Precalculus - Math 100

Lecture Notes
Fall 2017

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Important:

Only SFU students enrolled in the fall 2017 Math 100 - Precalculus course are permitted to use this booklet.

Veselin Jungic

To my sons, my best teachers.

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Preface

The purpose of this *Lecture Notes* booklet is to be an additional learning resource for students who are taking the Math 100 precalculus course at Simon Fraser University. Please note that this booklet is not a replacement for the course textbook.

The images on the front page and the back page are created by Simon Roy.

No project such as this can be free from errors and incompleteness. I will be grateful to everyone who points out any typos, incorrect statements, or sends any other suggestion on how to improve this manuscript.

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Chapter 1

Introduction

1.1 Recommendations for Success in Mathematics

Written by Dr. Petra Menz, Department of Mathematics, Simon Fraser University

The following is a list of various categories gathered by the Department of Mathematics. This list is a recommendation to all students who are thinking about their well-being, learning, and goals, and who want to be successful academically.

Tips for Reading these Recommendations:

- Do not be overwhelmed with the size of this list. You may not want to read the whole document at once, but choose some categories that appeal to you.
- You may want to make changes in your habits and study approaches after reading the recommendations. Our advice is to take small steps. Small changes are easier to make, and chances are those changes will stick with you and become part of your habits.
- Take time to reflect on the recommendations. Look at the people in your life you respect and admire for their accomplishments. Do you believe the recommendations are reflected in their accomplishments?

Habits of a Successful Student:

- **Acts responsibly:** This student
  - reads the documents (such as course outline) that are passed on by the instructor and acts on them.
  - takes an active role in their education.
  - does not cheat and encourages academic integrity in others.
- **Sets goals:** This student
  - sets attainable goals based on specific information such as the academic calendar, academic advisor, etc..

  - is motivated to reach the goals.
  - is committed to becoming successful.
  - understands that their physical, mental, and emotional well-being influences how well they can perform academically.

- **Is reflective:** This student
  - understands that deep learning comes out of reflective activities.
  - reflects on their learning by revisiting assignments, midterm exams, and quizzes and comparing them against posted solutions.
  - reflects why certain concepts and knowledge are more readily or less readily acquired.
  - knows what they need to do by having analyzed their successes and their failures.

- **Is inquisitive:** This student
  - is active in a course and asks questions that aid their learning and build their knowledge base.
  - seeks out their instructor after a lecture and during office hours to clarify concepts and content and to find out more about the subject area.
  - shows an interest in their program of studies that drives them to do well.

- **Can communicate:** This student
  - articulates questions.
  - can speak about the subject matter of their courses, for example by explaining concepts to their friends.
  - takes good notes that pay attention to detail but still give a holistic picture.
  - pays attention to how mathematics is written and attempts to use a similar style in their written work.
  - pays attention to new terminology and uses it in their written and oral work.

- **Enjoys learning:** This student
  - is passionate about their program of study.
  - is able to cope with a course they dont like because they see the bigger picture.
  - is a student because they made a positive choice to be one.
  - reviews study notes, textbooks, etc..
  - works through assignments individually at first and way before the due date.
  - does extra problems.
  - reads course related material.

- **Is resourceful:** This student

  - uses the resources made available by the course and instructor such as the Math Workshop, the course container on WebCT, course websites, etc..
  - researches how to get help in certain areas by visiting the instructor, or academic advisor, or other support structures offered through the university.
  - uses the library and internet thoughtfully and purposefully to find additional resources for a certain area of study.

- **Is organized:** This student
  - adopts a particular method for organizing class notes and extra material that aids their way of thinking and learning.

- **Manages his/her time effectively:** This student
  - is in control of their time.
  - makes and follows a schedule that is more than a timetable of course. It includes study time, research time, social time, sports time, etc..

- **Is involved:** This student
  - is informed about their program of study and their courses and takes an active role in them.
  - researches how to get help in certain areas by visiting the instructor, or academic advisor, or other support structures offered through the university.
  - joins a study group or uses the support that is being offered such as a Math Workshop (that accompanies many first and second year math courses in the Department of Mathematics) or the general SFU Student Learning Commons Workshops.
  - sees the bigger picture and finds ways to be involved in more than just studies. This student looks for volunteer opportunities, for example as a Teaching Assistant in one of the Mathematics Workshops or with the MSU (Math Student Union).

How to Prepare for Exams:

- Start preparing for an exam on the FIRST DAY OF LECTURES!
- Come to all lectures and listen for where the instructor stresses material or points to classical mistakes. Make a note about these pointers.
- Treat each chapter with equal importance, but distinguish among items within a chapter.
- Study your lecture notes in conjunction with the textbook because it was chosen for a reason.
- Pay particular attention to technical terms from each lecture. Understand them and use them appropriately yourself. The more you use them, the more fluent you will become.
- Pay particular attention to definitions from each lecture. Know the major ones by heart.
- Pay particular attention to theorems from each lecture. Know the major ones by heart.

- Pay particular attention to formulas from each lecture. Know the major ones by heart.
- Create a cheat sheet that summarizes terminology, definitions, theorems, and formulas. You should think of a cheat sheet as a very condensed form of lecture notes that organizes the material to aid your understanding. (However, you may not take this sheet into an exam unless the instructor specifically says so.)
- Check your assignments against the posted solutions. Be critical and compare how you wrote up a solution versus the instructor/textbook.
- Read through or even work through the paper assignments, online assignments, and quizzes (if any) a second time.
- Study the examples in your lecture notes in detail. Ask yourself, why they were offered by the instructor.
- Work through some of the examples in your textbook, and compare your solution to the detailed solution offered by the textbook.
- Does your textbook come with a review section for each chapter or grouping of chapters? Make use of it. This may be a good starting point for a cheat sheet. There may also be additional practice questions.
- Practice writing exams by doing old midterm and final exams under the same constraints as a real midterm or final exam: strict time limit, no interruptions, no notes and other aides unless specifically allowed.
- Study how old exams are set up! How many questions are there on average? What would be a topic header for each question? Rate the level of difficulty of each question. Now come up with an exam of your own making and have a study partner do the same. Exchange your created exams, write them, and then discuss the solutions.

Getting and Staying Connected:

- Stay in touch with family and friends:
  - A network of family and friends can provide security, stability, support, encouragement, and wisdom.
  - This network may consist of people that live nearby or far away. Technology in the form of cell phones, email, facebook, etc. is allowing us to stay connected no matter where we are. However, it is up to us at times to reach out and stay connected.
  - Do not be afraid to talk about your accomplishments and difficulties with people that are close to you and you feel safe with, to get different perspectives.
- Create a study group or join one:
  - Both the person being explained to and the person doing the explaining benefit from this learning exchange.

  - Study partners are great resources! They can provide you with notes and important information if you miss a class. They may have found a great book, website, or other resource for your studies.

- Go to your faculty or department and find out what student groups there are:
  - The Math Student Union (MSU) seeks and promotes student interests within the Department of Mathematics at Simon Fraser University and the Simon Fraser Student Society. In addition to open meetings, MSU holds several social events throughout the term. This is a great place to find like-minded people and to get connected within mathematics.
  - Student groups or unions may also provide you with connections after you complete your program and are seeking either employment or further areas of study.

Staying Healthy:

- A healthy mind, body, and soul promote success. Create a healthy lifestyle by taking an active role in this lifelong process.
- Mentally:
  - Feed your intellectual hunger! Choose a program of study that suits your talents and interests. You may want to get help by visiting with an academic advisor: math_advice@sfu.ca.
  - Take breaks from studying! This clears your mind and energizes you.
- Physically:
  - Eat well! Have regular meals and make them nutritious.
  - Exercise! You may want to get involved in a recreational sport.
  - Get out rain or shine! Your body needs sunshine to produce vitamin D, which is important for healthy bones.
  - Sleep well! Have a bed time routine that will relax you so that you get good sleep. Get enough sleep so that you are energized.
- Socially:
  - Make friends! Friends are good for listening, help you to study, and make you feel connected.
  - Get involved! Join a university club or student union.

References:

Thien, S. J. Bulleri, A. The Teaching Professor. Vol. 10, No. 9, November 1996. Magna Publications.

Costa, A. L. and Kallick, B. 16 Habits of Mind.
http://www.instituteforhabitsofmind.com/what-are-habits-mind.

1.2 Lecture 1: Welcome to Math 100

1. **Quote.** Designed to prepare students for first year Calculus courses. Topics include language and notation of mathematics; problem solving; algebraic, exponential, logarithmic and trigonometric functions and their graphs. Prerequisite: Pre-Calculus 11 or Foundations of Mathematics 11 (or equivalent) with a grade of at least B or Pre-Calculus 12 (or equivalent), with a grade of at least C and SFU FAN credit, or SFU FAN X99 course with a grade of at least B-, or achieving a satisfactory grade on the Simon Fraser University Quantitative Placement Test. Students with credit for MATH 150 or 151 or 154 or 157 may not take MATH 100 for further credit. MATH 100 may not be counted towards the mathematics minor, major or honours degree requirements.

   SFU Calendar.

2. **My responsibilities:**

   (a) To create conditions under which each of you will successfully complete Math 100 to the best of your abilities. This includes:

      i. Providing all necessary learning material

      ii. Making all assessments reasonably challenging and doable

      iii. Being available, face-to-face or through the Canvas discussion board, whenever you need extra help

   (b) To be always on time for lectures and office hours

   (c) To be fair

   (d) To meet individual learning needs of each of you to the best of my abilities

3. **Your responsibilities:**

   (a) To attend **ALL** lectures

   (b) To be an active participant in the class. This includes:

      i. Taking notes

      ii. Asking questions in the class

      iii. Answering my questions using your i-clicker or

      iv. Discussing questions with your neighbours when you are directed to do so

   (c) To be always on time for lectures

   (d) To familiarize yourselves with **ALL** content that is posted in the Math 100 Canvas container and to daily check for any updates

(e) To do all required work: assignments, Canvas quizzes, Loncapa assignmnets

(f) To be absolutely ready for all in-class quizzes, midterms, and the final exam

(g) To come to see me during my office hours whenever you need my help

(h) To post questions and answers on the Canvas discussion board

(i) To study

4. **Quote.** Calculus is the study of how things change. It provides a framework for modelling systems in which there is change, and a way to deduce the predictions of such models.

   Daniel Kleitman, Emeritus Professor of Applied Mathematics, Combinatorics, Operations Research, MIT

5. **Must know!** All material from Chapter 0 from your textbook. This includes:

   0.1 The real line

   0.2 Algebra of real numbers

   0.3 Inequalities, intervals, and absolute values

6. **Examples:**

Example 1.

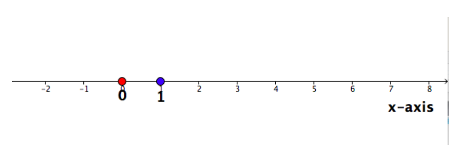
- **The real line in slow motion:**

(a) Choose a line

(b) Choose a point, call it 0 (= origin)

Figure 1.1: Note that we start by choosing any line in the plane and any point 0 on that line.

(a) Choose another point, call it 1



(b) The real line

Figure 1.2: Note that by choosing the point 1 we divide all other points on the line, excluding 0, in two sets: Those that are on the same side of 0 as the point 1, and those that are not. This is how we get "positive" and "negative" rays with 0 as their initial point.

Example 2.
- The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
- The set of whole numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$
- The set of integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- The set of rational numbers: $\mathbb{Q} = \{\frac{p}{q} : p \text{ and } q \text{ are integers}, q \neq 0\}$
- The set of irrational numbers: $\mathbb{I} = \{x : x \text{ cannot be expressed as a ratio of two integers}\}$
- The set of real numbers: $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$

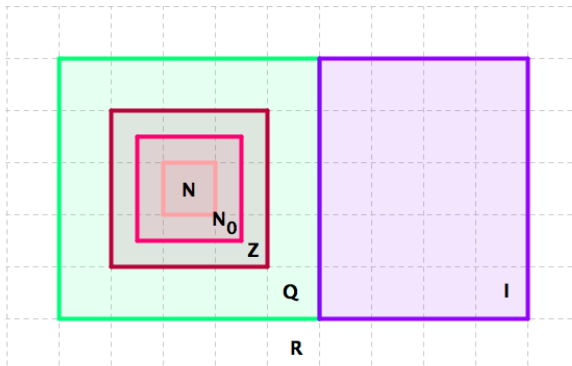


Figure 1.3: $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q}$; $\mathbb{Q} \cap \mathbb{I} = \emptyset$; $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$

Example 3. Order of operations:

$2 + 3 \cdot 4 =$

$2 + 3 \cdot 4 - 6 \div 3 =$

$(2 + 3) \cdot (4 - 6 \div 3) =$

Example 4. Inequalities:

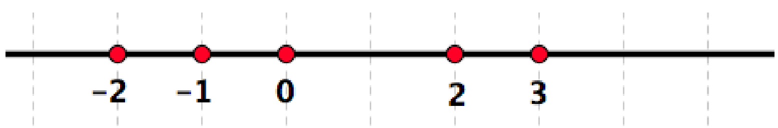
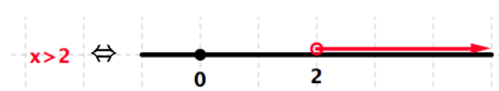


Figure 1.4: Which statement is **NOT TRUE**: $2 < 3$; $-1 > -2$; $3 \geq 2$; $-1 > 0 < 2$; $-1 \leq 0$?

Example 5. A variable in mathematics = A symbol or an object that represents elements of a certain set.

(a) Variable: Passport



(b) Variable: x

Figure 1.5: The same, just a bit different.

Example 6. Intervals

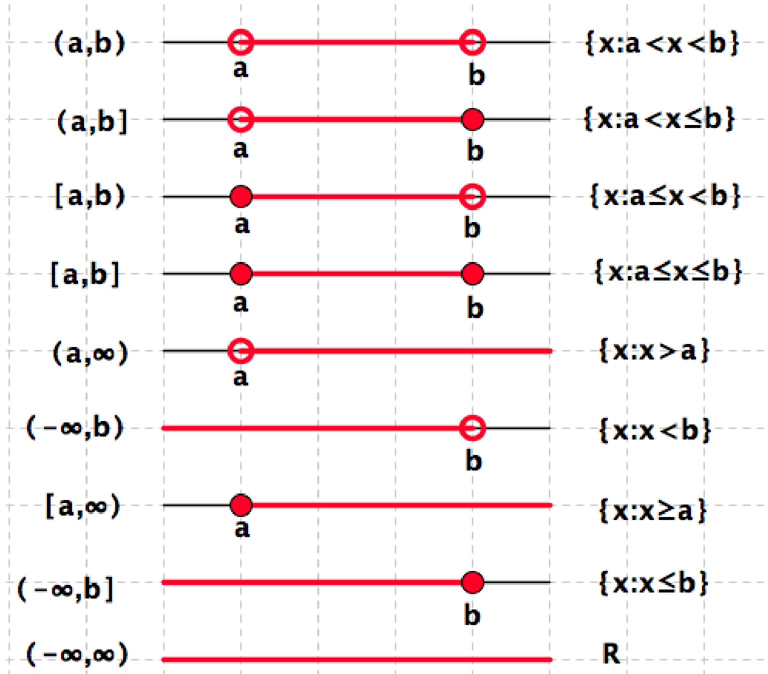


Figure 1.6: Must know!

Chapter 2

Functions

2.1 Lecture 2: Functions

1. **Quote.** The highest function of love is that it makes the loved one a unique and irreplaceable being.

   Thomas Eugene "Tom" Robbins, American author, 1932–.

2. The most important term in this course:

   FUNCTION

3. **What does the word *function* mean?**

   From Oxford Dictionaries:

noun:
   (a) An activity that is natural to or the purpose of a person or thing:
   '*bridges perform the function of providing access across water*'
   (b) Mathematics - A relation or expression involving one or more variables:
   '*the function* $(bx + c)$'
   (c) A thing dependent on another factor or factors:
   '*class shame is a function of social power*'
   (d) A large or formal social event or ceremony:
   '*he was obliged to attend party functions*'

verb:
   (a) Work or operate in a proper or particular way:
   '*her liver is functioning normally*'
   (b) (function as) Fulfil the purpose or task of (a specified thing):
   '*the museum intends to function as an educational and study centre*'

4. **Origin?** Mid 16th century: from French *fonction*, from Latin *functio*(n-), from *fungī* 'perform'.

5. **How often?**

   - Oxford Dictionaries list the word *function* among the top 1000 frequently used English words.

- The website http://www.wordfrequency.info/free.asp?s=y, lists the frequency of the word *function* in American English as 1449 (noun) and 3270 (verb).

6. **Why 'Functions' in Mathematics? To avoid confusion.**

Example 1. Reposition one and remove four matches to spell a thing from which good matches are made.

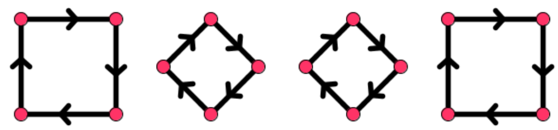


Figure 2.1: Wood?

Example 2. Definition of **pen** in Oxford Dictionaries:

(a) An instrument for writing or drawing with ink.
(b) A small enclosure in which sheep, pigs, or other farm animals are kept.
(c) A penitentiary; a prison.
(d) A female swan.

7. **Why 'Functions' in Mathematics? To avoid confusion.**

Example 1. Andrew was born in Vancouver.

Figure 2.2: Canadian or American?

Source of the confusion: The statement "Andrew was born in Vancouver" associates Andrew with more than one city.

Example 2. My cousin has three children, Chloe, Justin, and Zack. Chloe is 26 years old and Zack is 28 years old. Who is the oldest among the three siblings?

Source of the confusion: Not each child is associate with their age.

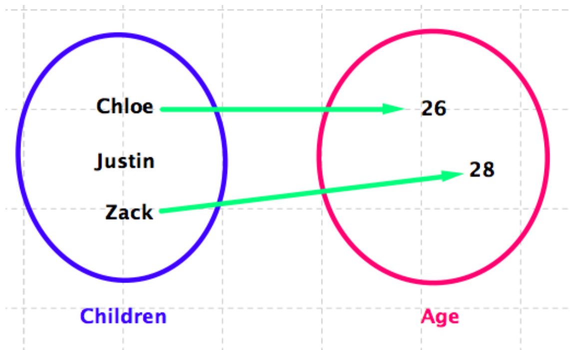


Figure 2.3: Who is the oldest?

8. **Function in Mathematics - Definition:** A **function** is a rule that assigns to each element in a set A exactly one element in a set B.

Example 1. Andrew was born in Vancouver, BC.

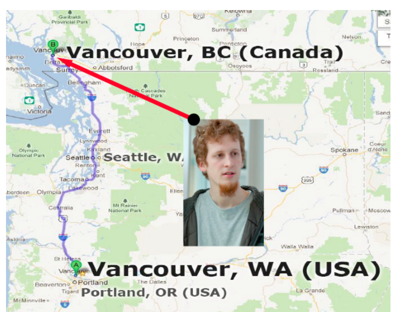


Figure 2.4: Canadian!

No confusion: The statement "Andrew was born in Vancouver, BC" associates Andrew with exactly one city.

Example 2. My cousin has three children, Chloe, Justin, and Zack. Chloe and Justin are twins and they are 26 years old and Zack is 28 years old. Who is the oldest among the three siblings?

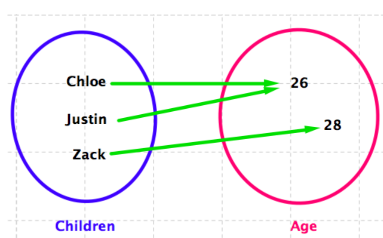


Figure 2.5: Zack is the oldest!

No confusion: Each child is associate with their age.

9. **Function in Mathematics - Definition:** A **function** is a rule that assigns to each element in a set A exactly one element in a set B.

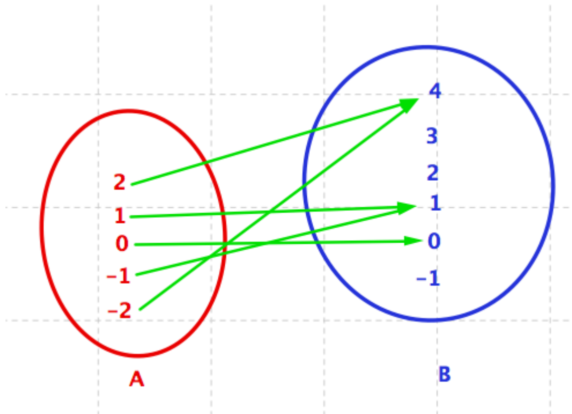


Figure 2.6: Function $f : A \to B$

Vocabulary and Notation:

- Function $f : A \to B$
- The set $A = \{-2, -1, 0, 1, 2\}$ is called **the domain of the function** f
- The set $\{0, 1, 4\}$ is called **the range of the function** f
- The rule f is given by:

$$-2 \to 4, \quad -1 \to 1, \quad 0 \to 0, \quad 1 \to 1, \quad 2 \to 4$$

- The rule f is given by:

$$f(-2) = 4, \quad f(-1) = 1, \quad f(0) = 0, \quad f(1) = 1, \quad f(2) = 4$$

- The rule f is given by:

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

- The rule f is given by:

$$f(x) = x^2, \quad x \in A = \{-2, -1, 0, 1, 2\}$$

10. **Function.**

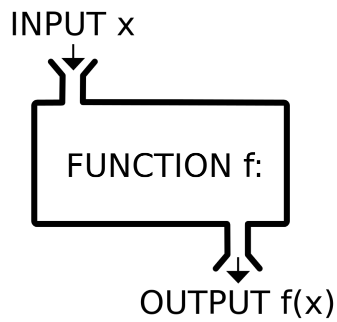


Figure 2.7: A Function Machine

11. **Vocabulary.**

- **Independent variable** – a variable that represents input numbers for a function;
- **Dependent variable** – a variable that represents output numbers;
- **Functional notation** – $y = f(x)$, which is read "y equals f of x," means that the output y correspondents to the input x by the rule f.
- **Range** – If $f : A \to B$ then the range of the function f is the set of all elements y in B so that there is an element x in the domain A such that $y = f(x)$.

  Example: If the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$ (where x is a real number,) then 9 is in the range of f and -9 is not in the range of f.

12. **Example.** Find the domain of each function and calculate $f(2)$, $f(-3)$, $f(1)$, $f(0)$, $g(x+1)$, and $g(x^2)$.

    (a) $f(x) = \dfrac{1}{x}$

    (b) $g(t) = \sqrt{t}$

13. **Piecewise Defined Functions.** The total annual fees in Canadian dollars for students enrolled in a science program at a major university in Montréal, Québec, are as follows:

    (a) $ 4,214.78, if the student is a resident of Québec,

    (b) $ 9,226.58, if the student is a Canadian citizen or permanent resident of Canada who is not a resident of Québec,

    (c) $ 33,325.78, if the student is not a Canadian citizen or permanent resident of Canada.

14. **Example.** Amir, Biljana, and Candice are enrolled in a math major program at the university mentioned above.

    - Amir was born and raised in Montréal, Québec. He lives in his parents house that is just a few blocks from the university campus.
    - Biljana represented Bosnia and Herzegovina at the International Math Olympiad earlier this year. She skypes her mom in Sarajevo every Sunday at 11:00 am.
    - Candice is a proud member of the Sliammon Nation. Her parents live on the Sliammon traditional territory that is just north from Powell River, B.C. Candice FaceTimes her mom every morning at 8:00 am.

    What is Candice's total annual fee for being enrolled in her program of study?

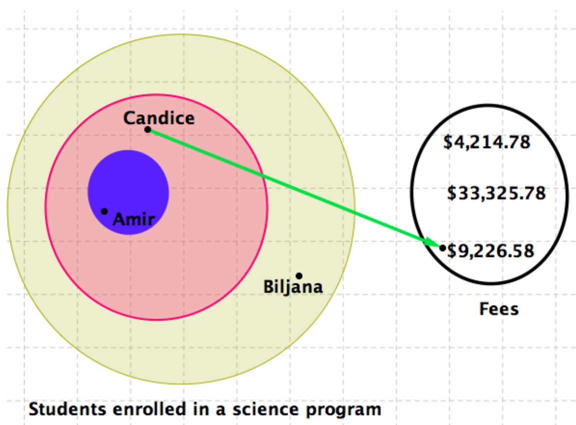


Figure 2.8: Candice's total annual fee

15. **The same, just a little bit different.** Let

$$\begin{aligned} A &= \{x|x \text{ is a student enrolled in a science program that is a resident of Québec}\} \\ B &= \{x|x \text{ is a student enrolled in a science program that is a not a Canadian citizen} \\ &\quad \text{or permanent resident of Canada}\} \\ C &= \{x|x \text{ is a student enrolled in a science program that is a Canadian citizen} \\ &\quad \text{or permanent resident of Canada but is not a resident of Québec}\} \end{aligned}$$

Let f be the rule that assigns to each student enrolled in a science program their total annual fee in Canadian dollars. Then

$$f(x) = \begin{cases} 4,214.78, & \text{if } x \in A \\ 33,325.78, & \text{if } x \in B \\ 9,226.58, & \text{if } x \in C \end{cases}$$

Example. What is $f(\text{Amir})$? What is $f(\text{Dr. J})$?

16. **Two observations and one important new term.**

(a) The domain of the function f is the set

$$A \cup B \cup C = \{x|x \text{ is a student enrolled in a science program}\}.$$

(b) The function f is defined by different expressions on different parts of the domain of f.

The function f is a so-called **piecewise defined function**.

17. **The most famous piecewise defined function.**

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Because of its importance and frequent use, the function f is usually written as

$$f(x) = |x|,$$

which is read "$f(x)$ equals the absolute value of x".

Example. For $f(x) = |x|$ evaluate

(a) $f(5) =$

(b) $f(0) =$

(c) $f(-5) =$

18. **Two questions that you have to ask.**

(a) What is the **domain** of the function $f(x) = |x|$?

(b) What is the **range** of the function $f(x) = |x|$?

19. **The meaning of things.** The function $f(x) = |x|$ is the rule that assigns to each point x on the number line its distance from the origin 0.

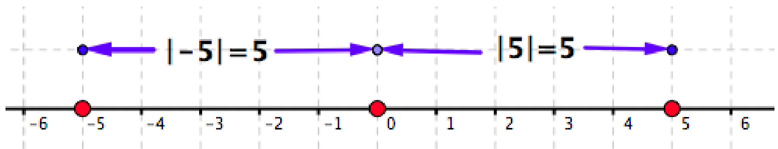


Figure 2.9: Absolute value = Distance from the origin

2.2 Lecture 3: The Coordinate Plane and Graphs

1. **Quote.** With me, everything turns into mathematics.

Figure 2.10: René Descartes, French philosopher, mathematician, and scientist, 1596–1650.

2. **Problem.** How to connect geometry and algebra?

3. **René Descartes:** "Any problem in geometry can easily be reduced to such terms that a knowledge of the length of certain straight lines is sufficient for construction." – 1637

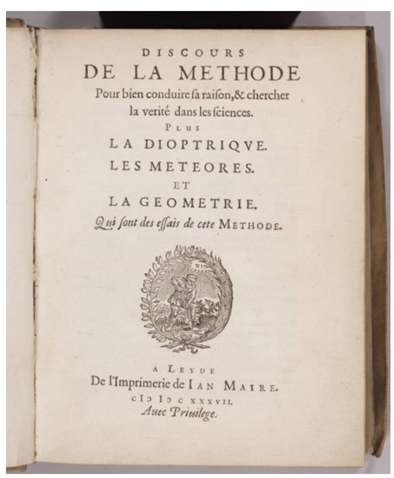


Figure 2.11: Discours de la methodé

4. **Coordinate plane:**

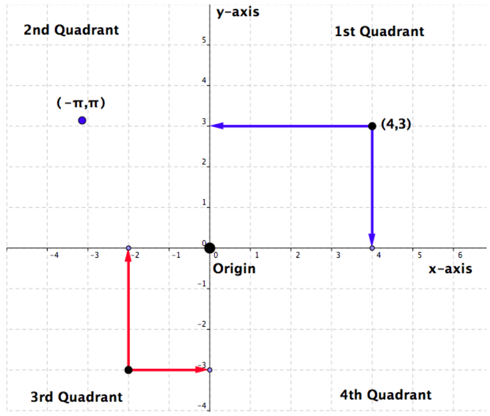


Figure 2.12: Each point in the plane is associated with a unique pair of real numbers representing its position in the grid of vertical and horizontal lines.

5. **Example:**

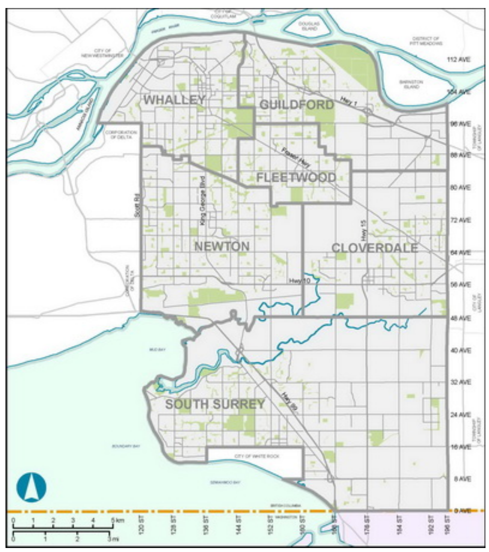


Figure 2.13: Each point in the city is associated with a unique pair of real numbers representing its position in the grid of avenues and streets.

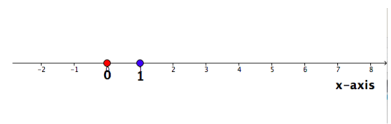
6. **The coordinate plane in slow motion:**

(a) Choose a line

(b) Choose a point, call it 0 (= origin)

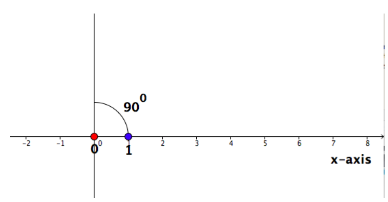
Figure 2.14: Note that we start by choosing any line in the plane and any point 0 on that line.

(a) Choose another point, call it 1

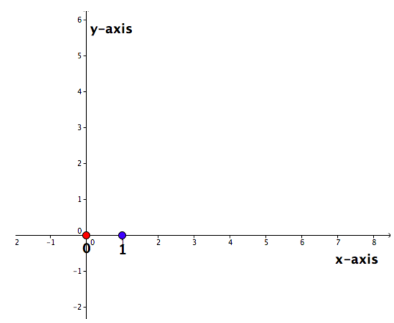


(b) The x-axis

Figure 2.15: Note that by choosing the point 1 we divide all other points on the line, excluding 0, in two sets: Those that are on the same side of 0 as the point 1, and those that are not. This is how we get "positive" and "negative" rays with 0 as their initial point.

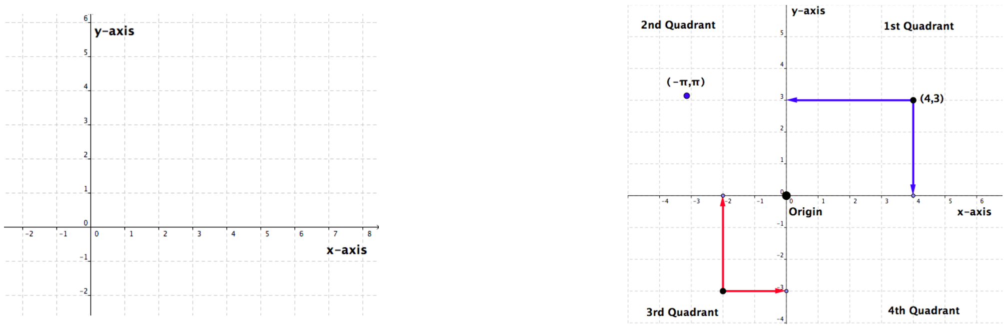


(a) The line perpendicular to the x-axis through 0



(b) The y-axis

Figure 2.16: The x- and y-axis are two mutually perpendicular number lines with the same origin.



(a) The grid

(b) The coordinate plane

Figure 2.17: The coordinate plane establishes an one-to-one correspondence between the points in the plane and the set of all ordered pairs (x, y) of real numbers.

7. **Exercise:** Locate on the coordinate plane the following points:

$$(0,0),\ (1.5,2),\ (-2,2.5),\ (-1,-2.5),\ (1.5,-2),(2,0),(0,2),(-2,0),(0,-2).$$

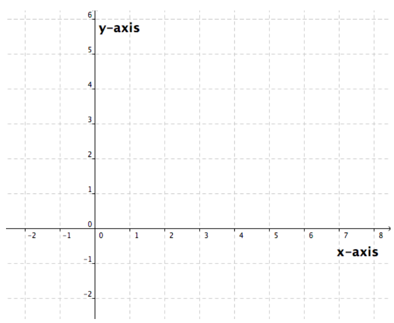


Figure 2.18: Locate the points.

8. **Must Know!** The **graph of a function** f is the set of all ordered pairs $(x, f(x))$ as x varies over the domain of f.

9. **The same, just a little bit different:** The **graph of a function** f is the set

$$\{(x, f(x))|x \in \text{domain of } f\}.$$

10. **Example:** Find the graph of the function

$$f(x) = x^2, \quad x \in \{-2,-1,0,1,2\}.$$

Solution 1:

$$\text{Graph of } f = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}.$$

Solution 2: Graph of f:

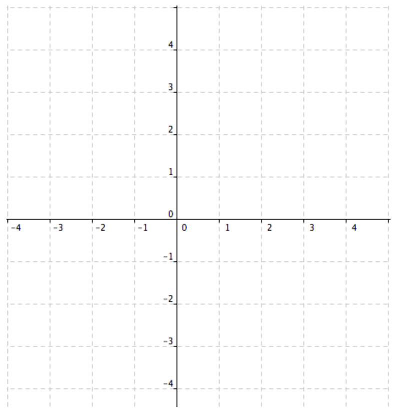


Figure 2.19: Graph of $f(x) = x^2$, $x \in \{-2, -1, 0, 1, 2\}$.

11. **Example:** Graph the function

$$g(x) = \begin{cases} -1 & \text{if} \quad x \in [-2, 0) \\ 1 & \text{if} \quad x = 0 \\ 3 & \text{if} \quad x \in (0, 2] \end{cases}$$

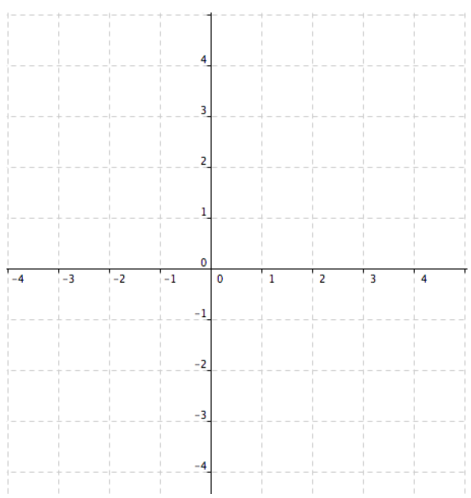
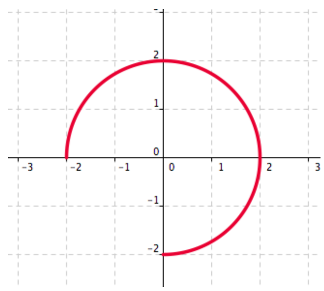
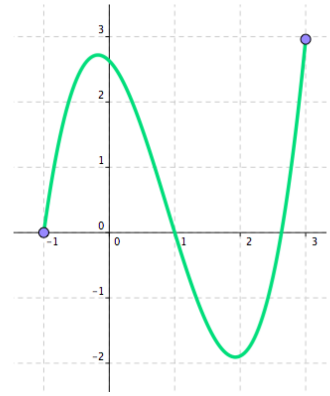


Figure 2.20: Graph of $g(x)$.

12. **Example:** Which of these curves represents a graph of a function?



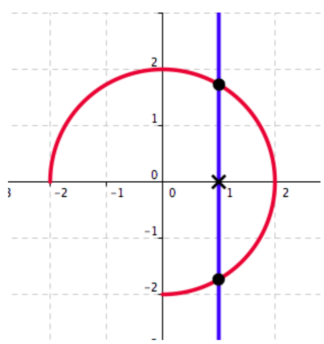
(a) Is this a graph of a function?



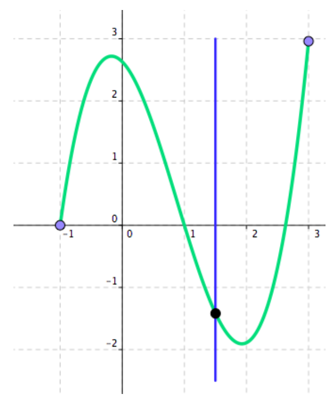
(b) Is this a graph of a function?

Figure 2.21: A **function** is a rule that assigns to each element in a set A exactly one element in a set B.

13. **Vertical Line Test:** Which of these curves represents a graph of a function?



(a) Not a graph of a function!



(b) This a graph of a function!

Figure 2.22: A set of points in the coordinate plane is the graph of a function if and only if every vertical line intersects the set in **at most one point**.

14. **Example:** Let the function f be given by its graph:

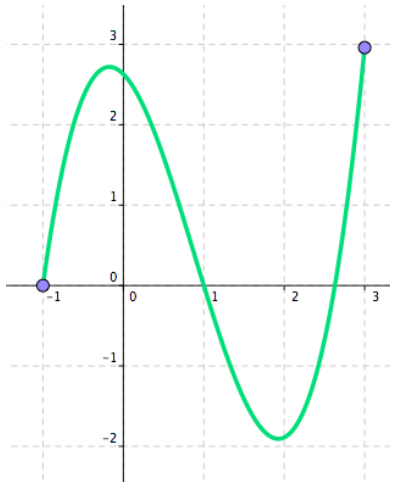


Figure 2.23: Graph of $f(x)$.

(a) Find the domain of f.

(b) Find the range of f.

(c) Find $f(-1)$, $f(1)$, and $f(3)$.

(d) Estimate $f(0)$ and $f(2)$.

(e) How many solutions does the equation $f(x) = 1$ have?

2.3 Lecture 4: Function Transformations and Graphs

1. **Quote.** After climbing a great hill, one only finds that there are many more hills to climb.

   Nelson Mandela, South African anti-apartheid revolutionary, politician, and philanthropist, 1918–2013

2. **Problem.** If we know the graph of the function $y = f(x)$, how can we get the graph of the function $y = -2f(x+2)+1$?

3. **Example:** My friend Pam lives in Calgary, AB. Her older son Brandon studies mathematics at Simon Fraser University, Burnaby, B.C., and her younger son Chris studies physics at the University of Waterloo, Waterloo, ON. Pam is very concerned that her sons are eating their meals regularly so to be able to follow their schedules she has made the following chart:

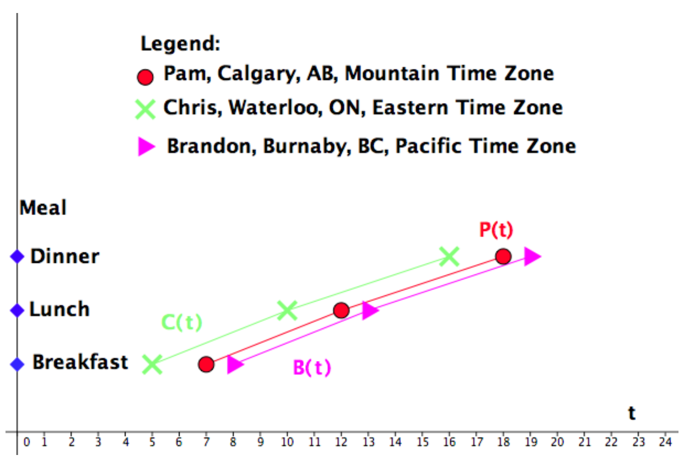


Figure 2.24: Pam's chart: What is "t"?

   What do you notice? How are the three graphs related?

4. **Example Continues:** Two tables and conclusion:

Mountain Time in hours	Eastern Time in hours	Pacific Time in hours
t	$t+2$	$t-1$

For example, if time in Calgary is $t = 5$ what is time in Waterloo? In Burnaby?

Chris		Pam		Brandon	
t	$C(t)$	t	$P(t)$	t	$B(t)$
5	Breakfast	7	Breakfast	8	Breakfast
10	Lunch	12	Lunch	13	Lunch
16	Dinner	18	Dinner	19	Dinner
$t =$ Mountain Time in hours					

Conclusion:

$$C(t) = P(t+2)$$
$$B(t) = P(t-1)$$

5. **Another look:**

$$C(t) = P(t+2)$$
$$B(t) = P(t-1)$$

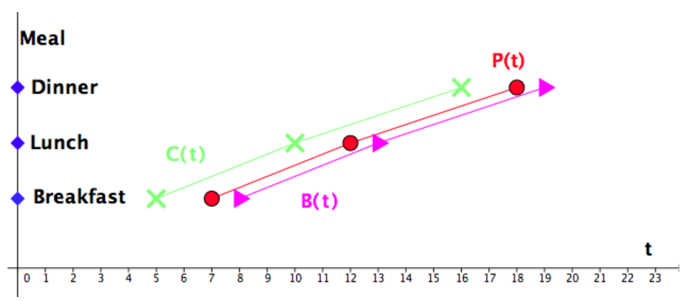


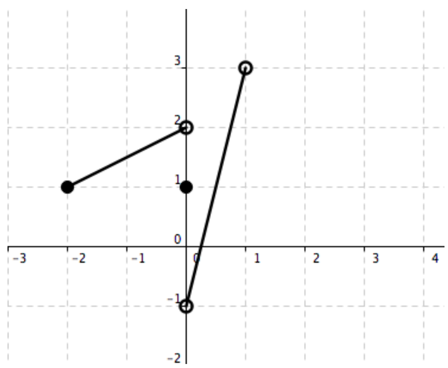
Figure 2.25: The graph of $B(t)$ is obtained by shifting the graph of $P(t)$ horizontally to the right by one unit. What about the graph of $C(t)$?

6. **Horizontal shift:** Let $f : [a, b] \to \mathbb{R}$ and let $m > 0$. Let the function g be defined by $g(x) = f(x - m)$ and let the function h be defined by $h(x) = f(x + m)$.

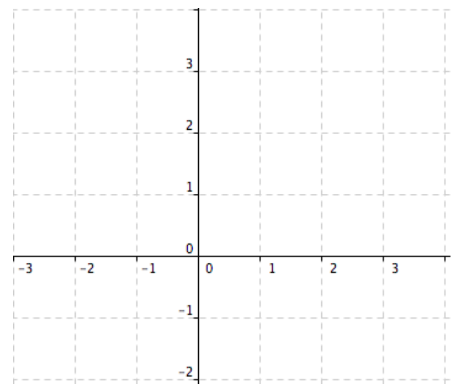
   (a) The domain of the function g is the interval $[a + m, b + m]$ and the graph of $g(x)$ is obtained by shifting the graph of $f(x)$ horizontally to the right by m units.

   (b) The domain of the function h is the interval $[a - m, b - m]$ and the graph of $h(x)$ is obtained by shifting the graph of $f(x)$ horizontally to the left by m units.

7. **Example:** The function f is given by its graph below. Draw the graph of the function g that is defined by $g(x) = f(x+1)$.



(a) The graph of the function f.



(b) The graph of the function $g(x) = f(x+1)$

Figure 2.26: What is the domain of the function f? What is the domain of the function g?

8. **Example:** Pam is very proud of her two sons. Pam thought that she knew her sons' strengths and weaknesses, but she was a bit surprised to find out the following two patterns in their academic records:

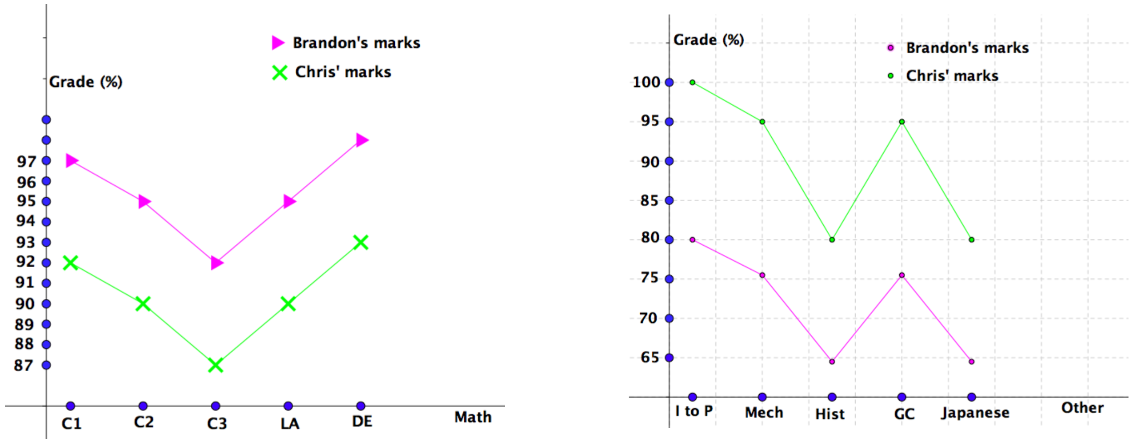
Math			Other subjects		
	Chris	Brandon		Chris	Brandon
Calculus 1	92%	97%	Intro to Physics	100%	80%
Calculus 2	90%	95%	Mechanics	95%	76%
Calculus 3	87%	92%	History	80%	64%
Linear Algebra	90%	95%	General Chemistry	95%	76%
Differential Equations	93%	98%	Japanese 100	80%	64%

Do you see the two patterns?

(a) If x is a math course, let $b(x)$ be Brandon's course grade and let $c(x)$ be Chris' course grade. Then:

(b) If x is one of other courses, let $\mathcal{B}(x)$ be Brandon's course grade and let $\mathcal{C}(x)$ be Chris' course grade. Then:

9. **Patterns:**



(a) $b(x) = c(x) + 5$ – vertical shift

(b) $\mathcal{B}(x) = 0.8 \cdot \mathcal{C}(x)$ – vertical stretch

Figure 2.27: Vertical shift and vertical stretch

10. **Vertical shift:** Let $f : [a, b] \to \mathbb{R}$ and let $n > 0$. Let the function g be defined by $g(x) = f(x) + n$ and let the function h be defined by $h(x) = f(x) - n$.

    (a) The domain of the function g is the interval $[a, b]$ and the graph of $g(x)$ is obtained by shifting the graph of $f(x)$ vertically upwards by n units.

    (b) The domain of the function h is the interval $[a, b]$ and the graph of $h(x)$ is obtained by shifting the graph of $f(x)$ vertically downwards by n units.

11. **Vertical stretch:** Let $f : [a, b] \to \mathbb{R}$ and let $p > 0$. Let the function g be defined by $g(x) = p \cdot f(x)$.

    The domain of the function g is the interval $[a, b]$ and the graph of $g(x)$ is obtained by vertically stretching the graph of $f(x)$ by a factor of p.

12. **Example:** The function f is given by the following graph:

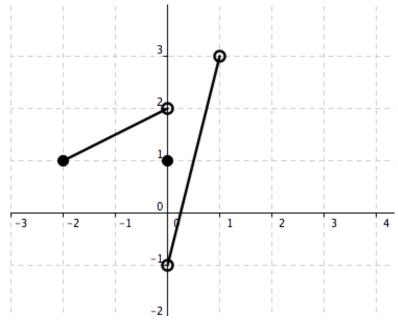


Figure 2.28: The function $y = f(x)$.

Draw the graph of the function

(a) $g(x) = f(x) - 1$

(b) $h(x) = 2 \cdot f(x)$

(a) $g(x) = f(x) - 1$

(b) $h(x) = 2 \cdot f(x)$

Figure 2.29: How would one obtain the graph of the function $i(x) = 2 \cdot f(x) - 1$?

13. **Example:** Last October the whole Pam's family got together for Thanksgiving at her sister Adraina's house in Victoria, B.C. Chris and Brandon decided to use this opportunity to run the Victoria Marathon, their first marathon races ever. Brandon took his training much more seriously than Chris, so his passing times during the race and his overall time were better than his brother's:

Clock	Distance covered (in km)	
	Brandon	Chris
9 : 00 am	0	0
10 : 00 am	10	8
11 : 00 am	20	16
12 : 00	30	24
1 : 00 pm	40	32
1 : 13 : 12 pm	42.2	33.76
2 : 00 pm	−−	40
2 : 16 : 30 pm	−−	42.2

14. **Mathematize this!** Note that Brandon ran the whole race at the constant rate of 10 km/h and that Chris' rate was 8 km/h.

    Let t represent the "running time" in hours, i.e., the time that has elapsed since the start of the race. Then the distances covered during the race, up to the time t (in hours,) for the two runners are given by

Brandon: $\mathscr{B}(t) = 10t,\ t \in [0, 4.22]$

Chris: $\mathscr{C}(t) = 8t,\ t \in [0, 5.28]$

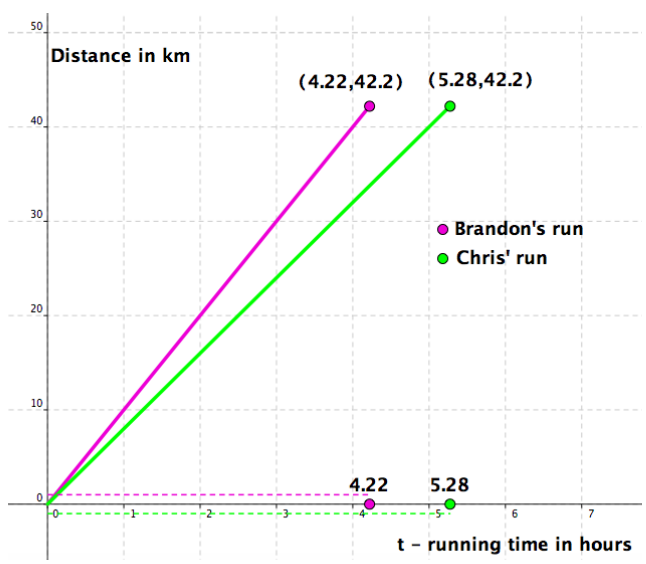


Figure 2.30: Graphs representing Brandon and Chris' distances covered during the race as functions of their running times

15. **Observation:** Note that
$$10 = 10 \cdot \frac{8}{8} = 8 \cdot \frac{10}{8} = 8 \cdot \frac{5}{4}.$$
It follows that, for $t \in [0, 4.22]$,
$$\mathscr{B}(t) = 10t = 8 \cdot \frac{5}{4}t = \mathscr{C}\left(\frac{5}{4}t\right).$$

16. **Closer look:**
$$\mathscr{B}(t) = \mathscr{C}\left(\frac{5}{4}t\right), \ t \in [0, 4.22].$$

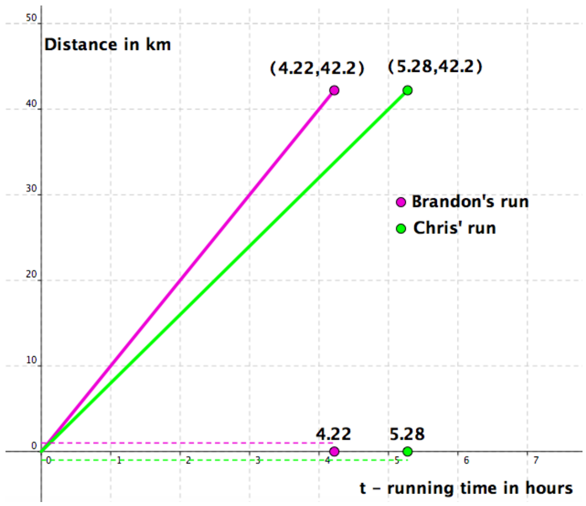


Figure 2.31: The graph of $\mathscr{B}(t)$ is obtained by horizontally stretching the graph of $\mathscr{C}(t)$ by a factor of $\mathbf{\frac{4}{5}}$.

17. **Horizontal stretch:** Let $f : [a, b] \to \mathbb{R}$ and let $q > 0$. Let the function g be defined by $g(x) = f(q \cdot x)$.

    The domain of the function g is the interval $\left[\frac{1}{q} \cdot a, \frac{1}{q} \cdot b\right]$ and the graph of $g(x)$ is obtained by horizontally stretching the graph of $f(x)$ by a factor of $\frac{1}{q}$.

18. **Example:** The function f is given by the following graph:

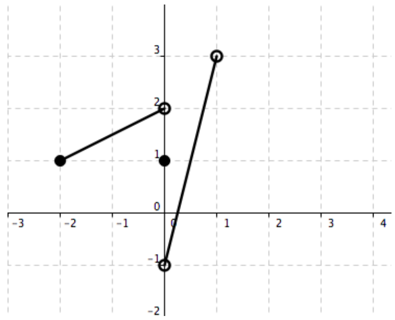
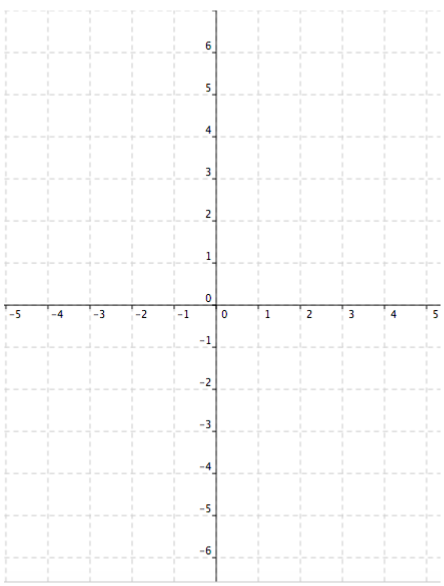


Figure 2.32: The function $y = f(x)$.

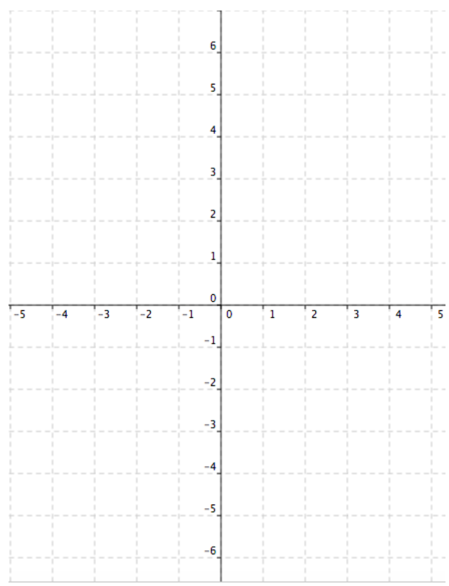
Draw the graph of the function

(a) $g(x) = f(2x)$

(b) $h(x) = f\left(\frac{1}{2} \cdot x\right)$



(a) $g(x) = f(2x)$



(b) $h(x) = f\left(\frac{1}{2} \cdot x\right)$

Figure 2.33: Horizontal stretch ($q = 1/2$) and horizontal contraction ($q = 2$)

19. **Reflections:** The function f is given by the following graph:

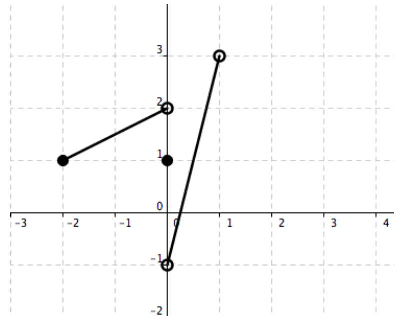
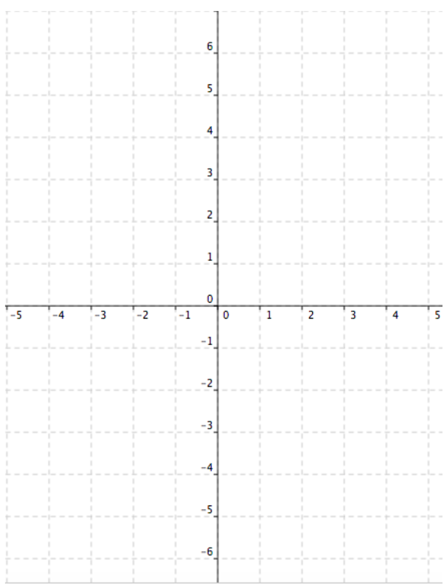


Figure 2.34: The function $y = f(x)$.

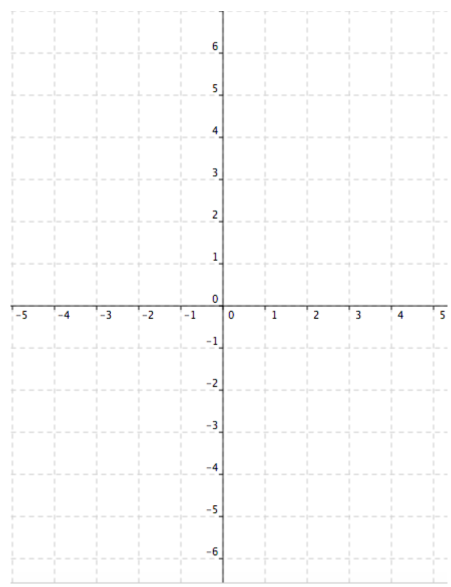
Draw the graph of the function

(a) $g(x) = -f(x)$

(b) $h(x) = f(-x)$



(a) $g(x) = -f(x)$



(b) $h(x) = f(-x)$

Figure 2.35: Reflections across the x- and y-axis

20. **Summary:** Assume that a graph of the function $f : [a, b] \to \mathbb{R}$ is given and assume that m, n, p, and q are positive real numbers.

New function	Effect	New Domain	Action*
$g(x) = f(x - m)$	Horizontal shift to the right by m units	$[a + m, b + m]$	$(x, y) \mapsto (x + m, y)$
$g(x) = f(x + m)$	Horizontal shift to the left by m units	$[a - m, b - m]$	$(x, y) \mapsto (x - m, y)$
$g(x) = f(x) + n$	Vertical shift upwards by n units	$[a, b]$	$(x, y) \mapsto (x, y + n)$
$g(x) = f(x) - n$	Vertical shift downwards by n units	$[a, b]$	$(x, y) \mapsto (x, y - n)$
$g(x) = pf(x)$	Vertical stretch by a factor of p	$[a, b]$	$(x, y) \mapsto (x, py)$
$g(x) = f(qx)$	Horizontal stretch by a factor of $1/q$	$[a/q, b/q]$	$(x, y) \mapsto (x/q, y)$
$g(x) = -f(x)$	Reflection across the x-axis	$[a, b]$	$(x, y) \mapsto (x, -y)$
$g(x) = f(-x)$	Reflection across the y-axis	$[-b, -a]$	$(x, y) \mapsto (-x, y)$

* (x, y) is a point on the graph of the function f.

21. **Example:** The function f is given by the following graph:

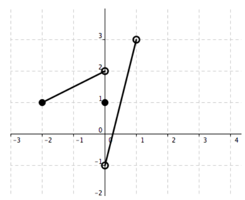
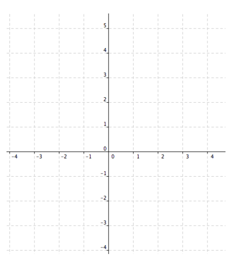
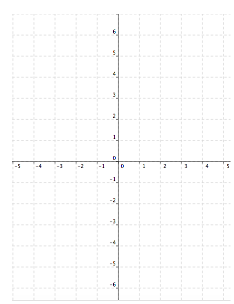


Figure 2.36: The function $y = f(x)$.

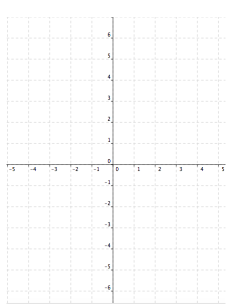
Draw the graph of the function $y = -2f(x + 2) + 1$.



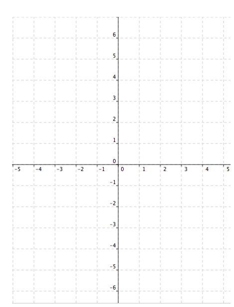
(a) $g(x) = f(x + 2)$



(b) $h(x) = 2f(x + 2)$



(a) $i(x) = -2f(x + 2)$



(b) $y = -2f(x + 2) + 1$

22. **Even and odd functions:**

Even: We say that $f : [-a, a] \to \mathbb{R}$ is an even function if

$$f(-x) = f(x) \text{ for all } x \in [-a, a].$$

Odd: We say that $f : [-a, a] \to \mathbb{R}$ is an odd function if

$$f(-x) = -f(x) \text{ for all } x \in [-a, a].$$

23. **Example:** Check if the given function is even, odd or, neither.

    (a) $f(x) = x^2$

    (b) $g(x) = x^3$

    (c) $h(x) = x^3 + x^2$

24. **Example:** Check if the given function is even, odd or, neither.

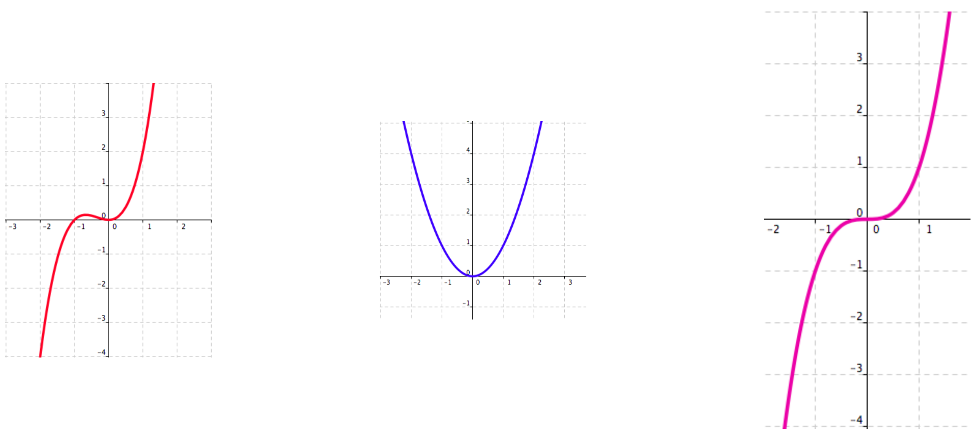


Figure 2.39: Even, odd, or neither?

2.4 Lecture 5: Composition of Functions

1. **Quote.** No one is an artist unless he carries his picture in his head before painting it, and is sure of his method and composition.

   Oscar-Claude Monet, a founder of French Impressionist painting, 1840 – 1926

2. **Problem:** If $f(x) = x^2$ and $g(x) = 2x$ what is $g(f(3))$?

3. **Example:** The Northwest Coast people have a long tradition of making various products for every day use from cedar roots. Generally, one can divide this procedure in two phases:

   (a) Gathering and processing cedar roots.

   (b) Weaving of an object.

   Note: Different parts of the cedar are used to make different objects. For example, the inner cedar bark is commonly used for making traditional hats and skirts. The very outer bark is used to make canoe balers. Some hats are made from spruce roots.

4. **Two illustrations and two questions:**

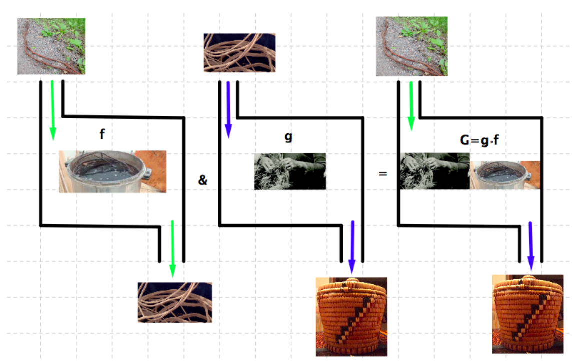


Figure 2.40: The function f represents the procedure in which raw cedar roots are inputs and processed cedar roots are outcomes. The function g represents the procedure of weaving a basket. The function $G = g \circ f$, the composition of f and g, represents the procedure in which raw cedar roots are inputs and baskets are outcomes.

   (a) **Question:** Does $f \circ g$ make sense?

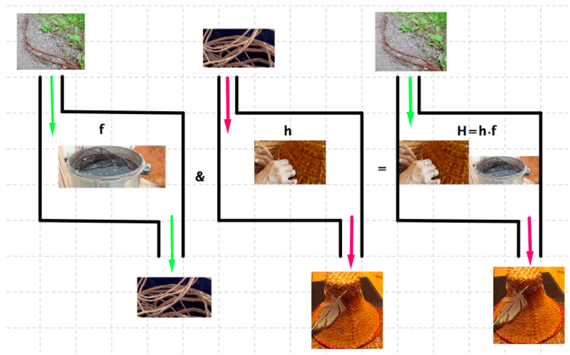


Figure 2.41: The function f represents the procedure in which raw cedar roots are inputs and processed cedar roots are outcomes. The function h represents the procedure of weaving a hat. The function $H = g \circ f$, the composition of f and h, represents the procedure in which raw cedar roots are inputs and hats are outcomes.

(b) **Question:** Is it true that

$$g \circ f = h \circ f \ ?$$

5. **Composition of Functions:** Let $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ be two functions. **We define a new function** $F = [a, b] \to \mathbb{R}$ by

$$F(x) = g(f(x)), x \in [a, b].$$

The function F is called **the composition of** g **and** f and it is **denoted** by $F = g \circ f$.

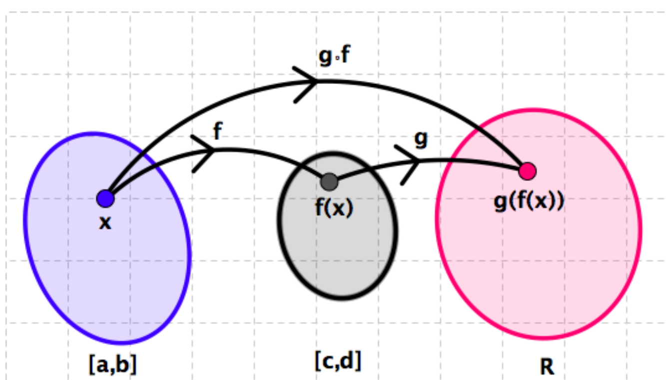


Figure 2.42: The function composition of the functions f and g associates x from the domain of f with $g(f(x))$ from the range of g.

6. **Example:** If $f(x) = x^2$ and $g(x) = 2x$ what is $g(f(3))$? What is $f(g(3))$?

7. **Example:** Find $(g \circ f)(x)$ and $(f \circ g)(x)$ and their domains if

   (a) $f(x) = 3x + 2$ and $g(x) = -\frac{3}{4}x + \frac{1}{2}$

   (b) $f(x) = 2x + 1$ and $g(x) = \frac{1}{x}$

8. **Example:** Sonoko is a Ph.D. candidate in the Department of Mathematics at Dalhousie University, Halifax, Nova Scotia. Recently her paper on the accessibility of the set of Fibonacci numbers was accepted in The Fibonacci Quarterly, the official publication of the Fibonacci Association. To celebrate this great news Sonoko invited her roommate Janice for a dinner at Presto Panini Cafe, Sonoko's favourite Italian restaurant.

   Sonoko ordered a bowl of penne pasta in pesto sauce for \$15 and Janice'c choice was a bowl of gnocchi in gorgonzola sauce with spinach and toasted walnuts that costed \$17. To celebrate this very special occasion the girls shared a piece of Presto Panini's famous tiramisu (\$8) and a 1/2 litter of Chianti (\$20.)

   In Nova Scotia sales taxes applied to foodservice are calculated as a single tax, called the Harmonized Sales Tax (HST), of 15% of the total food and beverage purchase.

   Since as an undergraduate student she used to work as a waitress, Sonoko is a very generous tipper. Her tipping policy is to give 8/5, the ratio of the sixth and the fifth terms of the Fibonacci sequence, of the HST amount to the server.

   Questions:

   (a) What was the amount of the HST on Sonoko and Janice's bill?

(b) What was the tip that Sonoko left for the server?

(c) Suppose that x is the amount in dollars of the total food and beverage purchase. Let $h(x)$ be the function that associates the amount of the HST to the amount x. Find an expression for $h(x)$.

(d) Let $f(t) = \frac{8}{5} \cdot t$. Express Sonoko's tipping policy in terms of the functions h and f.

9. **MUST KNOW - PREPARATION FOR CALCULUS:** Decomposition of functions: The process of breakdown of a function into less complex functions.

Figure 2.43: Decomposition of the *oreo function*

10. **Examples:** Express the given function as the composition of two or more functions:

(a) $F(x) = (2x + 1)^2$

(b) $G(x) = 3x^2 + 1$

(c) $H(x) = 3(2x + 1)^2 + 1$

11. **Identity function:** The **identity function** is the function I defined by

$$I(x) = x$$

for every real number x.

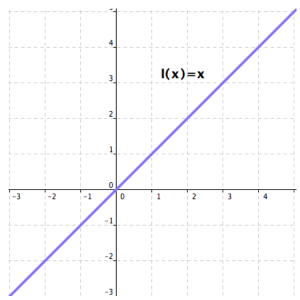


Figure 2.44: The identity function $I(x) = x$

12. **Important observation:** For any function f

$$f \circ I = I \circ f = f.$$

13. **Building a bank of functions: Linear functions.** A **linear function** is a function of the form

$$f(x) = mx + b$$

where m and b are given numbers.

14. **Examples:** Which of the following functions is NOT linear:

    (a) $I(x) = x$
    (b) $f(x) = -2x + 3$
    (c) $g(x) = x^2 + 1$
    (d) $h(x) = -1$

15. **Four graphs:**

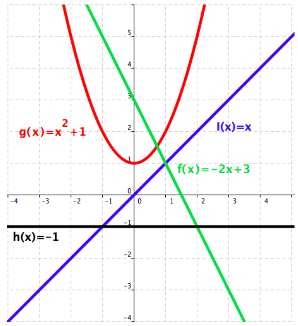


Figure 2.45: The graph of a linear function is a straight line

16. **Exercise:** Prove that the composition of two linear functions a linear function.

17. **Making new from old:** Let f and g be two real valued functions with the same domain D and let a and b be two real numbers. We **define** functions $af + bg$, $f \cdot g$, and $\frac{f}{g}$ in the following way:

$$\begin{aligned} (af + bg)(x) &= af(x) + bg(x) \\ (f \cdot g)(x) &= f(x) \cdot g(x) \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \text{ if } g(x) \neq 0 \end{aligned}$$

18. **Example:** Let $f(x) = 2x + 1$ and $g(x) = x - 3$. Evaluate

    (a) $(f - g)(2) =$

    (b) $(f \cdot g)(2) =$

    (c) $\left(\frac{f}{g}\right)(3) =$

2.5 Lecture 6: Inverse Functions

1. **Quote.** Just as we have two eyes and two feet, duality is part of life.

   Carlos Santana, a Mexican and American musician, 1947 –

2. **Problem:** Let $f(x) = 3x + 1$. For a given $y \in \mathbb{R}$ find x such that $f(x) = y$.

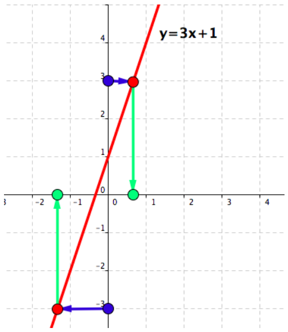
3. **Example:** Each student, faculty, and staff member at SFU is given a unique ID number. Who is the member of the SFU community with the ID number 401234567?

Figure 2.46: Simona Fraser's SFU ID number is 123456789. Who is the member of the SFU community with the ID number 401234567?

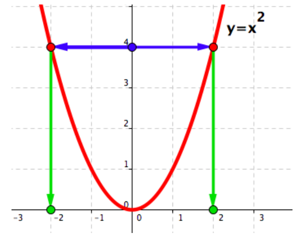
4. **Example:** Each member of the SFU community has a unique birthday date. Who is the member of the SFU community that was born on June 12?

Figure 2.47: There are at least two SFU community members born on June 12.

5. **Question:** For which functions "going backwards" is again a function?



(a) $y = 3x + 1$

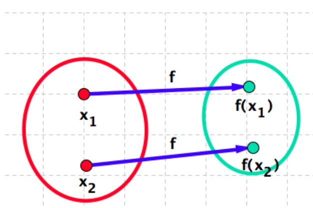


(b) $y = x^2$

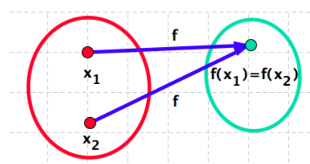
Figure 2.48: Going backwards

6. **One-to-one functions:** We say that a function f is **one-to-one** if different inputs have different outputs. In other words, f is one-to-one if

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$



(a) One-to-one



(b) **Not** one-to-one

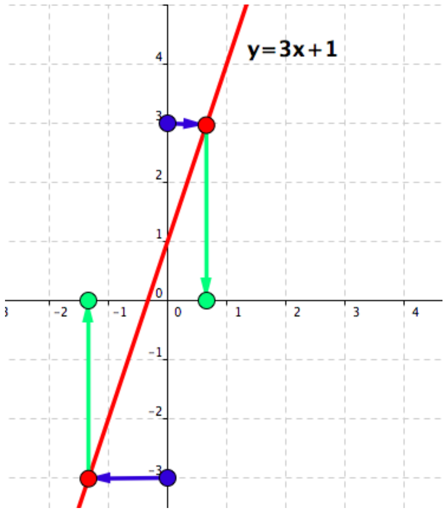
Figure 2.49: A function f is **one-to-one** if different inputs have different outputs.

7. **Example:** One-to-one or not one-to-one?

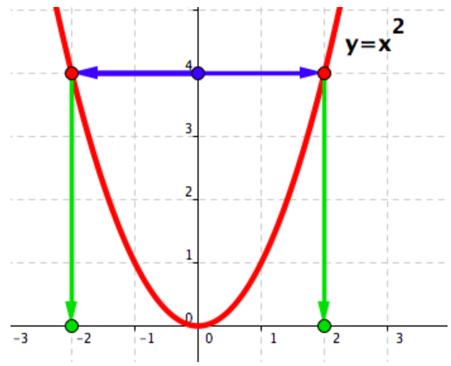
   (a) SFU ID Number Function?

   (b) Birthday Date Function?

8. **Example:**



(a) $f(x) = 3x + 1$

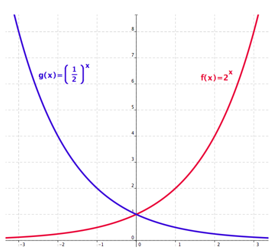


(b) $g(x) = x^2$

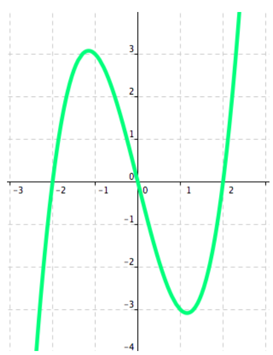
Figure 2.50: One-to-one or not one-to-one?

9. **Horizontal line test:** A graph represents an one-to-one function if and only if every horizontal line intersects that graph at most once.

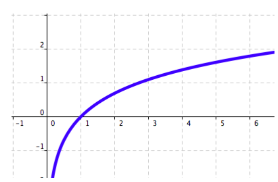
10. **Example:**



(a) A: Yes; B: No



(b) A: Yes; B: No

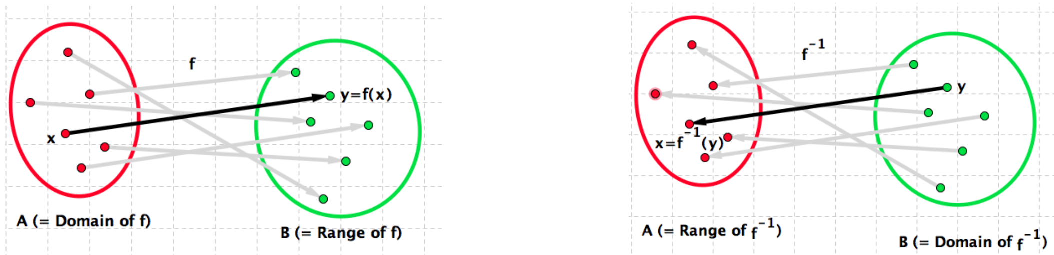


(c) A: Yes; B: No

Figure 2.51: Use the horizontal line test to determine if the graph represents an one-to-one function.

11. **Inverse function:** Let $f : A \to B$ be an one-to-one function and let the set B be the range of the function f. The **inverse function of** f, denoted f^{-1}, is the function defined on the domain B by:

    For any x in A and y in B, if $f(x) = y$ then $f^{-1}(y) = x$.



(a) f is one-to-one

(b) f^{-1} is the inverse of f

Figure 2.52: By definition: If $f(x) = y$ then $f^{-1}(y) = x$

12. **Example:** Let f be the "SFU ID Number" function. Then f is one-to-one and if "You" is a member of the SFU community then

$$f(\text{You}) = \text{Your SFU ID Number}.$$

    Does f^{-1} exist and if it does what is f^{-1}(Dr. J's SFU ID Number)?

13. **Example:** Let $f(x) = 3x + 1$.

    (a) Find the real number x for which $f(x) = 8$.

    (b) For a given $y \in \mathbb{R}$ find x such that $f(x) = y$.

14. **Four important facts:** Let f be an one-to-one function and let f^{-1} be its inverse function. Then:

    (a) For any x in the domain of f: $\boxed{(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x}$.
    Example: Check this fact for the function $f(x) = 3x + 1$.

    (b) For any x in the range of f: $\boxed{(f \circ f^{-1})(x) = f(f^{-1}(x)) = x}$.
    Example: Check this fact for the function $f(x) = 3x + 1$.

    (c) $\boxed{(f^{-1})^{-1} = f}$
    Example: Check this fact for the function $f(x) = 3x + 1$.

    (d) If the point (x, y) belongs to the graph of f then the point (y, x) belongs to the graph of the function f^{-1}. In other words, graphs of f and f^{-1} are symmetric with respect to the line $y = x$.

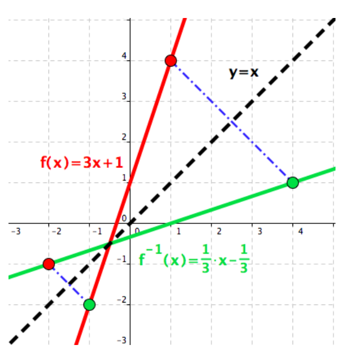
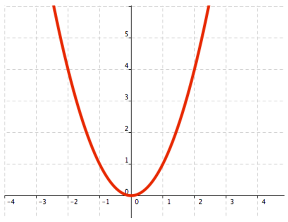
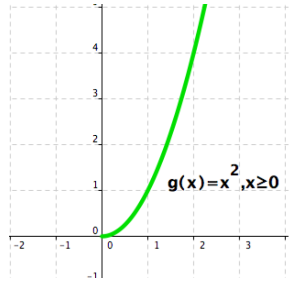


Figure 2.53: Graphs of f and f^{-1} are symmetric with respect to the line $y = x$

15. **About to meet an old friend:**



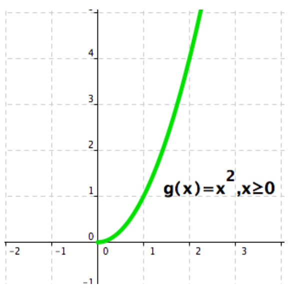
(a) $f(x) = x^2$, $x \in \mathbb{R}$



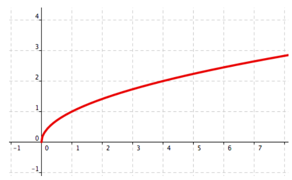
(b) $g(x) = x^2$, $x \geq 0$

Figure 2.54: Which function is one-to-one?

16. **An old friend:**



(a) $g(x) = x^2$, $x \geq 0$



(b) $g^{-1}(x) = \sqrt{x}$, $x \geq 0$

Figure 2.55: The function $g^{-1}(x) = \sqrt{x}$, $x \geq 0$, is the inverse function of the function $g(x) = x^2$, $x \geq 0$.

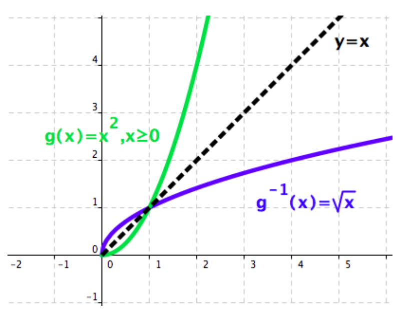


Figure 2.56: Graphs of g and g^{-1} are symmetric with respect to the line $y = x$

17. **Example:** Let $f(x) = x^2$, $x \in \mathbb{R}$, $g(x) = x^2$, $x \geq 0$, and $g^{-1}(x) = \sqrt{x}$. Investigate the following functions:

    (a) $g \circ g^{-1}$

    (b) $f \circ g^{-1}$

    (c) $g^{-1} \circ g$

    (d) $g^{-1} \circ f$

18. **Two important examples and a fact that you have to know:**

    (a) Let $f(x) = x$. Check that f is one-to-one and find its inverse function f^{-1}.

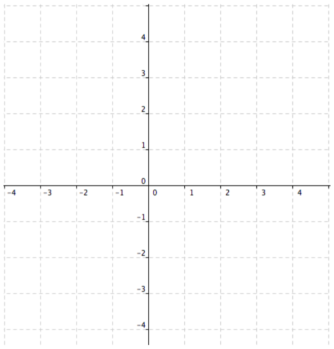


Figure 2.57: $f(x) = x$

    (b) Let $g(x) = \dfrac{1}{x}$. Check that g is one-to-one and find its inverse function g^{-1}.

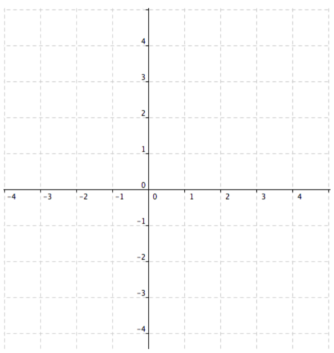


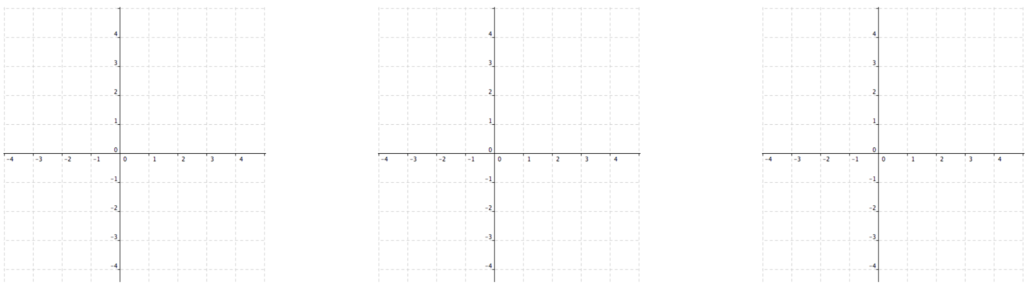
Figure 2.58: $f(x) = \frac{1}{x}$

    (c) **DO NOT CONFUSE $f^{-1}(x)$ AND $\dfrac{1}{f(x)}$: Those are two different mathematical objects.**

19. **Example:** Let $f(x) = \dfrac{x+2}{x+1}$.

(a) Find the domain of f.

(b) Sketch the graph of the function f:



(a) $g(x) = \frac{1}{x}$ (b) $h(x) = g(x+1)$ (c) $f(x) = h(x) + 1$

Figure 2.59: How to obtain the graph of the function $f(x) = \dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$ in three steps.

(c) Check that the function f is one-to-one.

(d) Find f^{-1}.

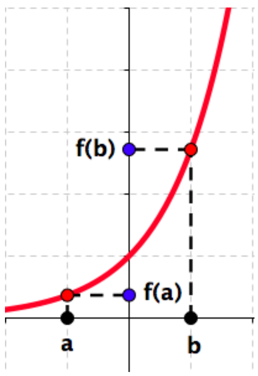
20. **Increasing and decreasing functions:**

    - A function f is **strictly increasing** on the interval I if for any $a, b \in I$

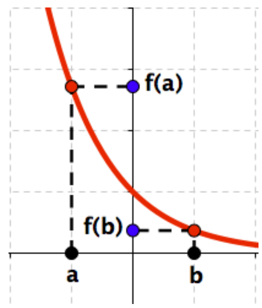
$$a < b \Rightarrow f(a) < f(b).$$

    - A function f is **strictly decreasing** on the interval I if for any $a, b \in I$

$$a < b \Rightarrow f(a) > f(b).$$



(a) $a < b \Rightarrow f(a) < f(b)$.



(b) $a < b \Rightarrow f(a) > f(b)$.

Figure 2.60: Which function is strictly increasing? Which function is strictly decreasing?

21. **Fact:** If a function f is strictly increasing (or strictly decreasing) on its domain then f is an one-to-one function.

22. **Exercise - A taste of calculus:** Determine the intervals where the function

$$f(x) = \frac{5}{3}x^3 - 5x, x \in [-2, 2]$$

is strictly increasing and where it is strictly decreasing.

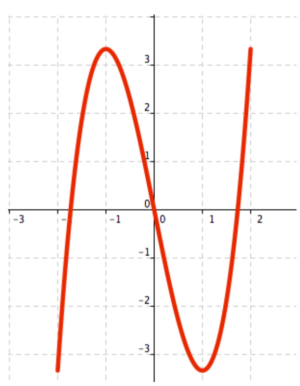


Figure 2.61: $f(x) = \frac{5}{3}x^3 - 5x, x \in [-2, 2]$

Chapter 3

Polynomials and Rational Functions

3.1 Lecture 7: Lines and Linear Functions

1. **Quote.** What is straight? A line can be straight, or a street, but the human heart, oh, no, it's curved like a road through mountains.

   Thomas Lanier "Tennessee" Williams III, American playwright and author, 1911 – 1983

2. **Problem:** A Toronto based cellular telephone service provider offers the following plan for \$55 per month:

Data:	1 GB included and \$5/250 MB for additional data
Talk:	Unlimited Canada-wide minutes
Messaging:	Unlimited international text, picture, and video messaging.

   If x is the number of MB of data used, what is the monthly bill before the applicable taxes?

3. **Example:** On August 15, 2017, the exchange rate between Canadian and US currency was:

   $$\$1 \text{ CAN} = \$0.78 \text{ US}.$$

   (a) Dr. J was attending a math related event in Seattle, WA, and he needed to stay overnight in Seattle. Dr. J booked a hotel room online on August 15, 2017, and was charged \$212.86 CAN. What was the price for the room in the US dollars?

(b) On August 15, 2017, Dr. J and his wife went to Bellingham, WA, to visit a friend who teaches mathematics at Western Washington University. Dr. J used his credit card to pay \$37.50 US for lunch at Harris Avenue Cafe. What was the cost of the lunch in Canadian dollars?

(c) Let f be the function that models the exchange rate from Canadian to US currency on August 15, 2017. Write the function f in terms of x, the amount of money in Canadian dollars.

(d) Let g be the function that models the exchange rate from US to Canadian currency on August 15, 2017. Write the function g in terms of x, the amount of money in US dollars.

4. **Linear Function:** A **linear function** is any function of the form

$$f(x) = mx + b$$

where m and b are given real numbers.

5. **Example:** for the given values of m and b write the corresponding linear function:

(a) $m = 0$, $b = 55$

(b) $m = \frac{1}{50}$, $b = 55$

(c) $m = 0.78$, $b = 0$

6. **Three important tasks:** Let a linear function $f(x) = mx + b$, $m, b \in \mathbb{R}$ be given.

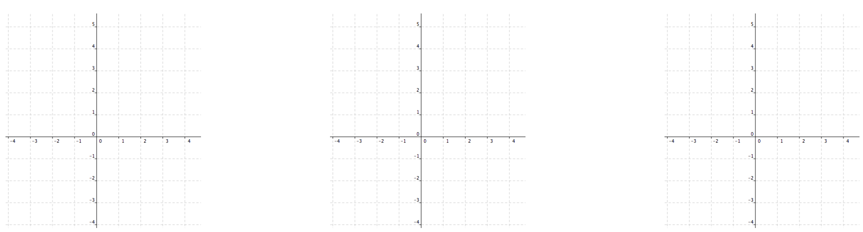
   (a) Determine the domain of f.

   (b) Determine the range of f.

   (c) Discuss the monotonicity of f.

7. **Fact:** The graph of a linear function is a straight line.

8. **Example:** Draw the graphs of the following functions:



(a) $f(x) = 2x + 1$ (b) $g(x) = -2x + 1$ (c) $h(x) = 1$

Figure 3.1: The graph of a linear function is a straight line.

9. **Functions and Graphs:**

Figure 3.2: "Are we to paint what's on the face, what's inside the face, or what's behind it..." - Pablo Picasso.

10. **Linear Functions and Lines:** Let m and b be given real numbers.

Linear function: $f(x) = mx + b$	Equation of the line: $y = mx + b$
Graph of $f = \{(x, y) : y = mx + b\}$	

11. **Big Question:** If $y = mx+b$ is the equation of a line, what is the meaning of the coefficients m and b?

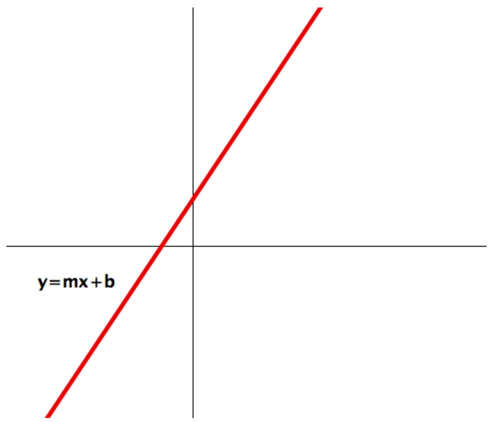


Figure 3.3: What is the meaning of the coefficients m and b?

12. **Two Old Friends:** Let $y = mx + b$ be the equation of a line.

- The number m is called **the slope of the line**. Also, for any two points (x_1, y_1) and (x_2, y_2), $x_1 \neq x_2$, on the line we have that

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Rise}}{\text{Run}}.$$

- The number b is called the y-**intercept**.

13. **Important:** To find the equation of a line you need to know at least two facts about that line:

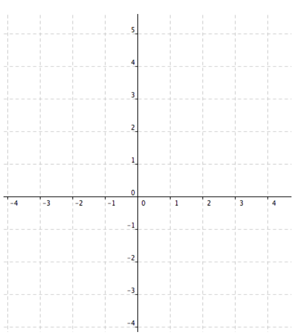
    (a) If the slope m and the y-intercept b are given then the equation of the line is:

    (b) If the slope m and one point (x_1, y_1) on the line are given then the equation of the line is:

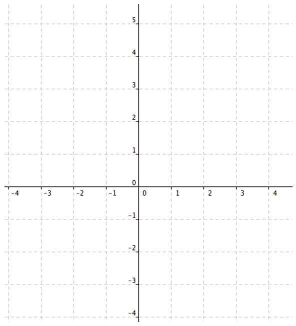
    (c) If two points (x_1, y_1) and (x_2, y_2) on the line are given then the equation of the line is:

14. **Examples:** Find the equation of the line such that

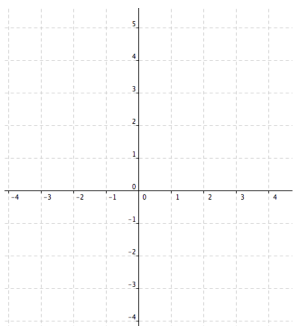
    (a) its slope is $m = -1.5$ and its y-intercept is $b = -1$



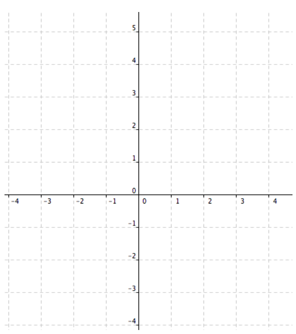
(b) its slope is $m = \dfrac{1}{3}$ and it passes through the point $(-1, 2)$



(c) it passes through the points $\left(-\dfrac{1}{2}, -\dfrac{1}{2}\right)$ and $\left(\dfrac{1}{3}, \dfrac{2}{3}\right)$



(d) it has no slope and passes through the point $\left(-\dfrac{1}{2}, -\dfrac{1}{2}\right)$



15. **Problem:** In 1930, the world record for the 1500-m run was 3:49.2 min (= 3.82 min.) In 1950, it was 3:43 min (= 3.72 min.) Let R represent the record in the 1500-m run and let t be the number of years since 1930.

    (a) Find a linear function $R(t)$ that fits the data.

    (b) Use the function of part (a) to predict the record in 2017.

    (c) Predict the year when the record may be 3 min.

16. **Parallel Lines:** Two non-vertical lines with slopes m_1 and m_2 are parallel if and only if

$$m_1 = m_2.$$

Example.

(a) Are the lines $2x - 4y = 3$ and $4x - 8y = -3$ parallel?

(b) Is the line that contains the points $(1, -3)$ and $(-2, 4)$ parallel to the line that contains the points $(-9, 1)$ and $(-2, -2)$?

(c) Find the equation of the line that contains the point $(3, 2)$ and is parallel to the line $3x + y = -3$.

(d) Is the line $x = 4$ parallel to the line $x = -4$?

17. **Perpendicular Lines.** Two non vertical lines with slopes m_1 and m_2 are perpendicular if and only if

$$m_1 \cdot m_2 = -1.$$

Examples.

(a) Are the lines $4x - 3y = 2$ and $4x + 3y = -7$ perpendicular?

(b) Is the line that contains the points $(-3, 2)$ and $(4, -1)$ perpendicular to the line that contains the points $(1, 3)$ and $(-2, -4)$?

(c) Find the equation of the line that contains the point $(2, -5)$ and is perpendicular to the line $y = \frac{5}{2}x - 4$.

(d) Is the line $y = 5$ perpendicular to the line $x = 5$?

3.2 Lecture 8: Quadratic Functions and Conics

1. **Quote.** Art is not a mirror to reflect the world, but a hammer with which to shape it.

   Vladimir Vladimirovich Mayakovsky, Russian poet, playwright, artist, and stage and film actor, 1893 – 1930

2. **Problem:** A hockey team plays in an arena that has a seating capacity of 15,000 spectators. With the ticket price set at $50, average attendance at recent games has been 10,500. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 100.

   Find a function that models the revenue in terms of ticket price.

3. **Quadratic Function:** A **quadratic function** is a function f of the form

$$f(x) = ax^2 + bx + c$$

   where a, b, and c are real numbers with $a \neq 0$.

4. **Example:** Which of the following functions is NOT a quadratic function:

   (a) $f(x) = x^2 + 2x + 1$

   (b) $f(x) = (x + 1)^2$

   (c) $f(x) = \dfrac{1}{x^2 + 2x + 1}$

   (d) $f(x) = 2x + x^2 + 1$

5. **Warm Up:**

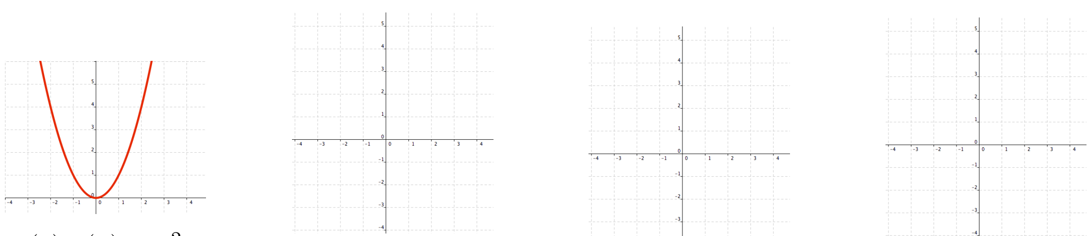
(a) Check that for any $a, b, c, x \in \mathbb{R}$, $a \neq 0$,

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

(b) **Complete the square**, i.e., re-write the function $f(x) = 2x^2 - 5x + 2$ in the form

$$f(x) = A(x + B)^2 + C.$$

(c) Draw a graph of the function $f(x) = 2x^2 - 5x + 2$.



(a) $g(x) = x^2$ (b) $h(x) = g\left(x - \frac{5}{4}\right)$ (c) $i(x) = 2h(x)$ (d) $f(x) = i(x) - \frac{9}{8}$

Figure 3.4: The graph of $f(x) = 2x^2 - 5x + 2 = 2\left(x - \frac{5}{4}\right)^2 - \frac{9}{8}$.

6. **All You Need to Know About Quadratic Functions:** Let a quadratic function

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}, a, b, c \in \mathbb{R}, a \neq 0$$

be given.

(a) What is the domain of the function f?

(b) Find the zeros of the function f, i.e., solve the equation $f(x) = 0$.

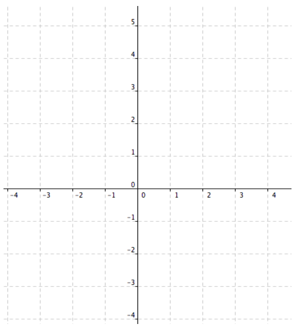
Summary:

$b^2 - 4ac > 0$	Two solutions: $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$
$b^2 - 4ac = 0$	One solution: $x = -\frac{b}{2a}$
$b^2 - 4ac < 0$	No solution

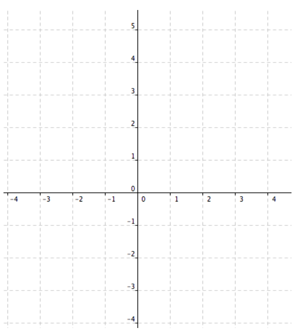
(c) Evaluate $f\left(-\frac{b}{2a}\right)$

(d) Check the following two facts and discuss the intervals of increase and decrease in each case:

i. If $a > 0$ then, for all $x \neq -\dfrac{b}{2a}$, $f(x) > -\dfrac{b^2 - 4ac}{4a}$.

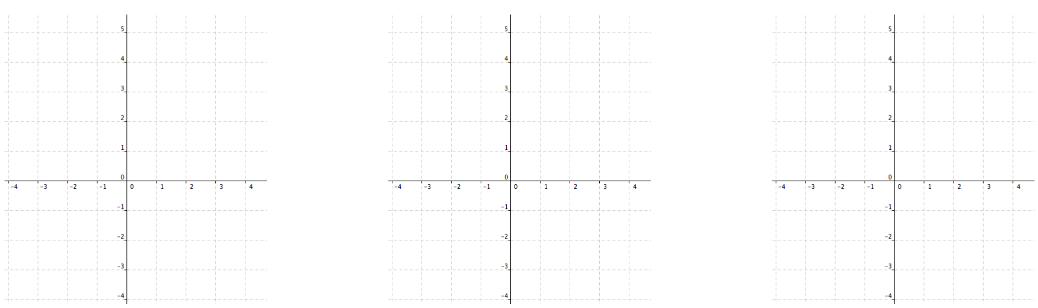


ii. If $a < 0$ then, for all $x \neq -\dfrac{b}{2a}$, $f(x) < -\dfrac{b^2 - 4ac}{4a}$.



(e) Putting everything together

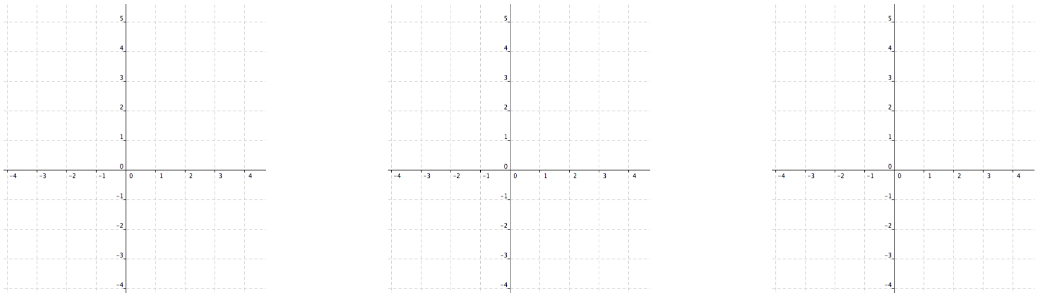
i. Let $a > 0$. Then



(a) Case: $b^2 - 4ac > 0$ (b) Case: $b^2 - 4ac = 0$ (c) Case: $b^2 - 4ac < 0$

Figure 3.5: If $a > 0$ then the graph of f is a 'smiley parabola'.

ii. Let $a < 0$. Then



(a) Case: $b^2 - 4ac > 0$ (b) Case: $b^2 - 4ac = 0$ (c) Case: $b^2 - 4ac < 0$

Figure 3.6: If $a < 0$ then the graph of f is a 'sad parabola'.

7. **Fast Motion:** Draw a graph of the function $f(x) = -x^2 + x + 2$.

The x-intercepts: solve $f(x) = 0$	The sign of a	The vertex $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$	The y-intercept

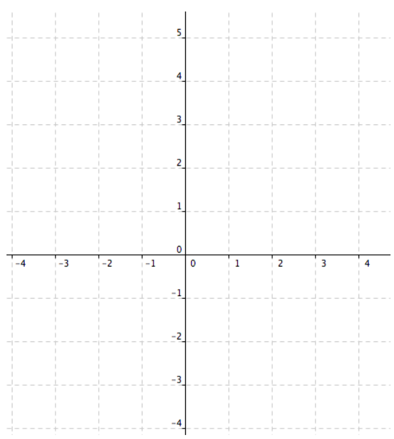


Figure 3.7: Graph of $f(x) = -x^2 + x + 2$

8. **Meet . . . Conics:**

YouTube 1
YouTube 2

9. **Circle:**

   (a) A circle is the set of all points in a plane that are at a given distance from a given point, the centre.The distance between any of the points and the centre is called the radius of the circle.

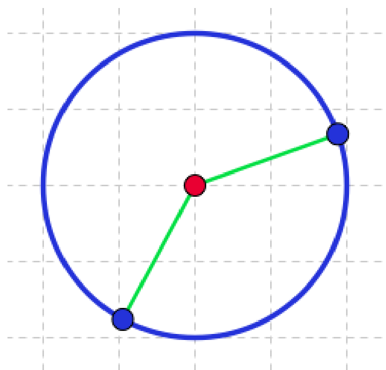


Figure 3.8: A circle

   (b) Evaluate the distance between the two points, $A = (x_1, y_1)$ and $B = (x_2, y_2)$, i.e., find the length of the line segment $\overline{AB}$.

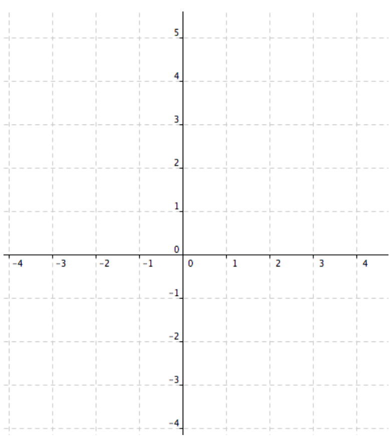


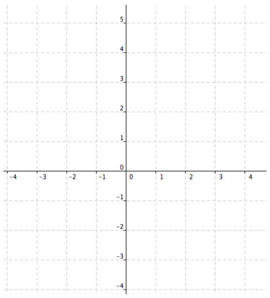
Figure 3.9: Pythagorean Theorem - The most famous theorem of all

   (c) The equation of the circle with the centre at the point (a, b) and the radius r is given by

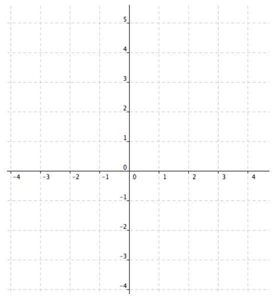
$$(x - a)^2 + (y - b)^2 = r^2.$$

10. **Examples:**

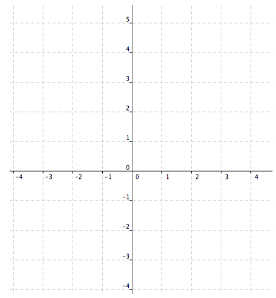
    (a) Draw the following circles:



(a) The unit circle: $x^2 + y^2 = 1$



(b) $(x-2)^2 + y^2 = 2$



(c) $(x-1)^2 + (y-1)^2 = 2$

Figure 3.10: Three circles

    (b) Write the equation of the circle with the centre at the point $(-1, 1)$ and radius $r = \pi$.

    (c) Find the centre and the radius of the circle

$$x^2 + 2x + y^2 - 6y = 0.$$

11. **Elipse:**

   (a) An ellipse is the set of all points in a plane with the property that the sum of the distances to two given points equals to the given number.

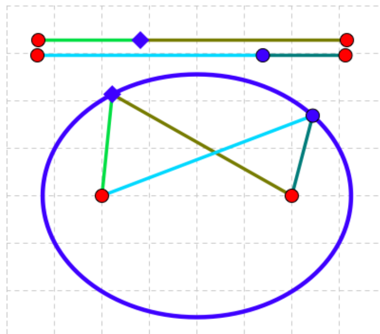


Figure 3.11: An ellipse

   (b) The standard form of the equation of an ellipse centered at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

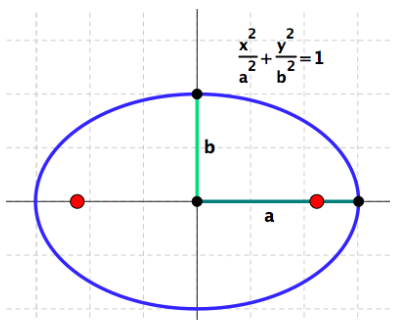


Figure 3.12: The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

12. **Hyperbola:**

   (a) A hyperbola is the set of all points in a plane with the property that the absolute value of the difference of the distances to two given points equals to the given number.

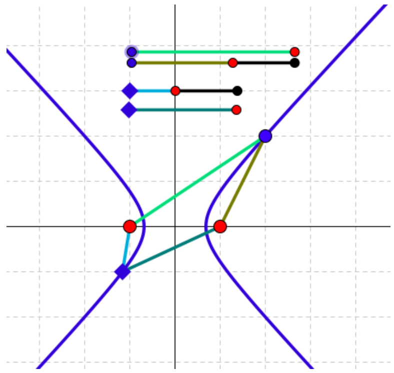


Figure 3.13: A hyperbola

   (b) The standard form of the equation of a hyperbola centered at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

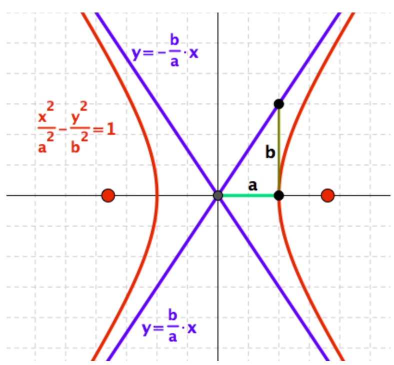


Figure 3.14: The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with its asymptotes $y = \pm\frac{b}{a} \cdot x$

13. **The Most Important Hyperbola:**

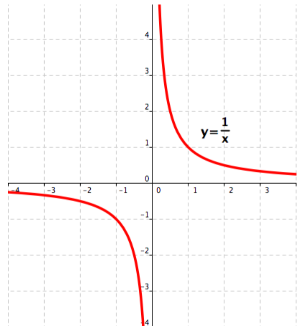
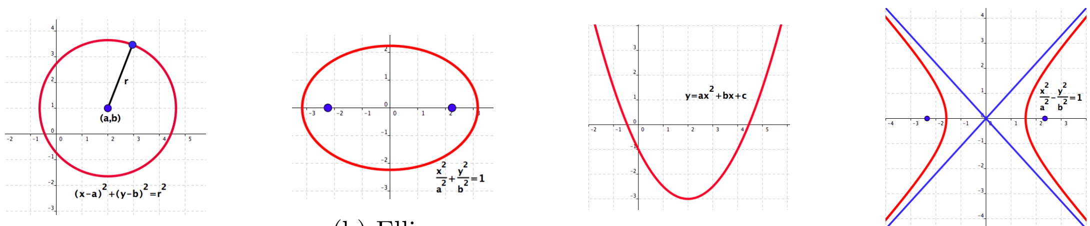


Figure 3.15: The hyperbola $y = \dfrac{1}{x}$ with its asymptotes the x- and y-axis

14. Conic Sections - Summary:



(a) Circle (b) Ellipse (c) Parabola (d) Hyperbola

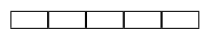
Figure 3.16: Conic Sections

3.3 Lecture 9: Power Functions

1. **Quote.** When the power of love overcomes the love of power the world will know peace.

   Jimi Hendrix, American rock guitarist, singer, and songwriter, 1942 – 1970

2. **Question:** How many different 5-digit numbers can you write by using only digits 1 and 2?

   What if you use the digits 1, 2, and 3?

3. **Positive Integer Exponent:** If x is a real number and m is a positive integer than

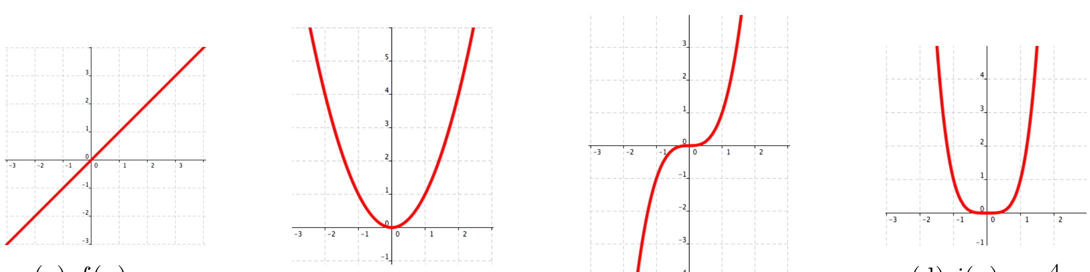
$$x^m = \underbrace{x \cdot x \cdot \ldots \cdot x}_{m-\text{times}}$$

4. **Power Function:** If m is a positive integer then the function

$$f(x) = x^m$$

   is called a **power function**.

5. **Example.**



(a) $f(x) = x$ (b) $g(x) = x^2$ (c) $h(x) = x^3$ (d) $i(x) = x^4$

Figure 3.17: Four power functions

6. **Facts About Power Functions:** Let a power function

$$f(x) = x^m, m \in \mathbb{N}$$

be given.

(a) What is the domain of the function f?

(b) What is the range of the function f?

(c) Find the zeros of the function f, i.e., solve the equation $f(x) = 0$.

(d) Determine the intervals of increase and decrease for the function f.

(e) **Summary:**

$f(x) = x^m, m \in \mathbb{N}$	m even	m odd
Domain		
Range		
$f(x) = 0$		
Increasing		
Decreasing		
Odd or even		

7. **Properties of Positive Integers Exponents:** Let x and y be real numbers and let m and n be positive integers. Then

$$\begin{aligned} x^m \cdot x^n &= x^{m+n} \\ (x^m)^n &= x^{mn} \\ x^m \cdot y^m &= (xy)^m \end{aligned}$$

8. **Examples:** Evaluate

(a) $3^4 \cdot 3^3 - (3^3)^2$

(b) $(3^3 + 6^3)^2 - 2 \cdot 18^3$

(c) $(x^n + y^n)^2 - 2 \cdot (xy)^n$, where $n \in \mathbb{N}$

9. **Detective J:** Evaluate:

(a) $\dfrac{3^5}{3^2} =$

(b) $\dfrac{3^2}{3^5} =$

(c) $\dfrac{3^5}{3^5} =$

(d) Let $x \neq 0$ and $m, n \in \mathbb{N}$. Then

$$\frac{x^m}{x^n} = \begin{cases} & \text{if} \quad m > n \\ \frac{1}{} & \text{if} \quad m < n \\ & \text{if} \quad m = n \end{cases}$$

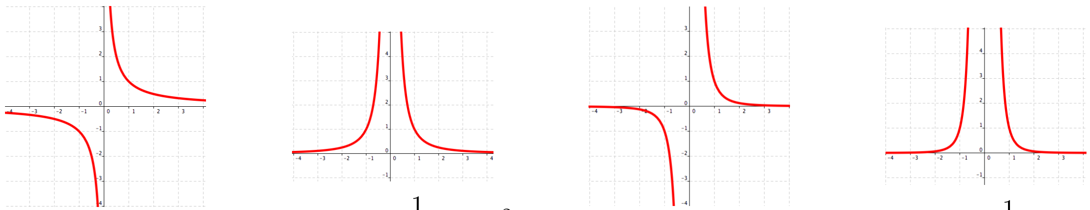
10. **Two Definitions:** Let $x \neq 0$ and $m \in \mathbb{N}$. Then, by definition

$$x^{-m} = \frac{1}{x^m}$$

and

$$x^0 = 1.$$

11. **More Power Functions:**



(a) $f(x) = \frac{1}{x} = x^{-1}$ (b) $g(x) = \frac{1}{x^2} = x^{-2}$ (c) $h(x) = \frac{1}{x^3} = x^{-3}$ (d) $i(x) = \frac{1}{x^4} = x^{-4}$

Figure 3.18: Four power functions

12. **More Facts About Power Functions:** Let a power function

$$f(x) = x^{-m} = \frac{1}{x^m}, m \in \mathbb{N}$$

be given.

(a) What is the domain of the function f?

(b) What is the range of the function f?

(c) Find the zeros of the function f, i.e., solve the equation $f(x) = 0$.

(d) Determine the intervals of increase and decrease for the function f.

(e) **Summary:**

$f(x) = x^{-m}, m \in \mathbb{N}$	m even	m odd
Domain		
Range		
$f(x) = 0$		
Increasing		
Decreasing		
Odd or even		

13. **More detective work:**

    (a) What is 2^3?

    (b) What is 2^{-3}?

    (c) What is $2^{\frac{22}{7}}$?

    (d) What is 2^{π}?

14. **Definition:** For $m \in \mathbb{N}\backslash\{1\}$ and $x \in \mathbb{R}$ we **define** $x^{\frac{1}{m}}$ as the real number for which

$$\left(x^{\frac{1}{m}}\right)^m = x$$

if such number exists. If there are two numbers with this property (this may happened if m is even) we choose one that is non-negative.

The number $x^{\frac{1}{m}}$ is called the m^{th} root of x

Notation:

$$x^{\frac{1}{m}} = \sqrt[m]{x}, m \geq 3$$
$$x^{\frac{1}{2}} = \sqrt{x}$$

15. **Examples:** Evaluate

    (a) $16^{\frac{1}{4}} =$

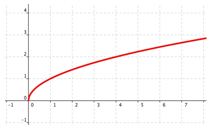
    (b) $\sqrt[4]{16} =$

    (c) $(-8)^{\frac{1}{3}} = \sqrt[3]{-8} =$

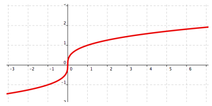
    (d) $4^{\frac{1}{2}} = \sqrt{4} =$

    (e) $(-4)^{\frac{1}{2}} = \sqrt{-4}$

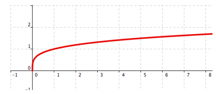
16. **Four more graphs:**



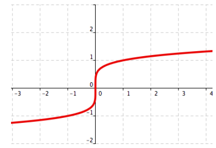
(a) $f(x) = x^{\frac{1}{2}} = \sqrt{x}$



(b) $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$



(c) $f(x) = x^{\frac{1}{4}} = \sqrt[4]{x}$



(d) $f(x) = x^{\frac{1}{5}} = \sqrt[5]{x}$

Figure 3.19: $y = x^{\frac{1}{m}} = \sqrt[m]{x}, m = 2, 3, 4, 5$

17. **Even More Facts About Power Functions:** Let a power function

$$f(x) = x^{\frac{1}{m}} = \sqrt[m]{x}, m \in \mathbb{N}\backslash\{1\}$$

be given.

(a) What is the domain of the function f?

(b) What is the range of the function f?

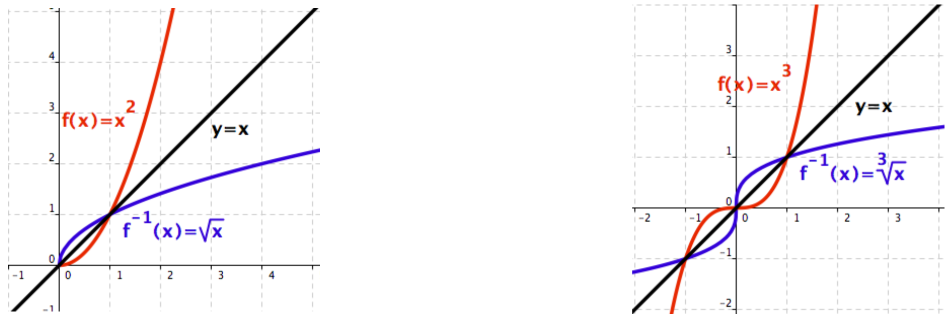
(c) Find the zeros of the function f, i.e., solve the equation $f(x) = 0$.

(d) Determine the intervals of increase and decrease for the function f.

(e) **Summary:**

$f(x) = \sqrt[m]{x}, m \in \mathbb{N}\backslash\{1\}$	m even	m odd
Domain		
Range		
$f(x) = 0$		
Increasing		
Decreasing		
Odd or even		

18. **Functions and their inverses:**



(a) $f(x) = x^2$, $x \geq 0$, and $f^{-1}(x) = \sqrt{x}$

(b) $f(x) = x^3$ and $f^{-1}(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$

Figure 3.20: Functions and their inverses

19. **Facts:**

    (a) If m is an even integer then $f(x) = x^m$, $x \geq 0$, is an one-to-one function and its inverse function is $f^{-1}(x) = x^{\frac{1}{m}} = \sqrt[m]{x}$. The domain of f^{-1} is the set of nonnegative real numbers.

    (b) If m is an odd integer then $f(x) = x^m$, $x \in \mathbb{R}$, is an one-to-one function and its inverse function is $f^{-1}(x) = x^{\frac{1}{m}} = \sqrt[m]{x}$. The domain of f^{-1} is the set of all real numbers.

20. **Definition:** If $m \in \mathbb{N}$, $n \in \mathbb{Z}$, and $x \in \mathbb{R}$ then

$$x^{\frac{n}{m}} = \left(x^{\frac{1}{m}}\right)^n$$

whenever this is defined.

21. **Must Know** Let x and y be real numbers and let m and n be rational numbers. Then

$$\begin{aligned}
x^m \cdot x^n &= x^{m+n} \\
(x^m)^n &= x^{mn} \\
x^0 &= 1 \\
\frac{x^m}{x^n} &= x^{m-n}, \ x \neq 0 \\
x^m \cdot y^m &= (xy)^m \\
\left(\frac{x}{y}\right)^m &= \frac{x^m}{y^m}, \ y \neq 0
\end{aligned}$$

whenever those numbers are defined.

22. **Example:** Simplify
$$\left(\frac{x^3y^2z}{x^2y^2z^3}\right)^3 =$$

3.4 Lecture 10: Polynomials

1. **Quote.** Truth is much too complicated to allow anything but approximations.

   John Von Neumann, Hungarian-American pure and applied mathematician, physicist, inventor, and polymath, 1903 – 1957

2. **Why polynomials:** The irrational number e is an important mathematical constant. It is approximately equal to 2.718282846. A scientific calculator gives that

   $$e^{\frac{1}{2}} = \sqrt{e} \approx 1.6487212707.$$

   Problem: Is it possible to quickly find a reasonable approximation of the number $\sqrt{e}$ by using only the four basic arithmetic operations: addition, subtraction, multiplication and division?

   Try this:

   - Let $p_1(x) = 1 + x$. Then $p_1\left(\frac{1}{2}\right) =$
   - Let $p_2(x) = 1 + x + \frac{1}{2} \cdot x^2$. Then $p_2\left(\frac{1}{2}\right) =$
   - Let $p_3(x) = 1 + x + \frac{1}{2} \cdot x^2 + \frac{1}{6} \cdot x^3$. Then $p_3\left(\frac{1}{2}\right) =$

3. **Polynomial:** A polynomial is a function p such that

   $$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

   where n is a nonnegative integer and $a_0, a_1, a_2, \ldots, a_n$ are given numbers with $a_n \neq 0$.

4. **Why polynomials:**

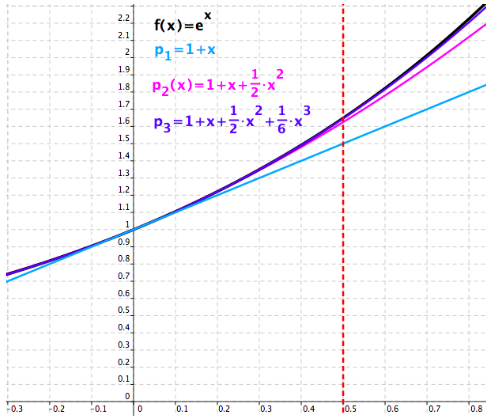


Figure 3.21: Polynomials as approximations of the function $f(x) = e^x$

5. **Vocabulary and Facts:** Let

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, a_n \neq 0,$$

be a polynomial.

- The number n is called the **degree** of the polynomial p and it is denoted by $\deg p = n$.
  Example. Determine the degree of the polynomial

  $$p(x) = 3x + 4 + 5x^3 - x^4 - x^2.$$

  (a) $\deg p = 1$
  (b) $\deg p = 2$
  (c) $\deg p = 3$
  (d) $\deg p = 4$

- If the number x_0 is such that

  $$p(x_0) = 0$$

  then x_0 is called **a zero** of the polynomial p.
  Example. Which of the following numbers is a zero of of the polynomial

  $$p(x) = x^3 - 2x^2 + x.$$

  (a) $x = 1$
  (b) $x = 2$
  (c) $x = 3$
  (d) $x = 4$

- If the number x_0 is a zero of the polynomial p then there is a polynomial $q(x)$ such that

  $$p(x) = (x - x_0) \cdot q(x).$$

Example. Check that

$$p(x) = x^3 - 2x^2 + x = (x-1)(x^2 - x).$$

- If the number x_0 is a zero of the polynomial p and if there are a number m and a polynomial $q(x)$ such that

$$p(x) = (x - x_0)^m \cdot q(x)$$

with $q(x_0) \neq 0$ then we say that x_0 is a **zero of order** m.

Example. Check that $x = 1$ is a zero of order 2 of the polynomial

$$p(x) = x^3 - 2x^2 + x = (x-1)(x^2 - x).$$

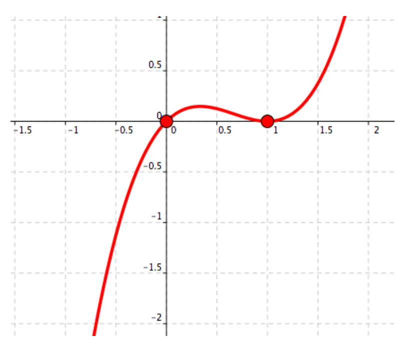
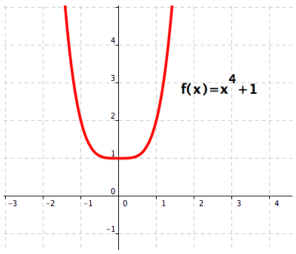


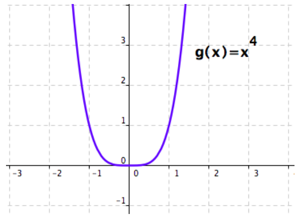
Figure 3.22: The zeros of $p(x) = x^3 - 2x^2 + x = x(x-1)^2$

- A polynomial of degree n can have at most n zeros.

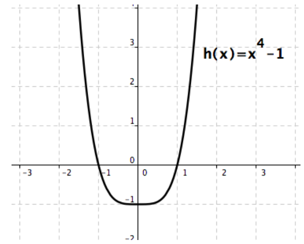
  Example.



(a) $f(x) = x^4 + 1$

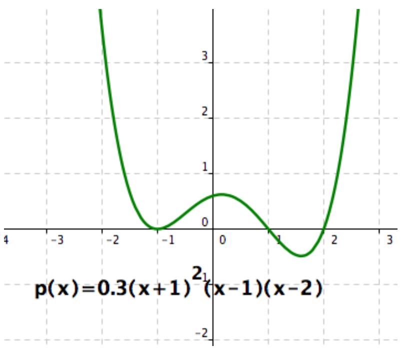


(b) $g(x) = x^4$

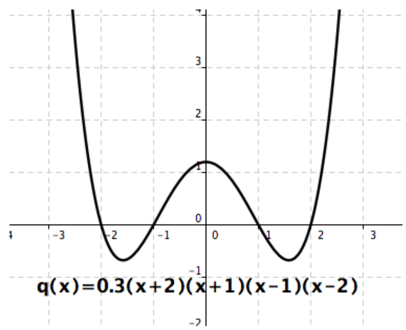


(c) $h(x) = x^4 - 1$
$= (x-1)(x+1)(x^2+1)$

Figure 3.23: Three polynomials of degree 4 - How many zeros?



(a) $p(x) = 0.3(x+1)^2(x-1)(x-2)$



(b) $q(x) = 0.3(x+2)(x+1)(x-1)(x-2)$

Figure 3.24: Two more polynomials of degree 4 - How many zeros?

6. **All You Need to Know About Polynomials:** Let

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n, a_n \neq 0,$$

be a polynomial.

(a) What is the domain of the function p?

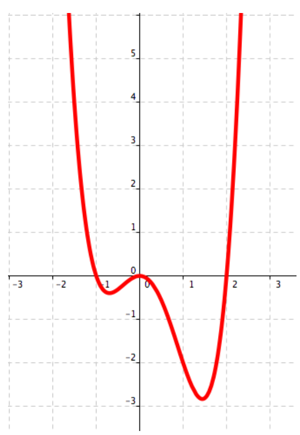
(b) Determine the behaviour of the polynomial p if $x \to \infty$ and $x \to -\infty$. The same ... just little bit different: What can you say about the ratio

$$\frac{p(x)}{a_nx^n} = \frac{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n}{a_nx^n}$$

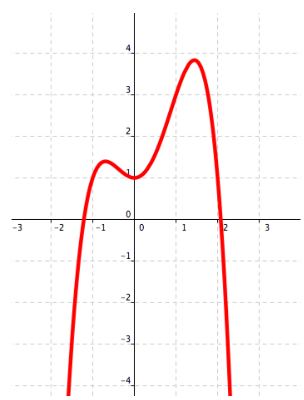
if x is such that $|x|$ is very big?

(c) What is the range of the function p?

(a) Odd degree with a positive leading coefficient



(b) Even degree with a positive leading coefficient



(c) Even degree with a negative leading coefficient

Figure 3.25: The range of a polynomial

(d) Find the zeros of the function p, i.e., solve the equation $p(x) = 0$.

Question: You know that p is a polynomial of an odd degree, with a positive leading coefficient, and that

$$p(-1) = 1, p(0) = -1, p(1) = 1, p(2) = 0.$$

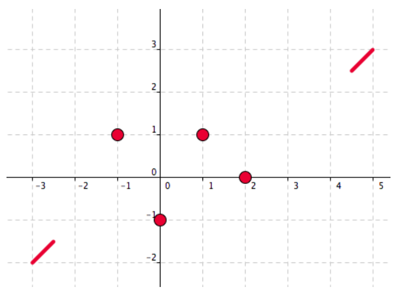


Figure 3.26: A mysterious polynomial of an odd degree with a positive leading coefficient

i. What can you tell about the number of zeros of the polynomial p?

ii. What can you tell about the degree of the polynomial p?

3.5 Lecture 11: Rational Functions

1. **Quote.** It has been said that man is a rational animal. All my life I have been searching for evidence which could support this.

   Bertrand Arthur William Russell, 3rd Earl Russell, British philosopher, logician, mathematician, historian, writer, social critic and political activist, 1872 – 1970

2. **Why rational functions:** Rational functions model data with singularities.

   Example: The distance between the City of Vancouver, B.C., and the City of Abbotsford is estimated to be 68.5 km.

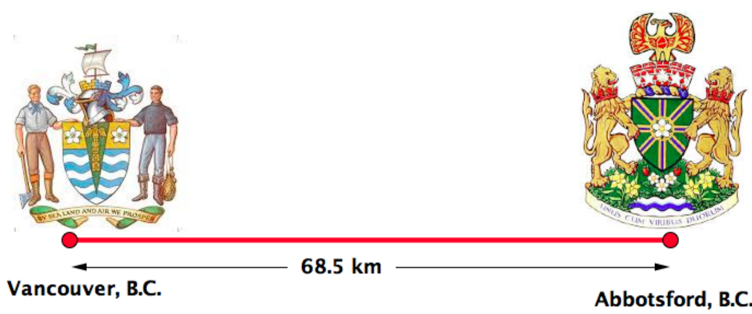


Figure 3.27: Assume that the distance between the two cities is 68.5 km

   Complete the table below, i.e., find the travelling time t from Vancouver, B.C., to Abbotsford, B.C., if the average speed s is given.

   Model: From

$$\text{Distance} = (\text{average speed}) \cdot (\text{travelling time})$$

   it follows that

$$t = \frac{68.5}{s}.$$

vehicle	$s =$ average speed in km/h	$t =$ travelling time in hours
Car	100	
Racing motorbike	300	
Racing bicycle	38	
Horse	44	
Walk	5.4	
Snail	10^{-3}	
Light	$1.079 \cdot 10^9$	

3. **Old friend:** Function $f(x) = \dfrac{1}{x}$.

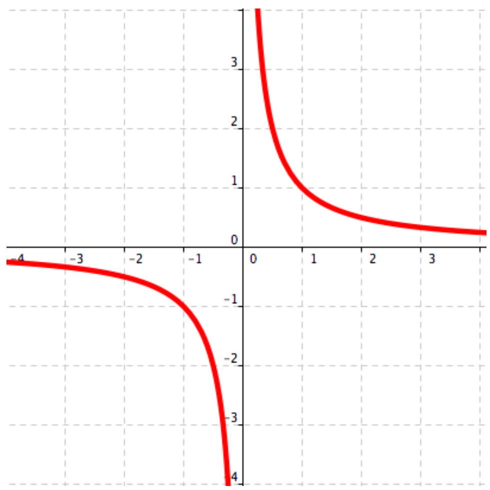


Figure 3.28: Function $f(x) = \dfrac{1}{x}$

Quick facts:

Domain:	$\mathbb{R}\backslash\{0\}$	VA:	$x = 0$
Range:	$\mathbb{R}\backslash\{0\}$	HA:	$y = 0$
This is a decreasing function			

How to introduce your friend $f(x) = \dfrac{1}{x}$ in a few words: His nickname is Inverse Proportion. He is very odd: if you give him less than one he will return you more than one, if you give him more than one, he will return less than one. If you take more than one he will ask for less than one, if you take less than one he will ask for more than one. Just make sure that you give something or that you take something. Otherwise…

4. **Rational functions.** Any function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials is called a **rational function**.

5. **Example:** Which of the following functions is **NOT** rational?

   (a) $f(x) = 3x^2 + 5x - 2$

   (b) $g(x) = \dfrac{1}{x}$

   (c) $h(x) = \dfrac{3x^3 - 5x^2 + 7x - 1}{4x^5 + 3x - 1}$

   (d) $i(x) = \frac{x}{x}$

   (e) $j(x) = x^{-0.5}$

6. **Quick facts:** Let a rational function

$$R(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$$

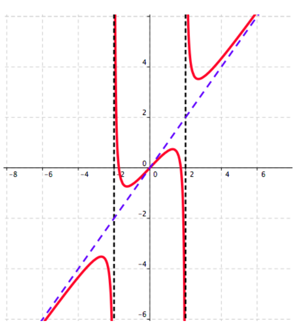
be given.

   (a) **Domain.** The function R is defined for all real numbers x for which its denominator is not equal to 0. In other words

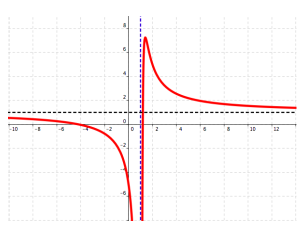
$$\text{Domain of } R = \{x \in \mathbb{R} : b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0 \neq 0\}.$$

   Note that the number of real numbers at which R is **NOT** defined is at most m.

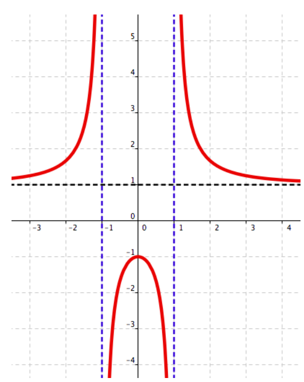
   (b) **Range.** Needs further investigation. Weird things may happen.



(a) $f(x) = \dfrac{x^3 - 3x}{x^2 - 4}$

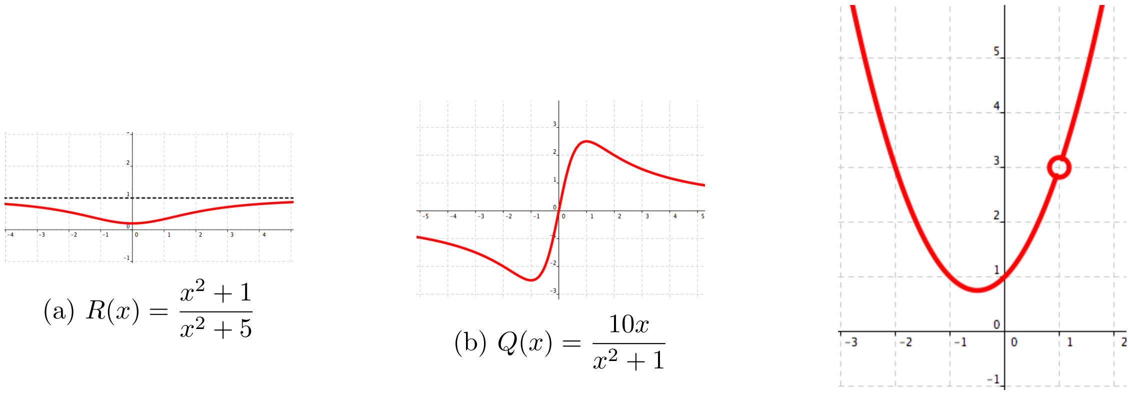


(b) $g(x) = \dfrac{x^2 + 3x - 5}{x^2 - 2x + 1}$



(c) $h(x) = \dfrac{x^2 + 1}{x^2 - 1}$

Figure 3.29: Three rational functions - What do you observe?



(a) $R(x) = \dfrac{x^2+1}{x^2+5}$ (b) $Q(x) = \dfrac{10x}{x^2+1}$ (c) $S(x) = \dfrac{x^3-1}{x-1}$

Figure 3.30: Three more rational functions - What do you observe?

(c) **Zeros:** If $R(x) = \dfrac{p(x)}{q(x)}$ is a rational function and $x_0 \in \mathbb{R}$ then

$$R(x_0) = 0 \Leftrightarrow (p(x_0) = 0 \text{ and } q(x_0) \neq 0).$$

Example: Let

$$S(x) = \frac{x-1}{x^2+1} \text{ and } T(x) = \frac{x-1}{x^2-1}.$$

Solve

i. $S(x) = 0$

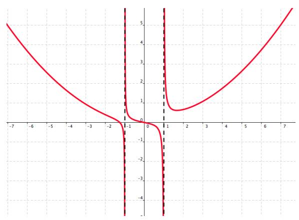
ii. $T(x) = 0$

(d) **Vertical asymptotes:** If the rational function $R(x)$ is **NOT** defined at the real number a then the vertical line $x = a$ is a possible vertical asymptote.

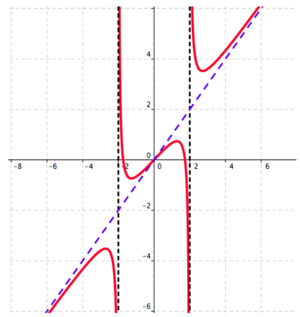
7. **BIG question:** If

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$$

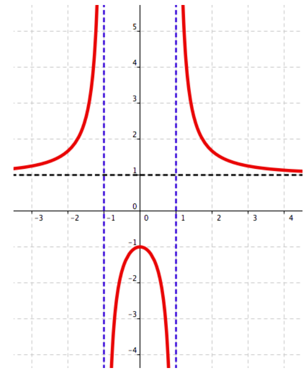
what can we tell about $R(x)$ if x is such that $|x|$ is huge?



(a) $f(x) = \dfrac{0.1 \cdot (x^4 + 2x)}{x^2 - 1}$



(b) $g(x) = \dfrac{x^3 - 3x}{x^2 - 4}$



(c) $h(x) = \dfrac{x^2 + 1}{x^2 - 1}$

Figure 3.31: What can we tell about $R(x)$ if x is such that $|x|$ is huge?

8. **Main tools:**

(a) What can we tell about the size of $\dfrac{1}{x^k}$ if k is a natural number and x is such that $|x|$ is huge?

(b) How to divide two polynomials? **Example:**

$$\begin{array}{r|l} & x^2 \qquad + 1 \\ \hline x^2 - 1 & x^4 \qquad + 2x \\ & -x^4 + x^2 \\ \hline & x^2 + 2x \\ & -x^2 \qquad + 1 \\ \hline & 2x + 1 \end{array}$$

Example: Divide the polynomial $p(x) = x^3 - 3x$ by the polynomial $q(x) = x^2 - 4$.

Example: Divide the polynomial $s(x) = x^2 + 1$ by the polynomial $t(x) = x^2 - 1$.

9. **A Taste Of Calculus.** Sketch the graphs of the following rational functions:

   (a) $f(x) = \dfrac{x^4 + 2x}{x^2 - 1}$

      i. **Domain:**

      ii. **Vertical asymptotes:**

      iii. **Zeros:**

      iv. **The y-intercept:**

      v. **The long range behaviour:**

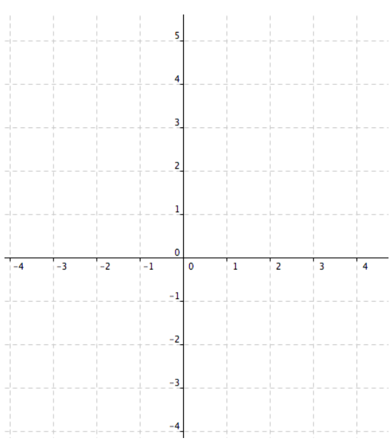


Figure 3.32: $f(x) = \dfrac{x^4 + 2x}{x^2 - 1}$

(b) $g(x) = \dfrac{x^3 - 3x}{x^2 - 4}$

i. **Domain:**

ii. **Vertical asymptotes:**

iii. **Zeros:**

iv. **The y-intercept:**

v. **The long range behaviour:**

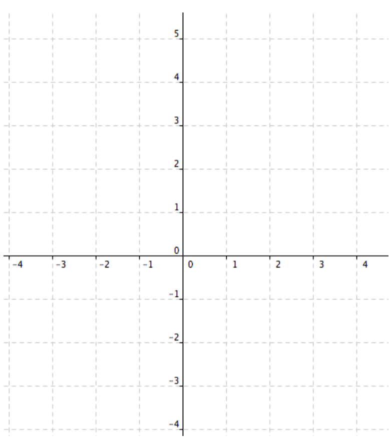


Figure 3.33: $g(x) = \dfrac{x^3 - 3x}{x^2 - 4}$

(c) $h(x) = \dfrac{x^2+1}{x^2-1}$

i. **Domain:**

ii. **Vertical asymptotes:**

iii. **Zeros:**

iv. **The y-intercept:**

v. **The long range behaviour:**

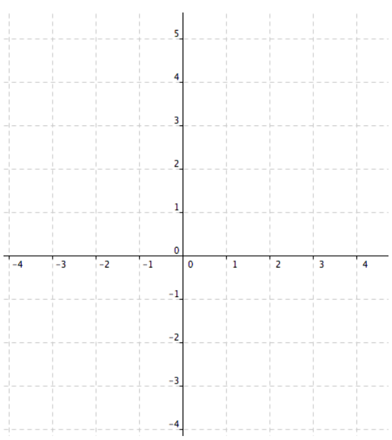


Figure 3.34: $h(x) = \dfrac{x^2+1}{x^2-1}$

Chapter 4

Exponential and Logarithmic Functions

4.1 Lecture 12: Logarithms as Inverses of Exponential Functions

1. **Quote.** Technology advances at exponential rates, and human institutions and societies do not. They adapt at much slower rates. Those gaps get wider and wider.

   Mitchell "Mitch" Kapor, A pioneer of the personal computing industry and advocate for social change, 1950 –

2. **Problem:** Solve
$$10^x = 3.$$

3. **Reminder:** For all $a, b \in (0,1) \cup (1, \infty)$ and all $x, y \in \mathbb{R}$:

   (a) $a^{x+y} = a^x \cdot a^y$

   (b) $a^{x-y} = \dfrac{a^x}{a^y}$

   (c) $(a^x)^y = a^{xy}$

   (d) $(ab)^x = a^x \cdot b^x$

4. Let
$$f(x) = 2^x \text{ and } g(x) = \left(\frac{1}{2}\right)^x.$$

   Complete the following table:

x	-3	-2	-1	0	1	2	3
$f(x)$							
$g(x)$							

5. **Two exponential functions:**

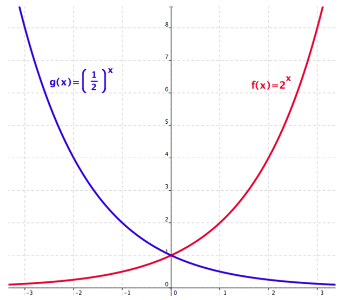
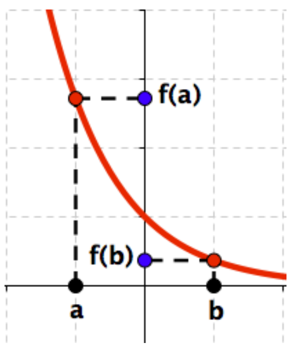
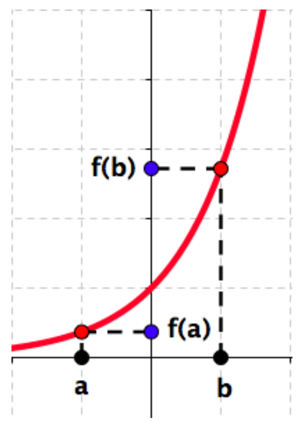


Figure 4.1: Two exponential functions: $f(x) = 2^x$ and $g(x) = \left(\dfrac{1}{2}\right)^x$.

6. **Graphs of exponential functions - general case:**



(a) $g(x) = a^x$, $0 < a < 1$



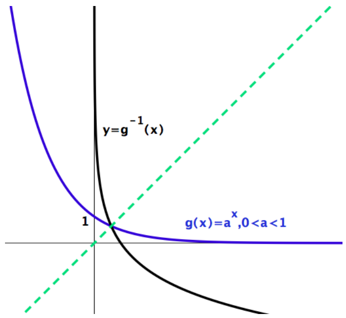
(b) $f(x) = a^x$, $a > 1$

Figure 4.2: Graphs of exponential functions: $0 < a < 1$ vs. $a > 1$

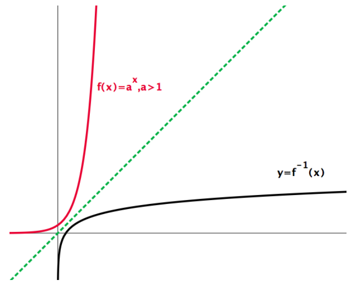
7. **Summary:**

$f(x) = a^x$	$0 < a < 1$	$a > 1$
Domain		
Range		
$f(x) = 0$		
$f(x) = 1$		
Increasing		
Decreasing		
One-to-one		
Odd or even		
Horizontal asymptote		

8. **Big question:** What is the inverse function of $f(x) = a^x$, $a > 0$, $a \neq 1$?



(a) $g(x) = a^x$, $0 < a < 1$, and its inverse



(b) $f(x) = a^x$, $a > 1$, and its inverse

Figure 4.3: Graph of the inverse function of an exponential function

9. **Logarithmic Function.** The inverse function of the exponential function $f(x) = a^x$, $a > 0$, $a \neq 1$, is called the **logarithmic function with base a** and it is denoted by

$$f^{-1}(x) = \log_a x.$$

10. **All You Need To Know.** For any $a > 0$, $a \neq 1$, any $x > 0$, and any $y \in \mathbb{R}$

$$\log_a x = y \iff a^y = x$$

11. **Example.** Determine $\log_2(16)$, $\log_2\left(\frac{1}{8}\right)$ and $\log_2(1)$.

12. **Example.** Can you find $\log_2(-32)$?

13. **Example.** Solve $10^x = 3$.

Note: If $a = 10$ then

$$\log_{10} x = \log x.$$

The logarithm base 10 is called the **common logarithm**.

14. **Logarithmic functions.**

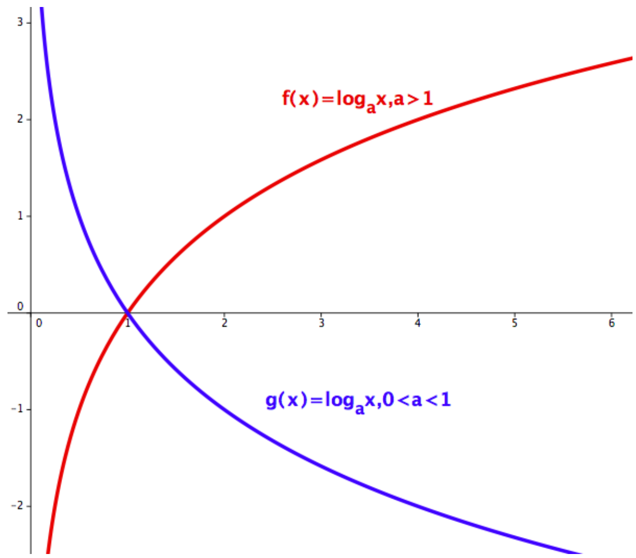


Figure 4.4: Graphs of logarithmic functions: $0 < a < 1$ vs. $a > 1$

15. **Summary:**

$f(x) = \log_a x$	$0 < a < 1$	$a > 1$
Domain		
Range		
$f(x) = 0$		
$f(x) = 1$		
Increasing		
Decreasing		
One-to-one		
$f^{-1}(x) =$		
Odd or even		
Vertical Asymptote		

16. **Remember f and f^{-1}?**

$$f^{-1}(f(x)) = x, \text{ for all } x \in \text{dom } f$$

$$f(f^{-1}(x)) = x, \text{ for all } x \in \text{dom } f^{-1}$$

17. **What about if $f(x) = a^x$ and $f^{-1}(x) = \log_a x$?**

    (a)
    $$f^{-1}(f(x)) = \log_a(a^x) = x, \text{ for all } x \in \text{dom } f = \mathbb{R}$$

    (b)
    $$f(f^{-1}(x)) = a^{\log_a x} = x, \text{ for all } x \in \text{dom } f^{-1} = (0, \infty)$$

18. **Example:** Evaluate

    (a) $10^{\log 2}$

    (b) $\log 10^2$

    (c) $\log 10^{-2}$

    (d) $10^{\log(-2)}$

19. **Conclusion.** (Credits: http://www.questgarden.com/01/08/2/051210131420/conclusion.htm)

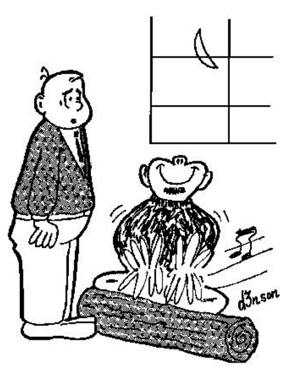


Figure 4.5: "I thought he said he was going to practice his logarithms all night."

4.2 Lecture 13: Rules for Logarithms

1. **Quote.** Let's not go and ruin it by thinking too much.

   Clinton Eastwood, Jr., American actor, film director, producer, and composer, 1930—

2. **Problem:** A day after your graduation you realize that the total amount of your student loan is $A = \$20,000$ at $i = 3\%$ annual interest compounded monthly. If your payment is $P = \$300$ a month, how long will it take to repay the loan?

3. **Reminder:** For any $a > 0$, $a \neq 1$, any $x > 0$, and any $y \in \mathbb{R}$

$$\log_a x = y \iff a^y = x$$

4. **The Properties of Logarithms.** For any positive real numbers x, y and a, $a \neq 1$:

$$\boxed{\log_a(xy) = \log_a x + \log_a y}$$

   (a) **Proof:**

   (b) **Example:** Solve

$$\log_3 x = \log_3 2 + \log_3(x^2 - 3)$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

(a) **Proof:**

(b) **Example:** Solve

$$\log(x-2) - \log x = 3$$

$$\log_a x^r = r \log_a x$$

(a) **Proof:**

(b) **Example:** Solve

$$2^{x-2} = 3^x$$

$$\log_a 1 = 0$$

(a) **Proof:**

(b) **Example:** Solve

$$\log_2(4x^2 - 12x + 10) = 0$$

$$\log_a a^x = x$$

(a) **Proof:**

(b) **Example:** Solve

$$\log 10^{x^2} + \log 10^{-12} = \log 10^x$$

Change of base formula

$$\log_a N = \frac{\log_b N}{\log_b a}$$

(a) **Proof:**

(b) **Example:** Solve

$$\log_{x^2} 10 + \log_x 100 = \log_2 1000$$

5. **MUST KNOW!**

   (a) $\log_a(xy) = \log_a x + \log_a y$

   (b) $\log_a \dfrac{x}{y} = \log_a x - \log_a y$

   (c) $\log_a x^r = r \log_a x$

   (d) $\log_a 1 = 0$

   (e) $\log_a a^x = x$

   (f) $\log_a N = \dfrac{\log_b N}{\log_b a}$

   whenever those numbers are defined.

6. **Exponents and logarithms** ... are part of life for many past and present students in Canada:

   (a) Loan balance B after n payments have been made:

   $$B = A(1+i)^n - \frac{P}{i}\left[(1+i)^n - 1\right].$$

   Here: A = the amount borrowed, i = the interest per the payment period, P = the payment.

   (b) Payment P needed to repay the amount A in N payments at the interest i per the payment period:

   $$P = \frac{iA}{1-(1+i)^{-N}}$$

   (c) Number N of payments P needed to repay the amount A at the interest i per the payment period:

   $$N = \frac{-\log(1-iA/P)}{\log(1+i)}$$

7. **Example** A day after your graduation you realize that the total amount of your student loan is $A = \$20,000$ at $i = 3\%$ annual interest compounded monthly. If your payment is $P = \$300$ a month, how long will it take to repay the loan?

4.3 Lecture 14: Exponential Growth

1. **Quote.** It's the whole issue with exponential growth, it's very slow in the beginning but over the long term it gets ridiculous.

   Drew Curtis, Founder and chief administrator of Fark.com, 1973 –

2. **Problem:** On Wednesday Dr. J assigned the weekly homework assignment that contained 64 problems. You would like to start working on the assignment right away, but because of other assignments and projects you decide to do only one problem on Wednesday, two problems on Thursday, and to keep doubling the number of problems that you solve per day all the way till Tuesday. Will you solve all homework problems before the following Wednesday morning when the weekly quiz is scheduled.

3. **Fact 1:**

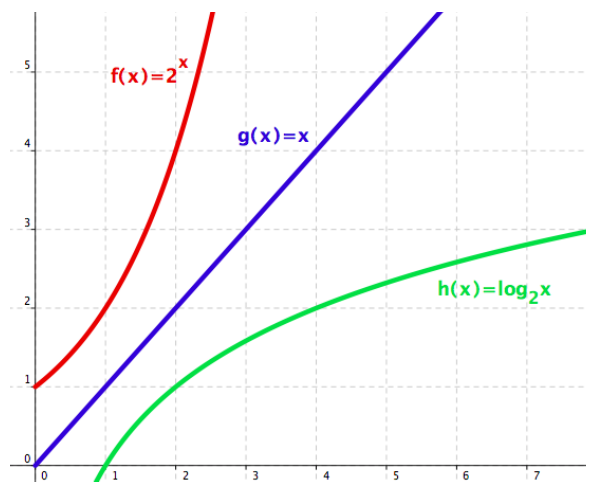


Figure 4.6: Fact: If $x > 0$ then $2^x > x > \log_2 x$

4. **Fact 2.** For any $n \in \mathbb{N}$ there is $a > 0$ such that

$$\text{if } x > a \text{ then } 2^x > x^n.$$

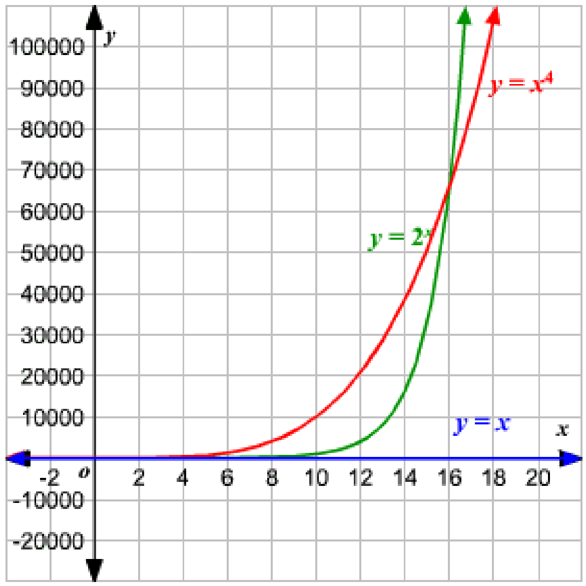


Figure 4.7: Fact: If $x > 16$ then $2^x > x^4$

5. **All you need to know about the exponential growth:** Consider the average change of the function f over the interval $[x, x+1]$:

$f(x) =$	x	x^2	2^x	$\log_2 x$
$\dfrac{f(x+1)-f(x)}{(x+1)-x} =$				

6. **Reminder.**

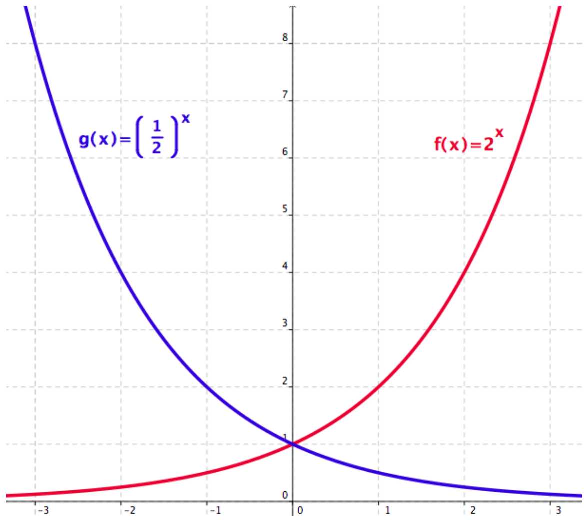


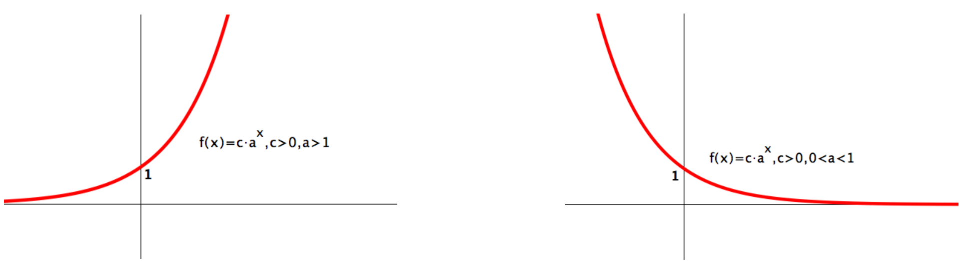
Figure 4.8: Two exponential functions: $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$.

7. **In general**... We are interested in functions of the form

$$f(x) = c \cdot a^x, \text{ where } a \in (0,1) \cup (1,\infty), c \in \mathbb{R}\backslash\{0\}.$$

(a) The domain of the function f:

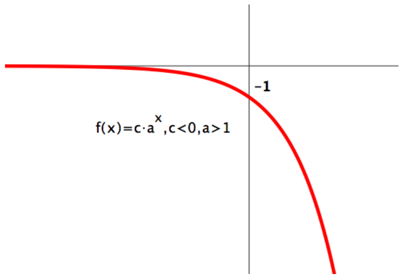
(b) The range of the function f:



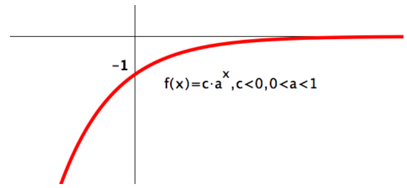
(a) $f(x) = c \cdot a^x$, $c > 0, a > 1$

(b) $f(x) = c \cdot a^x$, $c > 0, 0 < a < 1$

Figure 4.9: $f(x) = c \cdot a^x$, $c > 0$ - Range $= (0, \infty)$



(a) $f(x) = c \cdot a^x$, $c < 0, a >$



(b) $f(x) = c \cdot a^x$, $c < 0, 0 < a < 1$

Figure 4.10: $f(x) = c \cdot a^x$, $c < 0$ - Range $= (-\infty, 0)$

8. **Fact:** For any $c \in \mathbb{R} \backslash \{0\}$, and any $a \in (0,1) \cup (1, \infty)$,

$$f(x) = c \cdot a^x = c \cdot 2^{x \log_2 a}.$$

9. **What are exponential functions good for?**

- **Population Growth:** Population size N through time t is modeled by

$$N = N_0 2^{t/d}$$

where N_0 is the current population size and $d > 0$ is so-called *doubling time*, the time needed for the given population to double in size.

Example. Calculopolis had a population of 25000 in 1980 and the population of 30000 in 1990. What population can the Calculopolis planers expect in the year 2020 if the population grows at a rate proportional its size?

- **Compound Interest.** Suppose an amount A_0 is invested at an interest rate r.

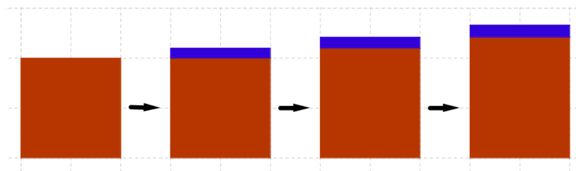


Figure 4.11: Compound Interest = "Interest on Interest".

If $A(t)$ is the amount after t years then

$$A(t) = A_0(1+r)^t \ .$$

If the interest is compound n times per year then after t years the investment is worth

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt} \ .$$

Example: If \$1000 is borrowed at 19% interest, find the amounts due at the end of 2 years if the interest is compounded

(a) annually

(b) quarterly

(c) monthly

(d) daily

4.4 Lecture 15: The Number e and Natural Logarithm

1. **Quote.** One of the first conditions of happiness is that the link between Man and Nature shall not be broken.

   Lev Nikolayevich Tolstoy, Russian writer, 1828 – 1910

2. **Question:** What is the most beautiful mathematical expression?

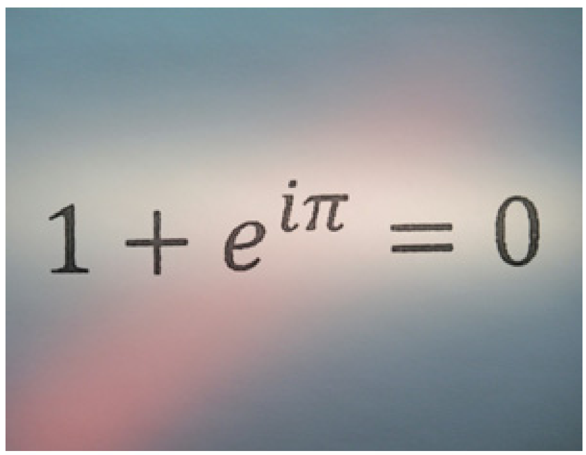


Figure 4.12: Euler's Identity

3. **Number e.**

   - $e \approx 2.71828182845904523536028747135266249775724709369995\ldots$
   - First time mentioned in 1618 by (almost certainly) William Oughtred, 1574 – 1660.
   - The notation e made its first appearance in a letter written by Leonhard Euler (1707 – 1783) in 1731.
   - What is e?

$$\text{If } x \to 0 \text{ then } (1+x)^{\frac{1}{x}} \to e$$

- Where is e?

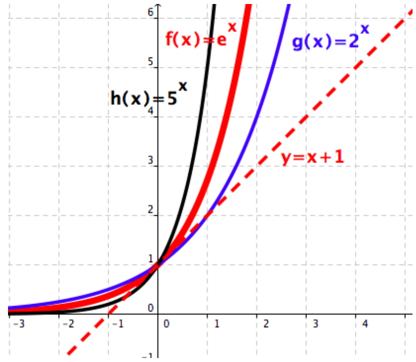


Figure 4.13: The function $f(x) = e^x$ is the exponential function with the property that the slope of its tangent line at the point $(1, 0)$ equals to 1.

- Does the number e play a role in your everyday life?

(a) Compound interest formula (b) Continuous compound interest formula

Figure 4.14: Two formulas

4. **Natural logarithm:** The inverse function of the function $f(x) = e^x$ is the function $f^{-1}(x) = \log_e x$. We write

$$\log_e x = \ln x, x > 0.$$

We say that, for $x > 0$, $\ln x$ is the **natural logarithm of** x.

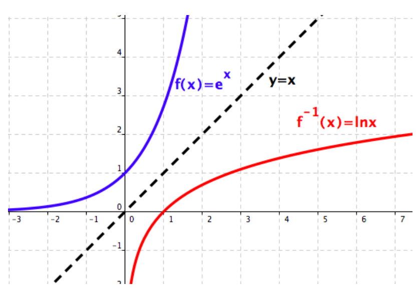


Figure 4.15: The function $f(x) = e^x$, its inverse function $f^{-1}(x) = \ln x$, and the line $y = x$.

5. **Summary:**

	$f(x) = e^x$	$f^{-1}(x) = \ln x$
Domain		
Range		
$f(x) = 0$		
$f(x) = 1$		
Increasing		
Decreasing		
One-to-one		
Inverse		
Odd or even		
Vertical Asymptote		

6. **All you need to know.** If $x > 0$ then

$$\ln x = y \Leftrightarrow x = e^y.$$

7. **Example:** Solve the equation :

   (a) $\ln(x+1) = 2$

   (b) $\ln(x+12) - \ln(x+2) = \ln x$

(c) $\dfrac{\ln(x^2+12)}{\ln(x+2)} = 2$

8. **Reminder:** From the fact that the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverse to each other it follows that:

$$\ln e^x = x, x \in \mathbb{R}$$

and

$$e^{\ln x} = x, x > 0.$$

9. **Example:** What is the meaning of the expression x^x, $x > 0$?

Evaluate: π^π

Chapter 5

Trigonometric Functions

5.1 Lecture 16: The Unit Circle and Radians

1. **Quote.** A circle may be small, yet it may be as mathematically beautiful and perfect as a large one.

   Isaac D'Israeli, British writer, scholar and man of letters, 1766 – 1848

2. **Puzzle:** Illustrated below is a quarter-circle, containing two semicircles of smaller circles. Prove that the red segment has the same area as the blue.

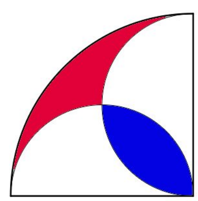


Figure 5.1: Red = Blue

3. **Unit Circle.** The unit circle is the circle of radius one centred at the origin $(0, 0)$.

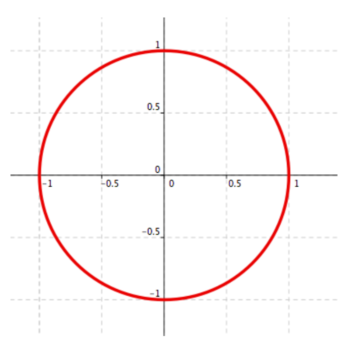


Figure 5.2: The unit circle: $x^2 + y^2 = 1$

4. **Example:** Which of the following points does not belong to the unit circle.

   (a) $(1,0)$

   (b) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

   (c) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

   (d) $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

   (e) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}\right)$

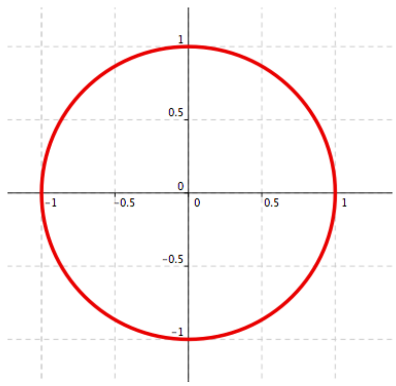


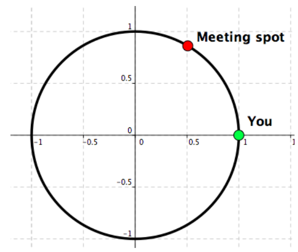
Figure 5.3: The unit circle: $x^2 + y^2 = 1$

5. **Will Go Round In Circles - One:**

Figure 5.4: Billy Preston (1946-2006): Will go round in circles

6. **Will Go Round In Circles - Two:**

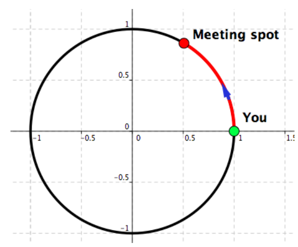
(a) You have just arrived



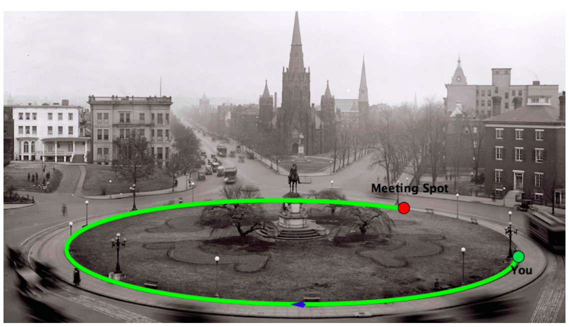
(b) You have just arrived

Figure 5.5: You have just arrived to the main city circle where you are about to meet your blind date.

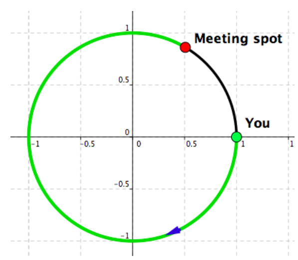
(a) You are on time



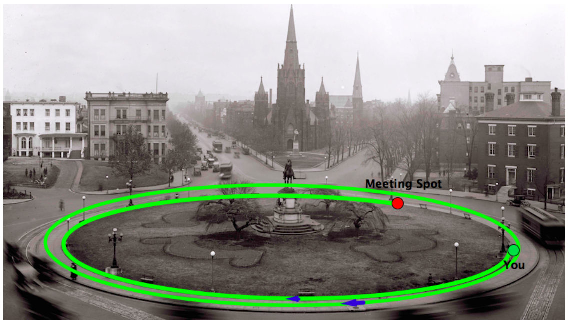
(b) You are on time



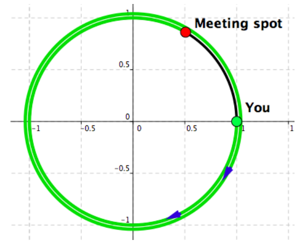
(a) You are early



(b) You are early



(a) Your date is late



(b) Your date is late

Observation: There are multiple ways of going from one point on the circle to another point on the circle, while staying on the circle all the time.

7. **Reminder:** The circumference of a circle with radius r is given by

$$C = 2r\pi.$$

The circumference of the unit circle is given by

$$C = 2\pi.$$

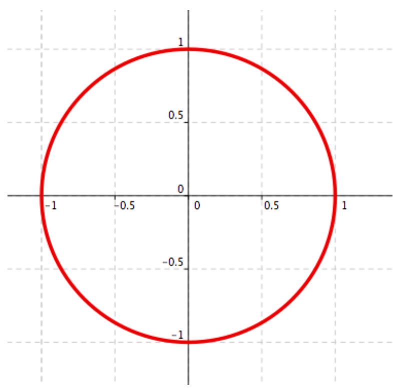
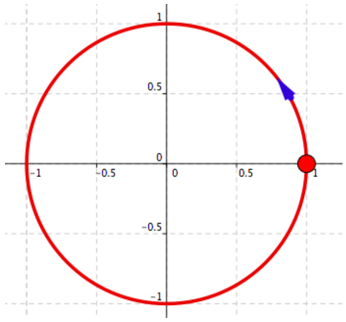
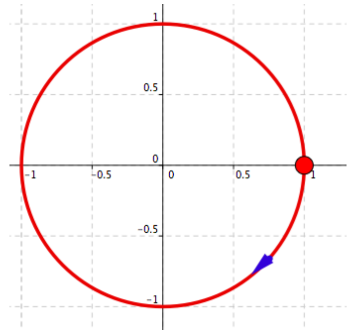


Figure 5.9: The circumference of the unit circle equals 2π.

8. **Two directions.**

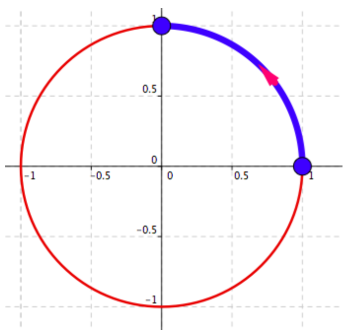


(a) Counterclockwise - Positive

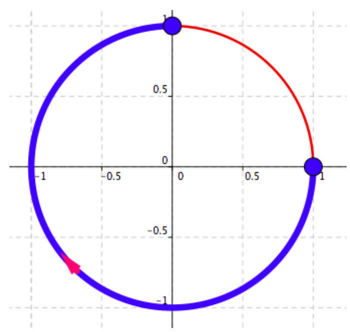


(b) Clockwise - Negative

9. **Example.** Sign and length = Signed length:



(a) Positive or negative? Length?



(b) Positive or negative? Length?

Figure 5.11: Same points – different numbers.

10. **Example.** Which number is associated with the blue arc oriented by the red arrow?

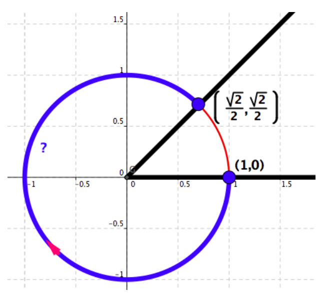


Figure 5.12: Think simple!

11. **Angles:** An angle is the figure formed by two rays, called the sides of the angle, sharing a common endpoint, called the vertex of the angle.

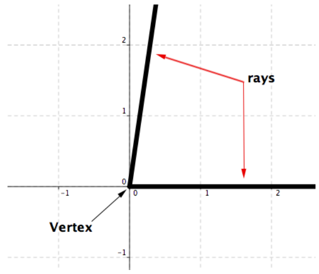
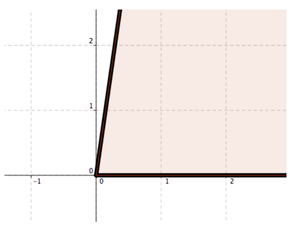
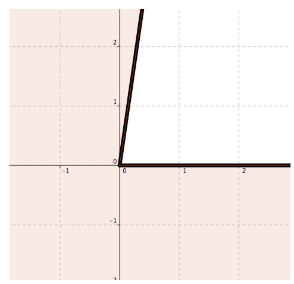


Figure 5.13: Two rays and the vertex

12. **One angle - Two regions.**





13. **Angles and numbers - One.** Consider an angle with the vertex at the origin and having the positive part of the x-axis as one ray. Angle's rays intersect the unit circle:

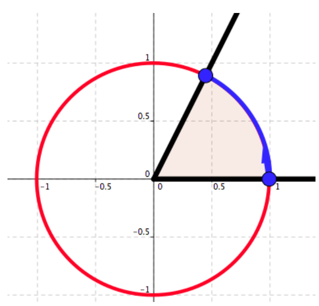


Figure 5.15: We associate the angle with the signed length of the arc on the unit circle between the intersection points taking the point $(1, 0)$ as the initial point.

14. **Angles and numbers - Two.** We say that the angle that is associated with the arc of the length 1 measures **1 radian**

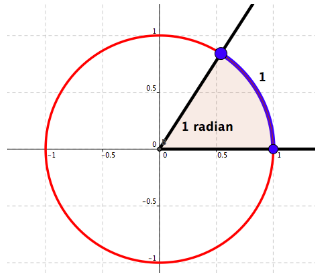
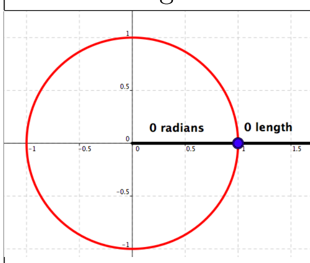
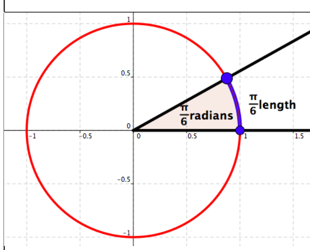
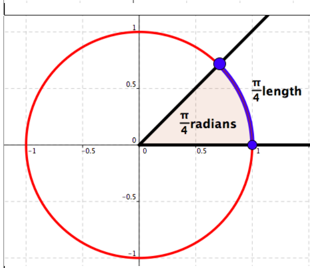
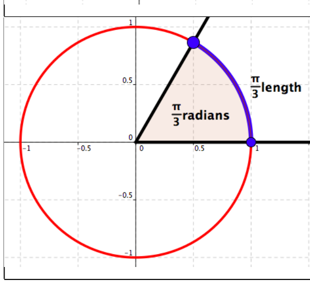
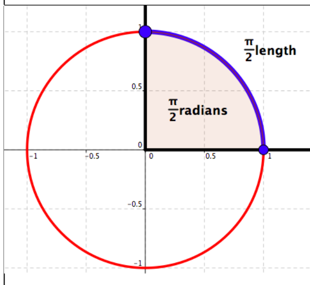
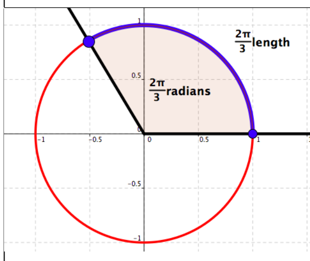
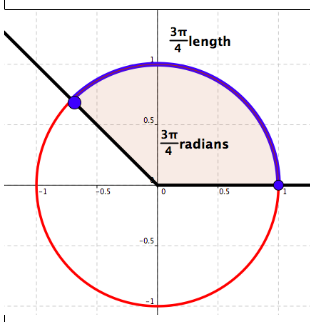
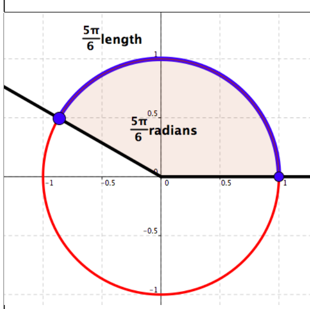
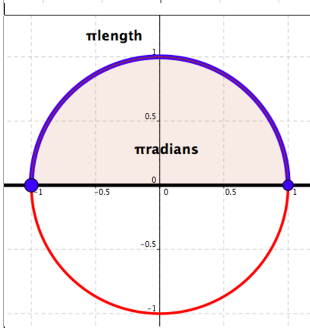
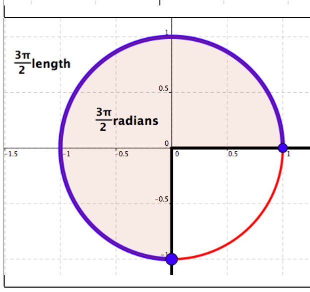


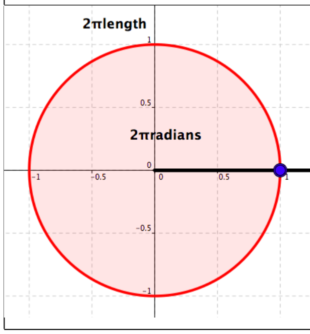
Figure 5.16: The angle that measures **1 radian**

.

15. **MUST KNOW!**

Angle	Measure in radians	Formerly known as:
	0 radians	$0°$
	$\frac{\pi}{6}$ radians	$30°$
	$\frac{\pi}{4}$ radians	$45°$
	$\frac{\pi}{3}$ radians	$60°$

Angle	Measure in radians	Formerly known as:
	$\frac{\pi}{2}$ radians	90°
	$\frac{2\pi}{3}$ radians	120°
	$\frac{3\pi}{4}$ radians	135°
	$\frac{5\pi}{6}$ radians	150°
	π radians	180°
	$3\frac{\pi}{2}$ radians	270°

Angle	Measure in radians	Formerly known as:
	2π radians	$360°$

16. **Angles and numbers - Three.** Starting at the point $(1, 0)$, to each real number x we associate the unique point X on the unit circle so that X is the end point of the arc of the signed length x.

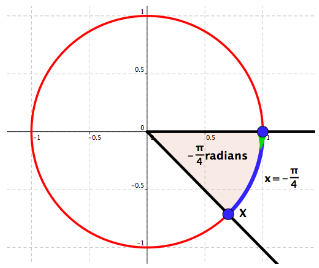
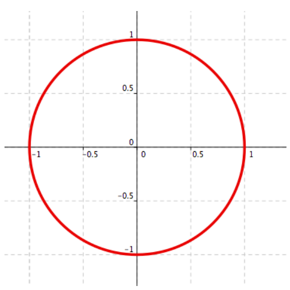
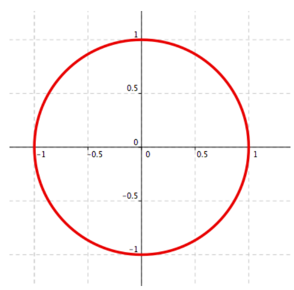


Figure 5.17: The angle that corresponds to the number $x = -\dfrac{\pi}{4}$

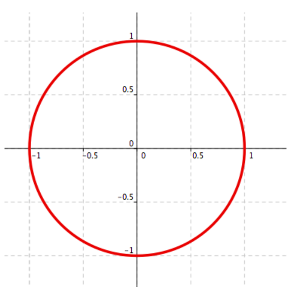
17. **Examples.** Draw the appropriate angles:



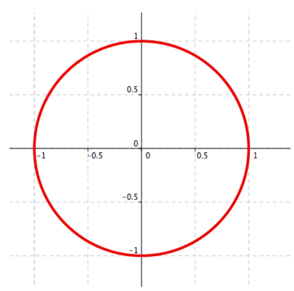
(a) $\alpha = \dfrac{4\pi}{3}$



(b) $\beta = -\dfrac{2\pi}{3}$



(a) $\gamma = \dfrac{9\pi}{4}$



(b) $\delta = -\dfrac{15\pi}{4}$

5.2 Lecture 17: Cosine and Sine

1. **Quote.** After exponential quantities the circular functions, sine and cosine, should be considered because they arise when imaginary quantities are involved in the exponential.

   Leonhard Euler, Swiss mathematician, physicist, astronomer, logician and engineer, 1707 – 1783

2. **Reminder:** We say that the angle that is associated with the arc of the length 1 measures **1 radian**.

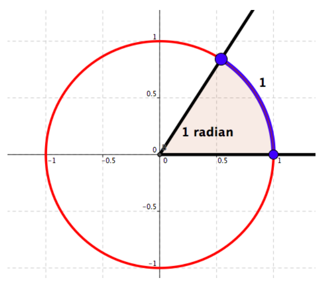
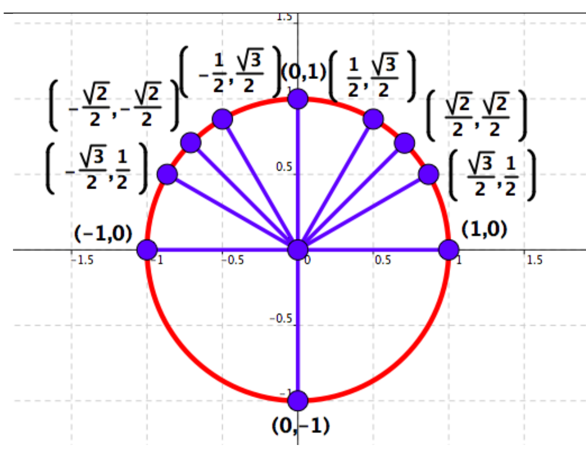


Figure 5.20: The angle that measures **1 radian**

.

3. **Reminder.**



x (radians)	the endpoint of the radius corresponding to x
0	$(1,0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0,1)$
$\frac{2\pi}{3}$	$\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$\frac{3\pi}{4}$	$\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
$\frac{5\pi}{6}$	$\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
π	$(-1,0)$
$\frac{3\pi}{2}$	$(0,-1)$
2π	$(1,0)$

4. **Script:**

$$\begin{aligned} \text{a real number } x \quad &\to \quad \text{the unique corresponding point on the unit circle } X = (a, b) \\ &\to \quad x \mapsto a \text{ and } x \mapsto b \\ &\to \quad \text{Call } a = \cos x \text{ and } b = \sin x \end{aligned}$$

5. **Script - Two:**

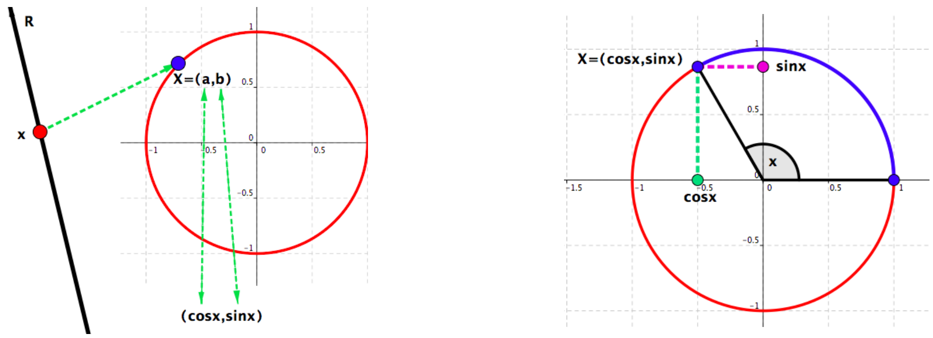
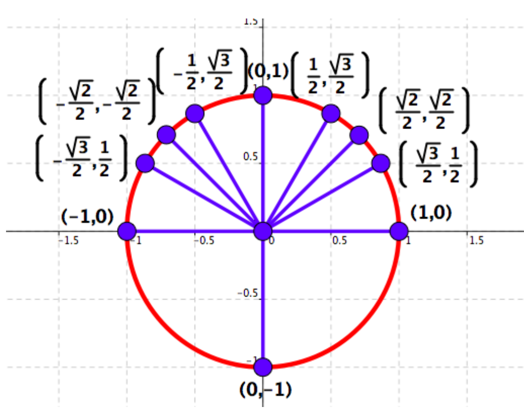


Figure 5.21: From x to $\cos x$ and $\sin x$

6. **Definition:** The **cosine** of x, denoted $\cos x$, is the first coordinate of the endpoint of the radius of the unit circle corresponding to x.

7. **Definition:** The **sine** of x, denoted $\sin x$, is the second coordinate of the endpoint of the radius of the unit circle corresponding to x.

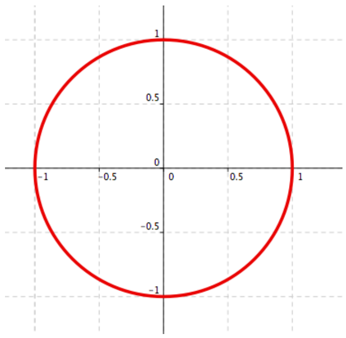
8. **Example:** Complete the following table:

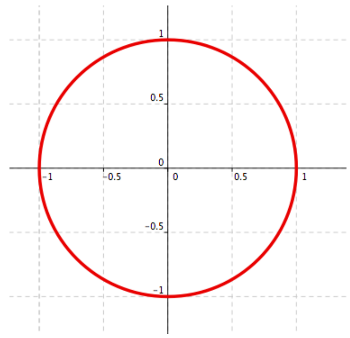


x (radians)	the endpoint of the radius corresponding to x	$\cos x$	$\sin x$
0	$(1, 0)$		
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$		
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$		
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$		
$\frac{\pi}{2}$	$(0, 1)$		
$\frac{2\pi}{3}$	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$		
$\frac{3\pi}{4}$	$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$		
$\frac{5\pi}{6}$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$		
π	$(-1, 0)$		
$\frac{3\pi}{2}$	$(0, -1)$		
2π	$(1, 0)$		

9. **Example.** Find

$$\cos\left(-\frac{\pi}{4}\right) \text{ and } \cos\left(\frac{7\pi}{4}\right)$$





10. **Five facts.** Let x be a real number. Then

    (a) $-1 \le \cos x \le 1$ and $-1 \le \sin x \le 1$

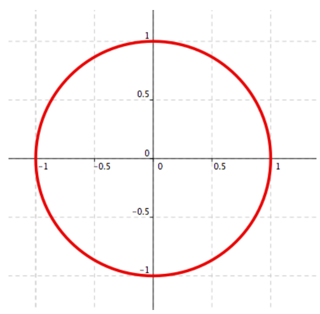


Figure 5.23: $-1 \le \cos x \le 1$ and $-1 \le \sin x \le 1$

.

    (b) The function $x \mapsto \cos x$ is even, i.e.,

$$\cos(-x) = \cos x$$

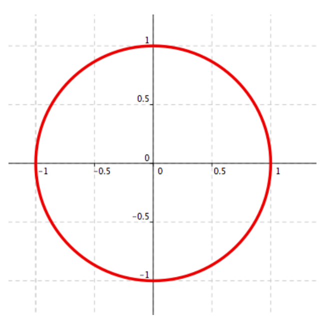


Figure 5.24: $\cos(-x) = \cos x$

.

(c) The function $x \mapsto \sin x$ is odd, i.e.,

$$\sin(-x) = -\sin x$$

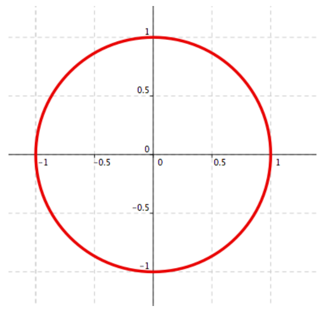


Figure 5.25: $\sin(-x) = -\sin x$

.

(d) $\cos^2 x + \sin^2 x = 1$

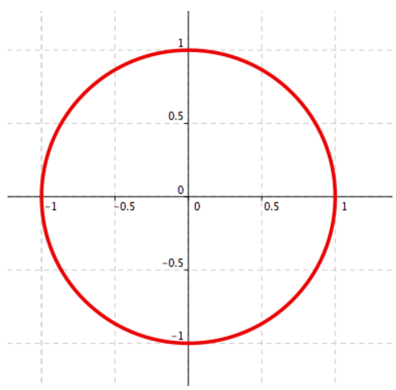


Figure 5.26: $\cos^2 x + \sin^2 x = 1$

(e) Functions $x \mapsto \cos x$ and $x \mapsto \sin x$ are **periodic**, i.e.,

$$\cos(x + 2\pi) = \cos x \text{ and } \sin(x + 2\pi) = \sin x.$$

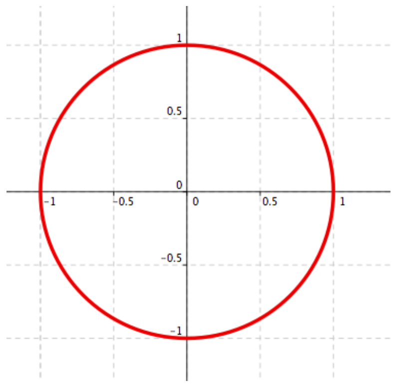


Figure 5.27: $\cos(x + 2\pi) = \cos x$ and $\sin(x + 2\pi) = \sin x$

11. **More facts:** Sign of $\cos x$ and $\sin x$ for $0 \le x < 2\pi$:

	$x = 0$	$0 < x < \frac{\pi}{2}$	$x = \frac{\pi}{2}$	$\frac{\pi}{2} < x < \pi$	$x = \pi$	$\pi < x < \frac{3\pi}{2}$	$x = \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
$\cos x$								
$\sin x$								

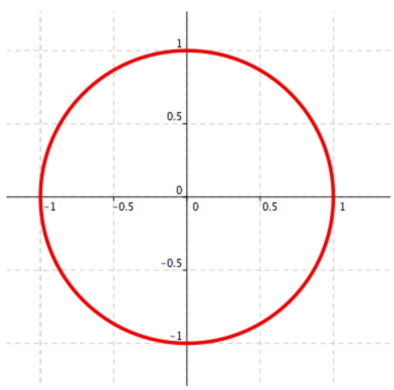


Figure 5.28: Where is $\cos x < 0$? Where is $\sin x > 0$?

12. **Two functions:**

    The function $f(x) = \cos x$:

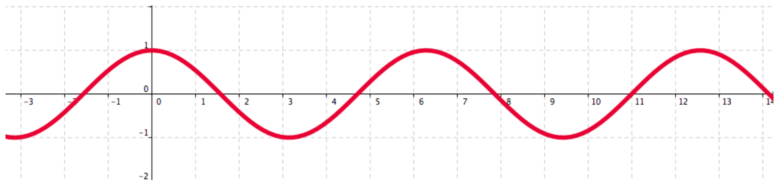


Figure 5.29: $f(x) = \cos x$

The function $g(x) = \sin x$:

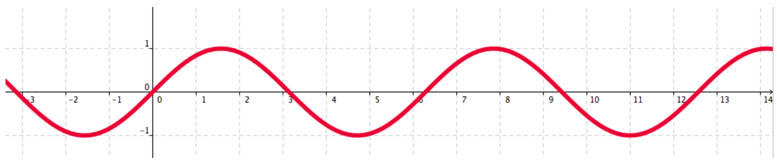


Figure 5.30: $f(x) = \sin x$

	$f(x) = \cos x$	$g(x) = \sin x$
Domain		
Range		
Zeros		
$y = 1$		
$y = -1$		
Parity		
Period		

5.3 Lecture 18: More Trigonometric Functions

1. **Quote.** What has to be overcome is not difficulty of the intellect but of the will.

   Ludwig Wittgenstein, Austrian-born philosopher, 1889 – 1951

2. **Reminder:** Solve

$$\cos x = 0.$$

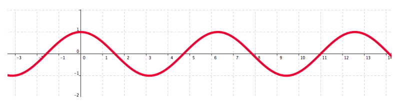


Figure 5.31: $x \mapsto y = \cos x$

.

3. **Definition.** The **tangent** function is defined as

$$\tan x = \frac{\sin x}{\cos x}.$$

4. **Example:**

x	$\sin x$	$\cos x$	$\tan x$
0			
$\frac{\pi}{6}$			
$\frac{\pi}{4}$			
$\frac{\pi}{3}$			
$\frac{\pi}{2}$			
$\frac{2\pi}{3}$			
$\frac{3\pi}{4}$			
$\frac{5\pi}{6}$			
π			
$\frac{3\pi}{2}$			
2π			

5. **Function** $f(x) = \tan x$**:**

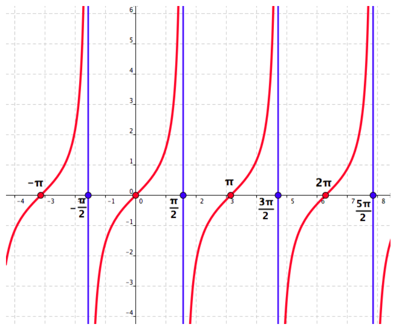


Figure 5.32: $x \mapsto f(x) = \tan x$

.

	$f(x) = \tan x$
Domain	
Range	
Zeros	
$\tan x > 0$	
$\tan x < 0$	
Vertical Asymptotes	
Parity	
Period	
Increasing	
Decreasing	

6. **Example:** Suppose that the real number θ is such that

$$\frac{\pi}{2} < \theta < \pi \text{ and } \cos\theta = -\frac{2}{3}.$$

Find the values of $\sin\theta$ and $\tan\theta$.

7. **Example:** Consider the line p given by the equation $y = mx$, $m \in (0, \infty)$. Let α be the smaller of the two positively oriented angles with the vertex at the origin, the positive part of the x axis being the initial ray, and the corresponding part of the line p being the terminal ray. Find $\cos\alpha$, $\sin\alpha$, and $\tan\alpha$.

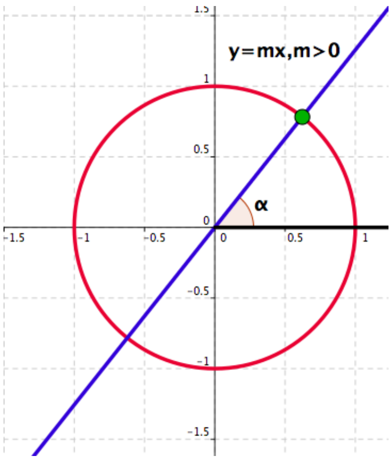


Figure 5.33: Find $\cos\alpha$, $\sin\alpha$, and $\tan\alpha$

.

8. **IMPORTANT Observation:** The slope of the line $y = mx$ equals to the tangent of a positively oriented angle between the line and the positive part of the x-axis.

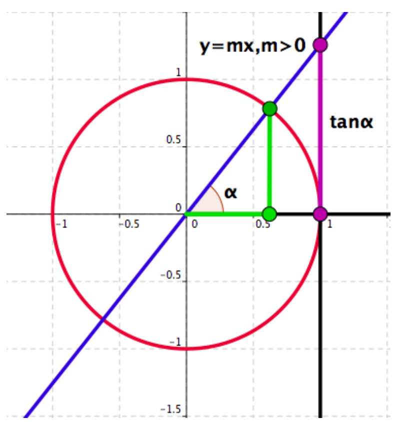


Figure 5.34: $m = \tan\alpha$

.

9. **More trigonometric functions.**

(a) **Cotangent:** The **cotangent** function is defined as

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}.$$

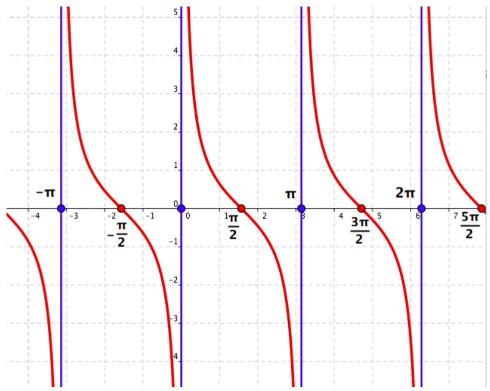
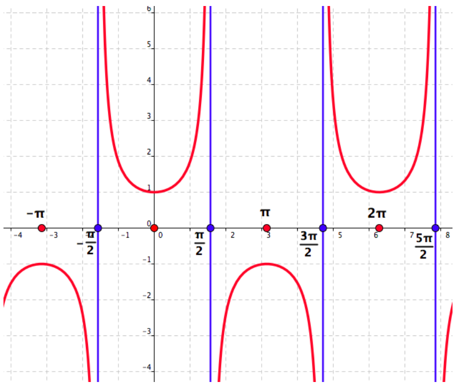


Figure 5.35: $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

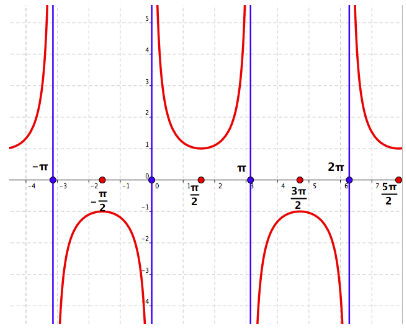
	$f(x) = \cot x$
Domain	
Range	
Zeros	
$\cot x > 0$	
$\cot x < 0$	
Vertical Asymptotes	
Parity	
Period	
Increasing	
Decreasing	

(b) **Secant and Cosecant:**

$$g(x) = \sec x = \frac{1}{\cos x} \text{ and } h(x) = \csc x = \frac{1}{\sin x}$$



(a) $\sec x = \dfrac{1}{\cos x}$



(b) $\csc x = \dfrac{1}{\sin x}$

	$f(x) = \sec x$	$g(x) = \csc x$
Domain		
Range		
Zeros		
Vertical Asymptotes		
$y = 1$		
$y = -1$		
Parity		
Period		

5.4 Lecture 19: Trigonometry In Right Triangle

1. **Quote.** The Bermuda Triangle got tired of warm weather. It moved to Alaska. Now Santa Claus is missing.

   Steven Alexander Wright, American comedian, actor, writer, and an Oscar-winning film producer, 1955–

2. **Problem:** Find x and y:

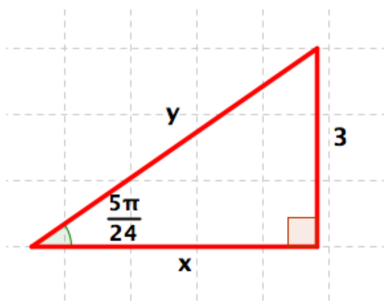


Figure 5.37: $x = ?$ $y = ?$

.

3. **Reminder.** We say that two triangles are **similar** if two angles of one triangle have measures equal to the measures of two angles of the other triangle.

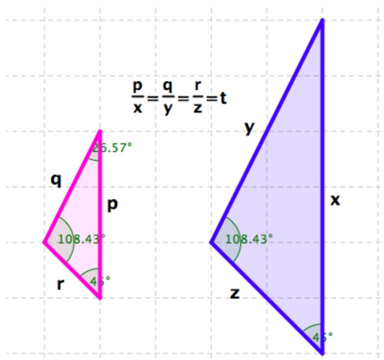


Figure 5.38: Similar triangles

.

4. **Similar Right Triangles:**

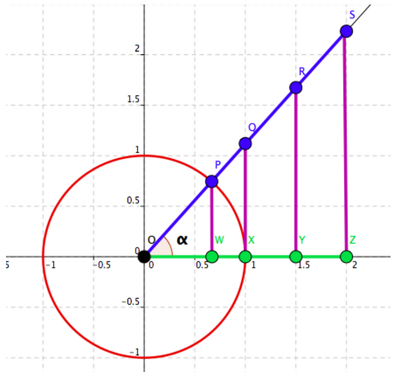


Figure 5.39: $\frac{\overline{ZS}}{\overline{OS}} = \frac{\overline{YR}}{\overline{OR}} = \frac{\overline{XQ}}{\overline{OQ}} = \frac{\overline{WP}}{\overline{OP}} = \frac{\sin\alpha}{1} = \sin\alpha$

.

5. **Right Triangle and Trigonometric Functions:** Let $\alpha \in \left(0, \frac{\pi}{2}\right)$ be an angle in a right triangle:

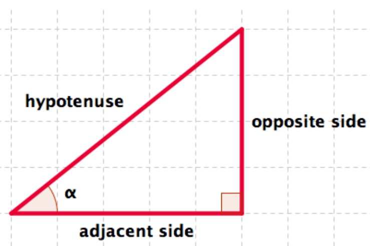


Figure 5.40: $\alpha \in \left(0, \frac{\pi}{2}\right)$

.

Then:

$$\cos\alpha = \frac{\text{adjacent side}}{\text{hypothenuse}}$$
$$\sin\alpha = \frac{\text{opposite side}}{\text{hypothenuse}}$$

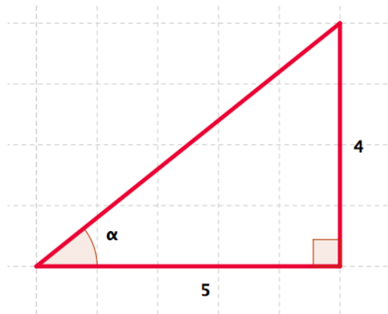
$$\tan\alpha =$$

$$\cot\alpha =$$

$$\sec\alpha =$$

$$\csc\alpha =$$

6. **Example:** Evaluate $\cos\alpha$, $\sin\alpha$, $\tan\alpha$, $\cot\alpha$, $\sec\alpha$, and $\csc\alpha$:



7. **Example:** Find x and y:

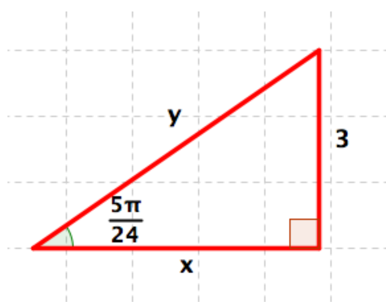


Figure 5.41: $x =?$ $y =?$

.

8. **A Summer Day in British Columbia:** It is a beautiful summer day and you and your visitor from Toronto, ON, are kayaking through Howe Sound, a network of fjords situated northwest of Vancouver, B.C. “This is The Stawamus Chief,” you inform your friend. “It is the second largest granite monolith in the world. For the Squamish nation The Chief is an important spiritual place. A Squamish legend says that it is a longhouse that spirit-beings turned into stone.”

   ”Wow,” you hear your friend exclaim. ”I wonder what the elevation of the highest point is.”

   ”Well, we are about 1200 metres away from the shore and I’d estimate that the angle of elevation is about $\frac{\pi}{6}$ so . . . ”

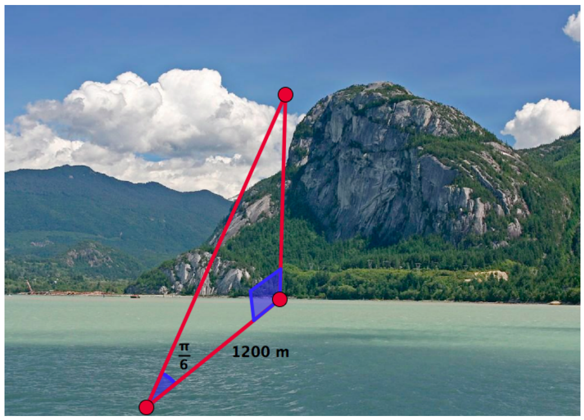


Figure 5.42: The Stawamus Chief, Squamish, B.C.

.

5.5 Lecture 20: Trigonometric Identities

1. **Quote.**

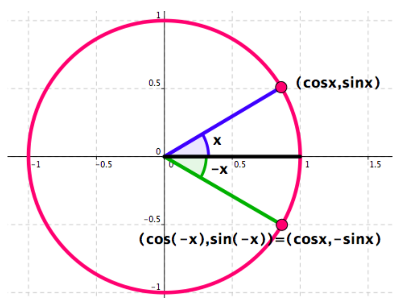
   I want the circle
   broken.

   (From the poem "The circle game".)

   Margaret Atwood, Canadian poet, novelist, literary critic, and essayist, 1939– .

2. **Problem:** Evaluate $\cos\dfrac{\pi}{12}$.

3. **Trigonometric Identities - Some are obvious.**



   For any real number x:

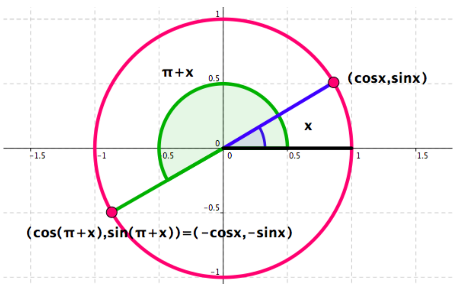
$$\begin{aligned}\cos^2 x + \sin^2 x &= 1\\ \cos(-x) &= \cos(x)\\ \sin(-x) &= -\sin(x)\end{aligned}$$

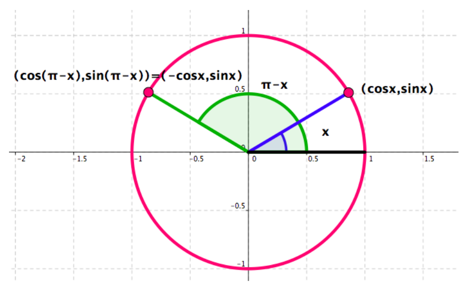
4. **Example:** If $x \in \left(\dfrac{\pi}{2}, \pi\right)$ and $\sin x = \dfrac{1}{3}$, find $\cos x$.

5. **Example:** Prove that any real number x such that $x \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$:

$$\tan^2 x + 1 = \sec^2 x.$$

6. **Trigonometric Identities - Some are easy to see.**





For any real number x:

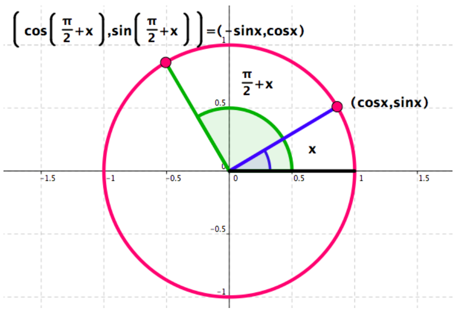
$$\begin{aligned} \cos(\pi + x) &= -\cos x \\ \sin(\pi + x) &= -\sin x \\ \cos(\pi - x) &= -\cos x \\ \sin(\pi - x) &= \sin x \end{aligned}$$

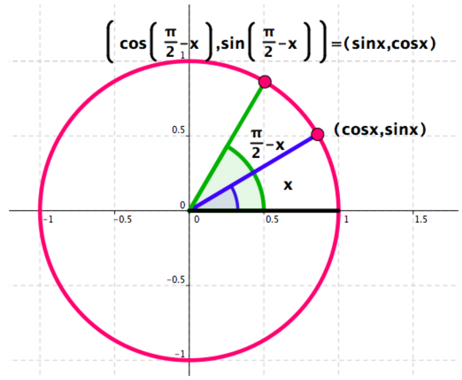
7. **Example:** Use the fact that

$$\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

to find $\sin\frac{11\pi}{12}$.

8. **Trigonometric Identities - Some need a bit of thinking.**





For any real number x:

$$\cos\left(\frac{\pi}{2}+x\right) = -\sin x$$
$$\sin\left(\frac{\pi}{2}+x\right) = \cos x$$
$$\cos\left(\frac{\pi}{2}-x\right) = \sin x$$
$$\sin\left(\frac{\pi}{2}-x\right) = \cos x$$

9. **Example:** Use the fact that

$$\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

to find $\cos\dfrac{7\pi}{12}$.

10. **Trigonometric Identities - Some need a bit more of thinking.**

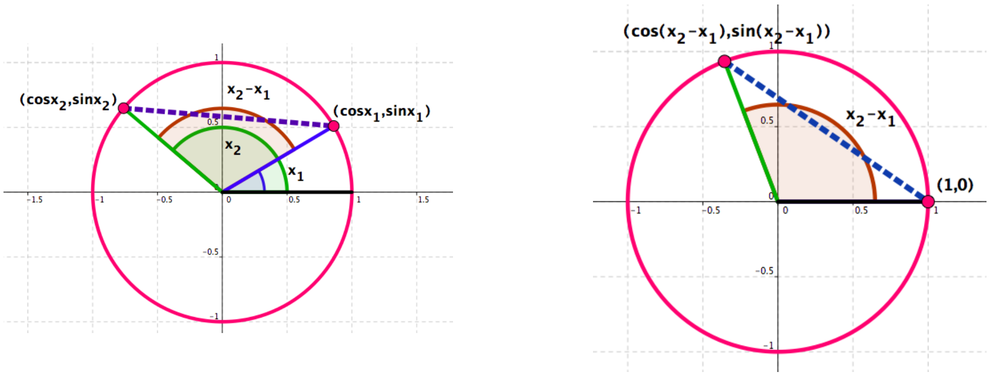


Figure 5.45: The distance between the points $(\cos x_2, \sin x_2)$ and $(\cos x_1, \sin x_1)$ is equal to the distance between the points $(\cos(x_2 - x_1), \sin(x_2 - x_1))$ and $(1, 0)$.

From

$$\begin{aligned} \text{Distance}\{(\cos x_2, \sin x_2), (\cos x_1, \sin x_1)\} &= \text{Distance } \{(\cos(x_2 - x_1), \sin(x_2 - x_1)), (1, 0)\} \\ (\cos x_2 - \cos x_1)^2 + (\sin x_2 - \sin x_1)^2 &= (\cos(x_2 - x_1) - 1)^2 + (\sin(x_2 - x_1) - 0)^2 \end{aligned}$$

11. **MUST KNOW! MUST MEMORIZE!** For any real number x:

$$\begin{aligned}
\cos(x_1 - x_2) &= \cos x_1 \cos x_2 + \sin x_1 \sin x_2 \\
\cos(x_1 + x_2) &= \cos x_1 \cos x_2 - \sin x_1 \sin x_2 \\
\sin(x_1 - x_2) &= \sin x_1 \cos x_2 - \cos x_1 \sin x_2 \\
\sin(x_1 + x_2) &= \sin x_1 \cos x_2 + \cos x_1 \sin x_2
\end{aligned}$$

12. **Example:** Evaluate $\cos \frac{\pi}{12}$.

13. **Trigonometric Identities - Some follow from what we already know.**

    (a) $\sin 2x = 2 \sin x \cos x$

    (b) $\cos 2x = \cos^2 x - \sin^2 x$

    (c) $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

    (d) $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

(e) $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$

(f) $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

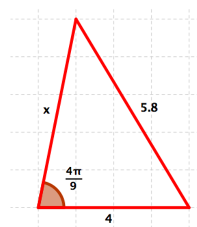
14. **Example:** Determine $\cos\frac{7\pi}{8}$, $\sin\frac{7\pi}{8}$, and $\tan\frac{7\pi}{8}$.

5.6 Lecture 21: The Law of Sines and the Law of Cosines

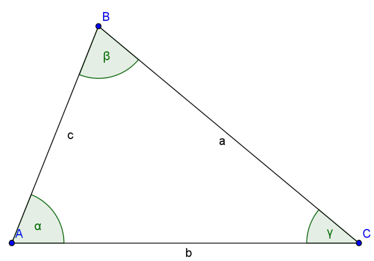
1. **Quote.** Live your life as though your every act were to become a universal law.

   Immanuel Kant, German philosopher, 1724–1804 .

2. **Problem:** Find x:



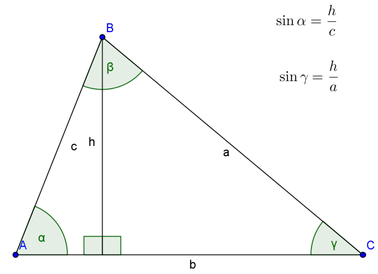
3. **Notation: Points, Sides and Angles of a Triangle**



   In a triangle ABC with points A, B and C,

   - α is the angle at A.
   - a is the side opposite A and α.
   - β is the angle at B.
   - b is the side opposite B and β.
   - γ is the angle at C.
   - c is the side opposite C and γ.

4. **The Law of Sines**



From the diagram,

$$\sin\alpha = \frac{h}{c} \text{ and } \sin\gamma = \frac{h}{a}.$$

We can calculate the length of h in two different ways:

$$h = c\sin\alpha \text{ and } h = a\sin\gamma.$$

Hence $c\sin\alpha = a\sin\gamma$. Similarly $b\sin\alpha = a\sin\beta$.

We write $D = a/\sin\alpha$. Since $b\sin\alpha = a\sin\beta$,

$$D = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

This is known as the Law of Sines.

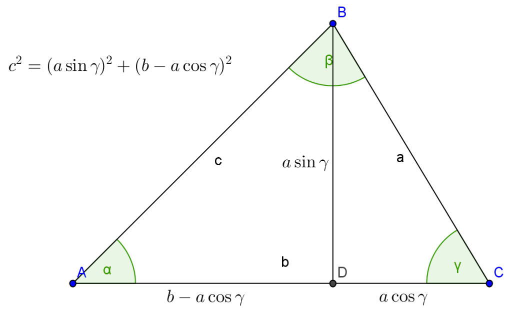
5. **Example:** In the triangle ABC with the usual notation, suppose that $\sin\alpha = 2\sqrt{2}/3$, that $a = 11$ and that $b = 9$. Find $\sin\beta$.

6. **Area of a Triangle:** Using the Law of Sines,

$$\text{Area}(ABC) = \frac{1}{2}ac\sin\beta.$$

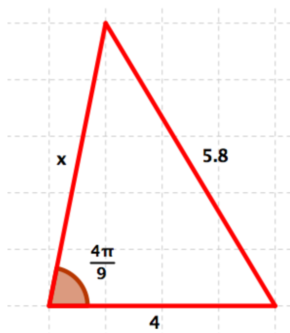
7. **Example:** In the triangle ABC with the usual notation, suppose that $\sin\alpha = 2/3$, that $b = 9$ and that the area of ABC is 24. Find c.

8. **The Law of Cosines**



$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

9. **Problem:** Find x:



Chapter 6

Practice Problems

1. Decide if the given rule is a function. If it is, find the value of the function if $x = \textit{yourself}$. If the given rule is not a function, explain clearly which condition in the definition of a function is not satisfied.

   a) Rule: Each student at SFU is associated with their SFU ID number.

   b) Rule: Each student at SFU is associated with the course that she or he is currently taking.

   c) Rule: Each student at SFU is associated with the car model that she or he owns.

   d) Rule: Each student at SFU is associated with their birthday.

2. We define a function f in the following way:

   $$\text{for every real number } x,\ f(x) \text{ is the largest integer that is less than or equal to } x$$

   (a) Find $f(0)$, $f(0.5)$, $f(0.75)$, $f(1)$, $f(1.5)$, $f(1.75)$, $f(2)$.

   (b) Find all solutions of the equation $f(x) = 1$.

   (c) Sketch a graph of the function f.

3. Let f be the function which associates to an integer n the number $\cos n\pi$ if n is even, and $\sin n\pi$ if n is odd.

   (a) Complete the following table:

n	-4	-3	0	3	11	16
$f(n)$						

   (b) Express the function f as a piecewise defined function.

   (c) State the domain and the range of the function f.

   (d) Draw a graph of f.

4. Sonoko is a Ph.D. candidate in the Department of Mathematics at Dalhousie University, Halifax, Nova Scotia. Recently her paper on the accessibility of the set of Fibonacci numbers was accepted in The Fibonacci Quarterly, the official publication of the Fibonacci Association. To celebrate this great news Sonoko invited her roommate Janice for a dinner at Presto Panini Cafe, Sonoko's favourite Italian restaurant.

   Sonoko ordered a bowl of penne pasta in pesto sauce for \$15 and Janice'c choice was a bowl of gnocchi in gorgonzola sauce with spinach and toasted walnuts that costed \$17. To celebrate this very special occasion the girls shared a piece of Presto Panini's famous tiramisu (\$8) and a 1/2 litter of Chianti (\$20.)

   In Nova Scotia sales taxes applied to foodservice are calculated as a single tax, called the Harmonized Sales Tax (HST), of 15% of the total food and beverage purchase.

   Since as an undergraduate student she used to work as a waitress, Sonoko is a very generous tipper. Her tipping policy is to give 8/5, the ratio of the sixth and the fifth terms of the Fibonacci sequence, of the HST amount to the server.

   (a) What was the amount of the HST on Sonoko and Janice's bill?

   (b) What was the tip that Sonoko left for the server?

   (c) Suppose that x is the amount in dollars of the total food and beverage purchase. Let $h(x)$ be the function that associates the amount of the HST to the amount x. Find an expression for $h(x)$.

   (d) Let $f(t) = \frac{8}{5} \cdot t$. Express Sonoko's tipping policy in terms of the functions h and f.

5. (a) Let $f(x) = 2\sqrt{x}$, $g(x) = 3^x$ and $h(x) = \cos x$.

      i. Find $h \circ g \circ f$. What is the domain of $h \circ g \circ f$?

      ii. Find $h \circ f \circ g$. What is the domain of $h \circ f \circ g$?

   (b) Given $F(x) = \sin(e^{x^2-x})$ find functions p, q and r such that $F = p \circ q \circ r$.

6. Let $f(x) = \dfrac{1}{x+1}$ and $g(x) = \ln x$. Find $g \circ f$. What is the domain of $g \circ f$? Find $f \circ g$. What is the domain of $f \circ g$?

7. Given $F(x) = \cos(e^{\sin\sqrt{x}})$ find functions p, q, r, and t such that $F = p \circ q \circ r \circ t$.

8. (a) Find the number t such that the line containing the points $(-3, t)$ and $(2, -4)$ is parallel to the line containing the points $(5, 6)$ and $(-2, 4)$.

   (b) Find an equation of the line that is perpendicular to the two lines in the first part of this question and passes through the point $(3, 0)$.

9. Suppose that you are given an open top box with the square base. Let x be the length of the side of the base and let the volume of the box be $V = 1$. Express the surface area of the box in terms of two power functions of x.

10. A hockey team plays in an arena that has a seating capacity of 15,000 spectators. With the ticket price set at \$50, average attendance at recent games has been 10,500. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 100.

    (a) Find a function that models the revenue in terms of ticket price.

    (b) Which attendance will give the maximum revenue? Justify your answer.

11. Let f be an **odd** function defined on the interval $[-2, 2]$ and such that

$$f(x) = \begin{cases} x & \text{if} \quad 0 \leq x \leq 1 \\ 2 & \text{if} \quad 1 < x \leq 2 \end{cases}$$

    (a) Complete the following table:

x	2	1.5	1	0.5	0	-0.5	-1	-1.5	-2
$f(x)$									

    (b) Draw the graph of the function f.

    (c) Draw the graph of the function $g(x) = -2f(x-1) + 1$

12. A particle is moving along the the curve $y = \sqrt{x}$ starting from the origin. What is the relationship between the x coordinate of the particle and the distance of the particle from the origin? For which x will the distance be equal to $\sqrt{12}$?

13. A particle is moving along the curve $y = \dfrac{1}{x}$, $x > 0$.

    (a) What is the relationship between the x coordinate of the particle and the distance of the particle from the origin?

    (b) For which x will the distance be equal to $\dfrac{\sqrt{17}}{2}$?

14. A rectangular sheet of cardboard measures a cm by b cm. Equal squares are cut out of each corner and the sides are turned up to form an open rectangular box. Express the volume of the box in terms of x, the length of the side of the square. Classify the obtained function (i.e., determine if is this a linear function, a power function, a polynomial, an exponential or a logarithmic function.) What is the domain of the function? Sketch the graph of the function under the assumption that $0 < b < a$.

15. Find the point on the line $2x + 3y = 1$ that is closest to the point $(1, -1)$.

16. Let

$$f(x) = \frac{x^4 + 4x^2 - 5}{x^2 + 2x + 1}.$$

    (a) Determine the domain of the function f.

    (b) Determine all vertical asymptotes.

    (c) Determine all zeros of the function f.

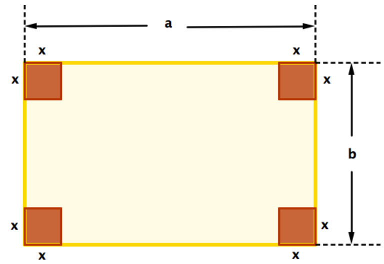


Figure 6.1: Equal squares are cut out of each corner

(d) Find a function of the form $g(x) = ax^n$, where a is a real number and n is an integer, which describes the behaviour of f near $\pm\infty$.

(e) Determine intervals where $f(x) > 0$ and where $f(x) < 0$.

(f) Sketch a graph of the function f.

17. Using the approximations $\log 2 \approx 0.301$ and $\log 5 \approx 0.699$, estimate each of the following to three decimal places. (You can check your answers with a calculator.)

(a) Estimate $\log 4$.

(b) Estimate $\log 8$.

(c) Estimate $\log 10$.

(d) Estimate $\log 16$.

(e) Estimate $\log 20$.

(f) Estimate $\log 25$.

(g) Estimate $\log 32$.

(h) Estimate $\log 40$.

(i) Estimate $\log 50$.

(j) Estimate $\log 64$.

(k) Estimate $\log 80$.

(l) Estimate $\log 100$.

(m) Estimate $\log 125$.

(n) Estimate $\log 160$.

(o) Estimate $\log 200$.

(p) Estimate $\log 250$.

18. The volume of a cone, shown below, is given by $V = \dfrac{r^2 h\pi}{3}$. Express the volume as a function of θ.

19. The swimming pool shown in the figure is 3-ft deep at the shallow end, 8-ft deep at the deepest end, 40-ft long, 30-ft wide, and the bottom is an inclined plane. Water is pumped into the pool.

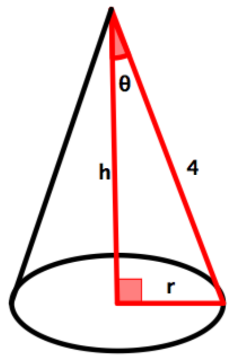


Figure 6.2: $V = V(\theta)$

(a) Express the volume of the water in the pool as a function of height h of the water measured from the bottom of the pool at the deep end. [Hint: The volume will be a piecewise-defined function.]

(b) Determine the domain and range of the function found in part (b).

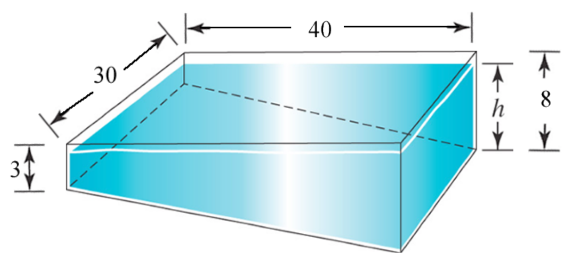


Figure 6.3: The swimming pool

20. Jayden has a rectangular yard that is 30ft by 34ft. He would like to add a sidewalk along the outside of two adjacent sides of the yard. The contractor quotes a price of $20 per square foot to put in the sidewalk.

(a) Draw a picture of the situation that shows the yard and adjoining sidewalk.

(b) If the sidewalk is only 1 ft wide, how many square feet of sidewalk would be have?

(c) What is the cost of a 1 ft wide sidewalk?

(d) Write a function $A(x)$ for the area (the number of square feet) of sidewalk installed if the side- walk is x ft wide.

(e) Write a function $C(x)$ for the cost of installing a sidewalk that is x ft wide.

(f) Use the function C to find the cost of installing a sidewalk 2 ft wide.

(g) If Jayden has $4, 000 to spend, what is the widest he can make the sidewalk to the nearest tenth of a foot?

21. Men's shoe size conversion is given by the following table

USA	6	7	8	9	10	11	12	13	14	15	16
EU	39	40	41	42	43	44	45	46	47	48	49
UK	5.5	6.5	7.5	8.5	9.5	10.5	11.5	12.5	13.5	14.5	15.5

Let the set A be given by

$$A = \{\text{Christian Bale, Koby Bryant, Steve Carell, Leonardo DiCaprio, Brad Pitt}\}.$$

Let functions f, g, and h be defined on the set A in the following way:

$$\begin{aligned} f(x) &= \text{the USA shoe size of the person } x \\ g(x) &= \text{the EU shoe size of the person } x \\ h(x) &= \text{the UK shoe size of the person } x \end{aligned}$$

It is known that

x	$f(x)$
Christian Bale	11
Koby Bryant	14
Steve Carell	10
Leonardo DiCaprio	12
Brad Pitt	9

(a) Draw tables for the functions g and h.

(b) Draw the graphs of the functions f, g, and h in the coordinate system in which the elements of the set A (in the alphabetical order) are represented as dots on the horizontal axis, and the vertical axis is a number line.

(c) Explain carefully how the three graphs are related.

(d) Based on your observation in part (b), express functions g and h in terms of the function f.

22. A function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ is given by its graph.

(a) Use the horizontal line test to determine that the function f is one-to-one.

(b) State the domain and range of the function f^{-1}.

(c) Draw a graph of the function f^{-1}.

23. Let f be a function on the domain $[-10, 1]$ which is defined by

$$f(x) = x^2 - 5x + 6.$$

Is f a one-to-one function? Justify your answer.

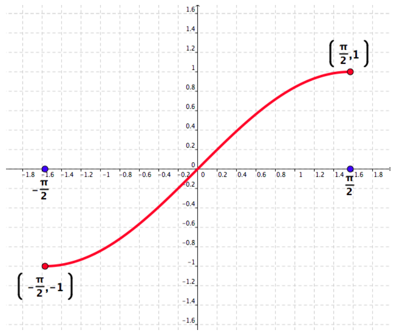


Figure 6.4: $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

24. The first several so-called Bernoulli polynomials are given as follows:

$$\begin{aligned}
B_0(x) &= 1 \\
B_1(x) &= x - \frac{1}{2} \\
B_2(x) &= x^2 - x + \frac{1}{6} \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30} \\
B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x \\
B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}
\end{aligned}$$

It is known that the Bernoulli polynomials satisfy the following relation:

$$B_n(x+1) - B_n(x) = nx^{n-1}.$$

Check this fact for $n = 1$, $n = 2$, and $n = 3$, i.e., check that

(a) $B_1(x+1) - B_1(x) = 1$,

(b) $B_2(x+1) - B_2(x) = 2x$,

(c) $B_3(x+1) - B_3(x) = 3x^2$.

25. The first few so-called Euler polynomials are given as follows:

$$\begin{aligned}
E_0(x) &= 1 \\
E_1(x) &= x - \frac{1}{2} \\
E_2(x) &= x^2 - x \\
E_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{4} \\
E_4(x) &= x^4 - 2x^3 + x \\
E_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}
\end{aligned}$$

It is known that the Euler polynomials satisfy the following property. If a real number h is such that $|h|$ is very small then, for any real number x,

$$\frac{E_n(x+h) - E_n(x)}{h} \approx nE_{n-1}(x).$$

Check this fact for $x = 1$, $n = 2$ and $n = 3$, i.e., check that if $|h|$ is small then

(a) $\dfrac{E_2(1+h) - E_2(1)}{h} \approx 2E_1(1)$

(b) $\dfrac{E_3(1+h) - E_3(1)}{h} \approx 3E_2(1)$.

26. Let

$$f(x) = 2\log(x+2) + 1.$$

(a) Determine the domain of the function f.

(b) Determine all zeros of the function f.

(c) Determine the vertical asymptote.

(d) Sketch a graph of the function f.

(e) Solve the equation $f(x) = 21$.

(f) Find the inverse function f^{-1} of the function f.

27. Each point on the unit circle corresponds to a unique angle α having the positive part of the x-axis as its initial ray and such that $0 \le \alpha < 2\pi$ (in radians.)

Complete the following table:

Point on the unit circle	Corresponding angle α	$\sin\alpha$	$\cos\alpha$
$(1,0)$	0		
$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$			
	$\frac{\pi}{4}$		
		$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$(0,1)$			
	$\frac{2\pi}{3}$		
		$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
	$\frac{5\pi}{6}$		
$(-1,0)$			
		0	-1

28. Assuming all terms are defined, simplify

$$\frac{\sin x}{\csc x}+\frac{1}{\sec^2 x}+\tan^2 x.$$

29. Let f be the function which associates to an integer n the number 1 if n is even, and -1 if n is odd. Find the following: $f(2), f(3), f(145), f(-5), f(0)$. Can you find a formula for f?

30. The number of gallons of paint needed to paint a house depends on the size of the house. A gallon of paint typically covers 250 ft^2. Thus the number of gallons of paint, n, is a function of the area to be painted, A ft^2. Write $n = f(A)$.

    (a) Find a formula for f.

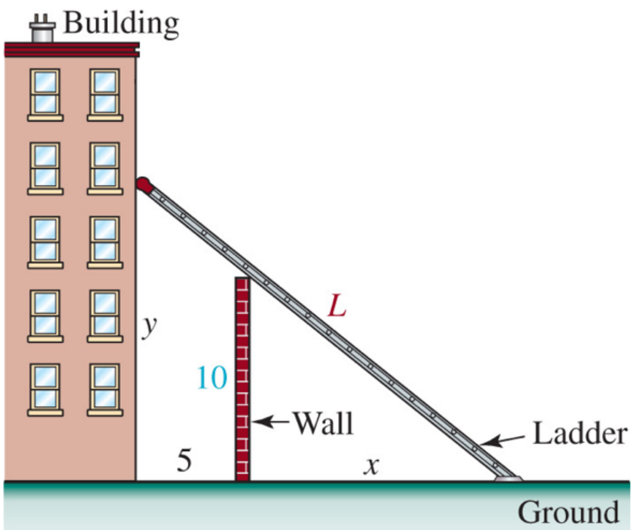
    (b) Explain in words what the statement $f(10,000) = 40$ tells us about painting houses.

31. A 10-ft wall stands 5 ft from a building and a ladder of variable length L, supported by the wall, is placed so it reaches from the ground to the building. Let y denote the vertical distance from the ground to where the tip of the ladder touches the building, and let x denote the horizontal distance from the wall to the base of the ladder.

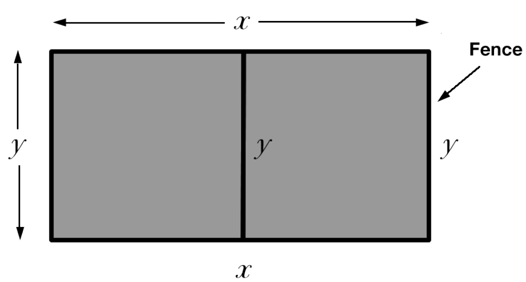
    (a) Find an expression for the height y as a function of x.

    (b) Find an expression for the length L as a function of x.

    (c) Determine the domain and range of the function $L(x)$ found in part (b).



32. A rancher intends to work off a rectangular plot of land that will have an area of 1000m^2. The plot will be fenced and divided into two equal portions by and additional fence parallel to two sides. (See diagram.)



    (a) Find the total length of fence used as a function of y.
    (b) What is the domain of the function found in part (a)?

33. An airplane is flying parallel to the ground at an altitude of 3 km. It flies directly over an observer on the ground and continues flying parallel to the ground. Let d be the distance from the observer to the airplane.
    (a) Draw a picture that represents the situation.
    (b) Identify and label all constants and variable quantities.
    (c) Find a relationship between d and the other variable quantities; express this relationship as an equation.

34. Two cars leave an intersection at the same time. One car travels north at the speed of 60 km/h and the other travels west at the speed of 50 km/h. Let t be time in hours since the cars left the intersection. Let d be the distance between the two cars at time t.
    (a) Draw a picture that represents the situation.
    (b) Express d as a function of t.

35. The **graph** of a function f is defined to be the set of all points (x, y) in the Cartesian plane satisfying the equation
$$y = f(x).$$
Sketch the graphs of the following functions.

(a) $f(x) = x + 1$

(b) $g(x) = x^2 - 1$

(c) $h(x) = \dfrac{x^2 - 1}{x - 1}$

(d) $i(x) = \begin{cases} 2x + 3 & \text{if } x \leq 0 \\ x^2 + 3 & \text{if } x > 0 \end{cases}$

(e) $j(x) = |x|$

(f) $k(x) = 2^{-x}$

(g) $l(x) = \log_{\frac{1}{2}} x$

(h) $m(x) = \sin x$

(i) $n(x) = \cos x$

(j) $o(x) = \tan x$

(k) $p(x) = x^3$

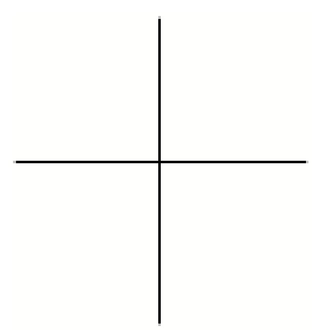
(l) $q(x) = \dfrac{1}{x}$

(m) $r(x) = e^x$

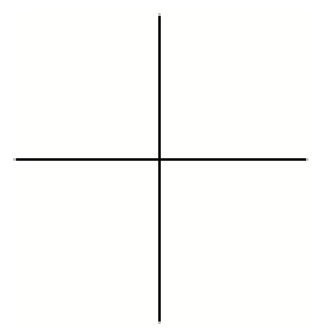
(n) $s(x) = \ln x$

36. Sketch the following angles in standard position and give the measure of the angle in degrees:

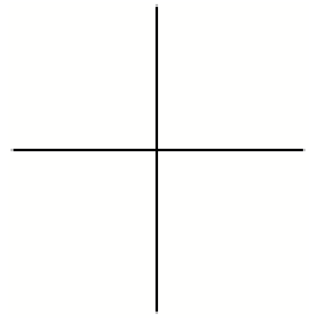
(a) $\dfrac{\pi}{2}$



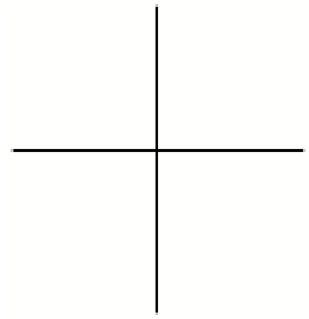
(b) $\dfrac{\pi}{4}$



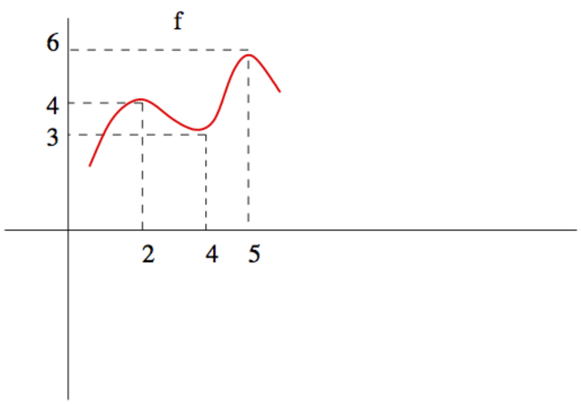
(c) $-\dfrac{5\pi}{6}$



(d) $\dfrac{13\pi}{3}$



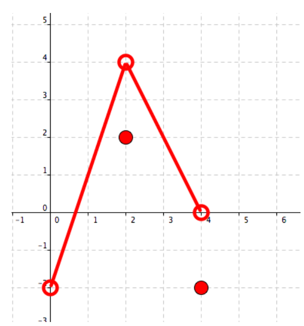
37. Given the graph of f sketch the graph of $\frac{1}{2}(f(x-4)-8)$.



38. If $f(-1)=1$, $f(2)=3$, $g(-1)=-5$ and $g(2)=17$ find $(f+g)(-1)$, $(fg)(-1)$ and $(f/g)(2)$.

39. Given the graph of f sketch the graph of the function

$$g(x) = 2 - \frac{1}{2} \cdot f(x-2).$$



40. (a) If $f(x)=2^x$, $g(t)=3\sqrt{t}$ and $h(\theta)=\sin\theta$ find $h\circ f\circ g$. What is the domain of $h\circ f\circ g$?

    (b) Given $F(x)=(1+2\sin x)^3$ find function p, q and r such that $F=p\circ q\circ r$.

    (c) Write $\sin^2\left(e^{x^2-x}\right)$ as a composition of (elementary) functions.

41. For each of the following functions draw its graph and determine intervals where the function is monotone, i.e., where the function increases and where the function decreases. Also, for each of the given functions in parts (a)-(d) find and simplify, if possible, the ratio

$$\frac{F(1+h)-F(1)}{h}$$

where F represents the appropriate function. If the function is one-to-one draw the graph of its inverse function.

   (a) $f(x) = x^2$
   (b) $g(x) = x^3$
   (c) $h(x) = e^x$
   (d) $i(x) = \sin x$
   (e) $j(x) = \sin x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

42. Find a formula for the inverse of:

   (a) $f(x) = \dfrac{x}{3x+1}$
   (b) $g(x) = e^{x^3+1} - 2$
   (c) $h(x) = 4(\log(x+2))^3 + 1$
   (d) $k(x) = \dfrac{2+e^x}{1-3e^x}$.

43. (a) Determine $\log_2(16)$, $\log_2\left(\frac{1}{8}\right)$ and $\log_2(1)$.
   (b) Can you find $\log_2(-32)$? Explain your reasoning.

44. Solve the equation $e^{x^3-3} - 9 = 0$ for x.

45. For any $-1 \leq x \leq 1$ and any $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$ we define the function $\sin^{-1} : [-1,1] \to \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ by

$$\sin^{-1} x = y \ \Leftrightarrow\ \sin y = x$$

Determine the following.

   (a) $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$
   (b) $\sin\left(\sin^{-1}\left(\dfrac{1}{3}\right)\right)$
   (c) $\sin^{-1}\left(\sin\left(\dfrac{3\pi}{4}\right)\right)$

46. **True or False.** Justify your answers.

   (a) If $f(s) = f(t)$ then $s = t$.
   (b) If f is an odd function and $f(3) = 6$ then $f(-3) = -6$.

(c) If $x_1 < x_2$ and g is a decreasing function then $g(x_1) > g(x_2)$.

(d) If f and g are functions then $f \circ g = g \circ f$.

(e) If the line $x = 1$ is a vertical asymptote of $y = f(x)$, then f is not defined at 1 .

(f) The graph of a function can have only one y-intercept

(g) Two lines with negative slopes could be perpendicular

47. Give the domain and a simplified expression for

(a) $m(h) = \dfrac{\sqrt{9+h} - 3}{h}$

(b) $h(x) = \dfrac{|3 - 6x|}{2x - 1}$. Sketch the graph of h.

48. The **average rate of change** of a function f over the interval $[x_1, x_2]$ is defined as

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Notice that this can be interpreted as the slope of the secant line through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 100,000\left(1 - \frac{t}{60}\right)^2 \quad 0 \leq t \leq 60 \ .$$

Find the average rate at which the water is flowing out of the tank between:

(a) 0 and 10 minutes,

(b) 40 and 50 minutes.

What is the average rate at which the water is flowing out of the tank over the entire 60 minute time period?

49. A 6-ft tall man walks away from a 15-ft lamppost and the man's shadow is cast on the ground.

(a) Draw a picture to represent this situation, and identify and label all constants and variables.

(b) Determine the relationship between the distance x from the man to the lamppost and length of the man's shadow.

50. Let s and t be real numbers such that $\log_2 s = 3/2$ and $\log_2 t = 4/3$.

(a) Calculate $\log_2(s^2t^3)$.

(b) Calculate $\log_4((st)^5)$

(c) Calculate $\log_2 \dfrac{s}{t}$

(d) Calculate $\dfrac{2^{s/2}-s}{s}$

(e) Find all solutions to the equation $s^x = t$.

51. Find all solutions to the equation:

(a) $x^4 - 2x^2 - 1 = 0$.

(b) $x^4 - x^2 - 12 = 0$.

(c) $2^{2x} - 7 \cdot 2^x - 8 = 0$.

(d) $4^{x+1} = 3$

(e) $\dfrac{\log(12x)}{\log(4x)} = 2$

(f) $\ln(x-1) - \ln x = \ln 3$

(g) $\log_3(\log_2 x) = 1$

(h) $1 + \sin x = 2\cos^2 x$ for values of x in the interval $(-\pi, \pi]$.

(i) $\sin x + \cos x = 1$ for values of x in the interval $(-\pi, \pi]$.

(j) $\sin 2x - \sin x = 0$ for values of x in the interval $(-\pi, \pi]$.

(k) $3\sin^4 x - 6\sin^2 x + 3 = 0$ or values of x in the interval $(-\pi, \pi]$.

52. Find the radius and the centre of the circle $x^2 + 6x + y^2 - 2y = 0$.

53. Find the amount that should be invested now at the annual interest rate of 4% compounded quarterly, in order to be worth \$100,000 in 10 years.

54. Use the identities

$$\sin^2 x+\cos^2 x = 1,\ \ \sin(x_1 \pm x_2) = \sin x_1 \cos x_2 \pm \cos x_1 \sin x_2,\ \text{ and }\ \cos(x_1 \pm x_2) = \cos x_1 \cos x_2 \mp \sin x_1 \sin x_2$$

to:

(a) To obtain the following identities

i. $\sin(2x) = 2\sin x \cos x$

ii. $\cos(2x) = \cos^2 x - \sin^2 x = 2(\cos^2 x - 1 = 1 - 2\sin^2 x$

iii. $\sin^2 x = \dfrac{1-\cos 2x}{2}$

iv. $\cos^2 x = \dfrac{1+\cos 2x}{2}$;

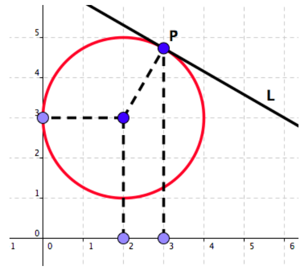
(b) To determine the exact values in each of the following:

i. $\sin \dfrac{\pi}{12}$

ii. $\cos \dfrac{7\pi}{12}$

iii. $\tan \dfrac{\pi}{12}$

iv. $\sin \dfrac{7\pi}{8}$, $\cos \dfrac{7\pi}{8}$, and $\tan \dfrac{7\pi}{8}$.



55. A tangent line to a circle at a point P on the circle is perpendicular to the line through P and the centre of the circle.

    (a) Find the equation of the circle in the diagram.
    (b) Find the coordinates of the point P.
    (c) Find the equation of the line through P and the centre of the circle.
    (d) Find the equation of the line L.

56. Consider a vertical post with a square cross-section of 1m by 1m set squarely on a cubical block 3m on a side. Supports from the floor to the sides of the post are desired. Find the relationship between the length of support L and the angle, θ, the support makes with the floor?

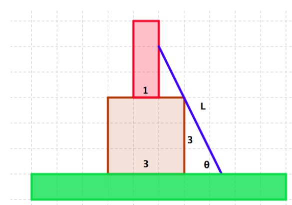


Figure 6.5: Vertical Post

57. Pam and Brian are standing 8m apart from each other, trying to read a sign which is some distance away from both of them. The angle formed by Pam's line of vision to the sign and her line of vision to Brian is $\pi/3$. The angle formed by Brian's line of vision to the sign and his line of vision to Pam is $\pi/6$. Find the distance between Pam and the sign.

58. Find the area of a regular octagon whose vertices are eight equally spaced points on the unit circle.

59. This question is based on the following diagram

    (a) Find the coordinates of the point Q.
    (b) Find the equation of the line through P and Q.

60. Ryan starts at point A located at the south bank of a river and wishes to reach point B located on the north bank of river (see below for a diagram). After some calculations, Ryan finds that the faster way to get to his destination is to use a boat to get from A to D and then

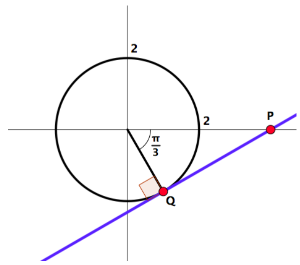


Figure 6.6: Circle and its tangent line

walk the rest of the way from D to B. The direct distance between A and B is 5 kilometres and the river is 3 kilometres wide. If Ryan's boat makes an angle of θ with the dashed line connecting the points A and C, write down an expression, in terms of θ, that describes the total distance Ryan must travel in this journey; i.e., the distance from A to D together with the distance from D to B. (Hint: what is the distance from C to B?)

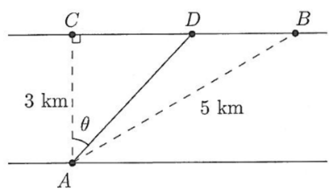


Figure 6.7: Ryan's trip

61. A rectangle has two vertices on the x-axes and two vertices on the semicircle whose equation is $y = \sqrt{25 - x^2}$. Find the area A of the rectangle as a function of x. What is the domain of the function $A = A(x)$ that you have obtain? What can you tell about about the values of the function $A(x)$ if x is close to 0? Close to 5?

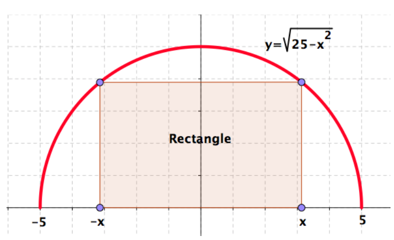


Figure 6.8: Area of the rectangle

62. Sketch the graph of a function f satisfying the following criteria:

- The domain of f is $(-4, \infty)$
- The range of f is $(-\infty, 5)$
- f is increasing on the interval $(-4, -1)$ and on the interval $(2, \infty)$
- The line $x = -4$ is a vertical asymptote to the graph of f
- The line $x = -3$ is a horizontal asymptote to the graph of f

- $f(0) = 1$

Chapter 7

Three Old Exams With Solutions

7.1 Exams

7.1.1 Midterm 1

1. **[6 marks]** In mathematics and computer science, the **ceiling function** $f(x) = \lceil x \rceil$ is defined in the following way:

   for every real number x, $f(x) = \lceil x \rceil$ is the **smallest integer** that is greater than or equal to x

   (a) Complete the following table:

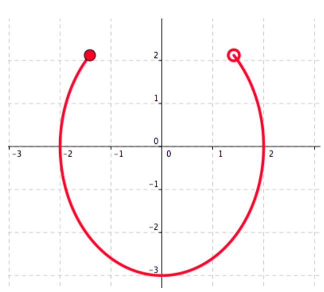
x	0	0.5	0.75	1	1.5	1.75	2	2.5
$f(x)$								

   (b) Find all solutions of the equation $f(x) = 1$.

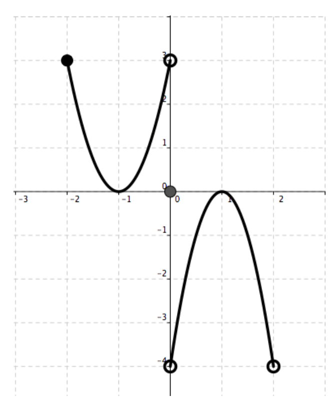
   (c) Sketch a graph of the function f.

2. **[6 marks]**

   (a) Use the vertical line test to determine which curve represents a graph of a function.



(a) Curve A



(b) Curve B

Circle the correct answer:

Function		Not a function	
A	B	A	B

(b) Let f be the function you picked above in the first part of this question.

i. Find the domain and the range of f.

ii. Complete the following table:

x	-2	-1	0	3
$f(x)$				

iii. Estimate $f(-0.75)$ and $f(1.99999)$.

iv. How many solutions does the equation $f(x) = 0$ have? Find all solutions of this equation.

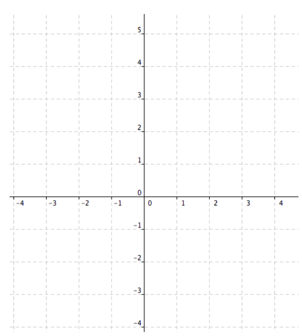
3. **[6 marks]** Let f be an **odd** function defined on the interval $[-2, 2]$ and such that

$$f(x) = \begin{cases} x & \text{if} \quad 0 \leq x \leq 1 \\ 2 & \text{if} \quad 1 < x \leq 2 \end{cases}$$

(a) Complete the following table:

x	2	1.5	1	0.5	0	-0.5	-1	-1.5	-2
$f(x)$									

(b) Draw the graph of the function f.



(c) Draw the graph of the function $g(x) = -2f(x-1) + 1$

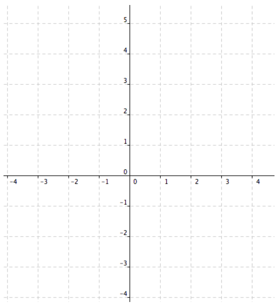
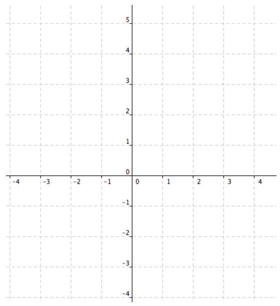
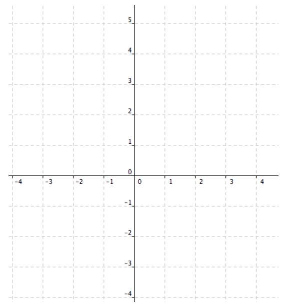
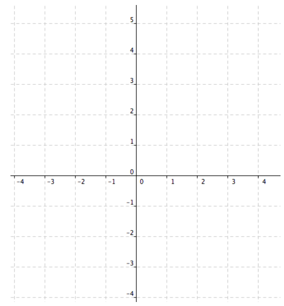
4. **[6 marks]**

(a) Let $f(x) = x + \dfrac{1}{x}$ and $g(x) = \dfrac{x+1}{x-1}$.

i. Find an expression for the function $(f \circ g)(x)$ and determine its domain.

ii. Find an expression for the function $(g \circ f)(x)$ and determine its domain.

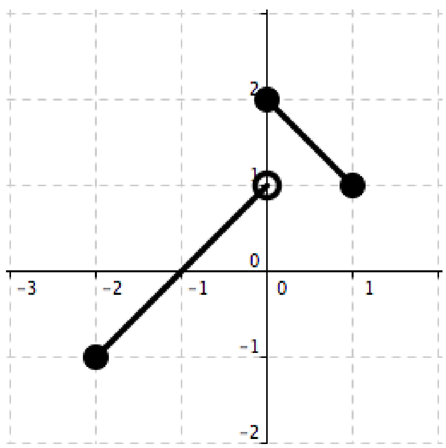
iii. Is it true that $g \circ f = f \circ g$? **Justify your answer!**

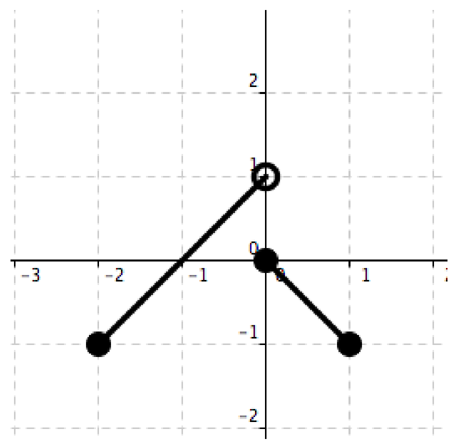
(b) Decompose the function $F(x) = 3|x^2 - 1| - 2$, i.e., find functions h, i, and j such that $F = j \circ i \circ h$.

5. **[6 marks]**

(a) Use the horizontal line test to determine which function below is one-to-one.



(a) Function A



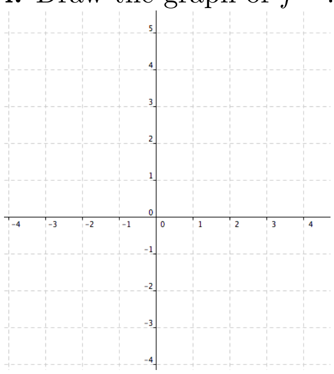
(b) Function B

Circle the correct answer:

One-to-one		Not one-to-one	
A	B	A	B

(b) Let f be the one-to-one function you picked above in the first part of this question.

i. Draw the graph of f^{-1}:



ii. Complete the following table and fill in all boxes:

Domain of f	$=$	
Range of f	$=$	
Domain of f^{-1}	$=$	
Range of f^{-1}	$=$	
$(f \circ f^{-1})(x)$	$=$	, for all $x \in [\square, \square]$
$(f^{-1} \circ f)(x)$	$=$	, for all $x \in [\square, \square]$
$((f^{-1})^{-1})(x)$	$=$	, for all $x \in [\square, \square]$

7.1.2 Midterm 2

1. **[6 marks]** Solve the following equations:

   (a) $4x + 7 - 2(x - 3) = 6x + 8 - 2(2x - 4)$

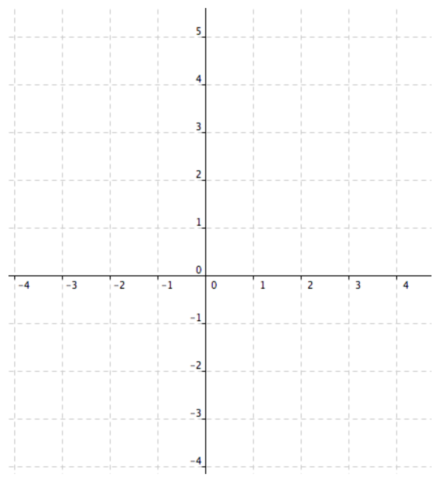
   (b) $4x^2 + 1 = 4x$

   (c) $5^{2x} - 5^x - 12 = 0$

2. **[6 marks]** Let the line l in the xy-plane be given by the equation $2x + y = 1$ and let P be the point with the coordinates $(-1, 1)$.

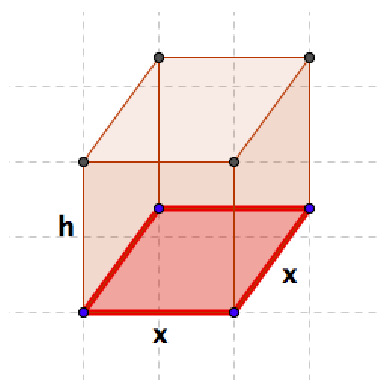
   (a) Find the point on the line l that is closest to the point P.

   (b) Draw the line l, the point P, and the point that you obtained in part (a).



3. **[6 marks]**

Suppose that you are given an open top box with the square base. Let x be the length of the side of the base and let the volume of the box be $V = 1$.



   (a) Express the surface area of the box in terms of two power functions of x.

   (b) What is the domain of the obtained function?

   (c) What will happen if x is very big? Very small?

4. **[6 marks]** Let $f(x) = \dfrac{x^2 + 2}{x^2 - x - 2}$.

(a) Determine the domain of f.

(b) Determine all zeros of f:

(c) Determine all vertical asymptotes of f:

(d) Determine the y-intercept of f:

(e) Determine the long range behaviour of f:

(f) Determine the intervals where $f(x) > 0$ and where $f(x) < 0$:

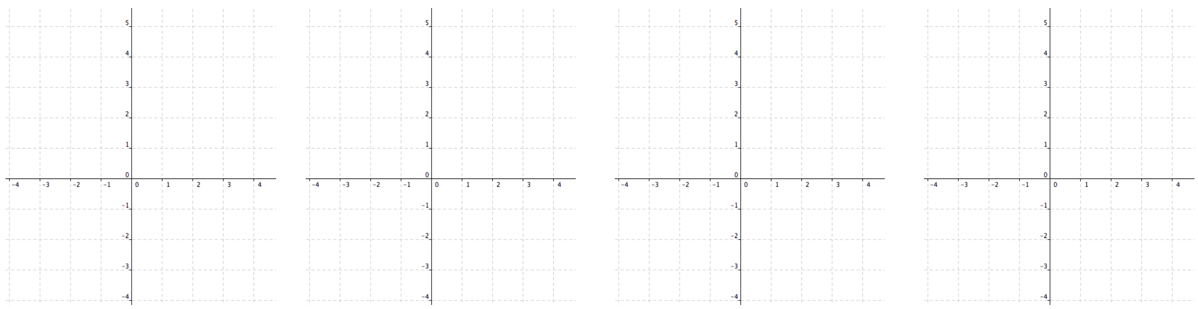
5. **[6]** Let $f(x) = 2\log(x+1) + 2$.

(a) Determine the domain of the function f.

(b) Determine all zeros of the function f.

(c) Find the inverse function f^{-1} of the function f.

(d) Starting with the graph of the function $y = \log x$ and by using the appropriate transformations, sketch a graph of the function f.



7.1.3 Final Exam

1. **[8 marks]** Let f be the function which associates to an integer n the number $\sin n\pi$ if n is even, and $\cos n\pi$ if n is odd.

(a) Complete the following table:

n	-2	-1	0	1	4	100	111
$f(n)$							

(b) Express the function f as a piecewise defined function:

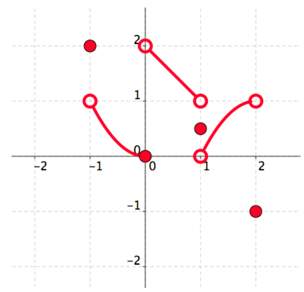
$$f(n) = \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

(c) State the domain and the range of the function f:

$$\text{Domain of } f \; = $$
$$\text{Range of } f \; = $$

(d) Draw a graph of f.

2. **[6 marks]** The function $y = f(x)$ is given by its graph:



(a) Complete the following table:

x	-2	-1	0	1	2
$f(x)$					

(b) Draw the graph of the function $g(x) = f(-x)$:

(c) Draw the graph of the function $h(x) = -f(x)$:

3. **[6 marks]** A soccer team plays in an arena that has a seating capacity of 10,000 spectators. With the ticket price set at \$75, the average attendance at recent games has been 5,500. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 50.

(a) Find a function that models the revenue in terms of the ticket price.

(b) Which price will give the maximum revenue? Justify your answer.

4. **[6 marks]** Let
$$f(x) = e^{\sqrt{1-x}} + 3.$$

(a) Find the domain of the function f.

(b) Decompose the function f, i.e., find functions p, q, r, and s such that $f = p \circ q \circ r \circ s$. **Note:** None of the functions chosen can be the identity function.

(c) Find a formula for the inverse of the function f.

5. **[12 marks]** Find all solutions to the equations:

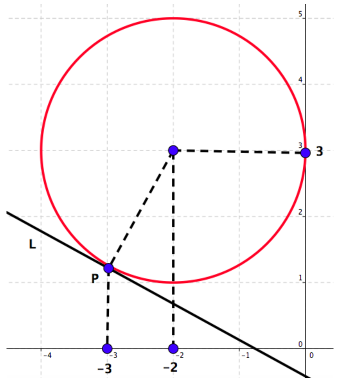
(a) $5^{2x} - 5^x - 12 = 0$.

(b) $[\log(10x)] \cdot [\log(100x)] = (\log x)^2$

(c) $\sin 2x - \sqrt{2} \cdot \sin x = 0$ for values of x in the interval $[0, 2\pi)$.

6. **[8 marks]** In this question use the fact that a tangent line to a circle at a point P on the circle is perpendicular to the line through P and the centre of the circle.

(a) Find the equation of the circle in the diagram.



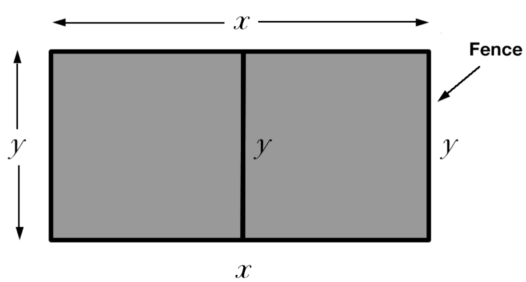
(b) Find the coordinates of the point P on the diagram above.

(c) Find the equation of the line through P and the centre of the circle.

(d) Find the equation of the tangent line L.

7. **[8 marks]**

A farmer intends to work off a rectangular plot of land that will have an area of 2000m^2. The plot will be fenced and divided into two equal portions by an additional fence parallel to two sides.



(a) Find the total length L of fence used as a function of y. Classify the obtained function $L = L(y)$ (i.e., determine if is this a linear function, a power function, a polynomial, a rational, an exponential, a logarithmic or a trigonometric function.)

(b) What is the domain of the function $L = L(y)$?

(c) Explain carefully what is that you can say about the total length $L = L(y)$ of fence used if y is a **large** positive number. What if y is a **small** positive number?

(d) Based on your answers in part (c) draw a rough sketch of the graph of the function $L = L(y)$. Is there a value of y that would yield the minimum length of the fence L?

8. **[7 marks]** The first few so-called Bernoulli polynomials are given as follows:

$$B_0(x) = 1 \qquad B_1(x) = x - \tfrac{1}{2}$$

$$B_2(x) = x^2 - x + \tfrac{1}{6} \qquad B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30} \qquad B_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x$$

It is known that the Bernoulli polynomials satisfy the following property: If h is such that $|h|$ is a very small positive number then, for any real number x and any positive integer n,

$$\frac{B_n(x+h) - B_n(x)}{h} \approx nB_{n-1}(x).$$

(a) Check the above fact for $x = 1$ and $n = 1$, i.e., check that if $|h| > 0$ is small then

$$\frac{B_1(1+h) - B_1(1)}{h} \approx B_0(1).$$

(b) Check the above fact for $x = 1$ and $n = 2$, i.e., check that if $|h| > 0$ is small then

$$\frac{B_2(1+h) - B_2(1)}{h} \approx 2B_1(1).$$

9. **[7 marks]** Using the approximations $\ln 2 \approx 0.6$ and $\ln 5 \approx 1.6$ and the properties of logarithms, estimate each of the numbers below. Each question is worth 1 point - You have to show all your work, i.e., it must be clear which properties you are using.

   (a) Estimate $\ln 8$.

   (b) Estimate $\ln 25$.

   (c) Estimate $\ln 10$.

   (d) Estimate $\ln 2.5$.

   (e) Estimate $\ln 40$.

   (f) Estimate $\ln 50$.

   (g) Estimate $\log_2 5$.

10. **[6 marks]** Laura and Ali are standing 10m apart from each other, trying to read a sign which is some distance away from both of them. The angle formed by Ali's line of vision to the sign and his line of vision to Laura is $\pi/6$. The angle formed by Laura's line of vision to the sign and her line of vision to Ali is $\pi/3$. Find the distance between Laura and the sign.

11. **[12 marks]** Use the identities $\sin^2 x + \cos^2 x = 1$, $\sin(x_1 \pm x_2) = \sin x_1 \cos x_2 \pm \cos x_1 \sin x_2$, and $\cos(x_1 \pm x_2) = \cos x_1 \cos x_2 \mp \sin x_1 \sin x_2$ to determine the exact values of each of the numbers below.

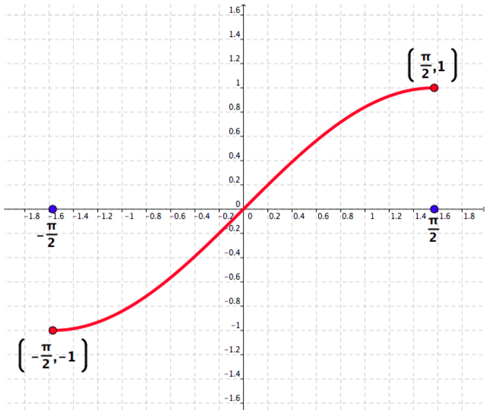
   (a) $\cos \frac{\pi}{12}$

   (b) $\sin \frac{7\pi}{12}$

   (c) $\tan \frac{\pi}{12}$

   (d) $\cot \frac{\pi}{12}$

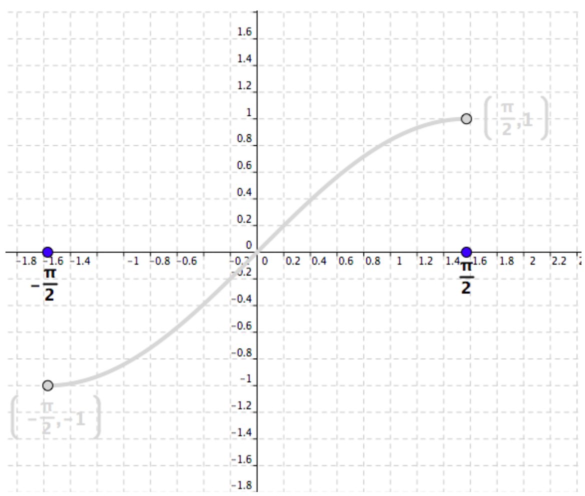
12. **[6 marks]** A function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ is given by $f(x) = \sin x$, for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This is the graph of the function f:



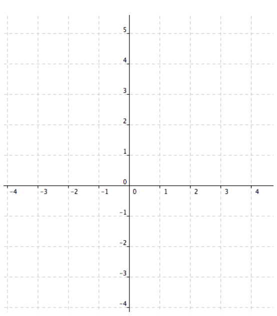
(a) Justify the fact that the function f has an inverse, i.e., justify the fact that the function f^{-1} exists. The function f^{-1} is called the inverse sine function and it is denoted by $f^{-1}(x) = \sin^{-1} x$.

(b) State the domain and range of the function f^{-1}.

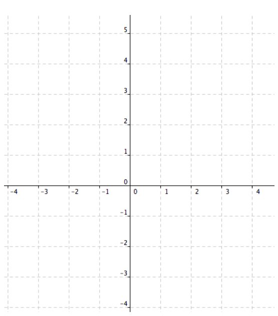
(c) Draw a graph of the function f^{-1}. For your convenience the graph of the function f is provided on the grid below.



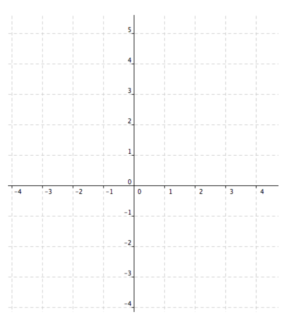
13. **[10 marks]** Sketch graphs of the following functions:



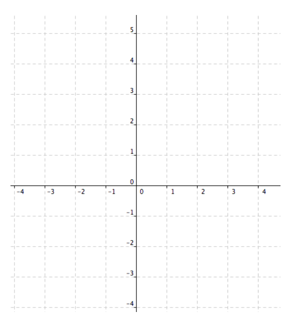
(a) $f(x) = x$



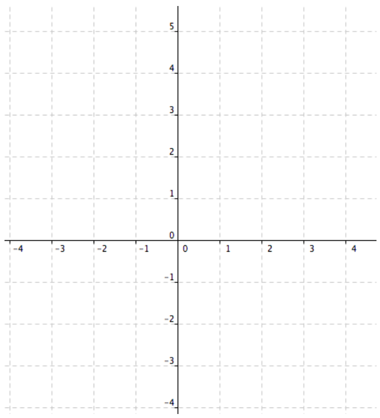
(b) $g(x) = |x|$



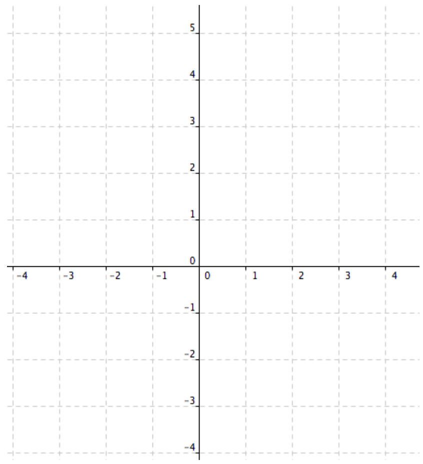
(c) $h(x) = x^2$



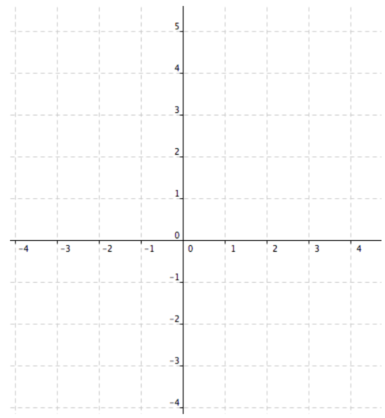
(d) $i(x) = x^3$



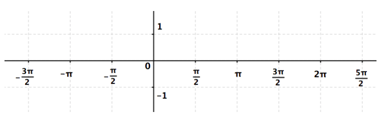
(a) $j(x) = x^{-1}$



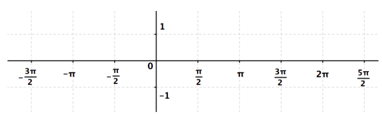
(b) $k(x) = e^x$



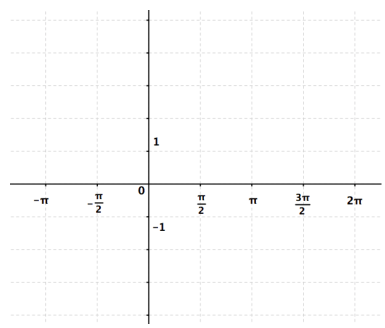
(c) $l(x) = \ln x$



(a) $m(x) = \cos x$



(b) $n(x) = \sin x$



(c) $o(x) = \cot x$

7.2 Solutions:

7.2.1 Midterm 1

1. **[6 marks]** In mathematics and computer science, the **ceiling function** $f(x) = \lceil x \rceil$ is defined in the following way:

for every real number x, $f(x) = \lceil x \rceil$ is the **smallest integer** that is greater than or equal to x

(a) Complete the following table:

Solution :

x	0	0.5	0.75	1	1.5	1.75	2	2.5
$f(x)$	0	1	1	1	2	2	2	3

(b) Find all solutions of the equation $f(x) = 1$.

Solution : We need to find all real numbers x with the property that $f(x) = \lceil x \rceil = 1$. In other words, we are looking for all real numbers such that the number 1 is the smallest integer that is greater than or equal to x. This gives the set

$$\{x : 0 < x \leq 1\} = (0, 1].$$

(c) Sketch a graph of the function f.
Solution :

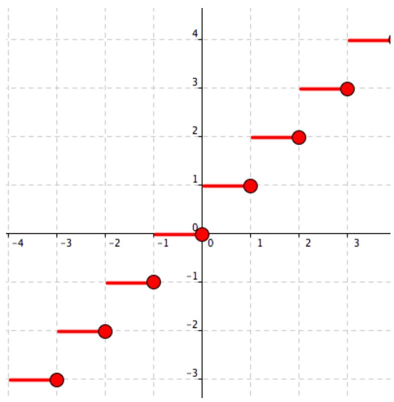
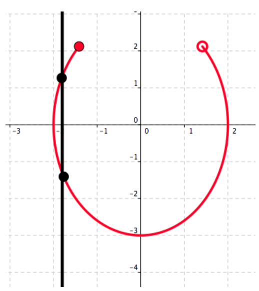


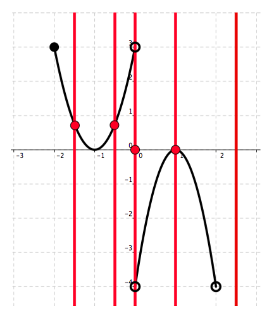
Figure 7.8: Graph of the function $f(x) = \lceil x \rceil$

2. **[6 marks]**

(a) Use the vertical line test to determine which curve represents a graph of a function.
Solution :



(a) Curve A



(b) Curve B

We note that there is a vertical line that intersects Curve A at two points. This means, by the vertical line test, that Curve A does not represent a graph of a function. We *see* that every vertical line intersects Curve B in at most one point. Hence, Curve B represents a graph of a function. Therefore:

Circle the correct answer:

Function		Not a function	
A	(B)	(A)	B

(b) Let f be the function you picked above in the first part of this question.

i. Find the domain and the range of f.
Solution : From the graph (b) we read that the function f is defined on the interval $[-2, 2)$. The range of the function is the interval $(-4, 3]$.

ii. Complete the following table:
Solution : We read from the graph:

x	-2	-1	0	3
$f(x)$	3	0	0	Not defined

iii. Estimate $f(-0.75)$ and $f(1.99999)$.
Solution : Based on the graph it appears that $f(-0.75) \approx 0.75$ and $f(1.99999) \approx -4$

iv. How many solutions does the equation $f(x) = 0$ have? Find all solutions of this equation.

Solution : From the graph we read that the equation $f(x) = 0$ has three solutions: -1, 0, and 1.

3. **[6]** Let f be an **odd** function defined on the interval $[-2, 2]$ and such that

$$f(x) = \begin{cases} x & \text{if} \quad 0 \leq x \leq 1 \\ 2 & \text{if} \quad 1 < x \leq 2 \end{cases}$$

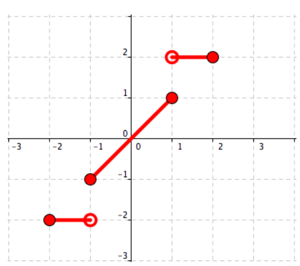
(a) Complete the following table:

Solution : To complete the table we use the given definition of the function f for $x \in [0, 2]$ and the fact that for odd functions we have $f(-x) = -f(x)$ for all x in the domain of f. For example, since, from the above definition, $f(2) = 2$, it follows that $f(-2) = -f(2) = -2$.

x	2	1.5	1	0.5	0	-0.5	-1	-1.5	-2
$f(x)$	2	2	1	0.5	0	-0.5	-1	-2	-2

(b) Draw the graph of the function f.

Solution : First we draw the graph of f for $x \in [0, 2]$ according to the given definition. Then we use the fact that f is an odd function and that its graph must be symmetric with the respect to the origin.

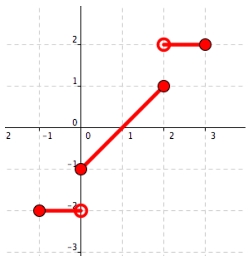


(c) Draw the graph of the function $g(x) = -2f(x - 1) + 1$

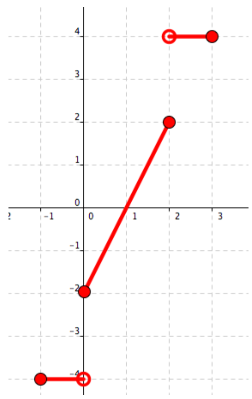
Solution :

4. **[6]**

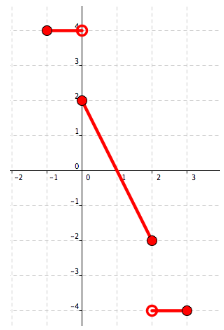
(a) Let $f(x) = x + \dfrac{1}{x}$ and $g(x) = \dfrac{x+1}{x-1}$.



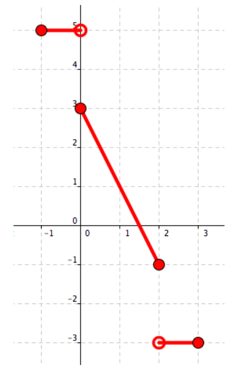
(a) Horizontal shift to the right by 1



(b) Vertical stretch by factor 2



(c) Reflection about the x-axis



(d) Vertical shift up by 1

i. Find an expression for the function $(f \circ g)(x)$ and determine its domain.
Solution : Note that the function g is not defined for $x = 1$. It follows that for all $x \neq 1$:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f\left(\frac{x+1}{x-1}\right) \\ &= \frac{x+1}{x-1} + \frac{1}{\frac{x+1}{x-1}} = \frac{x+1}{x-1} + \frac{x-1}{x+1} \\ &= \frac{(x+1)^2 + (x-1)^2}{(x+1)(x-1)} = \frac{x^2+2x+1+x^2-2x+1}{x^2-1} \\ &= \frac{2x^2+2}{x^2-1} = \frac{2(x^2+1)}{x^2-1}.\end{aligned}$$

Hence the domain of the function $f \circ g$ is the set $\{x : x \neq 1 \text{ and } x^2 - 1 \neq 0\} = \{x : x \neq 1 \text{ and } (x+1)(x-1) \neq 0\} = \{x : x \neq 1, x \neq -1\}$.

ii. Find an expression for the function $(g \circ f)(x)$ and determine its domain.
Solution : Note that the function f is not defined for $x = 0$. It follows that for all $x \neq 0$:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g\left(x + \frac{1}{x}\right) \\ &= \frac{x + \frac{1}{x} + 1}{x + \frac{1}{x} - 1} = \frac{\frac{x^2+1+x}{x}}{\frac{x^2+1-x}{x}} \\ &= \frac{x^2+x+1}{x^2-x+1}\end{aligned}$$

We observe that for all real numbers x

$$x^2 - x + 1 = x^2 - 2 \cdot \frac{1}{2} \cdot x + \frac{1}{4} + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0.$$

Hence the domain of the function $g \circ f$ is the set $\{x : x \neq 0\}$.

iii. Is it true that $g \circ f = f \circ g$? **Justify your answer!**
Solution : We see that the function $f \circ g$ is defined for $x = 0$ and the $g \circ f$ is not. Hence those two functions are not equal.

(b) Decompose the function $F(x) = 3|x^2 - 1| - 2$, i.e., find functions h, i, and j such that $F = j \circ i \circ h$.
Solution : We can take

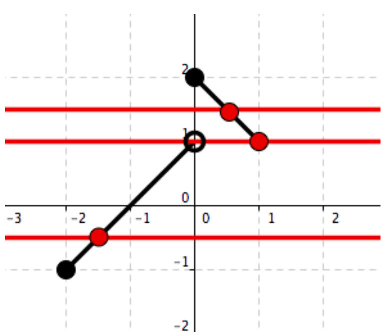
$$h(x) = x^2 - 1, \ i(x) = |x|, \text{ and } j(x) = 3x - 2.$$

Then

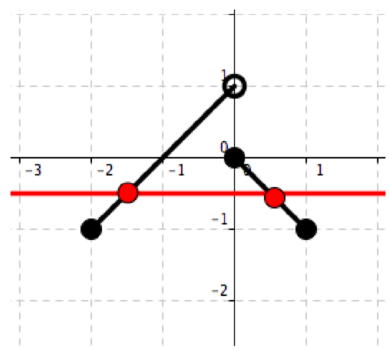
$$(j \circ i \circ h)(x) = j(i(h(x))) = j(i(x^2 - 1)) = j(|x^2 - 1|) = 3|x^2 - 1| - 2 = F(x).$$

5. **[6 marks]**

(a) Use the horizontal line test to determine which function below is one-to-one.
Solution : We note that there is a horizontal line that intersects the graph of Function B at two points. This means, by the horizontal line test, that Function B is not an one-to-one function. We *see* that every horizontal line intersects the graph of Function A in at most one point. Hence, Function A is an one-to-one function.



(a) Function A



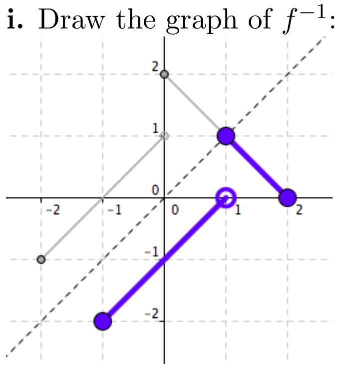
(b) Function B

Circle the correct answer:

One-to-one		Not one-to-one	
Ⓐ	B	A	Ⓑ

(b) Let f be the one-to-one function you picked above in the first part of this question.

i. Draw the graph of f^{-1}:



ii. Complete the following table and fill in all boxes:

Domain of f	$=[-2, 1]$
Range of f	$=[-1, 2]$
Domain of f^{-1}	$=[-1, 2]$
Range of f^{-1}	$=[-2, 1]$
$(f \circ f^{-1})(x)$	$=x$, for all $x \in [-1, 2]$
$(f^{-1} \circ f)(x)$	$=x$, for all $x \in [-2, 1]$
$((f^{-1})^{-1})(x)$	$=f(x)$, for all $x \in [-2, 1]$

7.2.2 Midterm 2

1. **[6 marks]** Solve the following equations:

   (a) $4x + 7 - 2(x - 3) = 6x + 8 - 2(2x - 4)$
   Solution: This is a linear equation. We note that the given equation is equivalent to the equation

   $$4x + 7 - 2x + 6 = 6x + 8 - 4x + 8 \Leftrightarrow 2x + 13 = 2x + 16.$$

   From

   $$2x + 13 = 2x + 16 \Leftrightarrow 2x - 2x = 16 - 13 \Leftrightarrow 0 = 3$$

   we conclude that the given equation has no solution.

   (b) $4x^2 + 1 = 4x$
   Solution: The given equation is a quadratic equation. From

   $$4x^2 + 1 = 4x \Leftrightarrow 4x^2 - 4x + 1 = 0 \Leftrightarrow (2x - 1)^2 = 0 \Leftrightarrow 2x - 1 = 0 \Leftrightarrow x = \frac{1}{2}.$$

   (c) $5^{2x} - 5^x - 12 = 0$
   Solution: We note that the substitution $t = 5^x$ gives a quadratic equation $t^2 - t - 12 = 0$. The quadratic formula gives

   $$t = \frac{1 \pm \sqrt{1 + 49}}{2} = \frac{1 \pm \sqrt{49}}{2} = \frac{1 \pm 7}{2}.$$

   Since $t = 5^x > 0$ we reject the negative solution of the quadratic equation above, $t = \frac{1 - 7}{2} = -3$. Hence

   $$t = \frac{1 + 7}{2} = 4 \Leftrightarrow 5^x = 4 \Leftrightarrow x = \log_5 4.$$

2. **[6 marks]** Let the line l in the xy-plane be given by the equation $2x + y = 1$ and let P be the point with the coordinates $(-1, 1)$.

   (a) Find the point on the line l that is closest to the point P.
   Solution: From

   $$2x + y = 1 \Leftrightarrow y = -2x + 1$$

   we conclude that all points on the line l are of the form $(x, -2x + 1)$, $x \in \mathbb{R}$. Hence the distance between a point $(x, -2x + 1)$ on the line and the point $(-1, 1)$ is given by

   $$d = \sqrt{(x - (-1))^2 + (-2x + 1 - 1)^2} = \sqrt{x^2 + 2x + 1 + 4x^2} = \sqrt{5x^2 + 2x + 1}.$$

   The quadratic function $f(x) = 5x^2 + 2x + 1$ achieves its minimum at the first coordinate of the vertex of its graph:

   $$x = -\frac{b}{2a} = -\frac{2}{10} = -\frac{1}{5}.$$

   Hence, the point on the line l that is closest to the point P is

   $$M = \left(-\frac{1}{5}, -2 \cdot \left(-\frac{1}{5}\right) + 1\right) = \left(-\frac{1}{5}, \frac{2}{5} + 1\right) = \left(-\frac{1}{5}, \frac{7}{5}\right).$$

   (b) Draw the line l, the point P, and the point that you obtained in part (a).

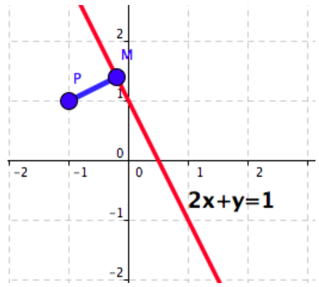
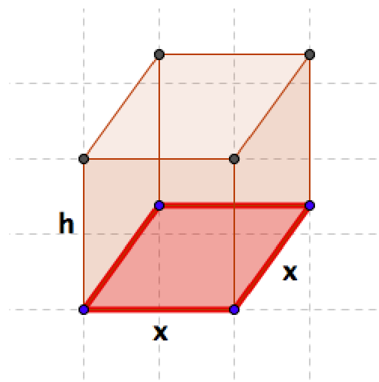


Figure 7.12: The point M is closest to the point P

3. **[6]**

Suppose that you are given an open top box with the square base. Let x be the length of the side of the base and let the volume of the box be $V = 1$.



(a) Express the surface area of the box in terms of two power functions of x.

Solution: The surface area is the sum of the areas of the square that is the bottom of the box and the areas of the four sides. Note that each side is a rectangle with sides x and h. Hence the surface area is given by $S = x^2 + 4hx$. Next we express the variable h in terms of the variable x.

The volume of the box is given as the product of the area of the base and the hight of the box. Since the base is a square with the side x we have that $V = x^2h$. It is given that $V = 1$ so

$$1 = x^2h \Rightarrow h = \frac{1}{x^2}.$$

It follows that

$$S = x^2 + 4 \cdot \frac{1}{x^2} \cdot x = x^2 + 4 \cdot \frac{1}{x}.$$

(b) What is the domain of the obtained function?

Solution: Observe that the expression for S obtained above is defined for all $x \neq 0$. Since x is the length of the side of the base we have that x must be a positive real number. Thus, the domain of the function S is the set $\{x \in \mathbb{R} | x > 0\}$.

(c) What will happen if x is very big? Very small?

Solution: If x is a large number then the term $\frac{4}{x}$ is very small and $S \approx x^2$. This means that the box will have a large base but that the hight of the box will be small.

If x is a small number then the term x^2 is very small and $S \approx \frac{4}{x}$. This means that the box will have a large hight but that the base of the box will be small.

4. **[6]** Let $f(x) = \dfrac{x^2+2}{x^2-x-2}$.

   (a) Determine the domain of f.

   Solution: Since f is a rational function, its domain is the set of all real numbers for which the denominator of f is not equal to 0. We solve the equation $x^2 - x - 2 = 0$ by using the quadratic formula (or any other method):

   $$x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}.$$

   Therefore, the domain of the function f is the set $\{x \in \mathbb{R} : x \neq -1, x \neq 2\}$. Note that

   $$f(x) = \frac{x^2+1}{x^2-x-2} = \frac{x^2+1}{(x+1)(x-2)}.$$

   (b) Determine all zeros of f:

   Solution: We observe that $x^2 \geq 0$ for all real numbers. Thus, $x^2 + 2 > 0$ for all real numbers which implies that the numerator of f is never equal to 0. This means that f has no zeros.

   (c) Determine all vertical asymptotes of f:

   Solution: The function f is not defined at $x = -1$ and $x = 2$. Since the numerator of f does not cancel at these values of x, we conclude that the vertical asymptotes are lines $x = -1$ and $x = 2$.

   (d) Determine the y-intercept of f:

   Solution: The y-intercept of f is given by

   $$f(0) = \frac{0^2+1}{0^2-0-2} = -\frac{1}{2}.$$

   (e) Determine the long range behaviour of f:

   Solution: We use the fact that for all x with large absolute values a polynomial in x behaves as its leading term. Thus

   $$f(x) = \frac{x^2+1}{x^2-x-2} \approx \frac{x^2}{x^2} = 1.$$

   This means that the line $y = 1$ is a horizontal asymptote to the graph of x.

   (f) Determine the intervals where $f(x) > 0$ and where $f(x) < 0$:

   Solution: From $f(x) = \dfrac{x^2+1}{x^2-x-2}$ and the fact that $x^2 + 1 > 0$ for all real number x we conclude that

   $$f(x) > 0 \Leftrightarrow x^2 - x - 2 > 0.$$

   The graph of the parabola $y = x^2 - x - 2 = (x+1)(x-2)$, a 'happy' parabola with zeros at $x = -1$ and $x = 2$, gives that

   $$x^2 - x - 2 > 0 \Leftrightarrow x \in (-\infty, -1) \cup (2, \infty) \text{ and } x^2 - x - 2 < 0 \Leftrightarrow x \in (-1, 2).$$

   Therefore

   $$f(x) > 0 \Leftrightarrow x \in (-\infty, -1) \cup (2, \infty) \text{ and } f(x) < 0 \Leftrightarrow x \in (-1, 2).$$

5. **[6]** Let $f(x) = 2\log(x+1) + 2$.

   (a) Determine the domain of the function f.

   Solution: The domain of the function f is given by

$$\{x \in \mathbb{R} : x + 1 > 0\} = \{x \in \mathbb{R} : x > -1\} = (-1, \infty).$$

   (b) Determine all zeros of the function f.

   Solution: We solve the equation $f(x) = 2\log(x+1) + 2 = 0$. It follows that

$$2\log(x+1) + 2 = 0 \Leftrightarrow 2\log(x+1) = -2 \Leftrightarrow \log(x+1) = -1 \Leftrightarrow x + 1 = 10^{-1}.$$

   Hence $x = -1 + 10^{-1} = -0.9$ is the only zero of the function f

   (c) Find the inverse function f^{-1} of the function f.

   Solution: To find f^{-1} we start with $y = 2\log(x+1) + 2$, interchange the roles of 'y' and 'x' and then solve

$$x = 2\log(y+1) + 2$$

   for y. It follows that

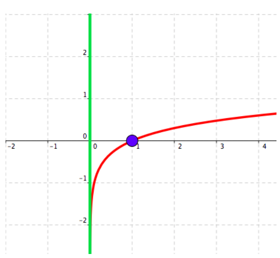
$$x = 2\log(y+1) + 2 \Leftrightarrow 2\log(y+1) = x - 2 \Leftrightarrow \log(y+1) = \frac{x-2}{2}$$

$$\Leftrightarrow y + 1 = 10^{\frac{x-2}{2}}.$$

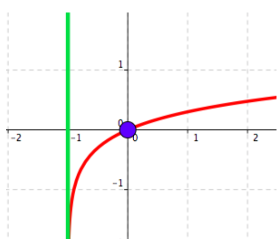
   Hence

$$f^{-1}(x) = -1 + 10^{\frac{x-2}{2}}.$$

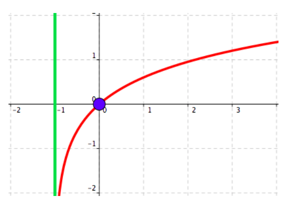
   (d) Starting with the graph of the function $y = \log x$ and by using the appropriate transformations, sketch a graph of the function f.



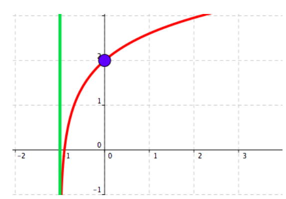
(a) $y = \log x$



(b) $y = \log(x+1)$



(c) $y = 2\log(x+1)$



(d) $y = f(x)$

7.2.3 Final Exam

1. **[8 marks]** Let f be the function which associates to an integer n the number $\sin n\pi$ if n is even, and $\cos n\pi$ if n is odd.

   (a) Complete the following table:

n	-2	-1	0	1	4	100	111
$f(n)$	$\sin(-2\pi) = 0$	$\cos(-\pi) = -1$	$\sin(0) = 0$	$\cos(\pi) = -1$	0	0	-1

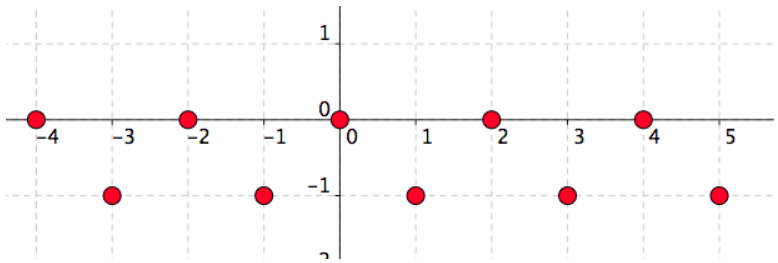
(b) Express the function f as a piecewise defined function:

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

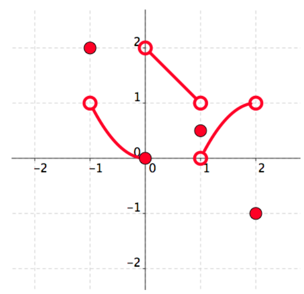
(c) State the domain and the range of the function f:

$$\begin{aligned} \text{Domain of } f &= \mathbb{Z} \\ \text{Range of } f &= \{-1, 0\} \end{aligned}$$

(d) Draw a graph of f.



2. **[6 marks]** The function $y = f(x)$ is given by its graph:



(a) Complete the following table:

x	-2	-1	0	1	2
$f(x)$	Not defined	2	0	0.5	-1

(b) Draw the graph of the function $g(x) = f(-x)$:

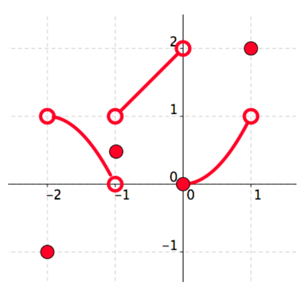


Figure 7.14: Function $y = g(x) = f(-x)$

(c) Draw the graph of the function $h(x) = -f(x)$:

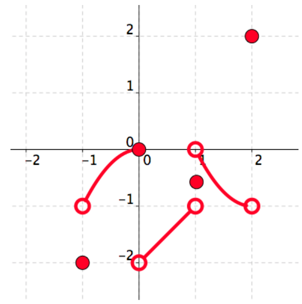


Figure 7.15: Function $y = h(x) = -f(x)$

3. **[6 marks]** A soccer team plays in an arena that has a seating capacity of 10,000 spectators. With the ticket price set at \$75, the average attendance at recent games has been 5,500. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 50.

   (a) Find a function that models the revenue in terms of the ticket price.

   Solution: Let x represent the ticket price in dollars and let $A(x)$ represent the average attendance at the price x. Observe that, since for each dollar the ticket price is lowered the average attendance increases by 50, the attendance with the ticket price of x dollars is given by

   $$A(x) = 5500 + 50(75 - x) = 5500 + 3750 - 50x = 9250 - 50x.$$

   The revenue is given as the product of the ticket price and the number of the people in the attendance. Hence the function

   $$R(x) = x \cdot A(x) = x(9250 - 50x) = -50x^2 + 9250x$$

   models the revenue $R(x)$ in terms of ticket price x.

   (b) Which price will give the maximum revenue? Justify your answer. **Solution:** Note that $R(x)$ is a quadratic function and that its graph is part of a parabola that opens downward. Thus the maximum revenue is achieved at the vertex of the parabola. The first coordinate of the vertex is given by

   $$x = -\frac{9250}{2 \cdot (-50)} = 92.50.$$

   Therefore the ticket price of $x = \$92.50$ will yield the maximum revenue.

   Note: From $R(x) = -50x^2 + 9250x = 50x(185 - x)$ we obtain that the maximum revenue is $R(92.50) = 50 \cdot 92.50 \cdot (185 - 92.50) = 4625 \cdot 92.50 = 427812.50$ dollars. Also, at the price of \$92.50 there will be $A(92.50) = 9250 - 50 \cdot 92.50 = 4625$ spectators in the arena.

4. **[6 marks]** Let

   $$f(x) = e^{\sqrt{1-x}} + 3.$$

(a) Find the domain of the function f.

Solution: The domain of the function F is given by

$$\{x \in \mathbb{R} : 1 - x \geq 0\} = \{x \in \mathbb{R} : x \leq 1\} = (\infty, -1].$$

(b) Decompose the function f, i.e., find functions p, q, r, and s such that $f = p \circ q \circ r \circ s$.
Note: None of the functions chosen can be the identity function.

Solution: If we take $s(x) = 1 - x$, $r(x) = \sqrt{x}$, $q(x) = e^x$, and $p(x) = x + 3$ then

$$(p \circ q \circ r \circ s)(x) = p(q(r(s(x)))) = p(q(r(1-x))) = p(q(\sqrt{(}x))) = p(e^{\sqrt{x}}) = e^{\sqrt{1-x}} + 3 = f(x).$$

(c) Find a formula for the inverse of the function f.

Solution: To find the inverse function of the function f we apply the following procedure. Let $y = e^{\sqrt{1-x}} + 3$. We interchange the roles of 'x' and 'y' to obtain

$$x = e^{\sqrt{1-y}} + 3$$

and then we solve this expression for 'y'. Thus

$$x = e^{\sqrt{1-y}} + 3 \Leftrightarrow x - 3 = e^{\sqrt{1-y}} \Leftrightarrow \ln(x-3) = \sqrt{1-y}$$

$$\Leftrightarrow ((\ln(x-3))^2 = 1 - y \Leftrightarrow y = 1 - ((\ln(x-3))^2.$$

Hence the inverse function of the function f is given by $f^{-1}(x) = 1 - ((\ln(x-3))^2$.

5. **[12 marks]** Find all solutions to the equations:

(a) $5^{2x} - 5^x - 12 = 0$.

Solution: Let $t = 5^x$. With this substitution the given equation becomes $t^2 - t - 12 = 0$. We solve this quadratic equation:

$$t^2 - t - 12 = 0 \Leftrightarrow (t-4)(t+3) = 0 \Leftrightarrow (t = 4 \text{ or } t = -3).$$

Since $t = 5^x > 0$, we reject the negative value $t = -3$. Thus

$$5^x = 4 \Leftrightarrow x = \log_5 4.$$

(b) $[\log(10x)] \cdot [\log(100x)] = (\log x)^2$

Solution: We use the fact that $\log(10x) = \log 10 + \log x = 1 + \log x$ and $\log(100x) = \log 100 + \log x = 2 + \log x$ to obtain

$$[\log(10x)] \cdot [\log(100x)] = (\log x)^2 \Leftrightarrow (1 + \log x) \cdot (2 + \log x) = (\log x)^2 \Leftrightarrow$$

$$2 + 3\log x + (\log x)^2 = (\log x)^2 \Leftrightarrow 2 + 3\log x = 0 \Leftrightarrow \log x = -\frac{2}{3} \Leftrightarrow x = 10^{-\frac{2}{3}}.$$

(c) $\sin 2x - \sqrt{2} \cdot \sin x = 0$ for values of x in the interval $[0, 2\pi)$.

Solution: We use the fact that $\sin 2x = 2 \sin x \cos x$ to obtain

$$\sin 2x - \sqrt{2} \cdot \sin x = 0 \Leftrightarrow 2 \sin x \cos x - \sqrt{2} \cdot \sin x = 0 \Leftrightarrow \sin x \cdot (2 \cos x - \sqrt{2}) = 0.$$

Hence

$$\sin 2x - \sqrt{2} \cdot \sin x = 0 \Leftrightarrow (\sin x = 0 \text{ or } 2 \cos x - \sqrt{2} = 0).$$

Finally, we observe that since, $x \in [0, 2\pi)$,

$$\sin x = 0 \Leftrightarrow x = 0 \text{ or } x = \pi$$

and

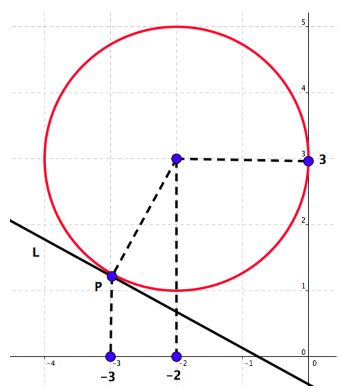
$$2 \cos x - \sqrt{2} = 0 \Leftrightarrow \cos x = \frac{\sqrt{2}}{2} \Leftrightarrow x = \frac{\pi}{4} \text{ or } x = \frac{7\pi}{4}.$$

Therefore, for $x \in [0, 2\pi)$,

$$\sin 2x - \sqrt{2} \cdot \sin x = 0 \Leftrightarrow x \in \left\{0, \frac{\pi}{4}, \pi, \frac{7\pi}{4}\right\}.$$

6. **[8 marks]** In this question use the fact that a tangent line to a circle at a point P on the circle is perpendicular to the line through P and the centre of the circle.

(a) Find the equation of the circle in the diagram.



Solution: We note that the centre of the circle is at the point $(-2, 3)$ and that its radius is $r = 2$. Hence the equation of the circle is

$$(x + 2)^2 + (y - 3)^2 = 4.$$

(b) Find the coordinates of the point P on the diagram above.

Solution: The first coordinate of the point P is $x = -3$. We find the second coordinate, y, by using the fact that the point is on the circle:

$$(-3 + 2)^2 + (y - 3)^2 = 4 \Leftrightarrow 1 + (y - 3)^2 = 4 \Leftrightarrow (y - 3)^2 = 3 \Leftrightarrow y - 3 = \pm\sqrt{3}.$$

Since the point P is below the centre of the circle, it follows that $y < 3$, i.e $y = 3 - \sqrt{3}$. Hence, $P = (-3, 3 - \sqrt{3})$.

(c) Find the equation of the line through P and the centre of the circle.

Solution: We use the equation of the line through two points, the centre $(-2,3)$ and the point $P(-3,3-\sqrt{3})$:

$$y-3=\frac{3-\sqrt{3}-3}{-3-(-2)}(x-(-2)) \Leftrightarrow y-3=\frac{-\sqrt{3}}{-1}(x+2) \Leftrightarrow y-3=\sqrt{3}(x+2).$$

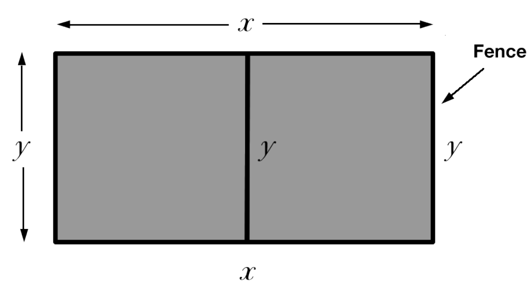
(d) Find the equation of the tangent line L.

Solution: The line L is perpendicular to the line through P and the centre of the circle. Hence its slope is $m=-\frac{1}{\sqrt{3}}$. The line is given by

$$y-(3-\sqrt{3})=-\frac{1}{\sqrt{3}}\cdot(x+3).$$

7. **[8 marks]**

A farmer intends to work off a rectangular plot of land that will have an area of 2000m^2. The plot will be fenced and divided into two equal portions by an additional fence parallel to two sides.



(a) Find the total length L of fence used as a function of y. Classify the obtained function $L=L(y)$ (i.e., determine if is this a linear function, a power function, a polynomial, a rational, an exponential, a logarithmic or a trigonometric function.)

Solution: The total length of the fence is given by $L=2x+3y$. From the fact that the area of the plot is 2000m^2 we conclude that

$$2000=xy \Rightarrow x=\frac{2000}{y}.$$

It follows that the total length L of fence used as a function of y is given by

$$L=L(y)=2\cdot\frac{2000}{y}+3y=\frac{4000}{y}+3y=\frac{4000+3y^2}{y}.$$

This is a rational function.

(b) What is the domain of the function $L=L(y)$?

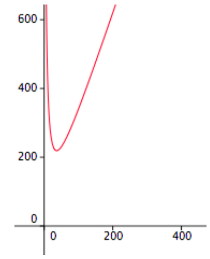
Solution: Since y represents the length of one side of the plot, we have that the domain of the function $L=L(y)$ is the set of all positive real numbers, $(0,\infty)$.

(c) Explain carefully what is that you can say about the total length $L=L(y)$ of fence used if y is a **large** positive number. What if y is a **small** positive number?

Solution: From $L(y)=\frac{4000}{y}+3y$ we observe that if y is large then $\frac{4000}{y}$ is small and $L(y)\approx 3y$ is large. If y is small then $L(y)\approx\frac{4000}{y}$ is large.

(d) Based on your answers in part (c) draw a rough sketch of the graph of the function $L = L(y)$. Is there a value of y that would yield the minimum length of the fence L?

Solution: Observe that $L(y) > 0$ for $y > 0$ and that for y very large or very small, $L(y)$ is very large. Hence the graph of the function L may look like one on the figure. The graph clearly suggests that there must be a value of y that will yield the minimum length of the fence L.



8. **[7 marks]** The first few so-called Bernoulli polynomials are given as follows:

$$B_0(x) = 1 \qquad B_1(x) = x - \tfrac{1}{2}$$

$$B_2(x) = x^2 - x + \tfrac{1}{6} \qquad B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30} \qquad B_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x$$

It is known that the Bernoulli polynomials satisfy the following property: If h is such that $|h|$ is a very small positive number then, for any real number x and any positive integer n,

$$\frac{B_n(x+h) - B_n(x)}{h} \approx nB_{n-1}(x).$$

(a) Check the above fact for $x = 1$ and $n = 1$, i.e., check that if $|h| > 0$ is small then

$$\frac{B_1(1+h) - B_1(1)}{h} \approx B_0(1).$$

Solution: We note that, for $|h| \neq 0$,

$$\frac{B_1(1+h) - B_1(1)}{h} = \frac{(1+h) - \frac{1}{2} - (1 - \frac{1}{2})}{h} = \frac{1 + h - \frac{1}{2} - 1 + \frac{1}{2}}{h} = \frac{h}{h} = 1 = B_0(1).$$

(b) Check the above fact for $x = 1$ and $n = 2$, i.e., check that if $|h| > 0$ is small then

$$\frac{B_2(1+h) - B_2(1)}{h} \approx 2B_1(1).$$

Solution: From, for $|h| \neq 0$,

$$\begin{aligned}
\frac{B_2(1+h) - B_2(1)}{h} &= \frac{(1+h)^2 - (1+h) + \frac{1}{6} - (1^2 - 1 + \frac{1}{6})}{h} \\
&= \frac{1 + 2h + h^2 - 1 - h + \frac{1}{6} - \frac{1}{6}}{h} \\
&= \frac{1 + 2h + h^2 - 1 - h}{h} = \frac{h(1+h)}{h} = 1 + h
\end{aligned}$$

and $B_1(1) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$ we conclude that, for a small value of $|h| \neq 0$,

$$\frac{B_2(1+h) - B_2(1)}{h} = 1 + h \approx 1 = 2\left(1 - \frac{1}{2}\right) = 2B_1(1).$$

9. **[7 marks]** Using the approximations $\ln 2 \approx 0.6$ and $\ln 5 \approx 1.6$ and the properties of logarithms, estimate each of the numbers below. Each question is worth 1 point - You have to show all your work, i.e., it must be clear which properties you are using.

   (a) Estimate $\ln 8$.
   Solution: Observe, $\ln 8 = \ln(2^3) = 3\ln 2 \approx 3 \cdot 0.6 = 1.8$.

   (b) Estimate $\ln 25$.
   Solution: Observe, $\ln 25 = \ln(5^2) = 2\ln 5 \approx 2 \cdot 1.6 = 3.2$.

   (c) Estimate $\ln 10$.
   Solution: Observe, $\ln 10 = \ln(2 \cdot 5) = \ln 2 + \ln 5 \approx 0.6 + 1.6 = 2.2$.

   (d) Estimate $\ln 2.5$.
   Solution: Observe, $\ln 2.5 = \ln \frac{25}{10} = \ln 25 - \ln 10 \approx 3.2 - 2.2 = 1$.

   (e) Estimate $\ln 40$.
   Solution: Observe, $\ln 40 = \ln(8 \cdot 5) = \ln 8 + \ln 5 \approx 1.8 + 1.6 = 3.4$.

   (f) Estimate $\ln 50$.
   Solution: Observe, $\ln 50 = \ln(25 \cdot 2) = \ln 25 + \ln 2 \approx 3.2 + 0.6 = 3.8$.

   (g) Estimate $\log_2 5$.
   Solution: Observe, $\log_2 5 = \dfrac{\ln 5}{\ln 2} \approx \dfrac{1.6}{0.6} \approx 2.66$.

10. **[6 marks]** Laura and Ali are standing 10m apart from each other, trying to read a sign which is some distance away from both of them. The angle formed by Ali's line of vision to the sign and his line of vision to Laura is $\pi/6$. The angle formed by Laura's line of vision to the sign and her line of vision to Ali is $\pi/3$. Find the distance between Laura and the sign.

    Solution: Note that $\dfrac{\pi}{3} + \dfrac{\pi}{6} = \dfrac{\pi}{2}$. It follows that the third angle in the triangle with the vertices "Laura", "Sign", and "Ali" equals $\dfrac{\pi}{2}$ and that this triangle is a right triangle with the right angle at the vertex "'Sign" and the hypothenuse of the length 10.

    Let x be the distance between Laura and the sign, Then

$$\cos\frac{\pi}{3} = \frac{x}{10} \Leftrightarrow \frac{1}{2} = \frac{x}{10} \Leftrightarrow x = 5.$$

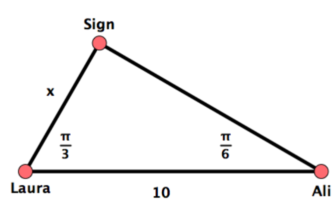


Figure 7.16: Laura. Ali, and Sign

11. **[12 marks]** Use the identities $\sin^2 x + \cos^2 x = 1$, $\sin(x_1 \pm x_2) = \sin x_1 \cos x_2 \pm \cos x_1 \sin x_2$, and $\cos(x_1 \pm x_2) = \cos x_1 \cos x_2 \mp \sin x_1 \sin x_2$ to determine the exact values of each of the numbers below.

(a) $\cos\dfrac{\pi}{12}$

Solution: Since $\dfrac{\pi}{12}=\dfrac{\pi}{4}-\dfrac{\pi}{6}$, it follows

$$\cos\frac{\pi}{12}=\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\cos\frac{\pi}{4}\cos\frac{\pi}{6}+\sin\frac{\pi}{4}\sin\frac{\pi}{6}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}.$$

(b) $\sin\dfrac{7\pi}{12}$

Solution: Since $\dfrac{7\pi}{12}=\dfrac{\pi}{2}+\dfrac{\pi}{12}$, it follows

$$\sin\frac{7\pi}{12}=\sin\left(\frac{\pi}{2}+\frac{\pi}{12}\right)=\sin\frac{\pi}{2}\cos\frac{\pi}{12}+\cos\frac{\pi}{2}\sin\frac{\pi}{12}=1\cdot\frac{\sqrt{6}+\sqrt{2}}{4}+0\cdot\sin\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}.$$

(c) $\tan\dfrac{\pi}{12}$

Solution: By definition, $\tan\dfrac{\pi}{12}=\dfrac{\sin\frac{\pi}{12}}{\cos\frac{\pi}{12}}$.

From

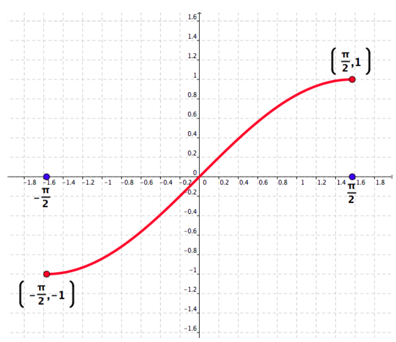
$$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\sin\frac{\pi}{4}\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\sin\frac{\pi}{6}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}.$$

It follows that $\tan\dfrac{\pi}{12}=\dfrac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}}=\dfrac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$.

(d) $\cot\dfrac{\pi}{12}$

Solution: By definition, $\cot\dfrac{\pi}{12}=\dfrac{1}{\tan\frac{\pi}{12}}=\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$.

12. **[6 marks]** A function $f:\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\to[-1,1]$ is given by $f(x)=\sin x$, for all $x\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$. This is the graph of the function f:



(a) Justify the fact that the function f has an inverse, i.e., justify the fact that the function f^{-1} exists. The function f^{-1} is called the inverse sine function and it is denoted by $f^{-1}(x)=\sin^{-1}x$.

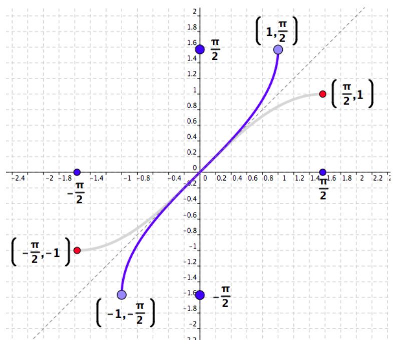
Solution: We observe that the graph of the function f passes the horizontal line test, i.e., each horizontal line intersects the graph of f at most one point. Hence, the function f is one-to-one, which implies that f has an inverse.

(b) State the domain and range of the function f^{-1}.

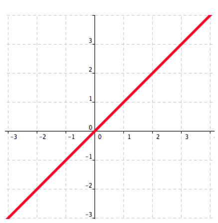
Solution: Since $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ we have that $f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, the domain of the function f^{-1} is the segment $[-1, 1]$ and its range is the segment $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(c) Draw a graph of the function f^{-1}.

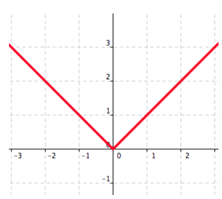
Solution:



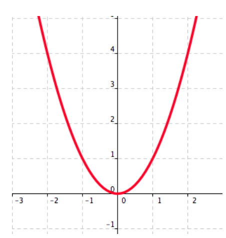
13. **[10 marks]** Sketch graphs of the following functions:



(a) $f(x) = x$

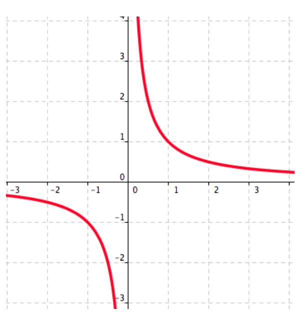


(b) $g(x) = |x|$

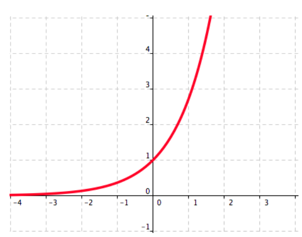


(c) $h(x) = x^2$

(d) $i(x) = x^3$

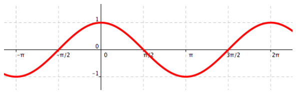


(a) $j(x) = x^{-1}$

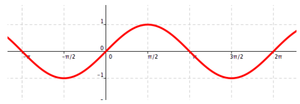


(b) $k(x) = e^x$

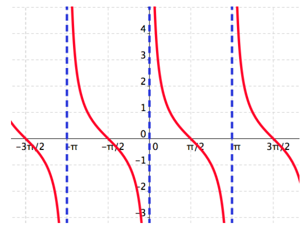
(c) $l(x) = \ln x$



(a) $m(x) = \cos x$



(b) $n(x) = \sin x$



(c) $o(x) = \cot x$