Finding the Inverse of a 3x3 Matrix

This worksheet is to accompany my video on 3x3 matrix inversion. You can print it and follow along the video. You can also do it without the video

Given the following matrix:

$$A = \begin{bmatrix} 8 & -1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 2 \end{bmatrix}$$

Find the inverse through a three step process: (1) find the determinant, (2) find the cofactor matrix, (3) find the adjoint and inverse of A

STEP 1: Find the determinant. Expand the second column

$$|A| = a_{21}\left|\overset{(-)}{C_{21}}\right| + a_{22}\left|\overset{(+)}{C_{22}}\right| + a_{32}\left|\overset{(-)}{C_{32}}\right|$$

$$|A| = (-1)\,(-1)\begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix} + 0\,(+1)\begin{vmatrix} 8 & 3 \\ 6 & 2 \end{vmatrix} - 0(-1)\begin{vmatrix} 8 & 3 \\ 4 & 1 \end{vmatrix}$$

$$|A| = -1\begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix} \underset{(-1)}{\overset{\text{cofactor sign}}{}} = 2$$

STEP 2: Find the Cofactor matrix

$$A = \begin{bmatrix} 8 & -1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} \underset{+}{\_\_\_\_\_} & \underset{-}{\_\_\_\_\_} & \underset{+}{\_\_\_\_\_} \\ \underset{-}{\_\_\_\_\_} & \underset{+}{\_\_\_\_\_} & \underset{-}{\_\_\_\_\_} \\ \underset{+}{\_\_\_\_\_} & \underset{-}{\_\_\_\_\_} & \underset{+}{\_\_\_\_\_} \end{bmatrix}$$

Cofactor Matrix: $C = \begin{bmatrix} 0 & -2 & 0 \\ 2 & -2 & -6 \\ -1 & 4 & 4 \end{bmatrix}$

adjoint $Adj A = C^T = \begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 4 \\ 0 & -6 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}C^T = \left(\frac{1}{2}\right)\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 4 \\ 0 & -6 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ -1 & -1 & 2 \\ 0 & -3 & 2 \end{bmatrix}$$

Inverse: $A^{-1} = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ -1 & -1 & 2 \\ 0 & -3 & 2 \end{bmatrix}$

Check your work using the test: $AA^{-1} = I$ where I is a $3x3$ Identity Matrix

$$
\begin{aligned}
AA^{-1} &= \begin{bmatrix} 8 & -1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ -1 & -1 & 2 \\ 0 & -3 & 2 \end{bmatrix} \\
&\text{sum-product rule} \\
&= \begin{bmatrix} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{bmatrix} \\
&\text{simplify} \\
&= \begin{bmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{bmatrix}
\end{aligned}
$$

You should get

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If not, go back and check your work. Usually it is a mistake involving the wrong sign on a number

For practice, show that if:

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 4 & 0 & 3 \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} -\frac{3}{5} & 0 & \frac{2}{5} \\ -\frac{4}{15} & \frac{1}{3} & \frac{1}{15} \\ \frac{4}{5} & 0 & -\frac{1}{5} \end{bmatrix} \\
\text{and if } &: \\
B &= \begin{bmatrix} 2 & 1 & 1 \\ 5 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} \text{ then } B^{-1} = \begin{bmatrix} 3 & -1 & -\frac{1}{3} \\ -5 & 2 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}
\end{aligned}
$$

$$\begin{bmatrix} \underline{\hspace{4cm}} & \underline{\hspace{4cm}} & \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} & \underline{\hspace{4cm}} & \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} & \underline{\hspace{4cm}} & \underline{\hspace{4cm}} \end{bmatrix}$$