

# ECON 301

## Two Variable Optimization

(with- and without- constraints)

### Review of Some Derivative Rules

1. Partial Derivative Rules:

$U = xy$	$\partial U/\partial x = U_x = y$	$\partial U/\partial y = U_y = x$
$U = x^a y^b$	$\partial U/\partial x = U_x = ax^{a-1}y^b$	$\partial U/\partial y = U_y = bx^a y^{b-1}$
$U = x^a y^{-b} = \frac{x^a}{y^b}$	$\partial U/\partial x = U_x = ax^{a-1}y^{-b}$	$\partial U/\partial y = U_y = -bx^a y^{-b-1}$
$U = ax + by$	$\partial U/\partial x = U_x = a$	$\partial U/\partial y = U_y = b$
$U = ax^{1/2} + by^{1/2}$	$\partial U/\partial x = U_x = a \left(\frac{1}{2}\right) x^{-1/2}$	$\partial U/\partial y = U_y = b \left(\frac{1}{2}\right) y^{-1/2}$

2. Logarithm (Natural log)  $\ln x$

(a) Rules of natural log

<i>If</i>	<i>Then</i>
$y = AB$	$\ln y = \ln(AB) = \ln A + \ln B$
$y = A/B$	$\ln y = \ln A - \ln B$
$y = A^b$	$\ln y = \ln(A^b) = b \ln A$

NOTE:  $\ln(A + B) \neq \ln A + \ln B$

(b) derivatives

<i>IF</i>	<i>THEN</i>
$y = \ln x$	$\frac{dy}{dx} = \frac{1}{x}$
$y = \ln(f(x))$	$\frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x)$

(c) Examples

<i>If</i>	<i>Then</i>
$y = \ln(x^2 - 2x)$	$dy/dx = \frac{1}{(x^2-2x)}(2x - 2)$
$y = \ln(x^{1/2}) = \frac{1}{2} \ln x$	$dy/dx = \left(\frac{1}{2}\right) \left(\frac{1}{x}\right) = \frac{1}{2x}$

3. The Number  $e$

if  $y = e^x$  then  $\frac{dy}{dx} = e^x$

if  $y = e^{f(x)}$  then  $\frac{dy}{dx} = e^{f(x)} \cdot f'(x)$

(a) Examples

$y = e^{3x}$	$\frac{dy}{dx} = e^{3x}(3)$
$y = e^{7x^3}$	$\frac{dy}{dx} = e^{7x^3}(21x^2)$
$y = e^{rt}$	$\frac{dy}{dt} = re^{rt}$

# Using Calculus For Maximization Problems

## One Variable Case

If we have the following function

$$y = 10x - x^2$$

we have an example of a *dome shaped* function. To find the maximum of the dome, we simply need to find the point where the slope of the dome is zero, or

$$\begin{aligned}\frac{dy}{dx} &= 10 - 2x = 0 \\ 10 &= 2x \\ x &= 5 \\ \text{and} \\ y &= 25\end{aligned}$$

## Two Variable Case

Suppose we want to maximize the following function

$$z = f(x, y) = 10x + 10y + xy - x^2 - y^2$$

Note that there are two unknowns that must be solved for:  $x$  and  $y$ . This function is an example of a *three-dimensional dome*. (i.e. the roof of *BC Place*)

To solve this maximization problem we use **partial derivatives**. We take a partial derivative for each of the unknown choice variables and set them equal to zero

$$\begin{aligned}\frac{\partial z}{\partial x} = f_x &= 10 + y - 2x = 0 && \text{The slope in the "x" direction} = 0 \\ \frac{\partial z}{\partial y} = f_y &= 10 + x - 2y = 0 && \text{The slope in the "y" direction} = 0\end{aligned}$$

This gives us a set of equations, one equation for each of the unknown variables. When you have the same number of independent equations as unknowns, you can solve for each of the unknowns.

rewrite each equation as

$$\begin{aligned}y &= 2x - 10 \\ x &= 2y - 10\end{aligned}$$

substitute one into the other

$$\begin{aligned}x &= 2(2x - 10) - 10 \\ x &= 4x - 30 \\ 3x &= 30 \\ x &= 10\end{aligned}$$

similarly,

$$y = 10$$

**REMEMBER:** To maximize (minimize) a function of many variables you use the technique of partial differentiation. This produces a set of equations, one equation for each of the unknowns. You then solve the set of equations simultaneously to derive solutions for each of the unknowns.

Second order Conditions (second derivative Test)

To test for a maximum or minimum we need to check the second partial derivatives. Since we have two first partial derivative equations ( $f_x, f_y$ ) and two variable in each equation, we will get four *second partials* ( $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ )

Using our original first order equations and taking the partial derivatives for each of them (a second time) yields:

$$f_x = 10 + y - 2x = 0 \quad f_y = 10 + x - 2y = 0$$

$$\begin{array}{ll} f_{xx} = -2 & f_{yy} = -2 \\ f_{xy} = 1 & f_{yx} = 1 \end{array}$$

The two partials,  $f_{xx}$ , and  $f_{yy}$  are the direct effects of a small change in  $x$  and  $y$  on the respective slopes in the  $x$  and  $y$  direction. The partials,  $f_{xy}$  and  $f_{yx}$  are the indirect effects, or the cross effects of one variable on the slope in the other variable's direction. For both *Maximums and Minimums*, the direct effects must outweigh the cross effects

## Rules for two variable Maximums and Minimums<sup>1</sup>

### 1. Maximum

$$\begin{array}{l} f_{xx} < 0 \\ f_{yy} < 0 \\ f_{yy}f_{xx} - f_{xy}f_{yx} > 0 \end{array}$$

### 2. Minimum

$$\begin{array}{l} f_{xx} > 0 \\ f_{yy} > 0 \\ f_{yy}f_{xx} - f_{xy}f_{yx} > 0 \end{array}$$

### 3. Otherwise, we have a *Saddle Point*

From our second order conditions, above,

$$\begin{array}{ll} f_{xx} = -2 < 0 & f_{yy} = -2 < 0 \\ f_{xy} = 1 & f_{yx} = 1 \end{array}$$

and

$$f_{yy}f_{xx} - f_{xy}f_{yx} = (-2)(-2) - (1)(1) = 3 > 0$$

therefore we have a maximum.

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<sup>1</sup>**Advanced Topic:** This section is optional for ECON 301. For most applications, the structure of the problem will make it clear that this is a max or min problem

## Example: Profit Max Capital and Labour

Suppose we have the following production function

$$q = f(K, L) = L^{\frac{1}{2}} + K^{\frac{1}{2}}$$

$q =$  Output  
 $L =$  Labour  
 $K =$  Capital

Then the profit function for a competitive firm is

$$\pi = Pq - wL - rK$$

$P =$  Market Price

*or*  $w =$  Wage Rate

$$\pi = PL^{\frac{1}{2}} + PK^{\frac{1}{2}} - wL - rK$$

$r =$  Rental Rate

First order conditions

	General Form
1. $\frac{\partial \pi}{\partial L} = \frac{P}{2}L^{-\frac{1}{2}} - w = 0$	$Pf_L - w = 0$
2. $\frac{\partial \pi}{\partial k} = \frac{P}{2}K^{-\frac{1}{2}} - r = 0$	$Pf_K - r = 0$

Solving (1) and (2), we get

$$L^* = \left(\frac{2w}{P}\right)^{-2} \quad K^* = \left(\frac{2r}{P}\right)^{-2}$$

Example: If  $P = 1000$ ,  $w = 20$ , and  $r = 10$

1. Find the optimal  $K$ ,  $L$ , and  $\pi$
2. Check second order conditions

## Example: Cobb-Douglas production function and a competitive firm

Consider a competitive firm with the following profit function

$$\pi = TR - TC = PQ - wL - rK \tag{1}$$

where  $P$  is price,  $Q$  is output,  $L$  is labour and  $K$  is capital, and  $w$  and  $r$  are the input prices for  $L$  and  $K$  respectively. Since the firm operates in a competitive market, the exogenous variables are  $P, w$  and  $r$ . There are three endogenous variables,  $K, L$  and  $Q$ . However output,  $Q$ , is in turn a function of  $K$  and  $L$  via the production function

$$Q = f(K, L)$$

which in this case, is the Cobb-Douglas function

$$Q = L^a K^b \tag{2}$$

where a and b are positive parameters. If we further assume decreasing returns to scale, then  $a + b < 1$ . For simplicity, let's consider the symmetric case where  $a = b = \frac{1}{4}$

$$Q = L^{\frac{1}{4}}K^{\frac{1}{4}} \quad (3)$$

Substituting Equation 3 into Equation 1 gives us

$$\pi(K, L) = PL^{\frac{1}{4}}K^{\frac{1}{4}} - wL - rK \quad (4)$$

The first order conditions are

$$\begin{aligned} \frac{\partial \pi}{\partial L} &= P \left(\frac{1}{4}\right) L^{-\frac{3}{4}}K^{\frac{1}{4}} - w = 0 \\ \frac{\partial \pi}{\partial K} &= P \left(\frac{1}{4}\right) L^{\frac{1}{4}}K^{-\frac{3}{4}} - r = 0 \end{aligned} \quad (5)$$

This system of equations define the optimal L and K for profit maximization. Rewriting the first equation in Equation 5 to isolate K

$$\begin{aligned} P \left(\frac{1}{4}\right) L^{-\frac{3}{4}}K^{\frac{1}{4}} &= w \\ K &= \left(\frac{4w}{P}L^{\frac{3}{4}}\right)^4 \end{aligned}$$

Substituting into the second equation of Equation 5

$$\begin{aligned} \frac{P}{4}L^{\frac{1}{4}}K^{-\frac{3}{4}} &= \left(\frac{P}{4}\right)L^{\frac{1}{4}}\left[\left(\frac{4w}{P}L^{\frac{3}{4}}\right)^4\right]^{-\frac{3}{4}} = r \\ &= P^4\left(\frac{1}{4}\right)^4w^{-3}L^{-2} = r \end{aligned}$$

Re-arranging to get L by itself gives us

$$L^* = \left(\frac{P}{4}w^{-\frac{3}{4}}r^{-\frac{1}{4}}\right)^2$$

Taking advantage of the symmetry of the model, we can quickly find the optimal K

$$K^* = \left(\frac{P}{4}r^{-\frac{3}{4}}w^{-\frac{1}{4}}\right)^2$$

$L^*$  and  $K^*$  are the firm's factor demand equations.

## Optimization with Constraints

### The Lagrange Multiplier Method

Sometimes we need to to maximize (minimize) a function that is subject to some sort of constraint. For example

$$\text{Maximize } z = f(x, y)$$

$$\text{subject to the constraint } x + y \leq 100$$

For this kind of problem there is a technique, or *trick*, developed for this kind of problem known as the *Lagrange Multiplier method*. This method involves adding an extra variable to the problem called the lagrange multiplier, or  $\lambda$ .

We then set up the problem as follows:

1. Create a new equation form the original information

$$L = f(x, y) + \lambda(100 - x - y)$$

*or*

$$L = f(x, y) + \lambda [Zero]$$

2. Then follow the same steps as used in a regular maximization problem

$$\begin{aligned}\frac{\partial L}{\partial x} &= f_x - \lambda = 0 \\ \frac{\partial L}{\partial y} &= f_y - \lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= 100 - x - y = 0\end{aligned}$$

3. In most cases the  $\lambda$  will drop out with substitution. Solving these 3 equations will give you the constrained maximum solution

### Example 1:

Suppose  $z = f(x, y) = xy$ . and the constraint is the one from above. The problem then becomes

$$L = xy + \lambda(100 - x - y)$$

Now take partial derivatives, one for each unknown, including  $\lambda$

$$\begin{aligned}\frac{\partial L}{\partial x} &= y - \lambda = 0 \\ \frac{\partial L}{\partial y} &= x - \lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= 100 - x - y = 0\end{aligned}$$

Starting with the first two equations, we see that  $x = y$  and  $\lambda$  drops out. From the third equation we can easily find that  $x = y = 50$  and the constrained maximum value for  $z$  is  $z = xy = 2500$ .

### Example 2:

Maximize

$$u = 4x^2 + 3xy + 6y^2$$

subject to

$$x + y = 56$$

Set up the Lagrangian Equation:

$$L = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$$

Take the first-order partials and set them to zero

$$\begin{aligned}L_x &= 8x + 3y - \lambda = 0 \\ L_y &= 3x + 12y - \lambda = 0 \\ L_\lambda &= 56 - x - y = 0\end{aligned}$$

From the first two equations we get

$$\begin{aligned}8x + 3y &= 3x + 12y \\x &= 1.8y\end{aligned}$$

Substitute this result into the third equation

$$\begin{aligned}56 - 1.8y - y &= 0 \\y &= 20\end{aligned}$$

therefore

$$x = 36 \quad \lambda = 348$$

### Example 3: Cost minimization

A firm produces two goods,  $x$  and  $y$ . Due to a government quota, the firm must produce subject to the constraint  $x + y = 42$ . The firm's cost functions is

$$c(x, y) = 8x^2 - xy + 12y^2$$

The Lagrangian is

$$L = 8x^2 - xy + 12y^2 + \lambda(42 - x - y)$$

The first order conditions are

$$\begin{aligned}L_x &= 16x - y - \lambda = 0 \\L_y &= -x + 24y - \lambda = 0 \\L_\lambda &= 42 - x - y = 0\end{aligned}\tag{6}$$

Solving these three equations simultaneously yields

$$x = 25 \quad y = 17 \quad \lambda = 383$$

### Example of duality for the consumer choice problem

#### Example 4: Utility Maximization

Consider a consumer with the utility function  $U = xy$ , who faces a budget constraint of  $B = P_x x + P_y y$ , where  $B$ ,  $P_x$  and  $P_y$  are the budget and prices, which are given.

The choice problem is

Maximize

$$U = xy\tag{7}$$

Subject to

$$B = P_x x + P_y y\tag{8}$$

The Lagrangian for this problem is

$$Z = xy + \lambda(B - P_x x - P_y y)\tag{9}$$

The first order conditions are

$$\begin{aligned} Z_x &= y - \lambda P_x = 0 \\ Z_y &= x - \lambda P_y = 0 \\ Z_\lambda &= B - P_x x - P_y y = 0 \end{aligned} \tag{10}$$

Solving the first order conditions yield the following solutions

$$x^M = \frac{B}{2P_x} \quad y^M = \frac{B}{2P_y} \quad \lambda = \frac{B}{2P_x P_y} \tag{11}$$

where  $x^M$  and  $y^M$  are the consumer's Marshallian demand functions.

### Example 5: Minimization Problem

Minimize

$$P_x x + P_y y \tag{12}$$

Subject to

$$U_0 = xy \tag{13}$$

The Lagrangian for the problem is

$$Z = P_x x + P_y y + \lambda(U_0 - xy) \tag{14}$$

The first order conditions are

$$\begin{aligned} Z_x &= P_x - \lambda y = 0 \\ Z_y &= P_y - \lambda x = 0 \\ Z_\lambda &= U_0 - xy = 0 \end{aligned} \tag{15}$$

Solving the system of equations for  $x$ ,  $y$  and  $\lambda$

$$\begin{aligned} x^h &= \left( \frac{P_y U_0}{P_x} \right)^{\frac{1}{2}} \\ y^h &= \left( \frac{P_x U_0}{P_y} \right)^{\frac{1}{2}} \\ \lambda^h &= \left( \frac{P_x P_y}{U_0} \right)^{\frac{1}{2}} \end{aligned} \tag{16}$$

## Application: Intertemporal Utility Maximization

Consider a simple two period model where a consumer's utility is a function of consumption in both periods. Let the consumer's utility function be

$$U(c_1, c_2) = \ln c_1 + \beta \ln c_2$$

where  $c_1$  is consumption in period one and  $c_2$  is consumption in period two. The consumer is also endowments of  $y_1$  in period one and  $y_2$  in period two.

Let  $r$  denote a market interest rate with the consumer can choose to borrow or lend across the two periods. The consumer's intertemporal budget constraint is

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$



## Method One: Find MRS and Substitute

Differentiate the Utility function

$$dU = \left(\frac{1}{c_1}\right) dc_1 + \left(\frac{\beta}{c_2}\right) dc_2 = 0$$

Rearrange to get

$$\frac{dc_2}{dc_1} = -\frac{c_2}{\beta c_1}$$

The MRS is the Absolute value of  $\frac{dc_2}{dc_1}$  :

$$MRS = \frac{c_2}{\beta c_1}$$

substitute into the budget constraint

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= c_1 + \frac{\beta c_1(1+r)}{1+r} = (1+\beta)c_1 \\ c_1^* &= \frac{y_1 + \frac{y_2}{1+r}}{(1+\beta)} \end{aligned}$$

Similarly, solving for  $c_2^*$  using the first order conditions

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= \frac{c_2}{\beta(1+r)} + \frac{c_2}{1+r} \\ (1+r)y_1 + y_2 &= \left(\frac{1}{\beta} + 1\right) c_2 \\ c_2^* &= \frac{(1+r)y_1 + y_2}{\frac{1}{\beta} + 1} \end{aligned}$$

## Method Two: Use the Lagrange Multiplier Method

The Lagrangian for this utility maximization problem is

$$L = \ln c_1 + \beta \ln c_2 + \lambda \left( y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right)$$

The first order conditions are

$$\begin{aligned} \frac{\partial L}{\partial \lambda} &= y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} = 0 \\ \frac{\partial L}{\partial c_1} &= \frac{1}{c_1} - \lambda = 0 \\ \frac{\partial L}{\partial c_2} &= \frac{\beta}{c_2} - \frac{\lambda}{1+r} = 0 \end{aligned}$$

Combining the last two first order equations to eliminate  $\lambda$  gives us

$$\begin{aligned} \frac{1/c_1}{\beta/c_2} &= \frac{c_2}{\beta c_1} = \frac{\lambda}{\frac{\lambda}{1+r}} = 1+r \\ c_2 &= \beta c_1(1+r) \quad \text{and} \quad c_1 = \frac{c_2}{\beta(1+r)} \end{aligned}$$

sub into the Budget constraint

$$y_1 + \frac{y_2}{1+r} = c_1 + \frac{\beta c_1(1+r)}{1+r} = (1+\beta)c_1$$
$$c_1^* = \frac{y_1 + \frac{y_2}{1+r}}{(1+\beta)}$$

Similarly, solving for  $c_2^*$  using the first order conditions

$$y_1 + \frac{y_2}{1+r} = \frac{c_2}{\beta(1+r)} + \frac{c_2}{1+r}$$
$$(1+r)y_1 + y_2 = \left(\frac{1}{\beta} + 1\right) c_2$$
$$c_2^* = \frac{(1+r)y_1 + y_2}{\frac{1}{\beta} + 1}$$

## Problems:

1. Maximize  $\ln x + y$  subject to  $2x + 3y = 12$  using the lagrange method
2. Skippy lives on an island where she produces two goods,  $x$  and  $y$ , according to the production possibility frontier  $100 = x^2 + y^2$ , and she consumes all the goods herself. Her utility function is

$$u = x \cdot y$$

Find her utility maximizing  $x$  and  $y$  as well as the value of  $\lambda$

3. A consumer has the following utility function:  $U(x, y) = xy^2$ , where  $x$  and  $y$  are quantities of two consumption goods whose prices are  $p_x$  and  $p_y$  respectively. The consumer also has a budget of  $B$ . Therefore the consumer's maximization problem is

$$xy^2 + \lambda(B - p_x x - p_y y)$$

- (a) From the first order conditions find expressions for  $x^*$  and  $y^*$ . These are the consumer's demand functions.
4. This problem could be recast as the following dual problem

$$\text{Minimize } p_x x + p_y y \text{ subject to } U_0 = xy^2$$

Find the values of  $x$  and  $y$  that solve this minimization problem.

5. Skippy has the following utility function:  $u = x^{\frac{1}{2}}y^{\frac{1}{2}}$  and faces the budget constraint:  $M = p_x x + p_y y$ .

  - (a) Suppose  $M = 120$ ,  $P_y = 1$  and  $P_x = 4$ . Find the optimal  $x$  and  $y$

## Finding the MRS from Utility functions

**EXAMPLE:** Find the total differential for the following utility functions

1.  $U(x_1, x_2) = ax_1 + bx_2$  where  $(a, b > 0)$
2.  $U(x_1, x_2) = x_1^2 + x_2^3 + x_1x_2$
3.  $U(x_1, x_2) = x_1^a x_2^b$  where  $(a, b > 0)$
4.  $U(x_1, x_2) = \alpha \ln c_1 + \beta \ln c_2$  where  $(\alpha, \beta > 0)$

Answers:

1.  $\frac{\partial U}{\partial x_1} = U_1 = a$     $\frac{\partial U}{\partial x_2} = U_2 = b$

and

$$dU = U_1 dx_1 + U_2 dx_2 = a dx_1 + b dx_2 = 0$$

If we rearrange to get  $dx_2/dx_1$

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = -\frac{U_1}{U_2} = -\frac{a}{b}$$

The MRS is the Absolute value of  $\frac{dx_2}{dx_1}$  :

$$MRS = \frac{a}{b}$$

2.  $\frac{\partial U}{\partial x_1} = U_1 = 2x_1 + x_2$     $\frac{\partial U}{\partial x_2} = U_2 = 3x_2^2 + x_1$

and

$$dU = U_1 dx_1 + U_2 dx_2 = (2x_1 + x_2) dx_1 + (3x_2^2 + x_1) dx_2 = 0$$

Find  $dx_2/dx_1$

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2} = -\frac{(2x_1 + x_2)}{(3x_2^2 + x_1)}$$

The MRS is the Absolute value of  $\frac{dx_2}{dx_1}$  :

$$MRS = \frac{(2x_1 + x_2)}{(3x_2^2 + x_1)}$$

iii)  $\frac{\partial U}{\partial x_1} = U_1 = ax_1^{a-1} x_2^b$     $\frac{\partial U}{\partial x_2} = U_2 = bx_1^a x_2^{b-1}$

and

$$dU = (ax_1^{a-1} x_2^b) dx_1 + (bx_1^a x_2^{b-1}) dx_2 = 0$$

Rearrange to get

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2} = -\frac{ax_1^{a-1} x_2^b}{bx_1^a x_2^{b-1}} = -\frac{ax_2}{bx_1}$$

The MRS is the Absolute value of  $\frac{dx_2}{dx_1}$  :

$$MRS = \frac{ax_2}{bx_1}$$

$$\text{iv) } \frac{\partial U}{\partial c_1} = U_1 = \alpha \left( \frac{1}{c_1} \right) dc_1 = \left( \frac{\alpha}{c_1} \right) dc_1 \quad \frac{\partial U}{\partial c_2} = U_2 = \beta \left( \frac{1}{c_2} \right) dc_2 = \left( \frac{\beta}{c_2} \right) dc_2$$

and

$$dU = \left( \frac{\alpha}{c_1} \right) dc_1 + \left( \frac{\beta}{c_2} \right) dc_2 = 0$$

Rearrange to get

$$\frac{dc_2}{dc_1} = -\frac{U_1}{U_2} = \frac{\left( \frac{\alpha}{c_1} \right)}{\left( \frac{\beta}{c_2} \right)} = -\frac{\alpha c_2}{\beta c_1}$$

The MRS is the Absolute value of  $\frac{dc_2}{dc_1}$  :

$$MRS = \frac{\alpha c_2}{\beta c_1} = (1 + r)$$

$$c_2 = \beta c_1 (1 + r) \quad \text{and} \quad c_1 = \frac{c_2}{\beta(1 + r)}$$