OPMT 5701
Optimization with Constraints

The Lagrange Multiplier Method

Sometimes we need to to maximize (minimize) a function that is subject to some sort of constraint. For example

$$\textit{Maximize} \quad z = f(x, y)$$

$$\textit{subject to the constraint} \quad x + y \leq 100$$

For this kind of problem there is a technique, or *trick,* developed for this kind of problem known as the *Lagrange Multiplier method.* This method involves adding an extra variable to the problem called the lagrange multiplier, or λ.

We then set up the problem as follows:

1. Create a new equation form the original information

$$L = f(x, y) + \lambda(100 - x - y)$$
$$or$$
$$L = f(x, y) + \lambda\,[Zero]$$

2. Then follow the same steps as used in a regular maximization problem

$$\frac{\partial L}{\partial x} = f_x - \lambda = 0$$
$$\frac{\partial L}{\partial y} = f_y - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 100 - x - y = 0$$

3. In most cases the λ will drop out with substitution. Solving these 3 equations will give you the constrained maximum solution

Example 1:

Suppose $z = f(x, y) = xy$. and the constraint is the one from above. The problem then becomes

$$L = xy + \lambda(100 - x - y)$$

Now take partial derivatives, one for each unknown, including λ

$$\frac{\partial L}{\partial x} = y - \lambda = 0$$
$$\frac{\partial L}{\partial y} = x - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 100 - x - y = 0$$

Starting with the first two equations, we see that $x = y$ and λ drops out. From the third equation we can easily find that $x = y = 50$ and the constrained maximum value for z is $z = xy = 2500$.

Example 2:

Maximize

$$u = 4x^2 + 3xy + 6y^2$$

subject to

$$x + y = 56$$

Set up the Lagrangian Equation:

$$L = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$$

Take the first-order partials and set them to zero

$$\begin{aligned} L_x &= 8x + 3y - \lambda = 0 \\ L_y &= 3x + 12y - \lambda = 0 \\ L_\lambda &= 56 - x - y = 0 \end{aligned}$$

From the first two equations we get

$$\begin{aligned} 8x + 3y &= 3x + 12y \\ x &= 1.8y \end{aligned}$$

Substitute this result into the third equation

$$\begin{aligned} 56 - 1.8y - y &= 0 \\ y &= 20 \end{aligned}$$

therefore

$$x = 36 \qquad \lambda = 348$$

Example 3: Cost minimization

A firm produces two goods, x and y. Due to a government quota, the firm must produce subject to the constraint $x + y = 42$. The firm's cost functions is

$$c(x, y) = 8x^2 - xy + 12y^2$$

The Lagrangian is

$$L = 8x^2 - xy + 12y^2 + \lambda(42 - x - y)$$

The first order conditions are

$$\begin{aligned} L_x &= 16x - y - \lambda = 0 \\ L_y &= -x + 24y - \lambda = 0 \\ L_\lambda &= 42 - x - y = 0 \qquad (1) \end{aligned}$$

Solving these three equations simultaneously yields

$$x = 25 \qquad y = 17 \qquad \lambda = 383$$

Example of duality for the consumer choice problem

Example 4: Utility Maximization

Consider a consumer with the utility function $U = xy$, who faces a budget constraint of $B = P_x x + P_y y$, where B, P_x and P_y are the budget and prices, which are given.

The choice problem is

Maximize

$$U = xy \tag{2}$$

Subject to

$$B = P_x x + P_y y \tag{3}$$

The Lagrangian for this problem is

$$Z = xy + \lambda(B - P_x x - P_y y) \tag{4}$$

The first order conditions are

$$\begin{aligned} Z_x &= y - \lambda P_x = 0 \\ Z_y &= x - \lambda P_y = 0 \\ Z_\lambda &= B - P_x x - P_y y = 0 \end{aligned} \tag{5}$$

Solving the first order conditions yield the following solutions

$$x^M = \tfrac{B}{2P_x} \quad y^M = \tfrac{B}{2P_y} \quad \lambda = \tfrac{B}{2P_x P_y} \tag{6}$$

where x^M and y^M are the consumer's Marshallian demand functions.

Example 5: Minimization Problem

Minimize

$$P_x x + P_y y \tag{7}$$

Subject to

$$U_0 = xy \tag{8}$$

The Lagrangian for the problem is

$$Z = P_x x + P_y y + \lambda(U_0 - xy) \tag{9}$$

The first order conditions are

$$\begin{aligned} Z_x &= P_x - \lambda y = 0 \\ Z_y &= P_y - \lambda x = 0 \\ Z_\lambda &= U_0 - xy = 0 \end{aligned} \tag{10}$$

Solving the system of equations for x, y and λ

$$\begin{aligned} x^h &= \left(\tfrac{P_y U_0}{P_x}\right)^{\frac{1}{2}} \\ y^h &= \left(\tfrac{P_x U_0}{P_y}\right)^{\frac{1}{2}} \\ \lambda^h &= \left(\tfrac{P_x P_y}{U_0}\right)^{\frac{1}{2}} \end{aligned} \tag{11}$$

Application: Intertemporal Utility Maximization

Consider a simple two period model where a consumer's utility is a function of consumption in both periods. Let the consumer's utility function be

$$U(c_1, c_2) = \ln c_1 + \beta \ln c_2$$

where c_1 is consumption in period one and c_2 is consumption in period two. The consumer is also endowments of y_1 in period one and y_2 in period two.

Let r denote a market interest rate with the consumer can choose to borrow or lend across the two periods. The consumer's intertemporal budget constraint is

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

Method One:Find MRS and Substitute

Differentiate the Utility function

$$dU = \left(\frac{1}{c_1}\right) dc_1 + \left(\frac{\beta}{c_2}\right) dc_2 = 0$$

Rearrange to get

$$\frac{dc_2}{dc_1} = -\frac{c_2}{\beta c_1}$$

The MRS is the Absolute value of $\frac{dc_2}{dc_1}$:

$$MRS = \frac{c_2}{\beta c_1}$$

substitute into the budget constraint

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= c_1 + \frac{\beta c_1(1+r)}{1+r} = (1+\beta)c_1 \\ c_1^* &= \frac{y_1 + \frac{y_2}{1+r}}{(1+\beta)} \end{aligned}$$

Similarly, solving for c_2^* using the first order conditions

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= \frac{c_2}{\beta(1+r)} + \frac{c_2}{1+r} \\ (1+r)y_1 + y_2 &= \left(\frac{1}{\beta} + 1\right) c_2 \\ c_2^* &= \frac{(1+r)y_1 + y_2}{\frac{1}{\beta} + 1} \end{aligned}$$

Method Two: Use the Lagrange Multiplier Method

The Lagrangian for this utility maximization problem is

$$L = \ln c_1 + \beta \ln c_2 + \lambda \left(y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right)$$

The first order conditions are

$$\begin{array}{l} \frac{\partial L}{\partial \lambda} = y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} = 0 \\ \frac{\partial L}{\partial C_1} = \frac{1}{c_1} - \lambda = 0 \\ \frac{\partial L}{\partial C_1} = \frac{\beta}{c_2} - \frac{\lambda}{1+r} = 0 \end{array}$$

Combining the last two first order equations to eliminate λ gives us

$$\begin{aligned} \frac{1/c_1}{\beta/c_2} &= \frac{c_2}{\beta c_1} = \frac{\lambda}{\frac{\lambda}{1+r}} = 1+r \\ c_2 &= \beta c_1(1+r) \qquad \textit{and} \qquad c_1 = \frac{c_2}{\beta(1+r)} \end{aligned}$$

sub into the Budget constraint

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= c_1 + \frac{\beta c_1(1+r)}{1+r} = (1+\beta)c_1 \\ c_1^* &= \frac{y_1 + \frac{y_2}{1+r}}{(1+\beta)} \end{aligned}$$

Similarly, solving for c_2^* using the first order conditions

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= \frac{c_2}{\beta(1+r)} + \frac{c_2}{1+r} \\ (1+r)y_1 + y_2 &= \left(\frac{1}{\beta} + 1\right) c_2 \\ c_2^* &= \frac{(1+r)y_1 + y_2}{\frac{1}{\beta} + 1} \end{aligned}$$

Problems:

1. Skippy lives on an island where she produces two goods, x *and* y, according the the production possibility frontier $200 = x^2 + y^2$, and she consumes all the goods herself. Her utility function is

$$u = x \cdot y^3$$

   FInd her utility maximizing x and y as well as the value of λ

2. A consumer has the following utility function: $U(x,y) = x(y+1)$, where x and y are quantities of two consumption goods whose prices are p_x and p_y respectively. The consumer also has a budget of B. Therefore the consumer's maximization problem is

$$x(y+1) + \lambda(B - p_x x - p_y y)$$

(a) From the first order conditions find expressions for x^* and y^*. These are the consumer's demand functions. What kind of good is y? In particular what happens when $p_y > B/2$?

3. This problem could be recast as the following dual problem

$$Minimize \quad p_x x + p_y y \text{ subject to } U^* = x(y+1)$$

Find the values of x and y that solve this minimization problem.

4. Skippy has the following utility function: $u = x^{\frac{1}{2}} y^{\frac{1}{2}}$ and faces the budget constraint: $M = p_x x + p_y y$.

(a) Suppose $M = 120$, $P_y = 1$ and $P_x = 4$. Find the optimal x and y

REVIEW:Partial Derivative Rules:

$U = xy$	$\partial U/\partial x = y$	$\partial U/\partial y = x$
$U = x^a y^b$	$\partial U/\partial x = ax^{a-1}y^b$	$\partial U/\partial y = bx^a y^{b-1}$
$U = x^a y^{-b} = \frac{x^a}{y^b}$	$\partial U/\partial x = ax^{a-1}y^{-b}$	$\partial U/\partial y = -bx^a y^{-b-1}$
$U = ax + by$	$\partial U/\partial x = a$	$\partial U/\partial y = b$
$U = ax^{1/2} + by^{1/2}$	$\partial U/\partial x = a\left(\frac{1}{2}\right)x^{-1/2}$	$\partial U/\partial y = b\left(\frac{1}{2}\right)y^{-1/2}$

Finding the MRS from Utility functions

EXAMPLE: Find the total differential for the following utility functions

1. $U(x_1, x_2) = ax_1 + bx_2$ where $(a, b > 0)$
2. $U(x_1, x_2) = x_1^2 + x_2^3 + x_1x_2$
3. $U(x_1, x_2) = x_1^a x_2^b$ where $(a, b > 0)$
4. $U(x_1, x_2) = \alpha \ln c_1 + \beta \ln c_2$ where $(\alpha, \beta > 0)$

Answers:
1. $\frac{\partial U}{\partial x_1} = U_1 = a \quad \frac{\partial U}{\partial x_2} = U_2 = b$
and

$$dU = U_1 dx_1 + U_2 dx_2 = adx_1 + bdx_2 = 0$$

If we rearrange to get dx_2/dx_1

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = -\frac{U_1}{U_2} = -\frac{a}{b}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{a}{b}$$

2. $\frac{\partial U}{\partial x_1} = U_1 = 2x_1 + x_2 \quad \frac{\partial U}{\partial x_2} = U_2 = 3x_2^2 + x_1$
and

$$dU = U_1 dx_1 + U_2 dx_2 = (2x_1 + x_2)dx_1 + (3x_2^2 + x_1)dx_2 = 0$$

Find dx_2/dx_1

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2} = -\frac{(2x_1 + x_2)}{(3x_2^2 + x_1)}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{(2x_1 + x_2)}{(3x_2^2 + x_1)}$$

iii) $\frac{\partial U}{\partial x_1} = U_1 = a x_1^{a-1} x_2^b \qquad \frac{\partial U}{\partial x_2} = U_2 = b x_1^a x_2^{b-1}$
and

$$dU = \left(a x_1^{a-1} x_2^b\right) dx_1 + \left(b x_1^a x_2^{b-1}\right) dx_2 = 0$$

Rearrange to get

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2} = -\frac{a x_1^{a-1} x_2^b}{b x_1^a x_2^{b-1}} = -\frac{a x_2}{b x_1}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{a x_2}{b x_1}$$

iv) $\frac{\partial U}{\partial c_1} = U_1 = \alpha \left(\frac{1}{c_1}\right) dc_1 = \left(\frac{\alpha}{c_1}\right) dc_1 \qquad \frac{\partial U}{\partial x_2} = U_2 = \beta \left(\frac{1}{c_2}\right) dc_2 = \left(\frac{\beta}{c_2}\right) dc_2$
and

$$dU = \left(\frac{\alpha}{c_1}\right) dc_1 + \left(\frac{\beta}{c_2}\right) dc_2 = 0$$

Rearrange to get

$$\frac{dc_2}{dc_1} = -\frac{U_1}{U_2} = \frac{\left(\frac{\alpha}{c_1}\right)}{\left(\frac{\beta}{c_2}\right)} = -\frac{\alpha c_2}{\beta c_1}$$

The MRS is the Absolute value of $\frac{dc_2}{dc_1}$:

$$\begin{aligned} MRS &= \frac{\alpha c_2}{\beta c_1} = (1+r) \\ c_2 &= \beta c_1 (1+r) \qquad and \qquad c_1 = \frac{c_2}{\beta(1+r)} \end{aligned}$$