its temperature has increased to 10°C.

(a) What is the temperature of the drink after 50 minutes?

(b) When will its temperature be 15°C?

Exercise 3.24 See Stewart's textbook on page 241 for an example involving *compound interest*.
■

3.9 Related rates

Oh, yes these problems can be nasty. Lots of students fear related rates problems. Why? Maybe because they are word problems, and students just don't like word problems. Having to change English into mathematics intimidates many people. It's as if when they hear it in words, the mathematical sides of their brain shut down.

Colin Adams, Mathematician, author of How to Ace Calculus: The Streetwise Guide

The Method of Related Rates:

(a) Organize (& visualize) the information:
   (i) Read the problem carefully!
   (ii) **Draw a diagram** (if possible).
   (iii) Introduce notation
   (iv) Identify:
      - Independent variables,
      - Dependent variables,
      - What you know,
      - What you want to know!

(b) Find a relationship (i.e. **an equation**) between dependent quantities (often geometry!).

(c) **Differentiate** (with respect to the independent variable - often time t) to find relationship between rates. (This usually involves the chain rule.)

(d) **Substitute** known information in and solve for the unknown rate.

Some identifying words to watch for:

- rate
- how fast
- velocity

Example 3.29 A spherical balloon is being inflated. The radius r of the balloon is increasing at the rate of 0.2 cm/s when $r = 5$ cm. At what rate is the volume V of the balloon increasing at that moment?

Example 3.30 A rocket is launched vertically and is tracked by a radar station located on the ground 5 km from the launch pad. Suppose that the elevation angle θ of the line of sight to the rocket is increasing at 3° per second when $\theta = 60^\circ$. What is the velocity of the rocket at that instant?
[link to applet]

Example 3.31 A man 6 ft tall walks with a speed of 8 ft/s away from a street light that is atop an 18-ft pole. How fast is the tip of his shadow moving along the ground when he is 100 ft from the light pole?
[link to applet]

Example 3.32 A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?
[link to applet]

Exercise 3.25 A lamp is located at point $(3,0)$ in the xy-plane. An ant is crawling in the first quadrant of the plane and the lamp casts its shadow onto the y-axis. How fast is the ant's shadow moving along the y-axis when the ant is at position $(1,2)$ and moving so that its x-coordinate is increasing at a rate of $1/3$ m/s and y-coordinate is decreasing at $1/4$ m/s? ■

Exercise 3.26 The minute hand on a watch is 6 mm long and the hour hand is 3 mm long. How fast is the distance between the tips of the hands changing at two o'clock? ■

3.10 Linear Approximation and Differentials

It is the mark of an instructed mind to rest satisfied with the degree of precision to which the nature of the subject admits and not to seek exactness when only an approximation of the truth is possible.

Aristotle, Greek philosopher, 384 BC - 322 BC.

Motivating Problem If $f(1) = 4$ and $f'(1) = 1$ use the linear approximation to $f(x)$ at $x = 1$ to approximate $f(2)$.

Idea. Instead of evaluating $f(x)$ evaluate $L(x)$ where L is the tangent line to the graph of $y = f(x)$ at a known point $(a, f(a))$ that is *close* to the point $(x, f(x))$.

Definition 3.10.1 — Linear Approximation. The linear function

$$L(x) = f(a) + f'(a)(x-a)$$

is called the **linearization** of f at a.

For x *close* to a we have that

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

and this approximation is called the **linear approximation** of f at a.

Example 3.33 If $f(1) = 4$ and $f'(1) = 1$ use the linear approximation to $f(x)$ at $x = 1$ to approximate $f(2)$.

Example 3.34 Use the linear approximation of the function $f(x) = \sqrt[3]{x}$ to find
(a) $\sqrt[3]{2}$
(b) $\sqrt[3]{26}$

Example 3.35 Use linear approximation to approximate $\sqrt{37}$. What is the accuracy of this approximation?

Definition 3.10.2 — Differential. Let f be a function differentiable at $x \in \mathbb{R}$. Let $\Delta x = dx$ be a (small) given number. The **differential** dy is defined as

$$dy = f'(x)\Delta x \,.$$

Important!

$$\begin{aligned} f(a+dx) &\approx L(a+dx) \\ f(a+dx) &\approx f(a)+f'(a)(a+dx-a) \\ f(a+dx) &\approx f(a)+f'(a)dx = f(a)+dy \\ \mathbf{dy} &\approx \mathbf{f(a+dx)-f(a)} \end{aligned}$$

Small differential means *good* approximation.

Example 3.36 Compare the values of Δy and dy if

$$y = f(x) = x^3 + x^2 - 2x + 1$$

and x changes from
(a) 2 to 2.05
(b) 2 to 2.01

Example 3.37 The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

Example 3.38 The equatorial radius of the earth is approximately 3960 mi. Suppose that a wire is wrapped tightly around the earth at the equator. Approximately how much must this wire be lengthened if it is to be strung all the way around the earth on poles 10 ft above the ground. (1 mi = 1760 yards = $1760 \cdot 3$ ft.)

Example 3.39 Suppose we're painting a circular-shaped deck of radius approximately $r = 5$m, and we need about 0.2 litres of paint per square meter. How much more or less paint do we need if the radius is 0.02 meters larger or smaller?

Exercise 3.27 Use differentials to estimate the amount of paint needed to apply a coat of paint 0.01 cm thick to a hemispherical dome with a diameter 50 m.

3.11 Summary

In this chapter we introduced the main topic of the course: the **derivative**. We also investigated some applications of the derivative.

Vocabulary

- derivative
- product rule, quotient rule, chain rule
- implicit differentiation, logarithmic differentiation,
- rate of change
- exponential growth and decay
- related rates,
- linear approximation, differential

Skills to acquire

- calculate the limit of a function using the differentiation rules developed
- interpret a rate of change problem from the natural and social sciences as a derivative
- convert a verbal problem into mathematical symbols, and solve using the techniques of this chapter
- compute the linear approximation/differential of a function at a point, and use this to estimate values of a function

4. Applications of the Derivative

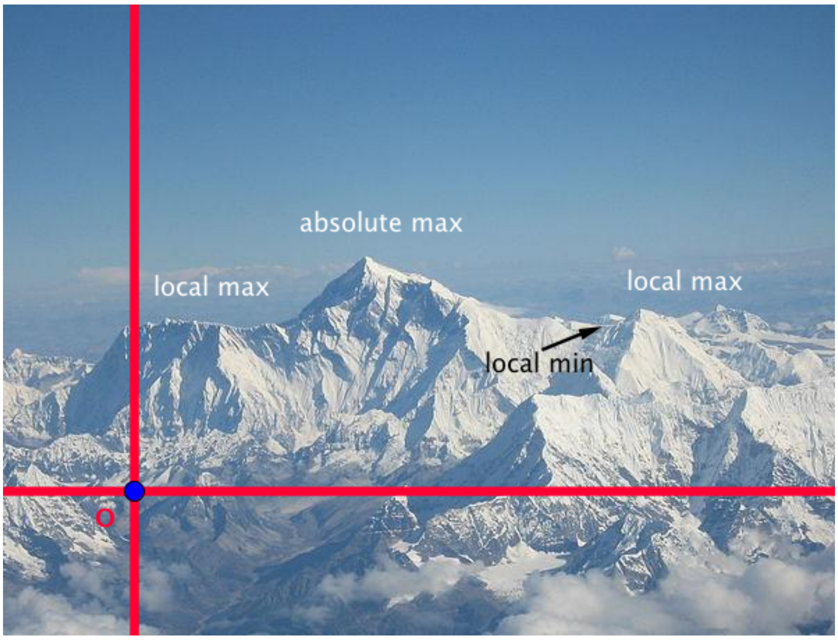


Image Source: http://commons.wikimedia.org/wiki/File:Mount_Everest_as_seen_from_Drukair.jpg

In this chapter we continue looking at applications of the derivative.

Topics we will cover are:

- optimization (determining maximum and minimum values of functions),
- using derivatives to determine the shape of graphs (curve sketching),
- revisit limits and develop L'Hospital's rule,
- approximating solutions to equations using Newton's Method.

4.1 Maximum and Minimum Values

I feel the need of attaining the maximum of intensity with the minimum of means. It is this which has led me to give my painting a character of even greater bareness.

Joan Miró, Catalan-Spanish artist, 1893 - 1983

Definition 4.1.1 A function f has an **absolute maximum** at c if

$$f(c) \geq f(x) \text{ for all } x \in D, \text{ the domain of } f\ .$$

The number $f(c)$ is called the **maximum value** of f on D.

A function f has an **absolute minimum** at c if

$$f(c) \leq f(x) \text{ for all } x \in D, \text{ the domain of } f\ .$$

The number $f(c)$ is called the **minimum value** of f on D.

Definition 4.1.2 A function f has a **local maximum** at c if

$$f(c) \geq f(x) \text{ for all } x \text{ in an open interval, in the domain, containing } c\ .$$

A function f has a **local minimum** at c if

$$f(c) \leq f(x) \text{ for all } x \text{ in an open interval, in the domain, containing } c\ .$$

Theorem 4.1.1 — Extreme Value Theorem. If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c,d \in [a,b]$.

Theorem 4.1.2 — (Fermat's Theorem. If f has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$.

Example 4.1 Find all local extrema of

(a) $f(x) = 3x^4 - 16x^3 + 18x^2, \quad -1 \le x \le 4$

(b) $f(x) = |x|, \quad -1 < x < 1$

Definition 4.1.3 A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 4.2 Find the maximum and minimum values of the function

$$f(x) = x^2 + 4x + 7, \; -3 \leq x \leq 0$$

Algorithm — Closed Interval Method. To find the **absolute** maximum and minimum values of a continuous function f on a closed interval $[a,b]$:

(a) Find the values of f at the critical numbers of f in (a,b).
(b) Find the values of f at the endpoints of the interval.
(c) The largest of the values from Step (a) and Step (b) is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 4.3 Find the maximum and minimum values of the given functions on the indicated closed intervals.

(a) $f(x) = x + \frac{4}{x}, \quad x \in [1,4]$

(b) $g(x) = 2 - \sqrt[3]{x}, \quad x \in [-1, 8]$

Exercise 4.1 Sketch the graph of the given function and locate its local and global extrema:

$$f(x) = \begin{cases} 1-x & \text{if } 0 \leq x < 2 \\ 2x-4 & \text{if } 2 \leq x \leq 3 \end{cases}$$

■

Exercise 4.2 Find the critical numbers of $f(t) = t^{3/4} - 2t^{1/4}$. ■

Exercise 4.3 Find the critical numbers of $f(p) = \frac{1}{p+2}$. ■

Exercise 4.4 Find the absolute maximum and minimum values of

$$f(x) = x^3 - 3x^2 + 1, \qquad -\frac{1}{2} \leq x \leq 4$$

■

Exercise 4.5 Find the absolute maximum and minimum values of

$$f(x) = x - \ln x, \qquad \left[\frac{1}{2}, 2\right]$$

■

Exercise 4.6 Show that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has no local extrema. ■

4.2 The Mean Value Theorem

The Mean Value Theorem is the midwife of calculus - not very important or glamorous by itself, but often helping to deliver other theorems that are of major significance.

Edwin Purcell and Dale Varberg, American mathematicians

Theorem 4.2.1 — Rolle's Theorem. (Michel Rolle, French mathematician, 1652-1719) Let f be a function that satisfies the following three hypotheses:
(a) f is continuous on the closed interval $[a,b]$.
(b) f is differentiable on the open interval (a,b).
(c) $f(a) = f(b)$.
Then there is a number c in (a,b) such that $f'(c) = 0$.

Example 4.4 Check if the following functions satisfy the hypotheses of Rolle's theorem.
(a) $f(x) = x^{1/2} - x^{3/2}$ on $[0,1]$.

(b) $f(x) = 1 - x^{2/3}$ on $[-1,1]$.

Example 4.5 Prove that the equation $x^3 + x - 1 = 0$ has exactly one solution.

- Use the **Intermediate Value Theorem** to show that a root exists.

- Argue the equation has no second solution using Rolle's Theorem. (Use a *contradiction* argument.)

Theorem 4.2.2 — The Mean Value Theorem. Let f be a function that satisfies the following hypotheses:

(a) f is continuous on the closed interval $[a,b]$.
(b) f is differentiable on the open interval (a,b).

Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Example 4.6 A car is driving along a rural road where the speed limit is 70 km/h. At 3:00 pm its odometer reads 18075 km. At 3:18 its reads 18100 km. Prove that the driver violated the speed limit at some instant between 3:00 and 3:18 pm.

Example 4.7 Let $f(x) = \frac{1}{4}x^3 + 1$.

(a) Show that the function $f(x)$ satisfies the hypotheses of the MVT on $[0,2]$.

(b) Find all values $x = c$ where the tangent line is parallel to the secant line of $f(x)$ joining the endpoints of the interval $[0,2]$.

Must Know! If $f'(x) = 0$ for all x in an interval (a,b), then f is constant on (a,b).

Fact. If $f'(x) = g'(x)$ for all x in an interval (a,b), then $f - g$ is constant on (a,b); that is,

$$f(x) = g(x) + c$$

where c is a constant.

Example 4.8 Prove the identity

$$\arcsin\left(\frac{x-1}{x+1}\right) = 2\arctan\left(\sqrt{x}\right) - \frac{\pi}{2}$$

Exercise 4.7 Show that the equation $x^4 = x + 1$ has **exactly** one solution in the interval $[1,2]$. ■

Exercise 4.8 A toll highway has a speed limit of 80 km/h. You enter it at 9:00 am and leave 90km later at at 10:00 am. Can you get a speeding ticket in the mail? ■

Exercise 4.9 Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be? ■

Exercise 4.10 Suppose that $f(x)$ is an odd function and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b,b)$ such that $f'(c) = \frac{f(b)}{b}$. ■

Exercise 4.11 Prove the identity:

$$2\sin^{-1} x = \cos^{-1}(1 - 2x^2), \qquad 0 \leq x \leq 1$$

■

4.3 How Derivatives Affect the Shape of a Graph

The spread of civilization may be likened to a fire; First, a feeble spark, next a flickering flame, then a mighty blaze, ever increasing in speed and power.

Nikola Tesla, American inventor and engineer, 1856 - 1943

4.3.1 What does f' say about f?

Theorem 4.3.1 — Increasing/Decreasing Test.

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.
(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example 4.9 Find the open intervals on the x-axis on which the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

is increasing and those on which it is decreasing.

Reminder If f has a local maximum or minimum at c then c must be a critical number of f (Fermat's theorem)

- But ... not every critical number corresponds to a local maximum or a local minimum!
- Need ... a test that will test us whether or not f is a local max/min at a critical number.

Theorem 4.3.2 — The First Derivative Test. Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.
(b) If f' changes from negative to positive at c, then f has a local minimum at c.
(c) If f' does not change sign at c, then f has no local minimum or maximum at c.

Example 4.10 Find all local extrema of the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

Example 4.11 Find all local extrema of

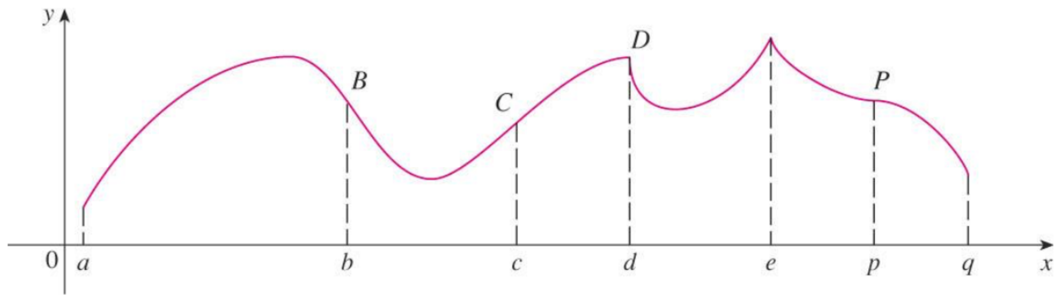
$$f(x) = |x|$$

4.3.2 What does f'' say about f?

Definition 4.3.1 If the graph of f lies above all of its tangent lines on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Theorem 4.3.3 — Concavity Test.
(a) If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward on I.
(b) If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward on I.

Example 4.12 Where is the following function concave up (CU)? Where is it concave down (CD)?



Definition 4.3.2 A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Theorem 4.3.4 — The Second Derivative Test.
Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

Example 4.13 Sketch the graph of the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

4.4 Indeterminate Forms and L'Hospital's Rule

"Proof". *Let $a = b$.*
$a^2 = ab$ *(Multiply both sides by a.)*
$a^2 + a^2 - 2ab = ab + a^2 - 2ab$ *(Add $a^2 - 2ab$ to both sides.)*
$2(a^2 - ab) = a^2 - ab$ *(Factor the left, and collect like terms on the right.)*
$2 = 1$ *(Divide both sides by $a^2 - ab$.)*

Indeterminate Forms.

(a) Indeterminate form of type $\frac{0}{0}$.

Example. Evaluate $\lim_{x\to 0} \dfrac{\sin kx}{x}$ for $k \in \mathbb{R}$.

(b) Indeterminate form of type $\frac{\infty}{\infty}$.

Example. Evaluate $\lim_{x\to \infty} \dfrac{ax+1}{bx+1}$ for $a, b \in \mathbb{R}$.

Theorem 4.4.1 — L'Hospital's Rule. Suppose that f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a.) Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

L'Hospital's Rule tells us that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives - *provided that the given conditions are satisfied.*

Important: Verify the conditions regarding the limits of f and g **before** using the rule.

(It's also valid for one-sided limits).

Example 4.14 Find

(a) $\displaystyle\lim_{x \to 0} \frac{e^x - 1}{\sin 2x}$

(b) $\displaystyle\lim_{x \to \infty} \frac{e^x}{x^2 + x}$

(c) $\displaystyle\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$

Indeterminate Form $0 \cdot \infty$:

! If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then it is not clear what the value of

$$\lim_{x\to a} f(x)g(x)$$

will be. This is called an indeterminate form of type $0 \cdot \infty$.

This can be handled by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

Notice this converts the given limit into an indeterminate form of type $0/0$ or ∞/∞.

Example 4.15 Find $\lim_{x\to\infty} x \ln\left(\frac{x-1}{x+1}\right)$.

Exercise 4.12 Find $\lim_{x\to 0^+} x \ln x$. ■

Exercise 4.13 Find $\lim_{x\to -\infty} xe^x$. ■

Indeterminate Form $\infty - \infty$:

! If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then the limit

$$\lim_{x\to a}[f(x) - g(x)]$$

is called an indeterminate form of type $\infty - \infty$.

This is handled by trying to convert the difference into a quotient (strategies: common denominator, rationalization, factoring out a common factor...) so that we get back to an indeterminate form of type $0/0$ of ∞/∞.

Example 4.16 Find $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

Example 4.17 Find $\lim_{x\to\infty} x - \ln x$.

Indeterminate Form $0^0, \infty^0, 1^\infty$:

Several indeterminate forms arise from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

(a) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0
(b) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0
(c) $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞

Each of these cases can be treated in either of 2 ways:

- Taking the natural logarithm:

$$\text{Let } y = [f(x)]^{g(x)}, \text{ then } \ln y = g(x)\ln f(x)$$

- Writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$$

Example 4.18 Find $\lim_{x\to 0}(\cos x)^{1/x^2}$

Example 4.19 Find $\lim_{x\to 0^+} x^x$

Exercise 4.14 Find $\lim_{x\to 0^+} x\cot x$. ■

Exercise 4.15 Find $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$. ■

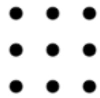
Exercise 4.16 Find a linear function $f(x)$ such that

$$\lim_{x\to 5}\frac{f(x)-3}{x-5}=1.$$

■

4.5 Summary of Curve Sketching

Puzzle. *Connect the nine dots with four (only four) straight lines without ever lifting your pen or pencil from the paper.*



Algorithm — Guideline for Sketching Curves.

(a) **Domain**

(b) **Intercepts:** For the x-intercepts set $y = 0$ and solve for x. For the y-intercept calculate $f(0)$.

(c) **Symmetry:**

  (i) Even function - symmetric about the y-axis.

  (ii) Odd function - symmetric about the origin.

  (iii) Periodic functions.

(d) **Asymptotes:**

  (i) Horizontal Asymptotes: $y = L$ if $\lim_{x\to\infty} f(x) = L$; $y = M$ is $\lim_{x\to-\infty} f(x) = M$.

  (ii) Vertical Asymptotes: $x = a$ if at least one of the following is true

$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$
$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty.$$

  (iii) Slant Asymptotes: $y = mx + b$ if

$$\lim_{x\to\infty} (f(x) - (mx + b)) = 0.$$

(e) **Intervals of Increase and Decrease:**
$f'(x) > 0$ on an interval I means f increasing $\nearrow$ on I.
$f'(x) < 0$ on an interval I means f decreasing $\searrow$ on I.

(f) **Local Maximum and Minimum Values:**
- First Derivative Test
- Second Derivative Test

(g) **Concavity and Points of Inflection:**
$f''(x) > 0$ on an interval I means f concave up $\smile$ on I.
$f''(x) < 0$ on an interval I means f concave down $\frown$ on I.
Inflection points occur where there is a switch in concavity.

(h) **Sketch the curve.**

Example 4.20 Sketch the graphs of the following functions f.In each case the first and second derivatives are already given.

(a) $f(x) = \dfrac{2+x-x^2}{(x-1)^2}$, $\quad f'(x) = \dfrac{x-5}{(x-1)^3}$, $\quad f''(x) = \dfrac{-2(x-7)}{(x-1)^4}$

(b) $f(x) = x^2 e^x, \qquad f'(x) = xe^x(2+x), \qquad f''(x) = (x^2+4x+2)e^x$

Example 4.21 In this example we will focus just on asymptotes in the guidelines outlined in (2).

Determine the asymptotes of the function $f(x) = \dfrac{x^2 + x - 1}{x - 1}$.

Exercise 4.17 Sketch the graph of the following function f. The derivatives are given for convenience.

$$f(x) = \frac{\cos x}{2+\sin x}, \quad f'(x) = -\frac{2\sin x + 1}{(2+\sin x)^2}, \quad f''(x) = -\frac{2\cos x(1-\sin x)}{(2+\sin x)^3}$$

(Use graphing software to check your answer.) ■

4.6 Optimization Problems

There is no branch of mathematics, however abstract, which may not someday be applied to the phenomena of the real world.

Nikolai Lobachevski, Russian mathematician, 1792 - 1856

Optimization is a term that may refer to:

- Finding absolute maximum/minimum of a function - *mathematics*
- Finding best possible solutions - *practical applications* (This requires converting to a math model that simulates the problem, and then solving the math optimization problem)

Steps for solving optimization problems:

(a) Organize the information:
- Read and understand the problem
- Draw a diagram
- Introduce notation

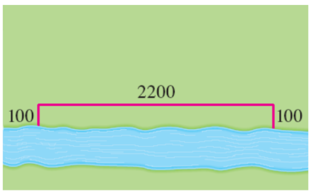
(b) Determine relationships:
- Express the function q that should be optimized, in terms of the introduced variables
- Express q in terms of just one variable; note its domain.

(c) Find the absolute max/min value of q
- open or closed interval?

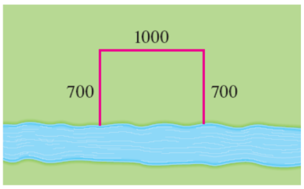
(d) Answer the question.

Example 4.22 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. She needs no fence along the fiver. What are the dimensions of the field that has the largest area?

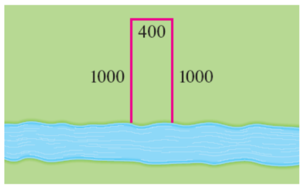
Here are three possible ways of laying out the 2400 ft of fencing:



Area = 100 · 2200 = 220,000 ft^2

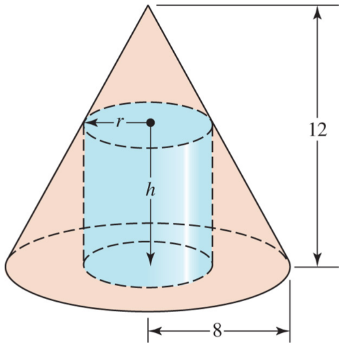


Area = 700 · 1000 = 700,000 ft^2

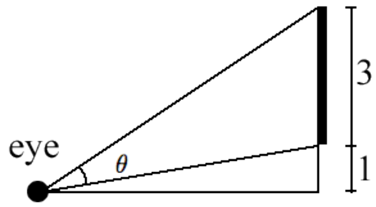


Area = 1000 · 400 = 400,000 ft^2

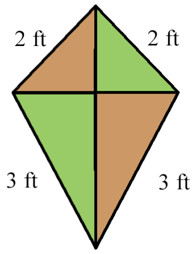
Example 4.23 Find the dimensions of the right circular cylinder with greatest volume that can be inscribed in a right circular cone of radius 8 cm and height 12 cm.



Example 4.24 A painting in an art gallery has height 3 m and is hung so that its lower edge is about 1 m above the eye of an observer. How far from the wall should the observer stand to get the best view? (i.e. the observer wants to maximize the angle θ subtended at the eye by the painting.)
[link to applet]
For an interesting paper connecting this problem to exponential functions click here. [Ćur06]

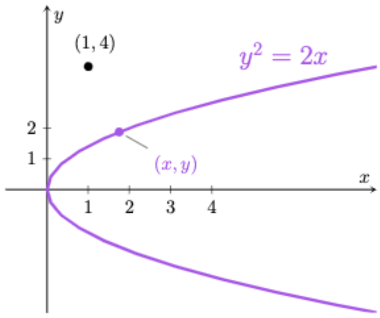


Example 4.25 The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths indicated in the figure. To maximize the area of the kite, how long should the diagonal pieces be?



Example 4.26 Maya is 2 km offshore on a boat and wishes to reach a coastal village which is 6 km down the straight shoreline from the point on the shore nearest to the boat. She can row at 2 km/hour and run 5 km/hour. Where she should land her boat to reach the village in the least amount of time?
[link to applet]

Exercise 4.18 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1,4)$.



■

Exercise 4.19 Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral. ■

Exercise 4.20 A store has been selling 200 of a particular item each week at \$350 each. A market survey indicates that, for each \$10 decrease in price, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. What price will maximize the revenue? ■

Sage Cell

http://goo.gl/vJkWC

4.7 Newton's Method

Sometimes, close enough is good enough.

Math Girl, Episode I - Differentials Attract

Newton's Method.

(a) **Problem:** Find a solution, say $x = r$, to

$$f(x) = 0.$$

(b) **Idea:**

(i) Let x_1 be a "good" estimate of r.

(ii) Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$. Look at the x-intercept of L, call it x_2. If x_1 is close to r then x_2 seems to be even closer to r, and we use x_2 as a second approximation to r.
How do we find x_2 in terms of x_1 and f?

(iii) Repeat this procedure to get a third approximation x_3 from our second approximation x_2:

If we keep repeating this process we obtain a sequence of approximations for r:

$$x_1, x_2, x_3, \ldots.$$

(iv) If the numbers x_n become closer and closer to r as n becomes large then we say that the sequence converges to r and we write

$$\lim_{n\to\infty} = r.$$

(c) **Method.**

(i) Begin with an **initial guess** x_1.

(ii) Calculate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

(iii) If x_n is known then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(iv) If x_n and x_{n+1} agree to k decimal places then x_n approximates the root r up to k decimal places and $f(x_n) \approx 0$.

Example 4.27 Solve

$$5x + \cos x = 5$$

for $x \in [0,1]$ correct to 6 decimal places.

Example 4.28 Use Newton's method to find $\sqrt{2}$ accurate to eight decimal places.

Example 4.29 Use Newton's method to solve $x^{1/3} = 0$ by taking $x_0 = 1$.

Example 4.30 Let $f(x) = x^3 + 3x + 1$.

(a) Show that f has **exactly one** root in the interval $(-\frac{1}{2}, 0)$. Explain.

(b) Use Newton's method to approximate the root that lies in the interval $(\frac{-1}{2}, 0)$. Stop when the next iteration agrees with the previous one at two decimal places.

(continued)

4.8 Summary

In this section we looked at applications of the derivative.

Vocabulary

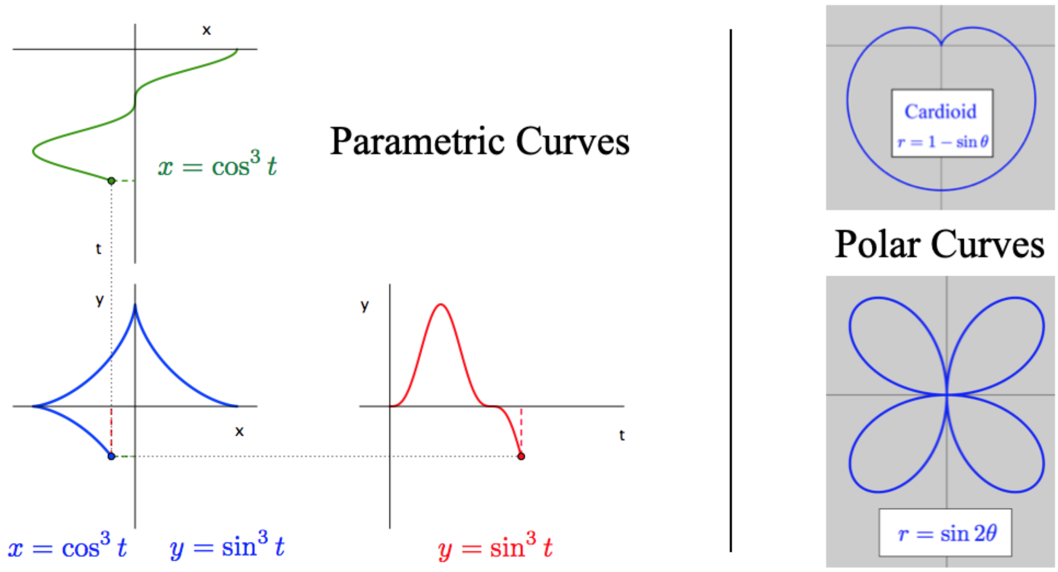
- local maximum/minimum/extremum,
- global maximum/minimum/extremum,
- critical point, inflection point,
- closed interval method,
- mean value theorem (MVT).
- indeterminate form, L'Hospital's Rule
- horizontal, vertical and slant asymptote
- Newton's Method

Skills to acquire

- calculate the local and global extrema of a function on a closed, semi-infinite, or infinite domain
- determine the critical points and inflection points of a function
- use the MVT to prove the existence of points with specific properties
- evaluate limits using L'Hospital's Rule
- use the techniques developed to provide an accurate sketch of a the graph of a function
- convert a verbal description of an optimization problem into mathematical symbols and solve
- approximate the solutions of an equation using Newton's Method

Part Three: Parametric Curves and Polar Coords

5. Parametric Curves and Polar Coordinates



In this chapter we consider curves defined by parametric equations, and also investigate a new coordinate system: polar coordinates.

Topics we will cover are:

- parametric curves: graphing, and derivatives,
- poloar coordinates, curves in polar coords.

5.1 Curves Defined by Parametric Equations

When you get to the top of the mountain, keep climbing.

Zen proverb

5.1.1 Parametric Curves

Motivating Problem Particle moving in plane

Motivating Problem In the xy-plane draw the set $P = \{(t^2, t) : t \in \mathbb{R}\}$.

Vocabulary Let I be an interval and let f and g be continuous on I.

(a) The set of points $C = \{(f(t), g(t)) : t \in I\}$ is called a **parametric curve**.
(b) The variable t is called a **parameter**.
(c) We say that the curve C is defined by **parametric equations**

$$x = f(t), \quad y = g(t).$$

(d) We say that $x = f(t)$, $y = g(t)$ is a **parametrization** of C.
(e) If $I = [a,b]$ then $(f(a), g(a))$ is called the **initial point** of C and $(f(b), g(b))$ is called the **terminal point** of C.

Example 5.1 Find two parametrizations of the unit circle $x^2 + y^2 = 1$.

Example 5.2 Find parametric equations for the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Example 5.3 Sketch the graph of the curve defined by parametric equations

$$x = \sin t, \quad y = \sin^2 t, \quad -\infty < t < \infty$$

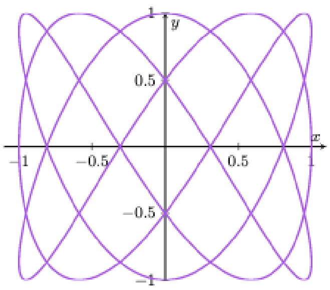
Notice: we distinguish between:

- A *curve*, which is a set of points;
- A *parametric curve*, where the points are traced in a particular way.

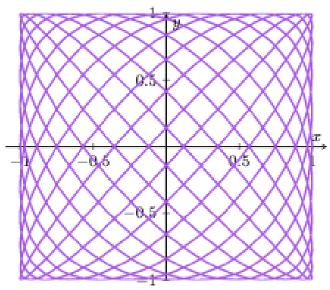
Some neat examples:

Lissajous Curve:

$$x = \cos at, \quad y = \sin bt, \quad t \in [0, 2\pi]$$



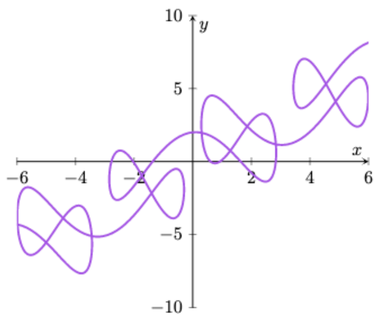
$a = 3,\ b = 5$



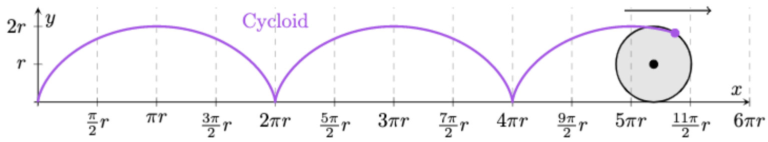
$a = 11,\ b = 13$

Knotty Curve:

$$x = t + 2\sin 2t, \quad y = t + 2\cos 5t, \quad t \in [-2\pi, 2\pi]$$



Example 5.4 — Cycloid. The curve traced by a point P on the edge of a rolling circle is called a **cycloid**. The circle rolls along a straight line without slipping or stopping. Find parametric equations for the cycloid if the line along which the circle rolls is the x-axis but always tangent to it, and the point P begins at the origin.



Exercise 5.1 Sketch the curves using parametric equations to plot points and indicate the direction in which the curve is traced. Then, find the Cartesian equation by eliminating t.

(a) $x = 1 + \sqrt{t}, \quad y = t^2 - 4t, \quad 0 \le t \le 5.$

(b) $x = t^2 - 2, \quad y = 5 - 2t, \quad -3 \le t \le 4.$

(c) $x = e^{2t}, \quad y = t + 1$

■

5.1.2 Derivatives of Parametric Curves

Theorem 5.1.1 — Derivatives of Parametric Curves. The derivative to the parametric curve $x = f(t)$, $y = g(t)$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}.$$

Example 5.5 Find the slope of the tangent line to the curve

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi.$$

at the point corresponding to $t = \pi/4$.

If we think of a parametric curve as being traced out by a moving particle, then

- $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are the **vertical** and **horizontal** velocities of the particle.
- the slope of the tangent line is the ratio of these velocities.

A curve has

- A **horizontal tangent** when $\frac{dy}{dt} = 0$ (provided $\frac{dx}{dt} \neq 0$).
- A **vertical tangent** when $\frac{dx}{dt} = 0$ (provided $\frac{dy}{dt} \neq 0$).

Further, it is useful to consider $\frac{d^2y}{dx^2}$ (recall the second derivative tells us about concavity):

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example 5.6 A curve C is defined by the parametric equations

$$x = t^2, \quad y = t^3 - 3t$$

a) Show that C has two tangents at the point $(3,0)$ and find their equations.
b) Find the points on C where the tangent is horizontal or vertical.
c) Determine where the curve is concave upward or concave downward.
d) Sketch the curve.

Example 5.7 Determine the points on the cycloid where the tangent line is horizontal or vertical.

Exercise 5.2 Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and values of the parameter for which the curve

$$x = t + \ln t, \quad y = t - \ln t$$

is concave up. ■

Exercise 5.3 Find the points on the curve where the tangent line is horizontal or vertical:

$$x = 10 - t^2, \quad y = t^3 - 12t$$

■

Exercise 5.4 Find equations of the tangent lines to the curve

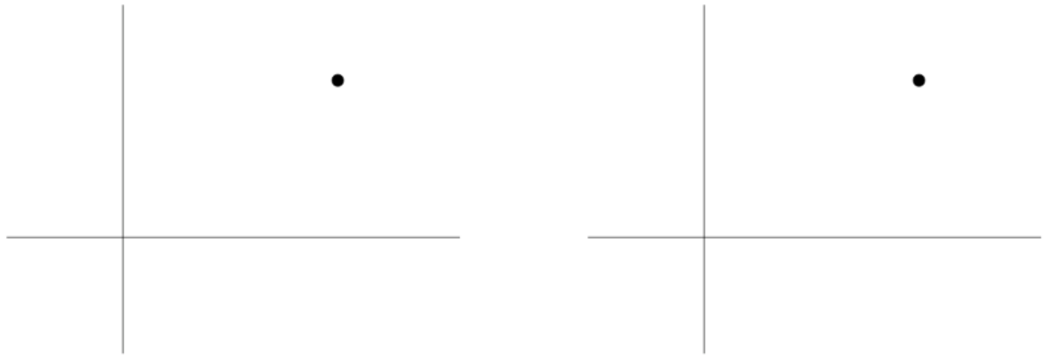
$$x = 2t + 5, \quad y = t^2 + 1$$

that pass through the point $(2, 1)$. ■

5.2 Polar Coordinates

Ancient math joke:
Q: What's a rectangular bear?
A: A polar bear after a coordinate transform.

Motivating Problem Given a point in the plane how can we describe its position?



Definition 5.2.1 — Polar Coordinate System.

(a) Choose a point in the plane. Call it O, which we also call the **pole**.
(b) Choose a ray starting at O. Call it the **polar axis**. (Usually taken as positive x axis.)
(c) Take any point P, except O, in the plane. Measure the distance $d(O,P)$ and call this distance r.
(d) Measure the angle between the polar axis and the ray starting at O and passing through P going from x in counterclockwise direction. Let θ be this measure in radians.
(e) There is a bijection between the plane and the set

$$\mathbb{R}^+ \times [0,2\pi) = \{(r,\theta) : r \in \mathbb{R}^+ \text{ and } \theta \in [0,2\pi)\}$$

This means that each point in the plane is uniquely determined by a pair $(r,\theta) \in \mathbb{R}^+ \times [0,2\pi)$.
(f) r and θ are called **polar coordinates** of P.

Example 5.8 Plot the points whose polar coordinates (r,θ) are given.
(a) $(1,\pi/4)$ (b) $(2,5\pi/4)$ (c) $(2,-\pi/3)$ (d) $(-1,5\pi/6)$

Example 5.9 Plot the three points whose polar coordinates are $(1, \pi/2)$, $(1, 5\pi/2)$, and $(-1, 3\pi/2)$.

Example 5.10 Plot the point given by the polar coordinates $(3, \pi/3)$. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

Example 5.11 Find the connection between **polar** and **Cartesian** coordinates.

Example 5.12 Convert the point $(2, \pi/6)$ from polar to Cartesian coordinates.

Example 5.13 Plot the point whose polar coordinates are $(2\sqrt{2}, 3\pi/4)$. Find the Cartesian coordinates of this point.

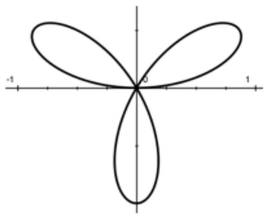
Example 5.14 Represent the point with Cartesian coordinates $(-1,1)$ in terms of polar coordinates.

Example 5.15 The cartesian coordinates of a point are $(-2\sqrt{3},-2)$. Find polar coordinates (r,θ) of this point, where $r>0$ and $0\le\theta<2\pi$).

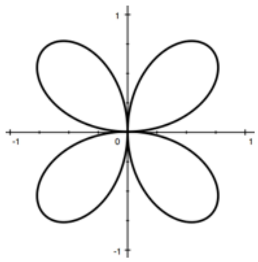
Example 5.16 Sketch the region in the plane consisting of points whose polar coordinates satisfy:

$$1\le r<2,\quad \frac{\pi}{4}\le\theta<\frac{3\pi}{4}.$$

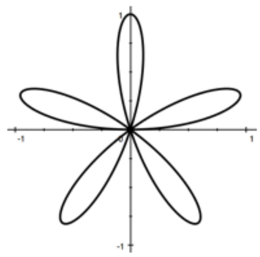
Polar Curves: The graph of a polar equation $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.
Some examples of polar curves are:



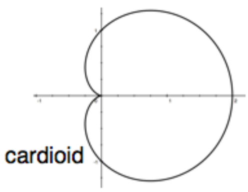
$r = \sin(3\theta)$



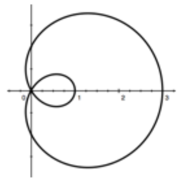
$r = \sin(2\theta)$



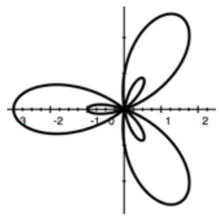
$r = \sin(5\theta)$



$r = 1 + \cos(\theta)$



$r = 1 + 2\cos(\theta)$



$r = 1 - 2\cos(3\theta)$

Example 5.17 What curve is represented by the polar equation $r = 3$?

Example 5.18 Sketch the graph of the curve $r = 2\sin\theta$.

Example 5.19 Sketch the graph of the curve $r = 2\cos 3\theta$.

Theorem 5.2.1 — Derivatives of Polar Curves. Suppose that $r = f(\theta)$ is a differentiable function of θ. Then from the parametric equations

$$x = r\cos\theta \quad y = r\sin\theta$$

it follows that

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Example 5.20 Find an equation of the tangent line to the curve $r = 2\cos 3\theta$ when $\theta = 2\pi/3$.

Example 5.21 Find an equation of the tangent line to the curve $r = 1 + \cos\theta$ if $\theta = \pi/6$.

Example 5.22 Consider the curve given by the polar equation

$$r = 1 + 2\sin\theta.$$

Find the point(s) on the curve which are furthest from the x-axis.

Exercise 5.5 Find a polar equation for the curve represented by the equation: $xy = 4$. ■

Exercise 5.6 Identify the curve by finding a Cartesian equation for the curve: $r = \tan\theta \sec\theta$. ■

Exercise 5.7 Find the points on the given curve where tangent lines are horizontal or vertical: $r = 3\cos\theta$. ■

5.3 Summary

In this chapter we introduced parametric curves, and how to do calculus on them. We also introduced a new coordinate system: the **polar coordinate** system. We looked at curves described by polar coordinates, and how to do calculus on them.

Vocabulary

- parametric curve, parametrization, parametric equations, parameter,
- initial point, terminal point,
- cycloid,
- cartesian equation vs parametric equation vs polar equation,
- derivative of parametric curve,
- pole, polar axis, polar coordinates,
- polar curve
- derivative of polar curves,

Skills to acquire

- graph a curve defined by parametric equations,
- identify points on a curve defined by parametric equations,
- compute the derivative of a curve defined by parametric equations,
- convert a parametrized curve to a cartesian equation, and visa-versa,
- graph a curve defined by a polar equation,
- identify points on a curve defined by a polar equation,
- compute the derivative of a curve defined by a polar equation,
- convert a polar curve to a cartesian equation, and visa-versa,
- use a graphing utility to plot curves defined by parametric equations or a polar equation.

IV Exam Preparation

6. Review Materials for Exam Preparation

For midterm review materials see Exam Resources in Canvas. There you will find a variety of review material including sample questions and practice exams.

In this chapter we include a summary of topics covered in the course, a concepts checklist for preparing for the final exam. You are encouraged to work through this checklist as the we progress through the term, rather than leave it for the week before the final exam.

6.1 End of Term Review Notes

1. **Special Limits:**
$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}(1+x)^{1/x}=e \qquad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$
2. **Definition of Derivative:** The derivative of a function f at a number a, denote by $f'(a)$, is
$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
if this limit exists.
3. **Differentiation Rules: (Chapter 3)**
   (a) **General Formulas:**

   $\frac{d}{dx}(c)=0$ (constant rule)

   $\frac{d}{dx}[x^n]=nx^{n-1}$ (power rule)

   $\frac{d}{dx}[cf(x)]=cf'(x)$ (scalar multiplication rule)

   $\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$ (sum and difference rule)

   $\frac{d}{dx}[f(x)g(x)]=f(x)g'(x)+f'(x)g(x)$ (product rule)

   $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$ (quotient rule)

   $\frac{d}{dx}[f(g(x))]=f'(g(x))g'(x)$ (chain rule)

   (b) **Exponential and Logarithmic Functions:**

   $\frac{d}{dx}(e^x)=e^x \qquad \frac{d}{dx}(a^x)=a^x\ln a$

   $\frac{d}{dx}(\ln|x|)=\frac{1}{x} \qquad \frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}$

   (c) **Trigonometric functions:**

   $\frac{d}{dx}(\sin x)=\cos x \qquad \frac{d}{dx}(\cos x)=-\sin x \qquad \frac{d}{dx}(\tan x)=\sec^2 x$

   (d) **Inverse Trigonometric Functions:**

   $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$

4. **Natural Growth Equation:** (Lecture 3.8) The solution of the initial-value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

is

$$y(t) = y_0 e^{kt} \ .$$

Radioactive Decay problems: Usually k is specified in terms of the **half-life** of the isotope

$$\tau = \frac{\ln 2}{k}.$$

This is the time required for half of any given quantity to decay.
Newton's Law of Cooling/Heating problems: The temperature T of an object is modeled by:

$$\frac{dT}{dt} = k(T - M) \quad \longrightarrow \quad T(t) = Ae^{kt} + M$$

where
   - M is the temperature of the surroundings (ambient temperature - which is constant)
   - k a constant (called the *heating/cooling constant*)

5. **Linear Approximation and Differentials:** (Lecture 3.10) The linear function

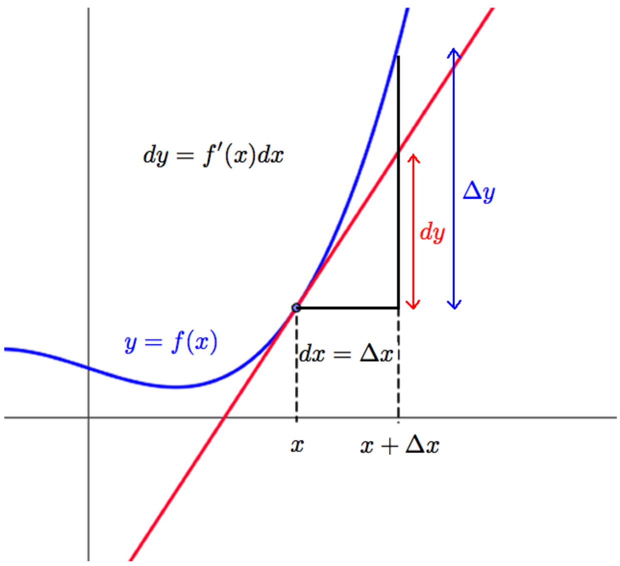
$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.
For x *close* to a we have that

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

and this approximation is called the **linear approximation** of f at a.
The **differential** dy is defined as

$$dy = f'(x)\Delta x = f'(x)dx \ .$$



6. **L'Hospital's Rule:** (Lecture 4.4) Suppose that f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a.) Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Exercise 6.1 Compute the following limit:

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x}$$

7. **Newton's Method for approximating solutions to $f(x) = 0$:** (Lecture 4.7)
   (i) Begin with an **initial guess** x_1.
   (ii) Calculate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

   (iii) If x_n is known then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   (iv) If x_n and $x_n + 1$ agree to k decimal places then x_n approximates the root r up to k decimal places and $f(x_n) \approx 0$.

Exercise 6.2 (a) Show that the equation $e^x = 5x$ has exactly two solutions.
(b) Use Newton's Method to find the two solutions to the equation in (a) to three decimal places.

8. **Increasing/Decreasing Test.**
   (a) If $f'(x) > 0$ on an interval, then f is increasing $\nearrow$ on that interval.
   (b) If $f'(x) < 0$ on an interval, then f is decreasing $\searrow$ on that interval.
9. **Definition.** A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.
10. **The First Derivative Test.** Suppose that c is a critical number of a continuous function f.
    (a) If f' changes from positive to negative at c, then f has a local maximum at c.
    (b) If f' changes from negative to positive at c, then f has a local minimum at c.
    (c) If f' does not change sign at c, then f has no local minimum or maximum at c.
11. **Concavity Test.**
    (a) If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward $\smile$ on I.
    (b) If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward $\frown$ on I.
12. **Definition.** A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward $\smile$ to concave downward $\frown$ or from concave downward $\frown$ to concave upward $\smile$ at P.

13. **The Second Derivative Test.** Suppose f" is continuous near c.
    (a) If $f'(c) = 0$ and f"$(c) > 0$ then f has a local minimum at c.
    (b) If $f'(c) = 0$ and f"$(c) < 0$ then f has a local maximum at c.
14. **The Mean Value Theorem.** (Lecture 4.2)) Let f be a function that satisfies the following hypotheses:
    (a) f is continuous on the closed interval $[a,b]$.
    (b) f is differentiable on the open interval (a,b).
    Then there is a number c in (a,b) such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
    or, equivalently,
$$f(b) - f(a) = f'(c)(b - a).$$
15. **Closed Interval Method for finding Absolute Extrema:** To find the **absolute** maximum and minimum values of a continuous function f on a closed interval $[a,b]$:
    (a) Find the values of f at the critical numbers of f in (a,b).
    (b) Find the values of f at the endpoints of the interval.
    (c) The largest of the values from Step 1 and Step 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Exercise 6.3 Find the absolute maximum and absolute minimum values of $f(x) = e^{-x} - e^{-2x}$ on the interval $[0,1]$. ■

16. **Derivatives of Parametric Curves:** (Lecture 5.1) The **derivative** to the parametric curve $x = f(t)$, $y = g(t)$ is given by
$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{g'(t)}{f'(t)}.$$
    The **second derivative** is given by
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3}.$$
17. **Derivative of Polar Curves:** (Lecture 5.2) Suppose that $r = f(\theta)$ is a differentiable function of θ. Then from the parametric equations
$$x = r\cos\theta \quad y = r\sin\theta$$
    it follows that
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Exercise 6.4 **True or False.** Justify your answers.
(a) If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$.

(b) If $f'(c) = 0$ then f has a local max/min at c.

(c) If $f'(x) < 0$ for $1 < x < 6$, then f is decreasing on $(1,6)$.

(d) If $f''(2) = 0$ then $(2, f(2))$ is an inflection point of f.

(e) If $f'(x) = g'(x)$ for $0 < x < 1$, then $f(x) = g(x)$ for $0 < x < 1$.

(f) There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

(g) If f is periodic then f' is periodic.

(h) If f is even then f' is even.

(i) If the parametric curve $x = f(t)$, $y = g(t)$ satisfies $g'(1) = 0$, then it has a horizontal tangent line when $t = 1$.

(j) The equations

$$r = 2,$$

$$x^2 + y^2 = 4$$

and

$$x = 2\sin(3t), \quad y = 2\cos(3t) \quad (0 \le t \le \frac{2\pi}{3})$$

all have the same graph.

End of Term Review Notes - Answers to Exercises:
6.1. limit is 0
6.2. (a) Consider $f(x) = e^x - 5x$ and notice $f(x) \to \infty$ as $x \to \pm\infty$. Also notice, $f'x) = 0$ at exactly one x-value. Now $f(0) > 0$, $f(1) < 0$, and $f(3) > 0$ so by the IVT f has at least two zeros.If f has more than two zeros then by the MVT $f'(x)$ must have at least two zeros, but this is not the case as noted above. Therefore, f has exactly two zeros.
(b) 0.25917 and 2.54264
6.3. max of $1/4$ at $= \ln 2$, and min of 0 at $x = 0$
6.4. (a) T (b) F (c) T (d) F (e) F (f) T (g) T (h) F (i) F (j) T

6.2 Final Exam Checklist

(Page numbers refer to Stewart's *Calculus* [Ste22].)

Definitions:

Chapter 2:

1. ☐ Limit (page 83)
2. ☐ Left-hand (right-hand) limit (page 86)
3. ☐ Infinite limit (pages 89 and 90)
4. ☐ Vertical asymptote (page 90)
5. ☐ Function continuous at a number (page 115)
6. ☐ Function continuous from the right (left) at a number (page 117)
7. ☐ Function continuous on an interval(page 117)
8. ☐ Limit at infinity (page 127 and 128)
9. ☐ Horizontal asymptote (page 128)
10. ☐ Tangent line (page 141)
11. ☐ The derivative of a function at a number (page 144)
12. ☐ Function differentiable on a set (page 156)
13. ☐ Vertical tangent line (page 159)

Chapter 3:

14. ☐ The number e (page 180)
15. ☐ Linear approximation (page 254)
16. ☐ Differential (page 256)

Chapter 4:

17. ☐ Absolute maximum/minimum (page 280)
18. ☐ Local maximum/minimum (page 280)
19. ☐ Critical number (page 284)
20. ☐ Function concave upward/downward (page 300)
21. ☐ Inflection point (page 301)

Chapter 10:

22. ☐ Parameter, parametric equations, parametric curve (page 662)
23. ☐ Initial and terminal points (page 663)
24. ☐ The Cycloid (page 666)
25. ☐ Polar coordinate system, pole, polar axis, polar coordinates (page 685)

Theorems/ Formulas/ Procedures:

Chapter 2:

1. ☐ $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a+} f(x) = L$ (page 87)
2. ☐ Limit Laws (Page 95-102)
3. ☐ Direct Substitute Property (page 97)
4. ☐ Theorem about the monotonicity and limits (page 101)
5. ☐ The Squeeze Theorem (page 101)
6. ☐ Continuity and combinations of functions (page 118)
7. ☐ Continuity of polynomials and rational functions (page 118)
8. ☐ Continuity and the combination of functions (page 120)
9. ☐ Continuity and the composition of functions (page 120, 121)
10. ☐ The Intermediate Value Theorem (page 122)
11. ☐ Limits at infinity of the power functions with negative rational exponents (page 130)
12. ☐ Relationship between differentiable and continuous functions (page 157)

Chapter 3:

13. ☐ Derivative of a constant function (page 174)
14. ☐ Power rule (pages 174 - 176)
15. ☐ The Constant Multiple Rule (page 177)
16. ☐ The Sum Rule (page 178)
17. ☐ The Difference Rule (page 178)
18. ☐ Derivative of the natural exponential function (page 180)
19. ☐ The Product Rule (page 185)
20. ☐ The Quotient Rule (page 186)
21. ☐ Derivatives of Trigonometric Functions (pages 193 and 194)
22. ☐ Special Trig Limits (pages 195 and 196)
23. ☐ The Chain Rule (page 200)
24. ☐ Derivative the exponential function $f(x) = b^x$ (page 204)
25. ☐ Derivatives of Logarithmic Functions (page 217 - 220)
26. ☐ Derivatives of Inverse Trigonometric Functions (page 223)
27. ☐ e as a limit (page 222)
28. ☐ The solution of the initial-value problem (page 239 and 240)

Chapter 4:

29. ☐ The Extreme Value Theorem (page 281)
30. ☐ Fermat's Theorem (page 282)
31. ☐ The Closed Interval Method (page 284)
32. ☐ Rolle's Theorem (page 290)
33. ☐ The Mean Value Theorem (page 291)
34. ☐ The relationship between two functions implied by the equality of their derivatives (page 294)
35. ☐ Increasing/Decreasing test (page 297)
36. ☐ The First Derivative Test (page 298)
37. ☐ Concavity Test (page 300)
38. ☐ The Second Derivative Test (page 302)
39. ☐ L'Hospital's Rule (page 310)
40. ☐ Newton's Method (page 352)

Chapter 10:

41. ☐ Derivative of parametric curve (page 673)
42. ☐ Second Derivative of parametric curve (page 674)
43. ☐ Derivative of a polar curve (page 698)

Examples:

Chapter 2:

1. ☐ Function with neither left-hand nor right-hand limits at the given point (Example 5 page 88)
2. ☐ Function with the left-hand and right hand limits that are not equal (Example 7 page 91)
3. ☐ Function with infinite left-hand and right hand limits (Examples 7,8 page 91)
4. ☐ Function with an infinite number of vertical asymptotes (Example 8 page 91)
5. ☐ Function $F = f \cdot g$ so that the limits of F and f at a exist and the limit of g at a does not exist. (Example 11 page 102)
6. ☐ Function $F = f + g$ so that the limit of F at a exists and the limits of f and g at a do not exist. (discover your own example)
7. ☐ Function $F = fg$ so that the limit of F at a exists and the limits of f and g at a do not exist. (discover your own example)
8. ☐ Function with a removable (infinite, jump) discontinuity. (Example 2 page 116)
9. ☐ Function with a graph that intersects its horizontal asymptote (page 128)
10. ☐ Function with two horizontal asymptotes (page 129)
11. ☐ Function that is continuous but not differentiable (Example 5 page 157)
12. ☐ Function with a vertical tangent line (page 159)
13. ☐ Function with a “corner" (page 159)

Chapter 3:

14. ☐ Function that is the same as its derivative (page 180)
    [Can you find THREE different functions with this property?]

Chapter 4:

15. ☐ Function f with no minimum, no maximum but such that $f'(a) = 0$ for some a. (Example 5 page 281)
16. ☐ Function with a local minimum that is not the global minimum. (Example 1 page 277)
17. ☐ Function(s) that does not satisfied the hypothesis of the Extreme Value Theorem. (page 282)
18. ☐ Function with a critical number but no maximum or minimum. (Example 5 page 281)
19. ☐ Sketch the graph of a function that is increasing for $x < 0$, decreasing for $0 < x < 1$, and increasing for $x > 1$. Also, suppose this function has a horizontal asymptote when $x \to -\infty$ and a slant asymptote when $x \to \infty$. (discover your own)
20. ☐ Function that is concave upward (concave downward) (page 300)
21. ☐ Function with an inflection point at which the first derivative equals 0. (discover your own)
22. ☐ Function with a local minimum at which the second derivative equals 0. (discover your own)

Limits:

Are you able to do most of the following:

1. ☐ Exercises 2.3: 1-34, 39-54
2. ☐ Exercises 2.6: 1-65
3. ☐ Exercises 3.3: 45-60
4. ☐ Exercises 4.4: 1-70

Differentiation:

Are you able to do most of the following:

1. ☐ Exercises 2.7: 5-48
2. ☐ Exercises 2.8: 21-32, 41-44, 47-52, 56-63
3. ☐ Exercises 3.2: 3-36, 43-52
4. ☐ Exercises 3.3: 1-34, 45-60
5. ☐ Exercises 3.4: 1-73
6. ☐ Exercises 3.5: 1-42, 48-52
7. ☐ Exercises 3.6: 1-56

Applications:

1. ☐ Tangent lines (throughout the text)
2. ☐ Velocity (throughout text)
3. ☐ The Intermediate Value Theorem (Section 2.5: Exercises 52-58)
4. ☐ Rates of change (Section 2.7: Exercises 49-56, Section 3.7: Exercises 5-28)
5. ☐ Exponential growth and decay (Section 3.8: all exercises)
6. ☐ Related rates (Section 3.9: all exercises)
7. ☐ Linear Approximation and Differentials (Section 3.10: Exercises 11-22, 41-48)
8. ☐ Maximum and minimum values (Section 4.1: Exercises 51-66, 73-77)
9. ☐ Mean Value Theorem (Section 4.2: Exercises 21-37)
10. ☐ Graphs of functions (Section 4.3 and Section 4.5)
11. ☐ Optimization (Section 4.7)
12. ☐ Newton's method (Section 4.8: Exercises 5-8, 11-22)

Parametric Curves:

1. ☐ Section 10.1: Exercises 1-30
2. ☐ Section 10.2: Exercises 1-34
3. ☐ Section 10.3: Exercises 15-50
4. ☐ Section 10.4: Exercises 63-72

6.3 Final Exam Practice Questions

Please refer to the final exam checklist (Section 6.2) for an indication of what you will need to know for the final exam (in short, the material from any section we covered this term will be on the exam, with the exception of Section 2.4).

To prepare for the exam you should: (i) read all sections of the textbook again; (ii) go through all the homework questions again and make sure you can do every single one of them, (iii) work through more problems from the textbook; (iv) work through the final exam checklist (this can be done in conjunction with (i) through (iii)). Only once you have done all this should you attempt the following questions.

The following is a list of practice exam questions. This will give you an idea of the types of questions you will be asked on the exam.

Instructions: No calculators, books, papers, or electronic devices shall be allowed within the reach of a student during the examination. Leave answers in "calculator ready" expressions: such as $3+\ln 7$ or $e^{\sqrt{2}}$.

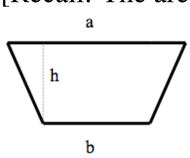
Questions 1-3 are on the statements of **definitions and theorems** and on your ability to give **examples** of functions with specified properties.

1. Define the following terms.
   (a) Limit
   (b) Function continuous at a number
   (c) Function continuous on an interval
   (d) Tangent line
   (e) The derivative of a function at a number
   (f) Function differentiable a on a set
   (g) The number e
   (h) Differential
   (i) Absolute maximum and absolute minimum
   (j) Local maximum and local minimus
   (k) Critical number
   (l) Function concave upward and concave downward
   (m) Inflection point
2. State the following theorems.
   (a) The Squeeze Theorem
   (b) The Intermediate Value Theorem
   (c) Fermat's Theorem
   (d) Extreme Value Theorem
   (e) Rolle's Theorem
   (f) The Mean Value Theorem
   (g) L'Hospital's Rule
3. Give an example for each of the following.
   (a) Function with an infinite number of vertical asymptotes.
   (b) Function $F = f \cdot g$ so that the limits of F and f at a exist and the limit of g at a does not exist.
   (c) Function with a removable discontinuity.
   (d) The most general form of a function with the property that its second derivative is the zero function.
   (e) Function that is continuous but not differentiable at a point.
   (f) Function with a critical number but no maximum or minimum.
   (g) Function with a local minimum at which the second derivative equals 0.

Questions 4-16 are **short answer** questions. The questions are given in no particular order.

4. Find the derivative $y' = \dfrac{dy}{dx}$ of each of the following:
   (a) $y = \cos^{-1}(x^2) - \ln(1+x^3)$ [Note: Another notation for $\cos^{-1}$ is arccos.]
   (b) $y = x^{\sin(x)}$.
   (c) $\arctan\left(\dfrac{y}{x}\right) = \frac{1}{2}\ln(x^2+y^2)$.
5. Let $f(x) = \tan x$. Find $f''(x)$, the second derivative of f.
6. Find the tangent line to the curve $y + x\ln y - 2x = 0$ at the point $(1/2, 1)$.
7. Evaluate $\displaystyle\lim_{x\to\infty} \frac{2x^3+3x-1}{1-2x^2+5x^3}$.
8. Evaluate $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2}$.
9. Let $f(x) = \dfrac{2x}{x^2+3}$.
   (a) Find the equation of the tangent line to the curve $y = f(x)$ at $x = 1$.
   (b) Use linear approximation to give an approximate value for $f(1.2)$.
10. A particle moves along the x-axis so that its position at time t is given by $x = t^3 - 4t^2 + 1$.
    (a) At $t = 2$, what is the particle's speed?
    (b) At $t = 2$, in what direction is the particle moving?
    (c) At $t = 2$, is the particle's speed increasing or decreasing?
11. The curve $y = x^4 e^{-x}$ has two inflection points. Find the x-coordinate of both points.
12. Find an equation of the slant asymptote to the curve $y = \dfrac{x^3+2x^2}{x^2+3x+2}$.
13. Find a number x_0 between 0 and π such that the tangent line to the curve $y = \sin x$ at $x = x_0$ is parallel to the line $y = -x/2$.
14. Evaluate $\displaystyle\lim_{x\to 0} x\sin\frac{1}{x}$.
15. Evaluate $\displaystyle\lim_{x\to 0} \frac{e^x-1-x}{x^2}$.
16. Evaluate $\displaystyle\lim_{x\to 0} x^{\sin x}$.

    Questions 17-33 are **full-solution** problems. Justify your answers and show all your work. The questions are given in no particular order.
17. Let $f(x) = \dfrac{5}{3x-1}$. Calculate $f'(2)$ directly from the *definition* of derivative.
18. A water-trough is 10m long and has a cross-section which is the shape of an isosceles trapezoid that is 30cm wide at the bottom, 80cm wide at the top, and has height 50cm. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30cm deep?
    [Recall: The area of an isosceles trapezoid as shown in the diagram is $A = \frac{1}{2}(a+b)h$.]



19. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05cm thick to a hemispherical dome with diameter 50m.
20. At 2:00 p.m.a car's speedometer reads 30 mi/h. At 2:10 p.m. it reads 50 mi/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 120 mi/h^2.
21. A piece of wire 10m long is cut into two pieces. One piece is bent into a square and the other is bent into an circle. How should the wire be cut so that the total area enclosed is minimum.
22. A turkey is put into an oven that has a constant temperature of 200°C. A thermometer

embedded in the turkey registers its temperature. When the turkey is put into the oven, the thermometer reads 20°C, and 30 minutes later it reads 30°C. The turkey will be ready to eat when the thermometer reads 80°C. How many minutes after being put into the oven will the turkey be ready to eat? Assume that the turkey's temperature satisfies Newton's law of cooling/heating.

23. Sketch the graph of the function
$$f(x) = \frac{-2x^2+5x-1}{2x-1}.$$

24. Let $f(x) = 2x^3 - 6x^2 + 3x + 1$.
    (a) First show that f has at least one zero in the interval $[2,3]$ and then use the first derivative of f to show that there is exactly one root of f between 2 and 3.
    (b) Use Newton's method to approximate the root of f in the interval $[2,3]$ by starting with $x_1 = 5/2$ and finding x_2.

25. Find the dimensions of the largest rectangle that can be inscribed inside a semicircular region of radius 5 such that one side of the rectangle is parallel to the base of the semicircular region.

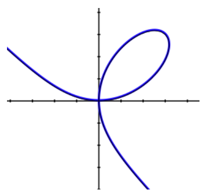
26. (a) A metal storage tank with fixed volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere, without a base. What dimensions will require the least amount of metal?
    (b) Suppose the metal for the hemisphere costs twice as much as the metal for the lateral sides. What are the dimensions for the tank that minimizes cost?
    (Recall: The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and the surface area is $4\pi r^2$.)

27. (a) Show that Newton's Method applied to the equation $x^2 - a = 0$ yields the iterative formula
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$
    and thus provides a method for approximating the square root $\sqrt{a}$ which uses only addition and multiplication.
    (b) Approximate $\sqrt{3}$ by taking $x_1 = 3/2$ and calculating x_2.

28. Find f if $f''(x) = 2 + \cos x$, $f'(0) = -1$ and $f(\pi/2) = 0$.

29. Sketch the curve which is given by the parametric equations
$$x = \cos(\pi t), \quad y = \sin(\pi t), \quad 1 \le t \le 2.$$
    Clearly label the initial and terminal points and describe the motion of the point $(x(t), y(t))$ as t varies in the given interval (i.e. indicate the direction the point is traveling).

30. A curve called the **folium of Descartes** is defined by the parametric equations
$$x = \frac{3t^2(t+1)}{3t^2+3t+1}, \quad y = \frac{-3t(t+1)^2}{3t^2+3t+1}, \quad -\infty < t < \infty.$$



    (a) Show that a Cartesian equation of this curve is $x^3 + y^3 = 3xy$.
    (b) Find the point on the curve corresponding to $t = -1/2$.
    (c) Find the equation of the tangent line to the curve at the point corresponding to $t = -1/2$.
    (d) Find the values of the parameter t which correspond to the point $(0,0)$ on the curve.

(e) Find equations of the tangent lines to the curve at the point $(0,0)$.

31. Consider the curve given by the parametric equations

$$x = 2\sin t, \ \ y = 4 + \cos t, \ \ 0 \leq t \leq 2\pi.$$

Determine the points on the curve which are closest to the origin and those which are furthest away.

32. Sketch the curve with the polar equation $r = 2\cos 4\theta$.

33. Consider the curve given by the polar equation

$$r = 1 + 2\sin(3\theta), \ \ 0 \leq \theta \leq 2\pi.$$

(a) Determine the points on the curve which are furthest from the origin.

(b) Find the slope of the tangent line to each of the points found in part (a).

Final Exam Practice Questions - Answers:

4. (a) $y' = -\dfrac{2x}{\sqrt{1-x^4}} - \dfrac{3x^2}{1+x^3}$ (b) $y' = x^{\sin x}(\frac{\sin x}{x} + \ln x \cos x)$ (c) $y' = \dfrac{x+y}{x-y}$
5. $f''(x) = 2\sec^2(x)\tan(x)$
6. $y = \frac{4}{3}x + \frac{1}{3}$
7. $2/5$
8. -2
9. (a) $y = \frac{1}{4}x + \frac{1}{4}$ (b) $f(1.2) \approx \frac{1}{4}(1.2) + \frac{1}{4} = \frac{11}{20} = 0.55$
10. (a) -4 (b) to the left (c) speed is decreasing
11. $x = 2$ and $x = 6$
12. $y = x - 1$
13. $2\pi/3$
14. 0 (Hint: use Squeeze Theorem since $\sin(1/x)$ is bounded)
15. $1/2$
16. 1
17. $-3/5$
18. $\frac{1}{30}$ m/min = $\frac{10}{3}$ cm/min
19. $\frac{5}{8}\pi \approx 2$ m^3
20. Hint: use Mean Value Theorem
21. The length of wire used to make the square should be $\frac{40}{\pi+4}$
22. $t = \frac{\ln(2/3)}{2\ln(17/18)} \approx 3.55$ hours $\approx$ 3 hours 33 minutes
24. (b) $x_2 = 16/7 \approx 2.285714286$
25. base is $4\sqrt{5}$ and height is $\sqrt{5}$
26. (a) hemisphere (b) $r = \frac{1}{2}\sqrt[3]{3V/\pi}$ and $h = \sqrt[3]{3V/\pi}$.
27. (b) $\frac{7}{4}$
28. $f(x) = -\cos x + x^2 - x + \pi/2(1-\pi/2)$
29. Consists of points on the unit semicircle in quadrants 3 and 4. Points are moving counterclockwise along curve with initial point $(-1,0)$ and terminal point $(1,0)$.
30. (b) $(3/2, 3/2)$ (c) $y = -x + 3$ (d) $t = -1, 0$ (e) There are two lines: the first is when $t = -1$ and the tangent is the horizontal line $y = 0$, the second occurs when $t = 0$ and is the vertical line $x = 0$.
31. closest point is $(0,3)$, furthest point is $(0,5)$
33. (a) The three points are $(r,\theta) = (3, \frac{\pi}{6}), (3, \frac{5\pi}{6}), (3, \frac{3\pi}{2})$. (b) slopes of tangents at these points are $-\sqrt{3}$, $\sqrt{3}$, and 0, respectively.

V

Appendix

Solutions to Exercises

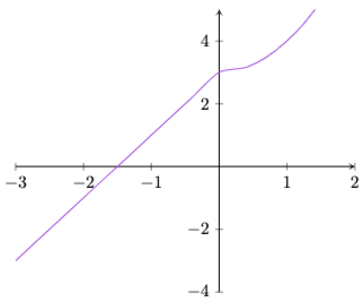
1.1 Review of Functions

1.1 One possible formula is

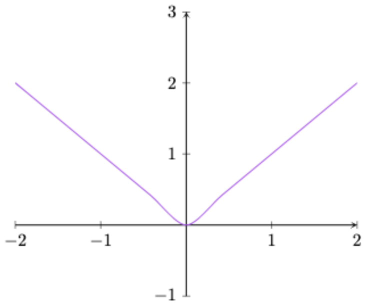
$$f(x) = \begin{cases} 2 & \text{if } x < 0 \\ |x-1|-1 & \text{if } 0 \leq x \leq 3 \\ 1 & \text{if } x > 3 \end{cases}$$

An expression that involves two lines, instead of the absolute value, is also possible.

1.2



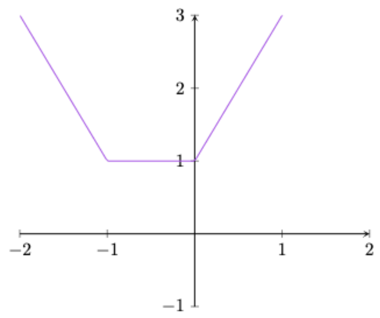
1.3 (a) $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$ and the graph is



(b) No, $g(x) \neq x$ for all x. In particular, if $x < 0$ then $g(x) = -x$. For example, think about what $g(-2)$ would be.

(c) $\sqrt{x^2} = |x|$ for all x.

1.4



1.2 Catalog of Essential Functions

1.5 $p(x) = a_0$ where a_0 is a real number.

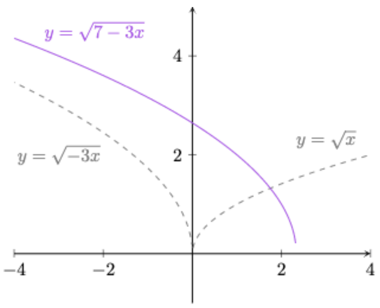
1.6 $p(x) = a_1x + a_0$ where a_0, a_1 are real numbers.

1.7 $p(x) = a_2x^2 + a_1x + a_0$ where a_0, a_1, a_2 are real numbers.

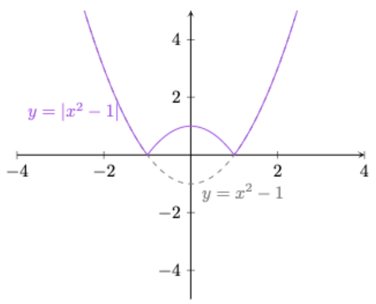
1.8 $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ where a_0, a_1, a_2, a_3 are real numbers.

1.3 New Functions from Old Functions

1.9 First horizontal compress by a factor of 3, and then flip about the y-axis, to obtain $y = \sqrt{-3x}$. Then horizontal shift to the right $7/3$ units (think about replacing x by $x - 7/3$ in the expression $\sqrt{-3x}$).

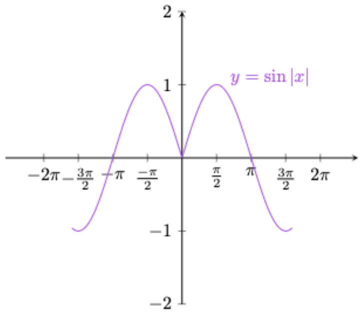


1.10 Any part of the graph of $f(x)$ that is below the x-axis is flipped over the x-axis when plotting $|f(x)|$.



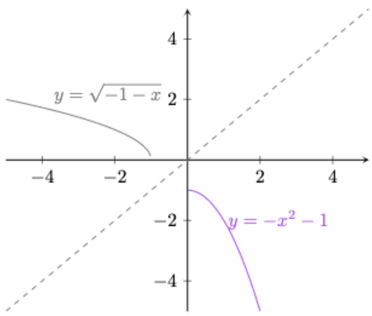
1.11 $(f \circ f)(x) = \frac{\left(\frac{x+1}{x+2}\right)+1}{\left(\frac{x+1}{x+2}\right)+2} = \frac{(2x+3)(x+2)}{(x+2)(3x+5)} = \frac{2x+3}{3x+5}$. The domain consists of all real numbers x except $x = -2, -5/3$. This is because, $x = -2$ is not in the domain of f, and $f(-5/3) = -2$, therefore neither can be in the domain of $f \circ f$.

1.12



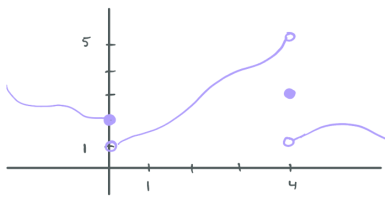
1.13 (a) $(f\circ g)(1) = f(g(1)) = f(4) = 2$
(b) $(g\circ g)(-2) = g(g(-2)) = g(1) = 4$
(c) $(g\circ f)(6) = g(f(6)) = g(6)$ but $g(6)$ is not defined, therefore $(g\circ f)(6)$ is not defined either.

1.4 Exponential Functions & Inverse Function and Logarithms
1.14 $e^{x^3-3} - 9 = 0 \quad \Rightarrow \quad x^3 - 3 = \ln 9 \quad \Rightarrow \quad x = \sqrt[3]{3+\ln 9}$.
1.15 $f^{-1}(x) = \frac{3x}{1-4x}$.
1.16 $f^{-1}(-9) = 8$, $f^{-1}(5) = 3$, $f^{-1}(4) = 1$,
1.17 $f^{-1}(x) = \sqrt[3]{\frac{3-x}{2}}$
1.18



1.19 $\ln\left(\frac{a^{1/3}}{b^{1/4}}\right)$

2.2 Limit of a Function
2.2 (a) 3 (b) 1 (c) left and right limits don't match, so two-sided limit does not exist. (d) 2 (e) 2 (f) 2
2.3 (a) 0 (b) -6 (c) -2 (d) left and right limits don't match, so two-sided limit does not exist. (e) ∞ (f) $-\infty$ (g) does not exist.
2.4



2.3 Calculating Limits Using Limit Laws
2.5 (a)$1+5(-1) = -4$ (b) does not exist, since the limit of g at 1 does not exist. (c) does not exist, since the limit of g at 2 is 0 but the limit of f is not 0. We have $\lim_{x\to 2^-} \frac{f(x)}{g(x)} = -\infty$ and $\lim_{x\to 2^+} \frac{f(x)}{g(x)} = \infty$, but the two-sided limit does not exist.
2.6 $\frac{2+3-4}{2-3} = -1$ (just plug the value into the function)
2.7 (a) $\frac{9-4}{2-3} = -5$ (b) $\lim_{x\to 2} \frac{(x-2)(x+2)}{2-x} = \lim_{x\to 2} -(x+2) = -4$.
2.8 $1+1 = 2$
2.9 Since $|x+3|$ is behaves differently on each side of -3, let's look at the one-sided limits first:

$$\lim_{x\to -3^-} \frac{-(x+3)}{(x+3)(x-2)} = \lim_{x\to -3} \frac{-1}{x-2} = 1/5,$$

$$\lim_{x\to -3^+} \frac{(x+3)}{(x+3)(x-2)} = \lim_{x\to -3} \frac{1}{x-2} = -1/5.$$

Since these are not equal, the two sided limit does not exist.
2.10 Since $\lim_{x\to 4^-} f(x) = 0$ and $\lim_{x\to 4^+} f(x) = 0$ then $\lim_{x\to 4} f(x) = 0$.
2.11 Since $-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$ then $-\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\sin\left(\frac{\pi}{x}\right) \le \sqrt{x^3+x^2}$. By the squeeze theorem

$$\lim_{x\to 0} -\sqrt{x^3+x^2} \le \lim_{x\to 0} \sqrt{x^3+x^2}\sin\left(\frac{\pi}{x}\right) \le \lim_{x\to 0} \sqrt{x^3+x^2}$$

$$0 \le \lim_{x\to 0} \sqrt{x^3+x^2}\sin\left(\frac{\pi}{x}\right) \le 0$$

Therefore,

$$\lim_{x\to 0} \sqrt{x^3+x^2}\sin\left(\frac{\pi}{x}\right) = 0$$

2.12 4 (otherwise the given limit couldn't be a finite number)
2.13 Since $\lim_{x\to 0} \frac{f(x)}{x^3} = \lim_{x\to 0} \frac{f(x)/x^2}{x}$ exists, and the denominator tends to 0, so must the numerator. Therefore, $\lim_{x\to 0} \frac{f(x)}{x^2} = 0$.
2.14 Consider $a = 0$, $f(x) = 1/x$ and $g(x) = -1/x$. Then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ both do not exist, but $\lim_{x\to 0} f(x) + g(x) = 0$.
2.15

(a) $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right) = \lim_{t\to 0} \frac{1-\sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t\to 0} \frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\left(\frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}\right) = \lim_{t\to 0} \frac{1-(1+t)}{t\sqrt{1+t}(1+\sqrt{1+t})} = \lim_{t\to 0} \frac{-1}{\sqrt{1+t}(1+\sqrt{1+t})} = -1/2$

(b) $\lim_{h\to 0} \frac{(2+h)^3 - 8}{h} = \lim_{h\to 0} \frac{(8+12h+6h^2+h^3)-8}{h} = \lim_{h\to 0} 12+6h+h^2 = 12$

(c) $\lim_{x\to 25} \frac{5-\sqrt{x}}{25x-x^2} = \lim_{x\to 25} \frac{5-\sqrt{x}}{x(25-x)}\left(\frac{5+\sqrt{x}}{5+\sqrt{x}}\right) = \lim_{x\to 25} \frac{25-x}{x(25-x)(5+\sqrt{x})} == \lim_{x\to 25} \frac{1}{x(5+\sqrt{x})} = 1/250$

(d) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{|x|}\right)$ does not exist, since $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{x}\right) = 0$ but $\lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x\to 0^-}\left(\frac{1}{x} + \frac{1}{x}\right) = -\infty$.

(e) $\lim_{t\to 1} \frac{t^4-1}{t^3-1} = \lim_{t\to 1} \frac{(t-1)(t^3+t^2+t+1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1} \frac{t^3+t^2+t+1}{t^2+t+1} = 4/3$

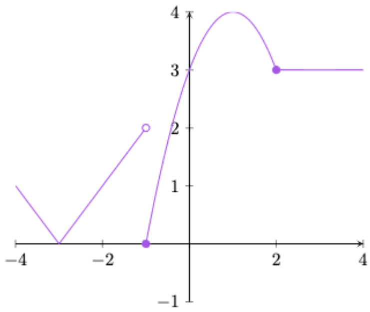
(f) $\lim_{x\to -6} \frac{2x+12}{|x+6|}$ does not exist, since $\lim_{x\to -6^+} \frac{2x+12}{|x+6|} = \lim_{x\to -6^+} \frac{2x+12}{x+6} = 2$ but $\lim_{x\to -6^-} \frac{2x+12}{|x+6|} = \lim_{x\to -6^-} \frac{2x+12}{-(x+6)} = -2$.

(g) $\lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h\to 0} \frac{3-(3+h)}{3h(3+h)} = \lim_{h\to 0} \frac{-1}{3(3+h)} = -1/9$

(h) Since $e^{-1} \le e^{\sin(\pi/x)} \le e$ then $\sqrt{x}e^{-1} \le \sqrt{x}e^{\sin(\pi/x)} \le \sqrt{x}e$ so by the squeeze theorem $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$.

2.5 Continuity

2.16 For any $a \in [-1,1]$ we have $\lim_{x\to a} f(x) = \lim_{x\to a} 1 - \sqrt{1-x^2} = 1 + \sqrt{1-a^2} = f(a)$, so f is continuous at $x = a$.
2.17 $\frac{e^x}{2+\cos x}$ is continuous on its domain, and π is in the domain, therefore, $\lim_{x\to\pi} \frac{e^x}{2+\cos x} = \frac{e^\pi}{2-1} = e^\pi$.
2.18 (a) $\mathbb{R}$ (b) $\{x \in \mathbb{R} : x \neq (2k+1)\pi$ for k an integer$\}$ (c) $\{x \in R : x > 0, x \neq 1\}$
2.19 Let $f(x) = 4x^3 - 6x^2 + 3x - 2$, then $f(1) = -1 < 0$ and $f(2) = 12 > 0$. Since f is continuous, by the IVT there is $c \in (1,2)$ such that $f(c) = 0$.
2.20 We want $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 + 2 = 2$. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x + c = c$, therefore $c = 2$.
2.21 There is a jump discontinuity at $x = -1$.



2.22 The function is continuous at all values of x except possible $x = \pi$. The one-sided limits are

$$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} a^3 - x^3 = a^3 - \pi^3$$

$$\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^+} a\sin x = a\sin\pi = 0$$

Therefore, f is continuous at π only when $a = \pi$.

2.6 Limits at Infinity

2.23

(a) $\lim_{x\to\infty} x^2 = \infty$

(b) $\lim_{x\to\infty} \frac{x^2+2x-1}{x^3+3} = 0$

(c) $\lim_{x\to\infty} \frac{x^4+5x^3-1}{x^2+x+1} = \infty$

(d) $\lim_{x\to\infty} e^x = \infty$

(e) $\lim_{x\to 0^-} e^{1/x} = 0$

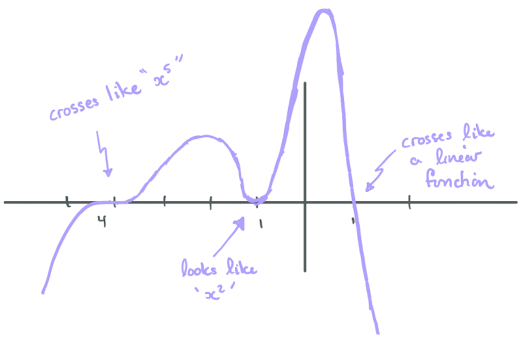
(f) $\lim_{x\to 4^+} \tan^{-1}\left(\frac{1}{4-x}\right) = \frac{-\pi}{2}$

(g) $\lim_{x\to\infty} \cos x$ does not exist

(h) $\lim_{x\to\infty} (x^2 - x) = \infty$

(i) $\lim_{x\to\infty} \frac{e^x}{x^2} = \infty$

2.14 Let $f(x) = (x+4)^5(x+1)^2(1-x)$, then we can work out a few values: $f(-5) = -96$, $f(-4) = 0$, $f(-3) = 16$, $f(-2) = 96$, $f(-1) = 0$, $f(0) = 1024$, $f(1) = 0$ and $f(2) = -69984$. The behaviour of how the graphs crosses the horizontal axis at the roots is given by the corresponding power:



2.8 Derivatives and Rates of Change

2.25 (a) $f(x) = \sqrt[4]{x}$, $a = 16$ (b) $f(x) = 2^x$, $a = 5$ (c) $f(x) = \tan x$, $a = \pi/4$

3.1 Derivatives of Polynomials and Exponential Functions

3.1 $y' = 4x^3 - 12x^2 + 8x = 4x(x-1)(x-2)$. The tangent line is horizontal when $y' = 0$ and this occurs at the points $(x,y) = (0,0),(1,1),(2,0)$.

3.2 The tangent line is $y = 2+2x$ and the normal line is $y = 2+(-1/2)x$.

3.3 The normal line is $y = (1/2)(x+1) = 1/2+(1/2)x$, and this intersects the parabola when $(1/2)+(1/2)x = x^2-1$. Therefore $2x^2 - x - 3 = 0$, or equivalently $(2x-3)(x+1) = 0$ and the solution are $x = -1, 3/2$. Hence, the normal intersects the parabola at $(3/2, 5/4)$.

3.4 The tangent line to $y = x^2$ at $x = a$ has equation $y = a^2 + 2a(x-a)$, which simplifies to $y = 2ax - a^2$. We want this tangent line to pass through the point $(0,-4)$, this means $-4 = 0 - a^2$, or $a = \pm 2$. Hence the two points on the parabola are $(\pm 2, 4)$.

3.5

n	$f^{(n)}(x)$
0	x^{-1}
1	$(-1)x^{-2}$
2	$(-1)(-2)x^{-3}$
3	$(-1)(-2)(-3)x^{-4}$
$\vdots$	$\vdots$
n	$(-1)^n(n!)x^{-n-1}$

3.6 A slope of 4 at $x=1$ means $2a+b=4$, a slope of -8 at $x=-1$ means $-2a+b=-8$. Solving these two equations for a and b we get $a=3$ and $b=-2$. If it passes through the point $(2,15)$ then $15=3(4)-2(2)+c$, and so $c=7$. Therefore, $y=3x^2-2x+7$.

3.7 $f'(x)=\begin{cases}2x & \text{if } x<1\\ 1 & \text{if } x>1\end{cases}$. Since $f'(x)\to 1$ as $x\to 1^+$, but $f'(x)\to 2$ as $x\to 1^-$, then f is not differentiable at $x=1$.

3.2 The Product and Quotient Rules

3.8 Simplify g first:

$$g(x)=\frac{3x^2+2\sqrt{x}}{x}=3x+2x^{-1/2} \quad \overset{diff}{\Longrightarrow} \quad g'(x)=3-x^{-3/2}$$

3.9

(a) $y(t)=(3t-e^t)\left(\frac{1}{\sqrt{t}}+\sqrt{t}\right) \quad \underset{rule}{\overset{prod}{\Longrightarrow}} \quad y'(t)=(3-e^t)\left(\frac{1}{\sqrt{t}}+\sqrt{t}\right)+(3t-e^t)\left(\frac{-1}{2}t^{-3/2}+\frac{1}{2}t^{-1/2}\right)$

(b) $f(x)=\dfrac{x^3-1}{x^2+7x+3} \quad \underset{rule}{\overset{quo}{\Longrightarrow}} \quad f'(x)=\dfrac{(3x^2)(x^2+7x+3)-(x^3-1)(2x+7)}{(x^2+7x+3)^2}$

(c) $y(t)=\dfrac{2}{at}=(2/a)t^{-1} \quad \overset{diff}{\Longrightarrow} \quad y'(t)=-(2/a)t^{-2}$

(d) $u(s)=\dfrac{s+\sqrt{s}}{s^{1/4}}=s^{3/4}+s^{1/4} \quad \overset{diff}{\Longrightarrow} \quad u'(s)=(3/4)s^{-1/4}+(1/4)s^{-3/4}$

3.10

$$y(t)=\frac{1+tf(t)}{\sqrt{t}} \quad \underset{rule}{\overset{quo}{\Longrightarrow}} \quad y'(t)=\frac{(\frac{d}{dt}(1+tf(t)))(\sqrt{t})-(1+tf(t))((1/2)t^{-1/2})}{t}$$
$$\underset{rule}{\overset{prod}{\Longrightarrow}} y'(t)=\frac{(f(t)+tf'(t))(\sqrt{t})-(1+tf(t))((1/2)t^{-1/2})}{t}$$
$$\overset{simplify}{\Longrightarrow} y'(t)=\frac{f(t)+tf'(t)}{\sqrt{t}}-\frac{1+tf(t)}{2t\sqrt{t}}$$

3.11 $y'=\dfrac{e^x(1+x^2)-e^x(2x)}{(1+x^2)^2}=\dfrac{e^x(x^2-2x+1)}{(1+x^2)^2}$. The slope of the tangent line at the point $(1,\frac{1}{2e})$ is 0, so the equation of the tangent line is $y=\frac{1}{2e}$.

3.12 By the product and chain rule: $P'(x)=F'(x)G(x)+F(x)G'(x)$ and $Q'(x)=\frac{F'(x)G(x)-F(x)G'(x)}{G^2(x)}$. From the diagram, $F'(2)=0$, $F'(4)=1/4$, $F'(5)=1/4$, $F'(7)=1/4$, $G'(2)=1/2$, $G'(4)$ does not exist, $G'(5)=-2/3$, $G'(7)=-2/3$. Therefore,

(a) $P'(2)=F'(2)G(2)+F(2)G'(2)=0+(3)(1/2)=3/2$

(b) $Q'(4)=\frac{F'(4)G(4)-F(4)G'(4)}{G^2(4)}$ does not exist

(c) $P'(5)=F'(5)G(5)+F(5)G'(5)\approx(1/4)(7/3)+(9/2)(-2/3)=-29/12$

(d) $Q'(7)=\frac{F'(7)G(7)-F(7)G'(7)}{G^2(7)}=\frac{(1/4)(1)-(5)(-2/3)}{1^2}=43/12$

3.3 Derivatives of Trigonometric Functions

3.13

n	$f^{(n)}(x)$
0	$\cos x$
1	$-\sin x$
2	$-\cos x$
3	$\sin x$
4	$\cos x$
$\vdots$	$\vdots$
24	$\cos x$
25	$-\sin x$
26	$-\cos x$
27	$\sin x$

3.14 $\lim_{x\to 0} \frac{\sin(7x)}{\sin(\pi x)} = \lim_{x\to 0} \frac{7}{\pi} \frac{\sin(7x)}{7x} \frac{\pi x}{\sin(\pi x)} = \frac{7}{\pi}$
3.15 $\lim_{x\to 0} x \cot x = \lim_{x\to 0} \frac{x}{\sin x} \cos x = 1$
3.16 Rewrite in terms of sine and cosine: $f'(x) = \frac{\cot x}{1+\csc x} = \frac{\cos x}{\sin x+1}$. Then by the quotient rule

$$f'(x) = \frac{-\sin x(\sin x+1) - \cos^2 x}{(\sin x+1)^2} = \frac{-(\sin^2 x + \cos^2 x + \sin x)}{(\sin x+1)^2} = \frac{-(1+\sin x)}{(\sin x+1)^2} = \frac{-1}{1+\sin x}$$

There is no horizontal tangent line since the derivative is never 0.

3.4 Chain Rule
3.17 $y' = \cos x + 2\sin x \cos x$, so $y'(0) = 1$, and the tangent line is $y = x$.
3.18 $f'(x) = -\cos(\cos(\tan(x))) \cdot \sin(\tan x) \cdot \sec^2(x)$

3.5 Implicit Differentiation
3.19 By the product rule $y' = \tan^{-1}(\sqrt{x}) + \frac{\sqrt{x}}{2(1+x)}$.
3.20 By implicit differentiation we have

$$-\sin(y^2) \cdot 2y\frac{dy}{dx} + 1 = e^y \frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$ we get

$$\frac{dy}{dx} = \frac{1}{e^y + 2y\sin(y^2)}$$

3.6 Derivatives of Logarithmic Functions
3.21 First take logarithms of both sides:

$$y\ln(x) = x\ln(y)$$

the use implicit differentiation

$$\frac{d}{dx} y\ln(x) = \frac{d}{dx} x\ln(y)$$

$$\frac{dy}{dx}\ln(x) + \frac{y}{x} = \ln(y) + \frac{x}{y}\frac{dy}{dx}$$

Now solve for $\frac{dy}{dx}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}} \\ &= \left(\frac{y}{x}\right)\frac{x\ln(y) - y}{y\ln(x) - x}\end{aligned}$$

3.22 For any x,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \left[\lim_{n\to\infty}\left(1+\frac{1}{n/x}\right)^{n/x}\right]^x = \left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^x = [e]^x = e^x$$

3.7 Rates of Change in the Natural and Social Sciences
3.23 Since

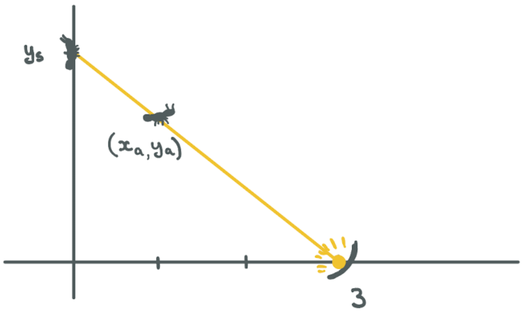
$$w(t) = |v(t)| = \begin{cases} v(t) & \text{if } v(t) \geq 0 \\ -v(t) & \text{if } v(t) < 0 \end{cases}$$

then

$$\begin{aligned} w'(t) = |v(t)| &= \begin{cases} v'(t) & \text{if } v(t) \geq 0 \\ -v'(t) & \text{if } v(t) < 0 \end{cases} \\ &= \begin{cases} a(t) & \text{if } v(t) \geq 0 \\ -a(t) & \text{if } v(t) < 0 \end{cases} \end{aligned}$$

This means $w'(t) > 0$ when both a and v are both positive, or both negative. Similarly, $w'(t) < 0$ when both a and v have opposite sign.

3.9 Related Rates
3.25 Let (x_a, y_a) be the coordinates of the ant at time t, and let y_s be the y-coordinate of the shadow on the y-axis at time t.



Then the line from the lamp to the ant is

$$y = 0 + \left(\frac{0-y_a}{3-x_a}\right)(x-3)$$

The shadow is on this line at the point $(0, y_s)$, therefore

$$y_s = \frac{3y_a}{3-x_a}$$

This gives us the relationship between x_a, y_a and y_s. Next we find the relationship between their rates by differentiating with respect to t:

$$\frac{dy_s}{dt} = \frac{3\frac{dy_a}{dt}(3-x_a)+3y_a\frac{dx_a}{dt}}{(3-x_a)^2}$$

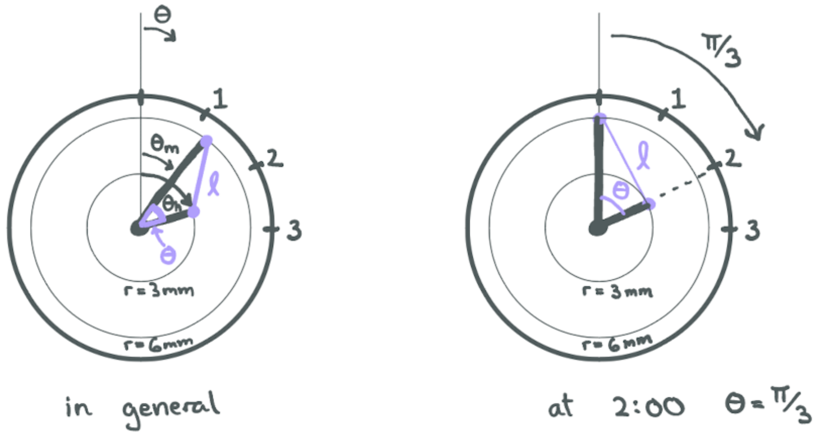
Now we plug in the known information: $x_a = 1$, $y_a = 2$, $\frac{dx_a}{dt} = 1/3$, $\frac{dy_a}{dt} = -1/4$.

$$\frac{dy_s}{dt} = \frac{3(-1/4)(3-1)+3(2)(1/3)}{(3-1)^2} = \frac{-3/2+2}{4} = \frac{1}{8}$$

Therefore, the shadow is moving up along the y-axis at a rate of $1/8$ m/s.

3.26 Let ℓ be the distance between the tips of the hands. We'll measure the angle representing the position of a hand from the vertical axis, in the clockwise direction. Let θ_h be the angle of the hour hand, and θ_m be the angle of the minute hand, and let θ be the angle between the hands:

$$\theta = \theta_h - \theta_m$$



Since the minute hand makes one full revolution per hour, and the hour hand makes one full revolution every 12 hours then

$$\frac{d\theta_m}{dt} = 2\pi \text{ rad/hr} \quad \text{and} \quad \frac{d\theta_h}{dt} = \frac{2\pi}{12} = \frac{\pi}{6} \text{ rad/hr}$$

Therefore, the rate of change of the angle between the hands is

$$\frac{d\theta}{dt} = \frac{d\theta_h}{dt} - \frac{d\theta_m}{dt} = \frac{\pi}{6} - 2\pi = \frac{-11\pi}{6} \text{ rad/hr}$$

The law of cosines gives us a relationship between the distance between the tips, ℓ, and the angle θ

$$\ell^2 = 3^2 + 6^2 - 2(3)(6)\cos\theta$$

$$\ell^2 = 45 - 36\cos\theta$$

Therefore, the rate of change of the ℓ is

$$2\ell\frac{d\ell}{dt} = 36\sin\theta\frac{d\theta}{dt}$$

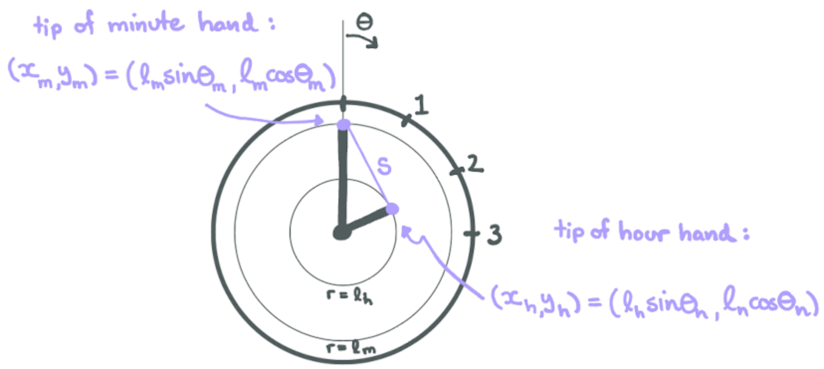
$$\frac{d\ell}{dt} = \frac{18}{\ell}\sin\theta\frac{d\theta}{dt}$$

At two o'clock $\theta = \frac{\pi}{3}$ and $\ell = \sqrt{45 - 36\cos\theta} = \sqrt{45 - 36\left(\frac{1}{2}\right)} = 3\sqrt{3}$, and we have

$$\frac{d\ell}{dt} = \frac{18}{\ell}\sin\theta\frac{d\theta}{dt} = \frac{18}{3\sqrt{3}}\sin\left(\frac{\pi}{3}\right)\left(\frac{-11\pi}{6}\right) = \frac{-11\pi}{2} \text{ mm/hr}$$

Alternate Method: Using the law of cosines is by-far the simpler method, however, we could solve this problem by using the distance formula between two points:

Let θ_h be the angle the hour hand makes with the vertical y-axis, and θ_m be the angle of the minute hand. Then the x and y coordinates of the tips of the hour and minute hands are as shown in the diagram (where $\ell_m = 6$mm and $\ell_h = 3$mm).



Moreover, since the minute hand makes one full revolution per hour, and the hour hand makes one full revolution every 12 hours then

$$\frac{d\theta_m}{dt} = 2\pi \text{ rad/hr} \quad \text{and} \quad \frac{d\theta_h}{dt} = \frac{2\pi}{12} = \frac{\pi}{6} \text{ rad/hr}$$

The distance between the tips satisfies

$$s^2 = (x_h - x_m)^2 + (y_h - y_m)^2 = (\ell_h \sin\theta_h - \ell_m \sin\theta_m)^2 + (\ell_h \cos\theta_h - \ell_m \cos\theta_m)^2$$

Differentiating with respect to t,

$$2s\frac{ds}{dt} = 2(\ell_h \sin\theta_h - \ell_m \sin\theta_m)\left(\ell_h \cos\theta_h \frac{d\theta_h}{dt} - \ell_m \cos\theta_m \frac{d\theta_m}{dt}\right)$$
$$- 2(\ell_h \cos\theta_h - \ell_m \cos\theta_m)\left(\ell_h \sin\theta_h \frac{d\theta_h}{dt} - \ell_m \sin\theta_m \frac{d\theta_m}{dt}\right)$$

Now plug in the known values (representing two o'clock): $\theta_m = 0$, $\theta_h = \frac{\pi}{3}$, $\ell_m = 6$, $\ell_h = 3$, $\frac{d\theta_m}{dt} = 2\pi$ rad/hr, and $\frac{d\theta_h}{dt} = \frac{2\pi}{12}$ rad/hr, and

$$s = \sqrt{(\ell_h \sin\theta_h - \ell_m \sin\theta_m)^2 + (\ell_h \cos\theta_h - \ell_m \cos\theta_m)^2} = \sqrt{\left(\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2} - 6\right)^2} = \sqrt{\frac{27}{4} + \frac{81}{4}} = 3\sqrt{3}$$

and we get

$$\begin{aligned}
\frac{ds}{dt} &= \frac{(3(\sqrt{3}/2))(3(1/2)(\pi/6) - 6(2\pi)) - (3/2 - 6)(3(\sqrt{3}/2)(\pi/6))}{3\sqrt{3}} \\
&= \frac{1}{3\sqrt{3}}\left(\frac{3\sqrt{3}}{2}\left(\frac{\pi}{4} - 12\pi\right) - \left(\frac{3}{2} - 6\right)\left(\frac{\sqrt{3}\pi}{4}\right)\right) \\
&= \frac{\pi\sqrt{3}}{3\sqrt{3}}\left(\frac{3}{2}\left(\frac{-47}{4}\right) - \left(\frac{-9}{2}\right)\left(\frac{1}{4}\right)\right) \\
&= \frac{\pi}{3 \cdot 8}(-132) \\
&= \frac{-11\pi}{2} \text{ mm/min}
\end{aligned}$$

3.10 Linear Approximation and Differentials

3.27 The amount of paint needed is the difference in volume from a hemisphere of radius 25m from one of radius 25m + 0.01cm. We'll use differentials to estimate the change in volume. The volume of a hemisphere of radius r is

$$V(r) = 2\pi r^3$$

Therefore,

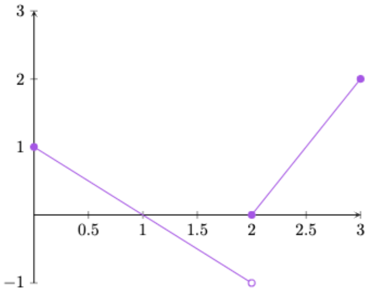
$$\Delta V \approx \frac{dV}{dt}\Delta r = 2\pi r^2 \Delta r$$

For $\Delta r = 0.01\text{cm} = 0.0001\text{m}$ and $r = 25\text{m}$,

$$\Delta V \approx 2\pi(25)^2 0.0001 \approx 0.3927\text{m}^3 \approx 392.7 \text{ litres}$$

4.1 Maximum and Minimum Values

4.1 Global max has value 2 and occurs at $x = 2$. There is no global min.



4.2 $f'(t) = (3/4)t^{-1/4} - (1/2)t^{-3/4}$. Setting this to 0 and solving for t we get the critical number:

$$(3/4)t^{-1/4} - (1/2)t^{-3/4} = 0$$
$$3t^{-1/4} - 2t^{-3/4} = 0$$
$$3t^{1/2} - 2 = 0$$
$$t = 4/9$$

4.3 $f'(p) = \frac{-1}{(p+2)^2}$. Since $f'(p) \neq 0$ for any p, there are no critical numbers.

4.4 $f'(x) = 3x^2 - 6x = 3x(x-2)$, and the critical numbers in $[-1/2, 4]$ are $x = 0, 2$. Comparing the values of f at the critical numbers and endpoints:

x	$f(x)$
$-1/2$	$1/8$
0	1
2	-3
4	17

Therefore, f has absolute max of 17 at $x = 4$ and an absolute min of -3 at $x = 2$.

4.5 $f'(x) = 1 - \frac{1}{x}$, and the critical number in $[1/2, 2]$ is $x = 1$. Comparing the values of f at the critical number and endpoints:

x	$f(x)$
$1/2$	$\frac{1}{2} - \ln\left(\frac{1}{2}\right) = \frac{1}{2} + \ln 2 \approx 1.19$
1	1
2	$2 - \ln 2 \approx 1.31$

Therefore, f has absolute max of $\frac{1}{2} - \ln\left(\frac{1}{2}\right)$ at $x = 1/2$ and an absolute min of 1 at $x = 1$.

4.6 $f'(x) = 101x^{100} + 51x^{50} + 1 > 0$ for $x \in \mathbb{R}$. Since f and f' are defined on $\mathbb{R}$ then if f were to have local extrema they must occur at critical points (Fermat's Theorem). But f has no critical points, therefore f has no local extrema.

4.2 Mean Value Theorem

4.7 Let $f(x) = x^4 - x - 1$. First notice $f(1) = -1 < 0$ and $f(2) = 12 > 0$, and since f is continuous then by the IVT there must be a root of f on this interval. Moreover, since $f'(x) = 4x^3 - 1 > 0$ for $x \in [1, 2]$, this means f is strictly increasing on the interval $[1, 2]$ and therefore it can have at most one root (this is a consequence of the MVT, since if it has at least two roots by the MVT there must be a point $x = c$ between them where $f'(c) = 0$. But this is impossible, since $f'(x) = 4x^3 - 1 > 0$ for $x \in [1, 2]$).

4.8 The average speed travelled along the highway by the car is $\frac{90 \text{ km}}{1 \text{ h}} = 90$ km/h, which exceeds the speed limit. By the MVT the vehicle has to be traveling the average speed at some instant between 9 and 10 am. Therefore, the car was speeding at some instant.

4.9 By the MVT there is a $c \in [0,2]$ such that $\frac{f(2)-f(0)}{2-0} = f'(c)$. It follows that

$$\frac{f(2)+3}{2} \leq 5$$

and so

$$f(2) \leq 7.$$

4.10 By the MVT there is a $c \in [-b,b]$ such that

$$\frac{f(b)-f(-b)}{b-(-b)} = f'(c)$$

and since f is odd,

$$\frac{f(b)+f(b)}{2b} = f'(c) \quad \text{hence} \quad \frac{f(b)}{b} = f'(c)$$

4.11 Let $f(x) = 2\sin^{-1}(x)$ and $g(x) = \cos^{-1}(1-2x^2)$. Then

$$f'(x) = \frac{2}{\sqrt{1-x^2}} \quad \text{and} \quad g'(x) = \frac{2x}{\sqrt{x^2-x^4}} = \frac{2x}{|x|\sqrt{x-x^2}}$$

For $x \in [0,1]$, $f'(x) = g'(x)$, therefore $f(x) = g(x)+c$ for some constant c. That is,

$$2\sin^{-1}(x) = \cos^{-1}(1-2x^2)+c$$

Setting $x = 0$ we see $c0$. Therefore,

$$2\sin^{-1}(x) = \cos^{-1}(1-2x^2).$$

4.4 L'Hospital

4.12

$$\begin{aligned}\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x} \\ &= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \quad \text{(by L'H)} \\ &= \lim_{x\to 0^+} -x \\ &= 0\end{aligned}$$

4.13

$$\begin{aligned}\lim_{x\to -\infty} xe^x &= \lim_{x\to -\infty} \frac{x}{e^{-x}} \\ &= \lim_{x\to -\infty} \frac{1}{-e^{-x}} \quad \text{(by L'H)} \\ &= \lim_{x\to -\infty} -e^x \\ &= 0\end{aligned}$$

4.14

$$\begin{aligned}\lim_{x\to 0^+} x\cot x &= \lim_{x\to 0^+} \frac{x}{\sin x}\cos x \\ &= 1\cdot 1 \\ &= 1\end{aligned}$$

4.15 Suppose the limit exits and let it be L.

$$L = \lim_{x\to 0^+} (1+\sin(4x))^{\cot x}$$

Taking logarithms of both sides:

$$\begin{aligned} \ln L &= \lim_{x\to 0^+} \cot x \cdot \ln(1+\sin(4x)) = \lim_{x\to 0^+} \cos x \left(\frac{\ln(1+\sin(4x))}{\sin x}\right) \\ &= \left(\lim_{x\to 0^+} \cos x\right)\left(\lim_{x\to 0^+} \frac{\ln(1+\sin(4x))}{\sin x}\right) \\ &= \lim_{x\to 0^+} \frac{4\cos(4x)}{(1+\sin(4x))\cos x} \quad \text{by L'H} \\ &= 4 \end{aligned}$$

Hence $L = e^4$.

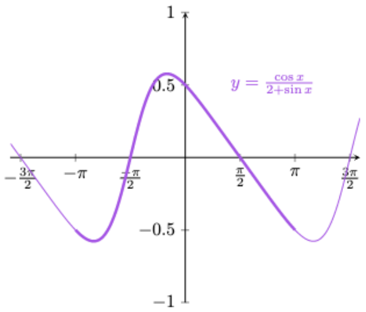
4.16 We are looking for $f(x) = ax+b$ which satisfies the limit. Since the limit exists the left-hand side must be of type $0/0$, and therefore by L'H we have

$$\lim_{x\to 5} f'(x) = a = 1$$

Furthermore, $f(5) = 3$ implies $5a+b = 3$, and since $a = 1$ then $b = -2$. Therefore, $f(x) = x-2$.

4.5 Summary of Curve Sketching

4.17 Go through all the steps for sketching the curve. We summarize some of the results here: The function is 2π-periodic. The local extrema occur at $x = -\frac{\pi}{6} + 2k\pi$ and $x = \frac{7\pi}{6} + 2k\pi$ (this is where $f'(x) = 0$. The inflection points occur at $x = \frac{(2k+1)\pi}{2}$.



4.6 Optimization

4.18 Consider an arbitrary point $(y^2/2, y)$ on the parabola, and let $f(y)$ be the square-distance between $(1,4)$ and this point (the y value that minimizes the square of the distance is the same y value that minimizes the distance).

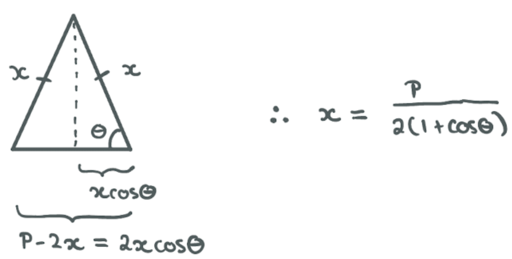
$$f(y) = (y^2/2-1)^2 + (y-4)^2$$

We want to find where f is minimized.

$$f'(y) = (y^2-2)y + 2(y-4) = y^3 - 8$$

The critical point is $y = 2$, and f is minimized at this point. The corresponding point on the parabola is $(2,2)$, and this is the closest point to $(1,4)$.

4.19 Let x be the length of one side of an isosceles triangle, and θ the angle as shown in the diagram. Let P be the fixed perimeter, then $2x + 2x\cos\theta = P$, hence $x = \frac{P}{2(1+\cos\theta)}$.



The area of the triangle is

$$A(\theta) = \frac{1}{2}(2x\cos\theta)(x\sin\theta) = x^2\cos\theta\sin\theta$$
$$= \frac{P^2}{4}\frac{\cos\theta\sin\theta}{(1+\cos\theta)^2}$$

We now maximize $A(\theta)$ over $0 \le \theta \le \frac{\pi}{2}$. The derivative is

$$A'(\theta) = \frac{P^2}{4}\frac{\cos(\theta)+\cos(2\theta)}{(\cos(\theta)+1)^3}$$

Setting to 0 we get

$$\cos 2\theta + \cos\theta = 0$$
$$2\cos^2\theta - 1 + \cos\theta = 0 \quad \text{(from double angle formula for cosine)}$$
$$2\cos^2\theta + \cos\theta - 1 = 0$$

From the quadratic formula

$$\cos\theta = -1 \text{ or } \frac{1}{2}$$

The first is impossible, since A is not defined there, hence the solution comes from the second:

$$\cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

Therefore, the triangle is equilateral.

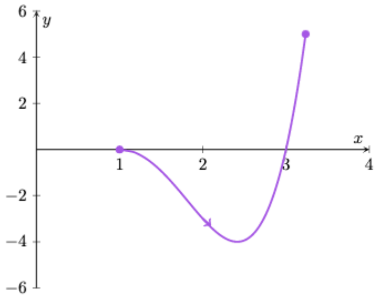
4.20 Let $P(x)$ be the price per unit that the store can charge and sell x units. In marketing this is called the demand function (or price function). Let $R(x)$ be the revenue generated by selling x units. Then

$$P(x) = 350 - \frac{10}{20}(x-200) = 450 - \frac{1}{2}x \quad \text{and} \quad R(x) = xP(x) = 450x - \frac{1}{2}x^2$$

R is maximum when $x = 450$, and therefore $P(450) = \$225$. Hence, selling for a price of \$225.00 will maximize revenue.

5.1 Curves Defined by Parametric Equations

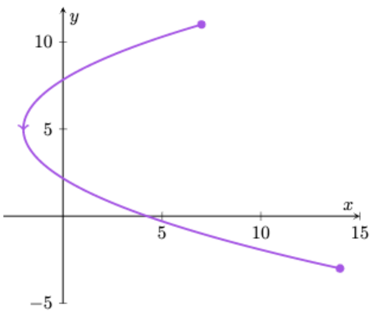
5.1 (a)



Eliminating the t we get the cartesian equation

$$y = (x-1)^4 - 4(x-1)^2 = (x-1)^2(x^2-2x-3)$$

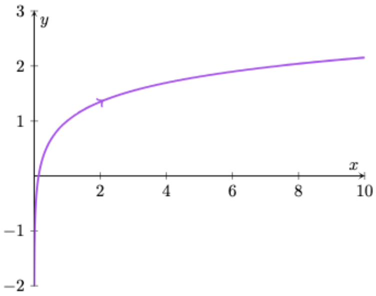
(b)



Eliminating the t we get the cartesian equation

$$x=\left(\frac{5-y}{2}\right)^2-2=\frac{1}{4}y^2-\frac{5}{2}y+\frac{17}{4}$$

(c)



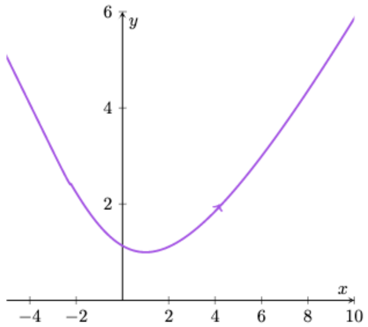
Eliminating the t we get the cartesian equation

$$x=e^{2(y-1)} \quad \text{or} \quad y=\frac{1}{2}\ln(x)+1$$

5.2

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1-\frac{1}{t}}{1+\frac{1}{t}}=\frac{t-1}{t+1} \quad \text{and} \quad \frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\frac{\frac{t+1-(t-1)}{(t+1)^2}}{1+\frac{1}{t}}=\frac{2t}{(t+1)^3}$$

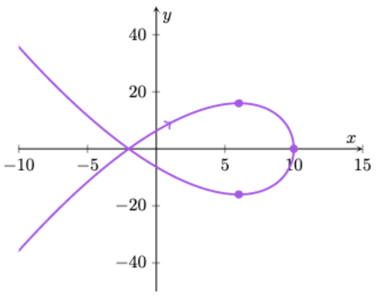
The domain of the parametrization is $t>0$, and for these values $\frac{d^2y}{dx^2}$ is positive. Therefore, the curve is concave up everywhere.



5.3 $x=10-t^2, \quad y=t^3-12t$

$$\frac{dy}{dt}=3t^2-12 \quad \text{and} \quad \frac{dx}{dt}=-2t$$

The tangent line is horizontal when $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=0$ and this is when $3t^2-12=0$, that is, when $t=\pm 2$. The corresponding points are $(x,y)=(6,\pm 16)$. The tangent line is vertical when $\frac{dx}{dy}=\frac{\frac{dx}{dt}}{\frac{dy}{dt}}=0$ and this is when $-2t=0$, that is, when $t=0$. The corresponding point is $(x,y)=(10,0)$.



5.4 $x = 2t+5, \quad y = t^2+!$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t$$

Consider an arbitrary value of t: $t = a$. The corresponding point on the curve is

$$(x,y) = (2a+5, a^2+1)$$

and the tangent line at this point is

$$\begin{aligned} y &= (a^2+1) + a(x-(2a+5)) \\ &= ax + (1-5a-a^2) \end{aligned}$$

We want to determine the value a for which this tangent line passes through the point $(2,1)$. Plugging $(2,1)$ into the equation of the tangent line:

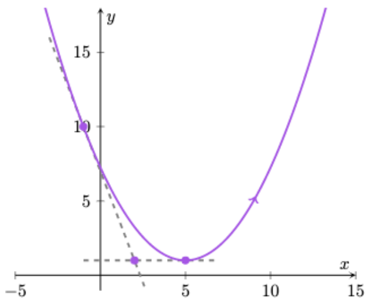
$$1 = 2a+1-5a-a^2$$

The solutions are $a = -3, 0$. Therefore, the corresponding points are

$$(x,y) = (-1,10), (5,1)$$

The tangent lines at these points are:

$$y = -3x+7 \quad (a=-3) \quad \text{and} \quad y = 1 \quad (a=0)$$



5.2 Polar Coordinates
5.5 Substitute $x = r\cos\theta$ and $y = r\sin\theta$ to get

$$r^2\cos\theta\sin\theta = 4$$

or

$$r^2 = \frac{4}{\cos\theta\sin\theta}$$

5.6 $r = \tan\theta\sec\theta = \frac{\sin\theta}{\cos^2\theta} \quad \Rightarrow \quad r\cos^2\theta = \sin\theta \quad \Rightarrow \quad r^2\cos^2\theta = r\sin\theta \quad \Rightarrow \quad x^2 = y$
Therefore, the cartesian equation is $y = x^2$.
5.7 $r = 3\cos\theta$, and $\frac{dr}{d\theta} = -3\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{(-3\sin\theta)\sin\theta + (3\cos\theta)\cos\theta}{(-3\sin\theta)\cos\theta + (3\cos\theta)\sin\theta} = \frac{\cos^2\theta - \sin^2\theta}{2\cos\theta\sin\theta} = \frac{\cos(2\theta)}{\sin(2\theta)}$$

Tangent line is horizontal when $\cos 2\theta = 0$, which is when $\theta = \frac{2k+1}{4}\pi$, and tangent line is vertical when $\sin 2\theta = 0$, which is when $\theta = \frac{k}{2}\pi$, for all integers k.

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