# 1 Derivatives: The Five Basic Rules

## 1.1 Nonlinear Functions

The term derivative means "slope" or rate of change. The five rules we are about to learn allow us to find the slope of about 90% of functions used in economics, business, and social sciences.

Suppose we have a function

$$y = f(x) \tag{1}$$

where $f(x)$ is a non linear function. For example:

$$\begin{array}{ll} 1 & y = x^2 \\ 2 & y = 3\sqrt{x} = 3x^{1/2} \\ 3 & y = ax + bx^2 + c \end{array} \tag{2}$$

Each equation is illustrated in Figure 1.

## 1.2 The Derivative

Given the general function

$$y = f(x)$$

the derivative of $y$ is denoted as

$$\frac{dy}{dx} = f'(x) \quad (= y')$$

The symbol $\frac{dy}{dx}$ is an abbreviation for "the change in $y$ ($dy$) FROM a change in $x$ ($dx$)"; or the "rise over the run". In other words, the slope.

## 1.3 The Five Rules

### 1.3.1 The Constant Rule

Given $y = f(x) = c$, where $c$ is an arbitrary constant, then

$$\frac{dy}{dx} = f'(x) = 0 \tag{3}$$

### 1.3.2 Power Function Rule

Suppose

$$y = ax^n \tag{4}$$

where $a$ and $n$ are any two constants. The power function rule states that the slope of the function is given by

$$\frac{dy}{dx} = f'(x) = anx^{n-1} \tag{5}$$

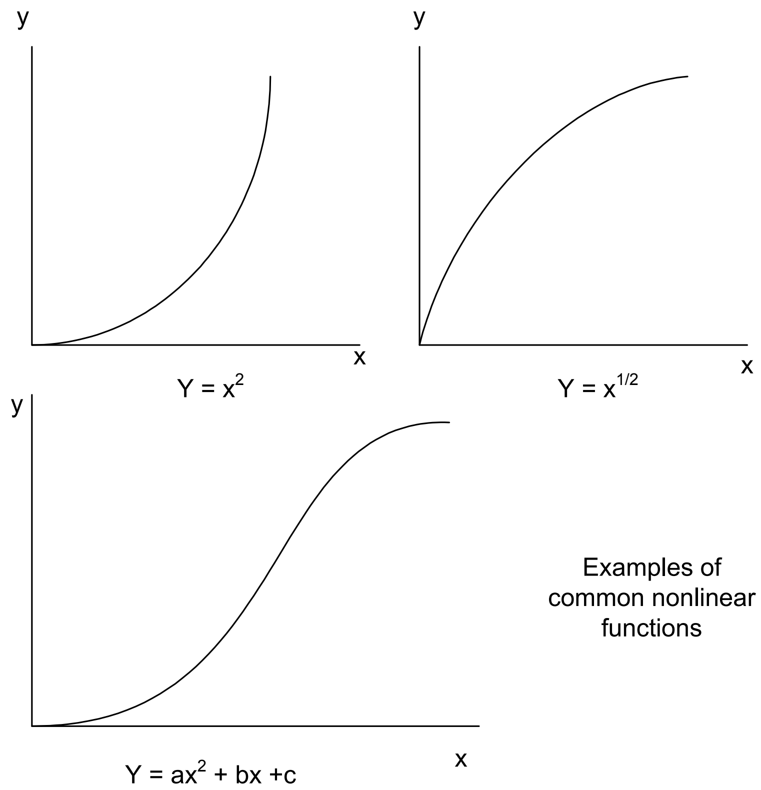
y
x
Y = x2
y
x
Y = x1/2
y
x
Y = ax2 + bx +c
Examples of
common nonlinear
functions

Figure 1:

This is probably the most commonly used rule in an introductory calculus course. Examples

$$
\begin{array}{ll}
y = x^2 & \frac{dy}{dx} = y' = 2x \\
y = 4x^3 & \frac{dy}{dx} = y' = 12x^2 \\
y = 5x^{1/3} & \frac{dy}{dx} = y' = (5)(1/3)x^{1/3-1} = \frac{5}{3}x^{-2/3} \\
y = x & \frac{dy}{dx} = y' = (1)x^{1-1} = (1)x^0 = 1
\end{array}
$$

Some functions that do not appear to be "power functions" can be manipulated to take the form of equation 4. For example, if

$$y = \frac{1}{x}$$

then it can also be written as

$$y = x^{-1}$$

thus

$$\frac{dy}{dx} = (-1)x^{-2}$$

Another example,

$$y = \sqrt{x}$$

which can also be written as

$$y = x^{1/2}$$

therefore, by equation 5,

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

### 1.3.3 Sum-difference Rule

If $y$ is a function created by adding or subtracting multiple functions such written as

$$y = f(x) \pm g(x)$$

where $f(x)$ and $g(x)$ are each functions similar to equation 4, then the derivative of $y$ ($y'$) is given by

$$y' = f'(x) \pm g'(x)$$

Example 1:

$$y = 4x^3 + 5x^2$$

the derivative is

$$y' = 12x^2 + 10x$$

Example 2:

$$
\begin{aligned}
y &= x^5 + 3x^{1/2} - 4x + 7 \\
y' &= 5x^4 + \frac{3}{2}x^{-1/2} - 4
\end{aligned}
$$

In each case we apply the power function rule (or constant rule) term-by-term

### 1.3.4 Product Rule

Suppose $y$ is a composite function created by multiplying two functions together

$$y = f(x)g(x)$$

the derivative is given by

$$\frac{dy}{dx} = f'g + fg' \tag{6}$$

Example:

$$y = (3x^2 + 4)(x^3 - 5x)$$

First, break this up into $f(x)$ and $g(x)$:

$$f(x) = 3x^2 + 4 \qquad g(x) = x^3 - 5x$$

then find the derivative for each

$$f'(x) = 6x \qquad g'(x) = 3x^2 - 5$$

Now re-combine the parts according to equation 6

$$\frac{dy}{dx} = f'g + fg' = [6x]\left[x^3 - 5x\right] + \left[3x^2 + 4\right]\left[3x^2 - 5\right]$$

then you simply collect terms and simplify

### 1.3.5 Quotient Rule

Suppose

$$y = \frac{f(x)}{g(x)}$$

then

$$\frac{dy}{dx} = \frac{f'g - fg'}{[g]^2} \tag{7}$$

Example

$$y = \frac{x^2 + 3}{2x - 1}$$

therefore $f = x^2 + 3$ and $g = 2x - 1$. The derivatives are

$$f' = 2x$$
$$g' = 2$$

Subsitute the componants into equation 7

$$\frac{dy}{dx} = \frac{f'g - fg'}{[g]^2} = \frac{(2x)(2x - 1) - \left(x^2 + 3\right)(2)}{(2x - 1)^2}$$

which (of course) can be further simplified