OPMT 5701
Constrained Optimization

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(Ch 12)

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1 Constrained Optimization

1.1 Method of Substitution

Consider the following Utility Max problem:

Max x_1,x_2

$$U = U(x_1,x_2) \tag{1}$$

Subject to:

$$B = P_1x_1 + P_2x_2 \tag{2}$$

Re-write Eq. 2

$$x_2 = \frac{B}{P_2} - \frac{P_1}{P_2}x_1 \tag{2`}$$

Now $x_2 = x_2(x_1)$ and $\frac{dx_2}{dx_1} = \frac{-P_1}{P_2}$

Sub into Eq. 1 for x_2

$$U = U(x_1, x_2(x_1)) \tag{3}$$

Eq. 3 is an unconstrained function of one variable, x_1

Differentiate, using the Chain Rule

$$\frac{dU}{dx_1} = \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2}\frac{dx_2}{dx_1} = 0$$

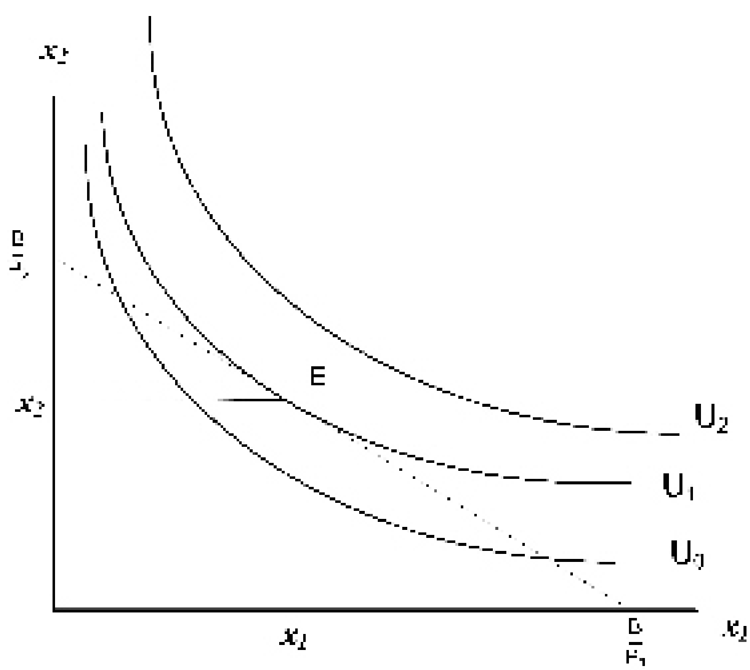
From Eq. 2‘ we know $\frac{dx_2}{dx_1} = -\frac{P_1}{P_2}$
Therefore:

$$\frac{dU}{dx_1} = U_1 + U_2\left(-\frac{P_1}{P_2}\right) = 0$$

OR

$$\frac{U_1}{U_2} = \frac{P_1}{P_2}$$

This is our usual condition that MRS(x_2, x_1) $= \frac{P_1}{P_2}$ or the consumer's willingness to grade equals his ability to trade.



The More General Constrained Maximum Problem
Max:

$$y = f(x_1, x_2) \tag{4}$$

Subject to:

$$g(x_1, x_2) = 0 \tag{5}$$

Take total differentials of Eq. 4 and Eq. 5

$$dy = f_1 dx_1 + f_2 dx_2 = 0 \tag{6}$$

$$dg = g_1 dx_1 + g_2 dx_2 = 0 \tag{7}$$

or Eq.6′

$$dx_1 = -\frac{f_2}{f_1} dx_2$$

Eq. 7′

$$dx_1 = -\frac{g_2}{g_1} dx_2$$

Subtract 6′ from 7′
$dx_1 - dx_1 = \left[-\frac{g_2}{g_1} - \left(-\frac{f_2}{f_1}\right)\right] dx_2 = \left(\frac{f_2}{f_1} - \frac{g_2}{g_1}\right) dx_2 = 0$
Therefore

$$\frac{f_2}{f_1} = \frac{g_2}{g_1}$$

Eq. 8: says that the level curves of the objective function must be tangent to the level curves of the constraint

1.2 Lagrange Multiplier Approach

Create a new function called the Lagrangian:

$$L = f(x_1, x_2) + \lambda g\left(x_1, x_2\right)$$

since $g(x_1, x_2) = 0$ when the constraint is satisfied

$$L = f(x_1, x_2) + zero$$

We have created a new independent variable λ (lambda), which is called the Lagrangian Multiplier.

We now have a function of three variables; $x_{1,}x_2$,and λ

Now we Maximize

$$L = f(x_1, x_2) + \lambda g(x_1, x_2)$$

First Order Conditions

$$\begin{array}{ll} L_\lambda = \frac{\partial L}{\partial \lambda} = g(x_1, x_2) = 0 & Eq.1 \\ L_1 = \frac{\partial L}{\partial x_1} = f_1 + \lambda g_1 = 0 & Eq.2 \\ L_2 = \frac{\partial L}{\partial x_2} = f_2 + \lambda g_2 = 0 & Eq.3 \end{array}$$

From Eq. 2 and 3 we get:

$$\frac{f_1}{f_2} = \frac{-\lambda g_1}{-\lambda g_2} = \frac{g_1}{g_2}$$

From the 3 F.O.C.'s we have 3 equations and 3 unknowns $(x_{1,}x_2, \lambda)$. In principle we can solve for x_1^*, x_2^*, and λ^*.

1.2.1 Example 1:

Let:

$$U = xy$$

Subject to:

$$10 = x + y \quad P_x = P_y = 1$$

Lagrange:

$$\begin{aligned} L &= f(x, y) + \lambda(g(x, y)) \\ L &= xy + \lambda(10 - x - y) \end{aligned}$$

F.O.C.

$$L_\lambda = 10 - x - y = 0 \qquad Eq.1$$
$$L_x = y - \lambda = 0 \qquad Eq.2$$
$$L_y = x - \lambda = 0 \qquad Eq.3$$

From (2) and (3) we see that:

$$\tfrac{y}{x} = \tfrac{\lambda}{\lambda} = 1 \ \ \underline{\text{or}} \ \ y = x \qquad Eq.4$$

From (1) and (4) we get:

$$10 - x - x = 0 \ \ or \ \ x^* = 5 \ \ and \ \ y^* = 5$$

From either (2) or (3) we get:

$$\lambda^* = 5$$

1.2.2 Example 2: Utility Maximization

Maximize

$$u = 4x^2 + 3xy + 6y^2$$

subject to

$$x + y = 56$$

Set up the Lagrangian Equation:

$$L = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$$

Take the first-order partials and set them to zero

$$\begin{aligned} L_x &= 8x + 3y - \lambda = 0 \\ L_y &= 3x + 12y - \lambda = 0 \\ L_\lambda &= 56 - x - y = 0 \end{aligned}$$

From the first two equations we get

$$\begin{aligned} 8x + 3y &= 3x + 12y \\ x &= 1.8y \end{aligned}$$

Substitute this result into the third equation

$$\begin{aligned} 56 - 1.8y - y &= 0 \\ y &= 20 \end{aligned}$$

therefore

$$x = 36 \qquad \lambda = 348$$

1.2.3 Example 3: Cost minimization

A firm produces two goods, x and y. Due to a government quota, the firm must produce subject to the constraint $x + y = 42$. The firm's cost functions is

$$c(x, y) = 8x^2 - xy + 12y^2$$

The Lagrangian is

$$L = 8x^2 - xy + 12y^2 + \lambda(42 - x - y)$$

The first order conditions are

$$\begin{aligned} L_x &= 16x - y - \lambda = 0 \\ L_y &= -x + 24y - \lambda = 0 \\ L_\lambda &= 42 - x - y = 0 \end{aligned} \tag{8}$$

Solving these three equations simultaneously yields

$$x = 25 \qquad y = 17 \qquad \lambda = 383$$

1.2.4 Example 4:

Max:

$$U = x_1 x_2$$

Subject to:

$$B = P_1 x_1 + P_2 x_2$$

Langrange:

$$L = x_1 x_2 + \lambda (B - P_1 x_1 - P_2 x_2)$$

F.O.C.

$$\begin{array}{cr} L_\lambda = B - P_1 x_1 - P_2 x_2 = 0 & \text{Eq. 1} \\ L_1 = x_2 - \lambda P_1 = 0 & \text{Eq. 2} \\ L_2 = x_1 - \lambda P_2 = 0 & \text{Eq. 3} \end{array}$$

From Eq. (2) and (3) $\left(\frac{x_2}{x_1} = \frac{P_1}{P_2} = MRS\right)$

$$\begin{aligned} x_2 &= \lambda P_1 \\ x_1 &= \lambda P_2 \end{aligned}$$

divide top equation by the bottom

$$\frac{x_2}{x_1} = \frac{\lambda P_1}{\lambda P_2}$$

Cancel the λ from top/bottom of RHS

$$\frac{x_2}{x_1} = \frac{P_1}{P_2}$$

Solve for x_1^*
From (2) and (3)

$$x_2 = \frac{P_1}{P_2} x_1$$

Sub into (1) and simplify

$$\begin{aligned}
B &= P_1x_1 + P_2x_2 \\
B &= P_1x_1 + P_2\left(\frac{P_1}{P_2}x_1\right) \\
B &= 2P_1x_1 \\
x_1^* &= \frac{B}{2P_1}
\end{aligned}$$

Substitute your answer for x_1^* into Eq 1

$$\begin{aligned}
B &= P_1x_1 + P_2x_2 \\
B &= P_1\left(\frac{B}{2P_1}\right) + P_2x_2 \\
B &= \frac{B}{2} + P_2x_2 \\
B - \frac{B}{2} &= P_2x_2 \\
\frac{B}{2} &= P_2x_2 \\
x_2^* &= \frac{B}{2P_2}
\end{aligned}$$

The solution to $x_1^*$and x_2^* are the Demand Functions for x_1 and x_2

1.2.5 Properties of Demand Functions

1. "Homogenous of degree zero" multiply prices and income by α

$$x_1^* = \frac{\alpha B}{2(\alpha P_1)} = \frac{B}{2P_1}$$

2. "For normal goods demand has a negative slope"

$$\frac{\partial x_1^*}{\partial P_1} = -\frac{B}{2P_1^2} < 0$$

3. "For normal goods Engel curve positive slope"

$$\frac{\partial x_1^*}{\partial B} = \frac{1}{2P_1} > 0$$

In this example x_1^* and $x_2^*$are both normal goods (rather than inferior or giffen)

Given:

$$U = x_1 x_2$$

And:

$$x_1^* = \frac{B}{2P_1} \quad \text{and} \quad x_2^* = \frac{B}{2P_2}$$

Substituting into the utility function we get:

$$\begin{aligned} U &= x_1^*, x_2^* = \left(\frac{B}{2P_1}\right)\left(\frac{B}{2P_2}\right) \\ U &= \left(\frac{B^2}{4P_1 P_2}\right) \end{aligned}$$

Now we have the utility expressed as a function of Prices and Income

$U^* = U(P_1 P_2, B)$ is "The Indirect Utility Function"

At $U = U_0 = \frac{B^2}{4P_1P_2}$ we can re-arrange to get:

$$\underbrace{B = 2P_1^{\frac{1}{2}} P_2^{\frac{1}{2}} U_0^{\frac{1}{2}}}_{\text{This is the "Expenditure Function"}}$$

1.3 Minimization and Lagrange

Min x, y

$$P_x X + P_y Y$$

Subject to

$$U_0 = U(x,y)$$

Lagrange

$$L = P_x X + P_y Y + \lambda(U_0 - U(x,y))$$

F.O.C.

$$\begin{aligned} L_\lambda = U_0 - U(x,y) = 0 & \quad \text{Eq. 1} \\ L_x = P_x - \lambda \tfrac{\partial U}{\partial x} = 0 & \quad \text{Eq. 2} \\ L_y = P_y - \lambda \tfrac{\partial U}{\partial y} = 0 & \quad \text{Eq. 3} \end{aligned}$$

From (2) and (3) we get

$$\underbrace{\frac{P_x}{P_y} = \frac{\lambda U_x}{\lambda U_y} = \frac{U_x}{U_y} = MRS}_{\text{(The same result as in the MAX problem)}}$$

Solving (1), (2), and (3), we get:

$$x^* = x(P_x, P_y, U_0) \qquad y^* = y(P_x, P_y, U_0) \qquad \lambda^* = \lambda(P_x, P_y, U_0)$$

1.3.1 Example (part 1)

Max

$$xy + \lambda(B - P_x x - P_y y)$$

F.O.C.'s

$$\underbrace{\begin{gathered} L_x = y - \lambda P_x = 0 \\ L_y = x - \lambda P_y = 0 \\ L_\lambda = B - P_x x - P_y y = 0 \end{gathered}}$$

$$x^* = \tfrac{B}{2P_x} \qquad y^* = \tfrac{B}{2P_y} \qquad \lambda^* = \tfrac{B}{2P_x P_y}$$

1.3.2 Example (part 2)

Min

$$P_x x + P_y y + \lambda(U_0 - xy)$$

F.O.C.'s

$$\begin{aligned} L_x &= P_x - \lambda y = 0 && (1) \\ L_y &= P_y - \lambda x = 0 && (2) \\ L_\lambda &= U_0 - xy = 0 && (3) \end{aligned}$$

First, use equations (1) and (2) to eliminate λ

$$\begin{aligned} P_x &= \lambda y \\ P_y &= \lambda x \end{aligned}$$

divide (1) by (2)

$$\begin{aligned} \frac{P_x}{P_y} &= \frac{\lambda y}{\lambda x} \\ \frac{P_x}{P_y} &= \frac{y}{x} \\ y &= \frac{P_x}{P_y} x \end{aligned}$$

Substitute into eq (3)

$$\begin{aligned} U_0 &= xy \\ U_0 &= x\left(\frac{P_x}{P_y} x\right) \\ U_0 &= \frac{P_x}{P_y} x^2 \\ x^2 &= \frac{P_y}{P_x} U_0 \\ x &= \sqrt{\frac{P_y}{P_x} U_0} = \frac{P_y^{\frac{1}{2}} U_0^{\frac{1}{2}}}{P_x^{\frac{1}{2}}} \end{aligned}$$

Follow the same procedure to find

$$y^* = \frac{P_x^{\frac{1}{2}} U_0^{\frac{1}{2}}}{P_y^{\frac{1}{2}}} \qquad \lambda^* = \frac{U_0^{\frac{1}{2}}}{P_x^{\frac{1}{2}} P_y^{\frac{1}{2}}}$$

1.4 Interpreting λ

Given Max

$$U(x, y) + \lambda (B - P_x x - P_y y)$$

By solving the F.O.C.'s we get

$$x^* = x(P_x, P_y, B) \qquad y^* = y(P_x, P_y, B) \qquad \lambda^* = \lambda(P_x, P_y, B)$$

Sub x^*, y^*, λ^* back into the Lagrange

$$L^* = U(x^*, y^*) + \lambda^* (B - P_x x^* - P_y y^*)$$

Differentiate with respect to the constant,B

$$\frac{\partial L^*}{\partial B} = U_x \frac{dx^*}{dB} + U_y \frac{dy^*}{dB} - \lambda^* P_x \frac{dx^*}{dB} - \lambda^* Py \frac{dy^*}{dB} + \lambda^* \frac{dB}{dB} + (B - P_x x^* - P_y y^*) \frac{d\lambda^*}{dB}$$

Or

$$\frac{\partial L^*}{\partial B} = \underbrace{(U_x - \lambda^* P_x)}_{=0} \frac{dx^*}{dB} + \underbrace{(U_y - \lambda^* P_y)}_{=0} \frac{dy^*}{dB} + \underbrace{(B - P_x x^* - P_y y^*)}_{=0} \frac{d\lambda^*}{dB} + \lambda^*$$

$\frac{\partial L^*}{\partial B} = \lambda^*$ $= \Delta$ in utility from Δ in the constant
$=$ Marginal Utility of Money

2 Extensions and Applications of Constrained Optimization

2.1 Homogenous Functions

2.1.1 Constant Returns to Scale

$\Longrightarrow$ Given

$$y = f(x_1, x_2, ...x_n)$$

if we change all the inputs by a factor of t, then

$$f(tx_1, tx_2, ...tx_n) = tf(x_1, x_2, ...x_n) = tY$$

ie. if we double inputs, we double output

$\Longrightarrow$ A constant returns to scale production function is said to be:

HOMOGENOUS of DEGREE ONE or **LINEARLY HOMOGENOUS**

2.1.2 Homogenous of Degree r

A function, $Y = f(x_1, ..., x_n)$ is said to be Homogenous of Degree r if

$$f(tx_1, tx_2, ...tx_n) = t^r f(x_1, x_2, ...x_n)$$

Example

Let f$(x_1, x_2) = x_1 x_2$

change all $x_i' s$ by t

$$\begin{aligned} f(tx_1, tx_2) &= (tx_1)(tx_2) \\ &= t^2(x_1 x_2) \\ &= t^2 f(x_1 x_2) \end{aligned}$$

Therefore $f(x_1, x_2) = x_1 x_2$ is homogenous of degree 2

2.1.3 Cobb-Douglas

Let output, $Y = f(K, L) = L^{\alpha} K^{1-\alpha}$ {where $0 \leq 1$}

Multiply K, L by t

$$
\begin{aligned}
f(tL, tK) &= (tL)^{\alpha}(tK)^{1-\alpha} \\
&= t^{\alpha+1-\alpha} L^{\alpha} K^{1-\alpha} \\
& \quad tL^{\alpha} K^{1-\alpha}
\end{aligned}
$$

Therefore L$^{\alpha} K^{1-\alpha}$is H.O.D one.
General Cobb-Douglas: y=L$^{\alpha} K^{\beta}$

$$
\begin{aligned}
f(tL, tK) &= (tL)^{\alpha}(tK)^{\beta} \\
&= t^{\alpha+\beta} L^{\alpha} K^{\beta}
\end{aligned}
$$

L$^{\alpha} K^{\beta}$ is homogenous of degree $\alpha + \beta$

2.1.4 Further properties of Cobb-Douglas

Given

$$y = L^{\alpha} K^{1-\alpha}$$

$$
\begin{aligned}
MP_L &= \frac{dY}{dL} = dL^{\alpha-1} K^{1-\alpha} = \alpha \left(\frac{K}{L}\right)^{1-\alpha} \\
MP_K &= \frac{dY}{dK} = (1-\alpha) L^{\alpha} K^{-\alpha} = (1-\alpha) \left(\frac{K}{L}\right)^{-\alpha}
\end{aligned}
$$

MP$_L$ and MP$_K$ are homogenous of degree zero

$$MP_L(tL, tK) = \alpha \left(\frac{tK}{tL}\right)^{1-\alpha} = \alpha \left(\frac{K}{L}\right)^{1-\alpha}$$

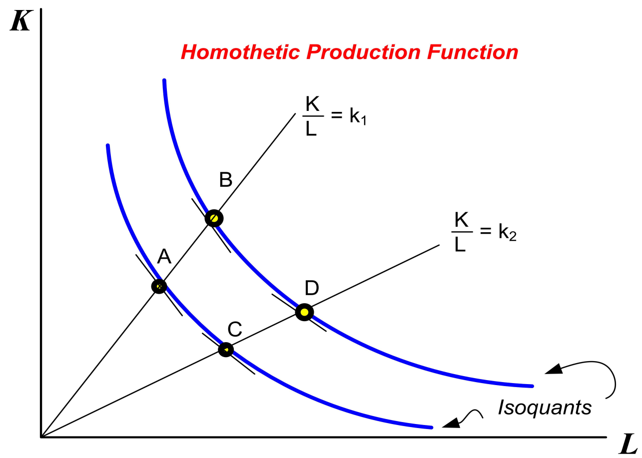
MP_L and MP_K depend only on the $\frac{K}{L}$ratio

2.2 The Marginal Rate of Technical Substitution

$$MRTS = \frac{MP_L}{MP_K} = \frac{\alpha(\frac{K}{L})^{1-\alpha}}{(1-\alpha)(\frac{K}{L})^{-\alpha}} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{K}{L}\right)$$

MRTS is homogenous of degree zero

The slope of the isoquant (MRTS) depends only on the $\frac{K}{L}$ ratio, not the absolute levels of K and L



Along any ray from the origin the isoquants are parallel. This is true for all homogenous functions regardless of the degree.

Given:

$$f(tx_1, ...tx_n) = t^r f(x_1, ...x_n)$$

Differentiate both sides with respect to x_1

$$\frac{df}{d(tx)}\frac{d(tx_1)}{dx_1} = t^r\frac{df}{dx_1}$$

But

$$\begin{aligned}\frac{d(tx_1)}{dx_1} &= t \\ \frac{df}{d(tx_1)}t &= t^r\frac{df}{dx_1}\end{aligned}$$

$$\frac{df}{d(tx_1)} = \frac{t^r}{t}\frac{df}{dx_1} = t^{r-1}\frac{df}{dx_1}$$

Therefore: For any function homogenous of degree r, that function's first partial derivatives are homogenous of degree $r - 1$.

2.3 Monotonic Transformations and Homothetic Functions

Let $y = f(x_1, x_2)$and Let $z = g(y)$

$\{$where $g`(y) > 0$ and $f(x_1, x_2)$ is H.O.D. r$\}$

$g(y)$ is a monotonic transformation of y

We know:

$$MRTS = -\frac{f_1}{fx} = \frac{dx_2}{dx_1}$$

Totally differentiate $z = g(y)$ and set $dz = 0$

$$dz = \frac{dg}{dy}\frac{dy}{dx_1}dx_1 + \frac{dg}{dy}\frac{dy}{dx_2}dx_2 = 0$$

or

$$\frac{dx_2}{dx_1} = \frac{-\left(\frac{dg}{dy_1}\right)\left(\frac{dy}{dx_1}\right)}{\left(\frac{dg}{dy_1}\right)\left(\frac{dy}{dx_2}\right)} = \frac{-\left(\frac{dy}{dx_1}\right)}{\left(\frac{dy}{dx_2}\right)} = \frac{-f_1}{f_2}$$

The slope of the level curves (isoquants) are invariant to monotonic transformations.

A monotonic transformation of a homogenous function creates a **homothetic function**

Homothetic functions have the same slope properties along a ray from the origin as the homogenous function.

However, homothetic functions are NOT homogenous.

Example: Let $f(x_1, x_2) = x_1, x_2$ {where $r = 2$}

Let:

$$\begin{aligned} z &= g(y) = \ln(x_1, x_2) \\ &= \ln x_1 + \ln x_2 \\ g(f(tx_1, tx_2)) &= \ln(tx_1) + \ln(tx_2) \\ &= 2\ln t + \ln x_1 + \ln x_2 \\ &\neq t^r \ln(x_1, x_2) \end{aligned}$$

Properties of Homothetic Functions

1. A homothetic function has the same shaped level curves as the homogenous function that was transformed to create it.
2. Homogenous production functions cannot produce U-shaped average cost curves, but a homothetic function can.

3. Slopes of Level Curves (ie. Indifference Curves)

   For homothetic functions the slope of their level curves only depend on the ratio of quantities.

   ie. <u>If</u>: $y = f(x_1, x_2)$ is homothetic

   Then: $\frac{f_1}{f_2} = g\left(\frac{x_2}{x_1}\right)$

2.4 Euler's Theorem

Let $f(x_1, x_2)$ be homogenous of degree r
Then $f(tx_1, tx_2) = t^r f(x_1, x_2)$
Differentiate with respect to t

$$\frac{df}{d(tx_1)}\frac{d\,(tx_1)}{dt} + \frac{df}{d(tx_2)}\frac{d(tx_2)}{dt} = rt^{r-1}f(tx_1, tx_2)$$

Since: $\frac{dtx_i}{dt} = x_i$ for all i

$$\frac{df}{d(tx_1)}x_1 + \frac{df}{d(tx_2)}x_2 = rt^{r-1}f(tx_1, tx_2)$$

This is true for all values of t, so let $t = 1$

$$\underbrace{\frac{df}{dx_1}x_1 + \frac{df}{dx_2}x_2 = f_1x_1 + f_2x_2 = rf(x_1, x_2)}_{\text{"Euler's Theorm"}}$$

If $y = f(L, K)$ is constant returns to scale
Then y=MP$_L L + MP_K K$ (Euler's Theorm)
Example: Let

$$y = L^{\alpha}K^{1-\alpha}$$

Where:

$$MP_L = \alpha L^{\alpha-1}K^{1-\alpha}$$

$$MP_K = (1-\alpha)L^{\alpha}K^{-\alpha}$$

From Euler's Theorm

$$\begin{aligned} y &= MP_L L + MP_K K = \left(\alpha L^{\alpha-1}K^{1-\alpha}\right)L + \left((1-\alpha)L^{\alpha}K^{-\alpha}\right)K \\ &= \alpha L^{\alpha-1}K^{1-\alpha} + (1-\alpha)L^{\alpha}K^{-\alpha} \\ &= \left[d + (1-\alpha)\right]L^{\alpha}K^{1-\alpha} \\ &= L^{\alpha}K^{1-\alpha} \\ &= y \end{aligned}$$

2.4.1 Euler's Theorm and Long Run Equilibrium

Suppose $q = f(K, L)$ is H.O.D 1

Then the profit function for a perfectly competitive firm is

$$\begin{aligned} \pi &= pq - rK - wL \\ \pi &= pf(K, L) - rK - wL \end{aligned}$$

F.O.C's

$$\begin{aligned} \frac{d\pi}{dL} &= pf_L - w = 0 \\ \frac{d\pi}{dK} &= pf_K - r = 0 \end{aligned}$$

$\{f_L = MP_L \qquad f_K = MP_K\}$

or $MP_L = \frac{w}{p}, MP_K = \frac{r}{p}$ are necessary conditions for Profit Maximization

Therefore, at the optimum

$$\pi^* = pf(K^* L^*) - wL^* - rK^*$$

From Euler's Theorem

$$f(K^* L^*) = MP_K K^* + MP_L L^*$$

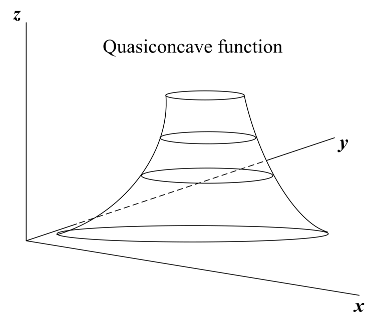
Substitute into π^*

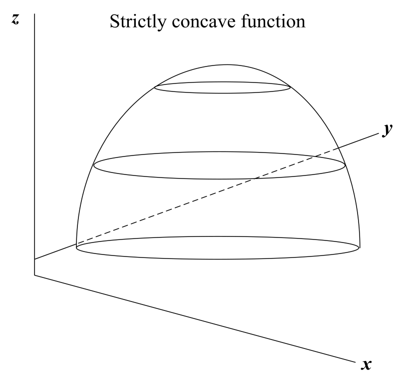
$$\pi^* = P\left[MP_K K^* + MP_L L^*\right] - wL^* - rK^*$$

OR

$$\pi^* = \underbrace{[wL^* + rK^*] - wL^* - rK^* = 0}_{\text{Long Run } \pi=0}$$

2.5 Concavity and Quasiconcavity





2.5.1 Concavity:

· Convex level curves and concave in scale

  · Necessary for unconstrained optimum

2.5.2 Quasi-Concavity:

· Only has convex level curves

  · Necessary for constrained optimum

Example:

1. Concave: $y = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$ is H.O.D. 2/3 (diminishing returns)

$$MRTS = \frac{x_2}{x_1}$$

2. Quasi -Concave: $y = x_1^2 x_2^2$ is H.O.D. 4 (increasing returns)

$$MRTS = \frac{x_2}{x_1}$$