MATH 301 Project: Roth's Theorem

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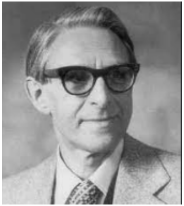
November 26th 2021

Who is Roth?



-Klaus Roth is a German-British mathematician. He was born on October 29th 1925 in Breslau, Germany and died on November 10th 2015 in Iverness, Scotland.
-In 1953, Roth proved what is now known as Roth's Theorem, a theorem that guarantees a 3-term arithmetic progression for a positive density using Fourier analytic mehtods.
-In 1958, Klaus won a Field's medal for his work in approximating algebraic numbers with rationals, an open problem which he solved in 1955. (O'Connor, 1998)

History Before Roth's Theorem

-Additive number theory determines the conditions needed to be imposed on a subset of integers.
-In order to determine if the subset of integers contains an arithmetic progression, the size of the set needs to be large enough.
-We will be considering the following upper density for $k \in \mathbb{Z}^+$:

$$\limsup_{N \to \infty} \frac{\left|A \cap [1, N]\right|}{N}$$

-In 1927, Bartel Van der Waerden determined $N(r, k)$ yielded a monochromatic arithmetic progression of length k on r colours.
-In 1936, Erdos and Turan made a stronger conjecture that the set of positive upper density contain large arithmetic progressions.
-In 1953, Roth solved for 3-term arithmetic progressions. (Lott, 2017)

History After Roth's Theorem

-In 1969, Szemeredi solved for $k -$ term arithmetic progessions.
-In 1972, Roth extended his work to solve for 4-term arithmetic progressions.
-In 1975, Szemeredi extended his work to solve for arithmetic progressions of arbitrary length.
-In the 1990s, Timothy Gowers developed new analytical machinery to work for both 4-term and arbitrarily long arithmetic progressions.
-Also in the 1990s, Ben Green showed the upper density contained a 3-term arithmetic progression for the set of prime numbers.
(Lott, 2017)

What is Roth's Theorem?

Theorem (Roth's Theorem):
For any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that for any $n \geq N$, if $A \subseteq [1, n]$ with $|A| > \epsilon n$, then A contains a 3-term arithmetic progression. (Roth, 1955)

Information needed to Prove Roth's Theorem

In order to Prove Roth's Theorem, we will be using the following ideas:

1. Discrete Fourier Analysis: Application of the Discrete Fourier Transform

2. Modular arithmetic on $\mathbb{Z}_n$ for the arbitrary 3-term arithmetic progression $x + y \equiv 2z \pmod{n}$, where $x \neq y \neq z$

3. Indicator Function of an arbitrary set A

4. Density of the function for an arbitrary set A and the size of its Fourier Coefficients:

$$\hat{A}(0) = \frac{|A|}{n} \text{ (The density of A in } [1, n])$$

(Robertson, 2021)

Approach for Roth's Theorem Summary:

1. Start by considering an arbitrary set holding for a positive density of an interval containing a finite number of elements.
We can show by contradiction that if this is not the case, we will get a positive density greater than 1, which contradicts the fact that the positive density is at most 1.
2. We compute the number of possible solutions in $\mathbb{Z}_n$ for our arbitrary 3-term arithmetic progression $x + y \equiv 2z \pmod{n}$
Computing the Discrete Fourier Transform of the function yields the arbitrary 3-term arithmetic progression in terms of its Fourier Coefficients.
We show that the Fourier Coefficients have to be small, which can be done by contradiction in assuming they were large.
3. We prove 2 claims regarding the density involving 3-term arithmetic progressions. (Robertson, 2021)
Note: Proving claims 1 and 2 will be sufficient to prove Roth's Theorem!

Helper Function for Pseudocode Proof of Roth's Theorem:

```
def Keytest(U, A)
d = |A| / |U|
Â = Calculate Â from indicator function of A
for k from 1 to n do
    if Â(k) ≥ δ²/10 then
        Return False
    end if
end for
Return True
```

The indicator function is calculated as follows:

$$\hat{A}(k) = \frac{1}{n} \sum_{x=1}^{n} A(x) \cdot w^{-xk} = \frac{1}{n} \sum_{x=1}^{n} A(x) \cdot e^{-2\pi i x k / n} \text{ (Robertson, 2021)}$$

Pseudocode Proof of Roth's Theorem:

```
def Roth ():
U = [1, n]
A = Subset of U with positive density
while True do
    if Keytest(U, A) = True then
        3-AP can be found
        Return 3-AP
    else
        An AP 'P' can be found in the motherset
        U^new = P[1,|P|]    // compressed in index
        A^new = [A ∩ P]    // matches with A^new
        U = U^new
        A = A^new
    end if
end while
```

Example From Pseudocode:

Suppose we took $n = 20$. This gives us $U = [1, 20]$, where U is our universal set.
So $U[1, 2, \ldots, 20]$
Let A and P be the following sets:
$A[3, 5, 6, 8, 11, 15, 17, 19]$, $P[2, 5, 8, 11, 14, 17, 20]$,
$A \cap P = \{5, 8, 11, 17\}$
$\implies \delta_N = \frac{|A|}{n} = \frac{8}{20} = \frac{2}{5} \; \frac{|A \cap P|}{|P|} = \frac{4}{7}$
Notice that $A \cap P > A \cap N$: $\frac{4}{7} > \frac{2}{5} \implies \frac{28}{35} > \frac{14}{35}$
We now make new variables to "prepare" for the next iteration:
$A^{new} = [A^{old} \cap P] \implies [2, 3, 4, 6]$
$U^{new} = [1, |P|] \implies [1, 2, 3, 4, 5, 6, 7]$
$\delta^{new} = \frac{A^{new}}{U^{new}}$
Note: A^{new} is the "compressed" version of $A^{old} \cap P$, where the key information is preserved: The relative positions stay the same, but the AP shrinks to a distance of 1. Additionally, U^{new} is the "compressed" version of the AP P.

Example From Pseudocode (Larger set):

Algorithm for Claims 1 and 2:

Link

Proof of Roth's Theorem (1):

Theorem (Roth's Theorem):
For any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that for any $n \geq N$, if $A \subseteq [1, n]$ with $|A| > \epsilon n$, then A contains a 3-term arithmetic progression. (Roth, 1955)
Proof:
Let $A \subseteq [1, n]$ with $|A| = \delta n$
Let B be the intersection of our set A and a finite interval.
Claim: If B is a partition of A, it has a positive density of at most 1
Contradiction Hypothesis: Suppose that if B is a partition of A, it has a positive density greater than 1.
Since B is a partition of $A \implies B$ must be smaller than A
$\implies$ The size of our interval dividing our partition must yield a positive result of at most 1, a contradiction (Since we assumed it could have a positive density greater than 1).

Proof of Roth's Theorem (2):

Consider the arbitrary 3-term arithmetic progression
$x + y \equiv 2z \pmod{n}, \; x \neq y \neq z$
Taking the Discrete Fourier Transform, we get the following equation M and error term E:

$$M = \frac{|A||B|^2}{n} + n^2 \sum_{\substack{j \in \mathbb{Z}_n \\ j \neq 0}} \hat{B}(j)\hat{A}(j)\hat{B}(-2j)$$

$$E = n^2 \sum_{j \in \mathbb{Z}_n} \hat{B}(j)\hat{A}(j)\hat{B}(-2j) \text{ (Robertson, 2021)}$$

We will be using this to consider the size of the Fourier Transform Coefficients.

Proof of Roth's Theorem (3):

We now attempt to prove the following Claims:
Claim 1: If the Fourier coefficients are all small $\left(\left|\hat{A}(j)\right| \leq \epsilon \text{ with } \epsilon \leq \delta^2/10 \text{ being the boundary for all } j \in \mathbb{Z}_n \setminus \{0\}\right)$, then A contains a 3-term AP.
Claim 2: If at least one Fourier Coefficient is larger than the boundary $\delta^2/10$ $\left(\hat{A}(k) \geq \frac{\delta^2}{10} \text{ for some k}\right)$, we can find an AP P in the mother set not in A such that the density of A relative to P is greater than the density of A relative to $[1, N]$:

$$\frac{|A \cap P|}{|P|} > \delta + \frac{\delta^2}{80} \text{ (Robertson, 2021)}$$

Proof of Roth's Theorem (4):

Structure of Claim 1:
Suppose all of the Fourier coefficients are small.

$$\implies \left(\left|\hat{A}(j)\right| \leq \epsilon \text{ with boundary } \epsilon \leq \delta^2/10 \text{ for all } j \in \mathbb{Z}_n \setminus \{0\}\right)$$

Calculating the Fourier Transform coefficients, we have that

$$E \leq \frac{1}{2m}|A||B|^2$$

$$M \geq \frac{1}{2m}|A||B|^2$$

Since we were given that M was the number of solutions to $x + y \equiv 2z \pmod{n}$, $x \neq y \neq z$ and $x, z \in B$ in the Fourier Transform shown on (2), we have a $3-$term AP in $\mathbb{Z}^+$ (Robertson, 2021)

Proof of Roth's Theorem (5):

Structure of Claim 2 (1):
Suppose at least one Fourier Coefficient is larger than the boundary $\delta^2/10$.
$\implies \left|\hat{A}(k)\right| > \epsilon$ for some $k \in \mathbb{Z}_n \setminus \{0\}$, where $\epsilon = \delta^2/10$.
We try to find an AP that satisfies

$$\frac{|A \cap P|}{|P|} \geq \delta + \frac{\epsilon}{8}$$

In order to find such P, we start from Q_n. Q_n has $2\left\lceil \frac{\sqrt{n}}{16} \right\rceil + 3$ terms This will approximate to $\sqrt{n}/8$ as n gets large. We define Q_n to start centered at 0, with $d < \sqrt{n}$. We have that

$$Q_n = [\ldots, -2d, -d, 0, d, 2d, \ldots]$$

And

$$\left|\hat{Q}_n(k)\right| > \frac{1}{16\sqrt{n}} = \frac{|Q_n|}{2n} \quad \text{(Robertson, 2021)}$$

Proof of Roth's Theorem (6):

Structure of Claim 2 (2):
We now want to find the "shift" value (a) that makes $A \cap Q_n$ large enough.
In order to find the shift value that makes $A \cap Q_n$ large enough, we define

$$g(a) = \sum_{j \in \mathbb{Z}} (A(j) - \delta) \cdot Q_n(a - j), \ \sum_{l \in \mathbb{Z}} g(l) = 0$$

After finding an a that satisfies

$$g(a) > \frac{|Q_n|}{4} \cdot \epsilon \text{ (This is strong enough!)}$$

We now have $P_n(j) = Q_n(a - j)$ (Robertson, 2021)
$\therefore$ We have successfully found our AP P.

Proof of Roth's Theorem (7):

Now that we have shown Claims 1 and 2, we start with applying Claims 1 and 2 assuming the Fourier Coefficients are small ($\epsilon = \delta^2/10$)) and apply recursion:
If Claim 1 applies, we have our 3-term AP and are done!
$\implies$ We assume Claim 2 applies, which means we can find an AP P_1
$\implies$ For an arbitrary long recursion k, we again assume Claim 2 applies and can find an AP P_k
($\because$ If Claim 1 applies, we have our 3-term AP and are done).
Recall that we saw earlier on that it is impossible for A to have a positive density exceeding 1.
$\implies$ Either Claim 1 must eventually apply, or the assumption that A contains a 3-term AP is wrong, a contradiction to Claim 1.
$\therefore$ We have proven Roth's Theorem. $\square$

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