Partial Derivatives, Monotonic Functions, and economic applications (ch 7)

Kevin Wainwright

October 3, 2012

1 Monotonic Functions and the Inverse Function Rule

If $x_1 < x_2$ and $f(x_1) < f(x_2)$ (for all x), then $f(x)$ is Monotonically increasing.

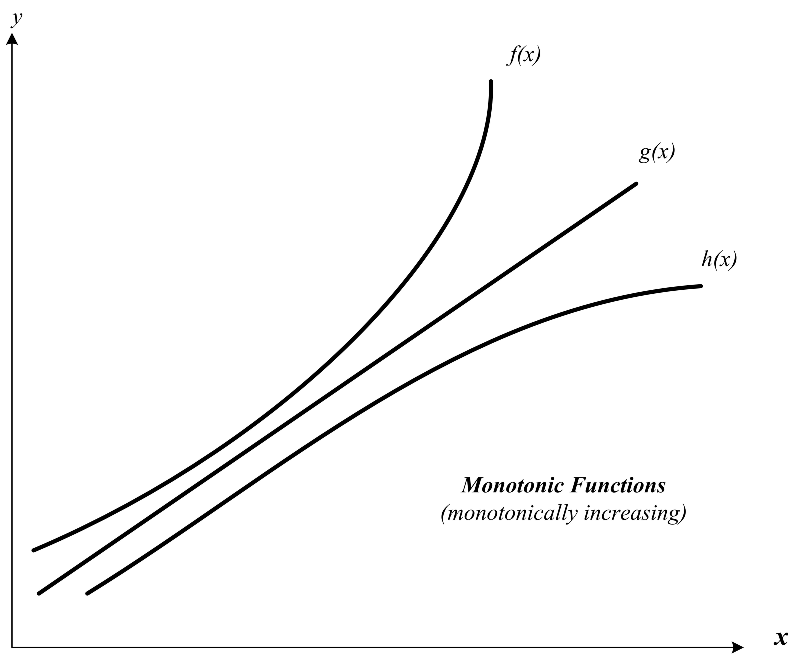
If $x_1 < x_2$ and $f(x_1) > f(x_2)$ then $f(x)$ is Monotonically decreasing.

If a function is Monotonic the an inverse function exists. Ie. If $y = f(x)$, then $x = f^{-1}(y)$.

Example $y = x^2 \quad (x \geqslant 0)$, $x = \sqrt{y}$

1.1 Derivative of Inverse Functions

If $y = f(x)$ and $x = f^{-1}(y)$, then $\frac{dy}{dx} = f'(x)$ and $\frac{dx}{dy} = \frac{1}{f'(x)}$



1.1.1 Example 1:

$$
\begin{aligned}
y &= 3x + 2 \Rightarrow \frac{dy}{dx} = 3 \\
x &= \frac{1}{3}y - \frac{2}{3} \Rightarrow \frac{dx}{dy} = \frac{1}{3} = \frac{1}{\frac{dy}{dx}}
\end{aligned}
$$

1.1.2 Example 2:

$$
\begin{aligned}
\text{If:} \quad & y = x^2 \text{ and } \frac{dy}{dx} = 2x \\
\text{then:} \quad & x = y^{1/2} \text{ and } \frac{dx}{dy} = \frac{1}{2}y^{-1/2} = \frac{1}{2y^{1/2}} \\
\text{so:} \quad & \frac{dx}{dy} = \frac{1}{2x} = \frac{1}{\frac{dy}{dx}}
\end{aligned}
$$

Application: Revenue Functions

$$\begin{aligned}\text{Demand Function} &: \quad Q = 10 - P \\ \text{Inverse Demand Function} &: \quad P = 10 - Q\end{aligned}$$

Average Revenue

$$AR = P = 10 - Q \text{ Inverse demand function}$$

Total Revenue

$$\begin{aligned}TR &= P \cdot Q = (10 - Q)Q = 10Q - Q^2 \\ TR &= 10Q - Q^2 \text{ is a quadratic function}\end{aligned}$$

Marginal Revenue

$$MR = \frac{d(TR)}{dQ} = 10 - 2Q$$

Given $AR = 10 - Q$ and $MR = 10 - 2Q$ MR falls twice as fast as AR.

Generally:

$$\begin{aligned}TR &= aQ - bQ^2 \text{ (general form quadratic)} \\ AR &= \frac{TR}{Q} = a - bQ \text{ (inverse demand function)} \\ MR &= \frac{d(TR)}{dQ} = a - 2bQ \text{ (1st derivative)}\end{aligned}$$

Graphically

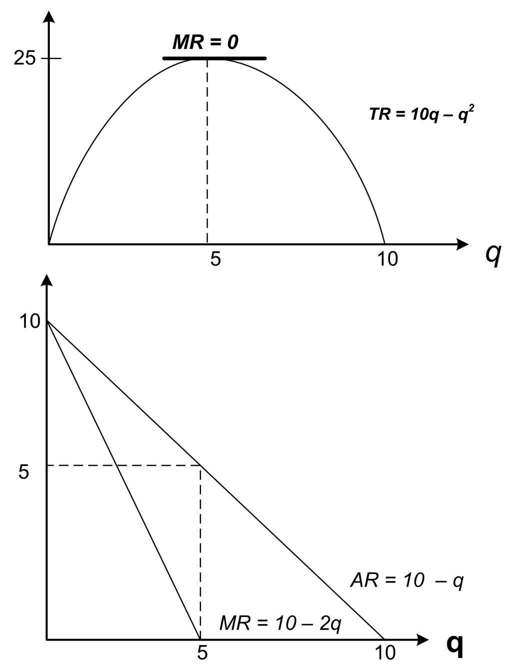
1. TR is at a MAX when $MR = 0$

2. $MR = 10 - 2Q = 0$

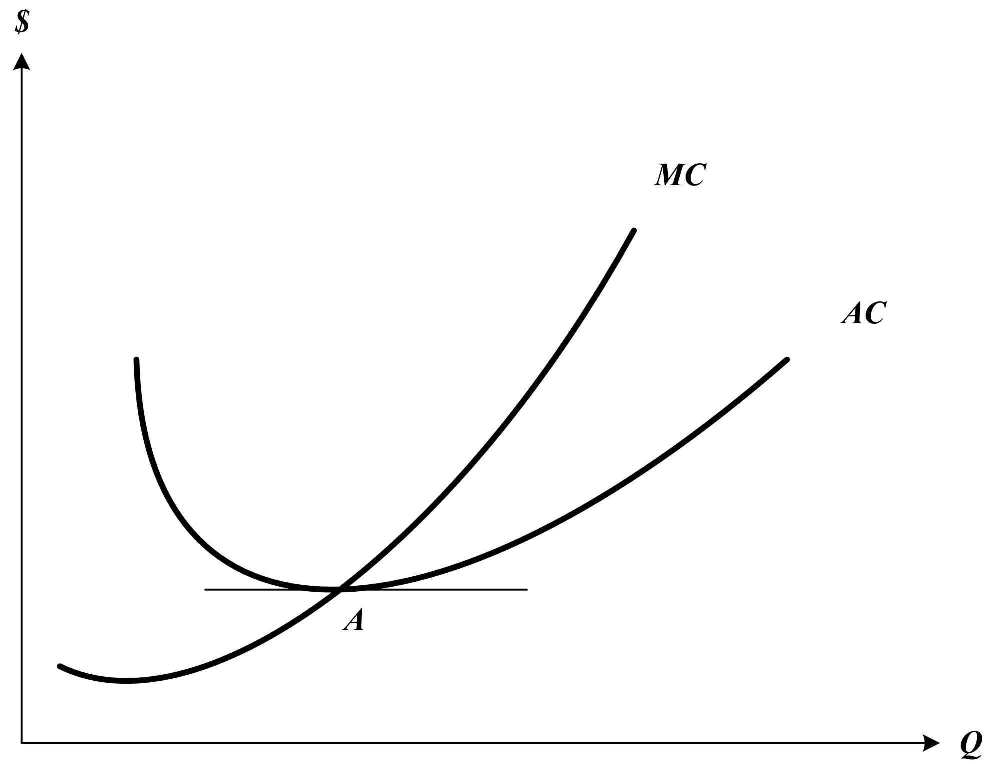
   $Q = 5$

3. $TR = 10Q - Q^2 = 25$

4. $AR = 10 - Q = 5$



1.1.3 Average cost and Marginal Cost



1. Total Cost = $C(Q)$
2. Marginal Cost = $\frac{dC(Q)}{dQ}$
3. Average Cost = $\frac{C(Q)}{Q}$
4. Average costs are minimized when the slop of AC=0 (point A)

$$\begin{aligned}
\text{Slope of AC} &= \frac{dAC}{dQ} = \frac{C'(Q)Q - C(Q)}{Q^2} \text{ Quotient Rule} \\
&= \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right] \text{ Factor out Q} \\
&= \frac{1}{Q}\left[MC - AC\right]
\end{aligned}$$

Slope of AC is:

1. (a) negative if MC $<$ AC
   (b) positive if MC $>$ AC
   (c) zero if MC $=$ AC

2 Multivariate Calculus

Single variable calculus is really just a "special case" of multivariable calculus. For the function $y = f(x)$, we assumed that y was the endogenous variable, x was the exogenous variable and everything else was a parameter. For example, given the equations

$$y = a + bx$$

or

$$y = ax^n$$

we automatically treated a, b, and n as constants and took the derivative of y with respect to x (dy/dx). However, what if we decided to treat x as a constant and take the derivative with respect to one of the other variables? Nothing precludes us from doing this. Consider the equation

$$y = ax$$

where

$$\frac{dy}{dx} = a$$

Now suppose we find the derivative of y with respect to a, *but TREAT x as the constant.* Then

$$\frac{dy}{da} = x$$

Here we just "reversed" the roles played by a and x in our equation.

2.1 Partial Derivatives

Suppose $y = f(x_1, x_2, ...x_n)$

ie. $y = 2x_1^2 + 3x_2 + 2x_1x_2$

What is the change in y when we change x_i $(i = 1, n)$ hold all other x's constant?

or: Find $\frac{\Delta y}{\Delta x_1} = \frac{\partial y}{\partial x_1} = f_1$ (holding $x_2, ...x_n$ fixed)

Rule: Treat all other variables as constants and use ordinary rules of differentation.

2.1.1 Example:

$$\begin{aligned} y &= 2x_1^2 + 3x_2 + 2x_1x_2 \\ \frac{dy}{dx_1} &= 4x_1 + 2x_2 (= f_1) \\ \frac{dy}{dx_2} &= 3 + 2x_1 (= f_2) \end{aligned}$$

2.2 Two Variable Case:

let $z = f(x, y)$, which means **"z is a function of x and y"**. In this case z is the endogenous (dependent) variable and both x and y are the exogenous (independent) variables.

To measure the the effect of a change in a single independent variable (x or y) on the dependent variable (z) we use what is known as the *PARTIAL DERIVATIVE*.

The partial derivative of z with respect to x measures the instantaneous change in the function as x changes while *HOLDING y constant*. Similarly, we would hold x constant if we wanted to evaluate the effect of a change in y on z. Formally:

- $\frac{\partial z}{\partial x}$ is the **"partial derivative"** of z with respect to x, treating y as a constant. Sometimes written as f_x.
- $\frac{\partial z}{\partial y}$ is the **"partial derivative"** of z with respect to y, treating x as a constant. Sometimes written as f_y.

The "∂" symbol ("bent over" lower case D) is called the "partial" symbol. It is interpreted in exactly the same way as $\frac{dy}{dx}$ from single variable calculus. The ∂ symbol simply serves to remind us that there are other variables in the equation, but for the purposes of the current exercise, these other variables are held constant.

EXAMPLES:

$$z = x + y \quad \partial z/\partial x = 1 \quad \partial z/\partial y = 1$$
$$z = xy \quad \partial z/\partial x = y \quad \partial z/\partial y = x$$
$$z = x^2y^2 \quad \partial z/\partial x = 2(y^2)x \quad \partial z/\partial y = 2(x^2)y$$
$$z = x^2y^3 + 2x + 4y \quad \partial z/\partial x = 2xy^3 + 2 \quad \partial z/\partial y = 3x^2y^2 + 4$$

- **REMEMBER:** When you are taking a partial derivative you treat the other variables in the equation as constants!

2.3 Rules of Partial Differentiation

Product Rule: given $z = g(x,y) \cdot h(x,y)$

$$\frac{\partial z}{\partial x} = g(x,y) \cdot \frac{\partial h}{\partial x} + h(x,y) \cdot \frac{\partial g}{\partial x}$$
$$\frac{\partial z}{\partial y} = g(x,y) \cdot \frac{\partial h}{\partial y} + h(x,y) \cdot \frac{\partial g}{\partial y}$$

Quotient Rule: given $z = \frac{g(x,y)}{h(x,y)}$ and $h(x,y) \neq 0$

$$\frac{\partial z}{\partial x} = \frac{h(x,y) \cdot \frac{\partial g}{\partial x} - g(x,y) \cdot \frac{\partial h}{\partial x}}{[h(x,y)]^2}$$
$$\frac{\partial z}{\partial y} = \frac{h(x,y) \cdot \frac{\partial g}{\partial y} - g(x,y) \cdot \frac{\partial h}{\partial y}}{[h(x,y)]^2}$$

Chain Rule: given $z = [g(x,y)]^n$

$$\frac{\partial z}{\partial x} = n\,[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial x}$$
$$\frac{\partial z}{\partial y} = n\,[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial y}$$

2.4 Further Examples:

For the function $U = U(x,y)$ find the the partial derivates with respect to x and y

for each of the following examples

Example 1

$$U = -5x^3 - 12xy - 6y^5$$

Answer:

$$\frac{\partial U}{\partial x} = U_x = 15x^2 - 12y$$
$$\frac{\partial U}{\partial y} = U_y = -12x - 30y^4$$

Example 2

$$U = 7x^2y^3$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = 14xy^3 \\ \frac{\partial U}{\partial y} &= U_y = 21x^2y^2\end{aligned}$$

Example 3

$$U = 3x^2(8x - 7y)$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = 3x^2(8) + (8x - 7y)(6x) = 72x^2 - 42xy \\ \frac{\partial U}{\partial y} &= U_y = 3x^2(-7) + (8x - 7y)(0) = -21x^2\end{aligned}$$

Example 4

$$U = (5x^2 + 7y)(2x - 4y^3)$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = (5x^2 + 7y)(2) + (2x - 4y^3)(10x) \\ \frac{\partial U}{\partial y} &= U_y = (5x^2 + 7y)(-12y^2) + (2x - 4y^3)(7)\end{aligned}$$

Example 5

$$U = \frac{9y^3}{x - y}$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = \frac{(x-y)(0) - 9y^3(1)}{(x-y)^2} = \frac{-9y^3}{(x-y)^2} \\ \frac{\partial U}{\partial y} &= U_y = \frac{(x-y)(27y^2) - 9y^3(-1)}{(x-y)^2} = \frac{27xy^2 - 18y^3}{(x-y)^2}\end{aligned}$$

Example 6

$$U = (x - 3y)^3$$

Answer:

$$\begin{aligned} \frac{\partial U}{\partial x} &= U_x = 3(x - 3y)^2(1) = 3(x - 3y)^2 \\ \frac{\partial U}{\partial y} &= U_y = 3(x - 3y)^2(-3) = -9(x - 3y)^2 \end{aligned}$$

2.5 A Special Function: Cobb-Douglas

The Cobb-douglas function is a mathematical function that is very popular in economic models. The general form is

$$z = x^a y^b$$

and its partial derivatives are

$\partial z/\partial x = a x^{a-1} y^b$ and $\partial z/\partial y = b x^a y^{b-1}$

Furthermore, the absolute value of the slope of the level curve of a Cobb-douglas is given by

$$\frac{\partial z/\partial x}{\partial z/\partial y} = MRS = \frac{a}{b}\frac{y}{x}$$

2.5.1 Example: Production Function

Let $Q = f(K, L)$

$f_L = \frac{dQ}{dK}$ = Marginal product of labour (Δ in Q from a Δ in L)

$f_K = \frac{dQ}{dK}$ = Marginal product of capital (Δ in Q from a Δ in K)

Let $Q = K^a L^b$ (Cobb-Douglas Technology)

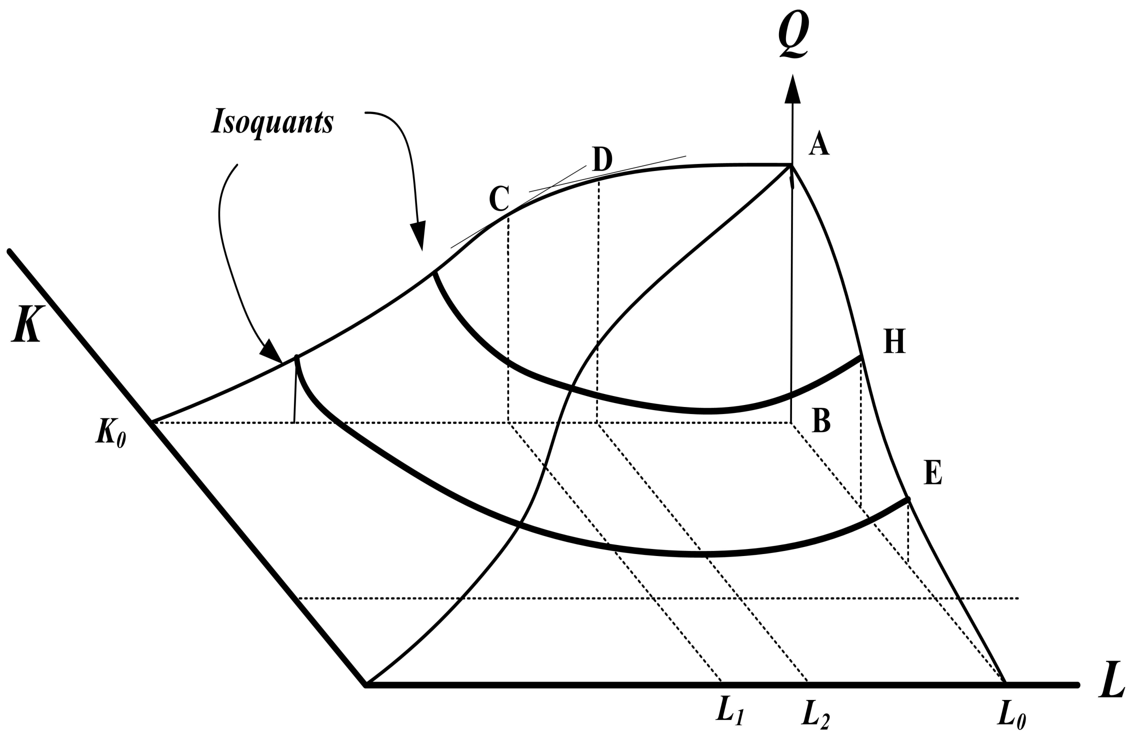
Then

$$\begin{aligned} MP_L &= bK^a L^{b-1} \qquad (\text{for } K = \bar{K}) \\ MP_K &= aK^{a-1} L^b \qquad (\text{for } L = \bar{L}) \end{aligned}$$

Isoquant: Δ's in K and L that keep $Q = \bar{Q}$
Then

$$\begin{aligned} \Delta L \cdot MP_L &= -MP_K \cdot \Delta K \\ &\text{or} \\ \Delta L \left(\frac{\partial Q}{\partial L}\right) &= \left(\frac{-\partial Q}{\partial K}\right) \Delta K \end{aligned}$$

$$\begin{aligned} \frac{\Delta K}{\Delta L} &= MRTS = \frac{MP_L}{MP_K} \\ &= \frac{bK^a L^{b-1}}{aK^{a-1} L^b} \\ &= \frac{b}{a} K^{(a-a+1)} L^{b-1-b} \\ &= \frac{b}{a} K^1 L^{-1} = \frac{b}{a} \frac{K}{L} \end{aligned}$$



Point C: $\frac{\partial Q}{\partial L} = MP_L$ at $L = L_1$ and $K = K_0$
Point D: $\frac{\partial Q}{\partial L}$ at $L = L_2$ and $K = K_0$
Point E: $\frac{\partial Q}{\partial K} = MP_K$ at $L = L_0$
MP_L = marginal product of labour
MP_K = marginal product of capital

3 National Income Model

Consider the linear model of a simple economy

$$
\begin{aligned}
Y &= C + I_0 + G_0 \\
C &= a + bY
\end{aligned}
$$

where Y, C are the endogenous variables and a, b, I_0 and G_0 are the exogenous variables and parameters.

In Equilibrium:

$$
\begin{aligned}
Y^e &= \frac{a + I_0 + G_0}{1-b} = \frac{a}{1-b} + \frac{I_0}{1-b} + \frac{G_0}{1-b} \\
C^e &= \frac{a + bI_0 + bG_0}{1-b} = \frac{a}{1-b} + \frac{bI_0}{1-b} + \frac{bG_0}{1-b}
\end{aligned}
$$

$\frac{\partial Y^e}{\partial G_0} = \frac{1}{1-b}$ $\quad \frac{\partial C^e}{\partial G_0} = \frac{b}{1-b}$ The Multipliers

What is $\frac{\partial Y^e}{\partial b}$?

$$
\begin{aligned}
Y^e &= (a + I_0 + G_0)(1-b)^{-1} \\
\frac{\partial Y^e}{\partial b} &= (a + I_0 + G_0)(1-b)^{-2}(-1)(-1) \qquad \text{Chain Rule} \\
\frac{\partial Y^e}{\partial b} &= +\left[\frac{a + I_0 + G_0}{(1-b)^2}\right] \\
&= \text{The income multiplier with respect to a change in the MPC}
\end{aligned}
$$