Indirect Utility Function

For any: { Utility function
Income
Set of prices

We obtain a set of optimally chosen quantities called ordinary or Marshallian demand functions:

$$q_1^* = D_1(p_1, p_2, Y)$$
$$q_2^* = D_2(p_1, p_2, Y)$$

So when we say $\underset{\{q_1, q_2\}}{MAX}\ u(q_1, q_2)$ s.t. $p_1 q_1 + p_2 q_2 \leq Y$

we get as a result:

$$MAX\ u(q_1^*(p_1, p_2, Y), q_2^*(p_1, p_2, Y))$$

$$\Rightarrow u^*(p_1, p_2, Y) \equiv V(p_1, p_2, Y)$$

* $V(p_1, p_2, Y)$ is called the "Indirect Utility Function".
  ↳ value of maximized utility given prices and income.

Direct utility: Utility from consumption $u(q_1, q_2)$

Indirect utility: utility given prices & income $V(q_1^*, q_2^*)$

EX

$$\max_{\{q_1,q_2\}} u(q_1,q_2) = q_1^{\frac{1}{2}} q_2^{\frac{1}{2}} \quad \text{s.t. } p_1 q_1 + p_2 q_2 = y$$

$$L(q_1,q_2,\lambda) = q_1^{\frac{1}{2}} q_2^{\frac{1}{2}} + \lambda(y - p_1 q_1 - p_2 q_2)$$

(i) $L_{q_1} = \frac{1}{2} q_1^{-\frac{1}{2}} q_2^{\frac{1}{2}} - \lambda p_1 = 0$

(ii) $L_{q_2} = \frac{1}{2} q_1^{\frac{1}{2}} q_2^{-\frac{1}{2}} - \lambda p_2 = 0$

(iii) $L_\lambda = y - p_1 q_1 - p_2 q_2 = 0$

using (i) and (ii) we obtain:

(iv) $\boxed{\frac{q_2}{q_1} = \frac{p_1}{p_2}}$ MRS = price ratio

* (iii) and (iv) are sufficient conditions for interior solution.

↳ (iii) says solution must be on the budget line.

↳ (iv) says slope of indifference curve at (q_1^*, q_2^*) must equal slope of the budget line.

Solution:

Sub (iv) into (iii) to get: $y - p_1 q_1 - p_2\left(\frac{q_1 p_1}{p_2}\right) = 0$

$\Rightarrow y - 2p_1 q_1 = 0$ or $\boxed{p_1 q_1 = y/2}$ Exp. on good 1 equals half of income!

Similarly, subbing (iv) into (iii) for x_1 yields:

$y - 2p_2 q_2 = 0$ or $\boxed{p_2 q_2 = y/2}$ Exp. on good 2 equals half of income!

THUS, $\boxed{q_1^* = \frac{y}{2p_1}, \quad q_2^* = \frac{y}{2p_2}}$

Indirect Utility Function

Now substitute q_1^* and q_2^* into the objective function $u(q_1, q_2)$ to get:

$$u\left(\frac{Y}{2p_1}, \frac{Y}{2p_2}\right) = \left(\frac{Y}{2p_1}\right)^{\frac{1}{2}} \left(\frac{Y}{2p_2}\right)^{\frac{1}{2}} = V$$

* V is not directly a function of individual demand but indirectly a function of prices and income.

REMARK:

Why bother calculating V?

Answer: It saves time for some purposes!

* If we want to calculate utility to compare or rank one set of prices and income with another set, we don't need to set up the Lagrange for each problem.

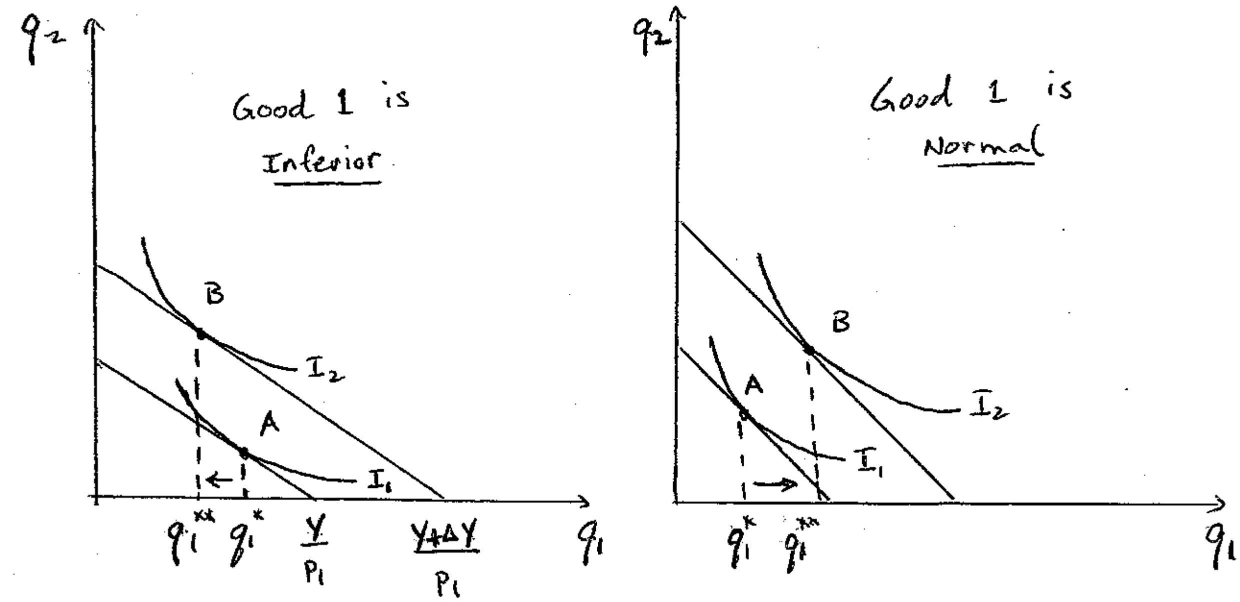
↳ given preferences $u(q_1, q_2)$, we solve individual demand functions using the general budget constraint to get q_1^* and q_2^*.

↳ Plug these into u to get $V(p_1, p_2, Y)$.

↳ We can then rank any set of prices and income!

Income and Substitution Effects

A change in consumer's budget (Y) involves a parallel shift in budget set. Since the shift preserves the market tradeoff (p_1/p_2) it <u>typically</u> has no affect on consumer's MRS (u_1/u_2).



Note: An increase in income from Y to Y+ΔY preserves the price ratio (p_1/p_2) in both cases.

The resulting equil'm in both cases still requires $MRS = u_1/u_2 = p_1/p_2$.

Thus, if p_1/p_2 hasn't changed then the MRS hasn't changed regardless of whether good 1 is normal/Inferior.

Exception

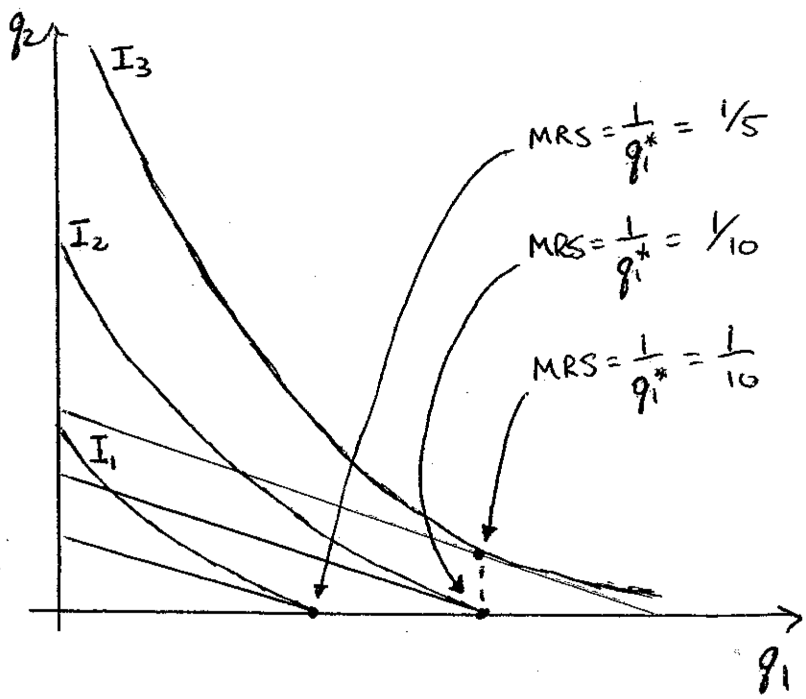
A change in income (ΔY) will affect the consumer's MRS at the optimal quantities when the chosen bundle is either initially or ultimately a corner solution.

EX $\quad u(q_1,q_2) = \ln q_1 + q_2 \quad \Rightarrow \quad \boxed{MRS(q_1,q_2) = \frac{u_1}{u_2} = \frac{1}{q_1}}$

2 solutions here:

$$\left.\begin{aligned} q_1^* &= p_2/p_1 \\ q_2^* &= \frac{Y-p_2}{p_2} \end{aligned}\right\} \quad \text{if } Y > p_2$$

$$\left.\begin{aligned} q_1^* &= Y/p_1 \\ q_2^* &= 0 \end{aligned}\right\} \quad \text{if } Y \leq p_2$$



Suppose $p_1 = \$1,\ p_2 = \$10$

(i) if $Y = \$5$

$\Rightarrow (q_1^*, q_2^*) = (5,0)$ since $Y \leq p_2$

(ii) if $Y = \$10$

$\Rightarrow (q_1^*, q_2^*) = (10,0)$ since $Y \leq p_2$

(iii) if $Y = \$20$

$\Rightarrow (q_1^*, q_2^*) = (10,1)$ since $Y > p_2$

A change in the price of one good holding constant income and all other prices has more complex effects!

(1) It alters the consumer's <u>real income</u> by changing the combination of goods that are affordable. This is the <u>income effect</u>.

(2) It alters the <u>relative price</u> facing consumers so the consumer faces a different market trade-off. This is the <u>substitution effect</u>.

NOTE: We <u>cannot</u> <u>observe</u> these effects <u>separately</u> b/c both occur simultaneously. But they are conceptually distinct and have potentially different implications for behavior.

We isolate effects on the demand for good 1 as follows:

(1) Δq_1 due to change in real income holding relative prices constant (IE)

(2) Δq_1 due to change in relative prices holding real income constant (SE).

Income Effect (IE)

Suppose money income (Y) falls. what's the impact?

(1) Total consumption [Falls]

(2) utility [Falls]

(3) qty of q_1 [Depends on Normal/Inferior]

(4) qty of q_2 [Depends on Normal/Inferior]

$\frac{dq_i}{dY} \gtreqless 0$ for $i = 1, 2$.

* In terms of (3) & (4), our prediction is ambiguous!

Substitution Effect (SE)

Suppose relative price of good 1 increases (i.e. $p_1/p_2 \uparrow$). but utility is held constant. what's the impact?

(1) after $p_1 \uparrow$, we have to $\uparrow Y$ to "cancel out" the decline in real income and return consumer to original level of utility. By $\uparrow Y$ we remove the IE.

(2) We then ask: What's the change in q_1 with higher prices for good 1 but with real income constant.

↳ Formally, what is $dq_1/dp_1 \big|_{u=u_0}$?

(3) Provided axiom of diminishing MRS applies $\Rightarrow \frac{dq_1}{dp_1}\Big|_{u=u_0} < 0$.
SE is always negative!

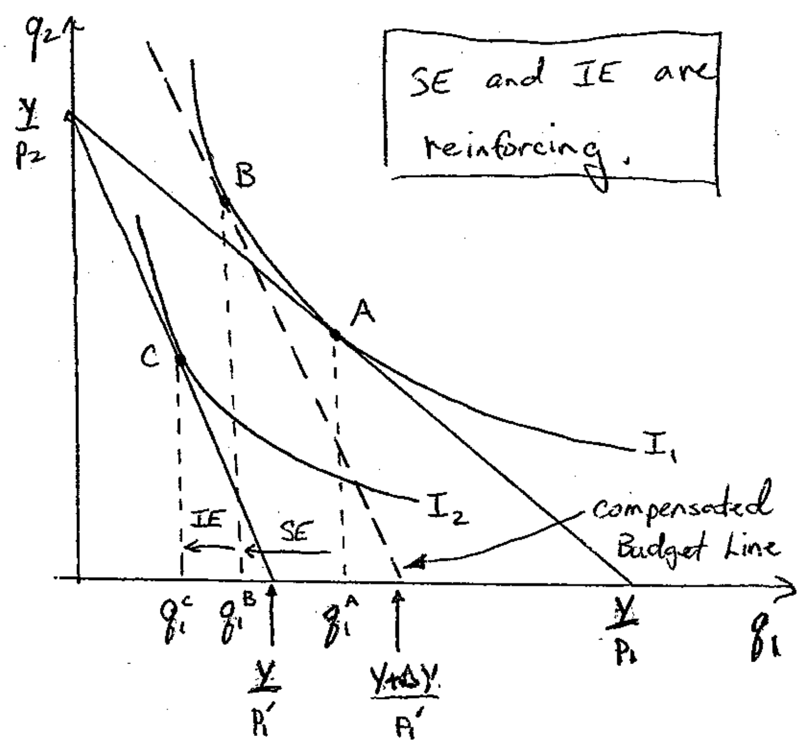
Types of Goods

The fact that sign of SE is negative ($dq_i/dp_i|_{u=u_0} < 0$) and sign of IE is ambiguous ($dq_i/dy \gtreqless 0$) gives rise to 3 types of goods.

(1) NORMAL GOODS

$$\frac{dq_1}{dy} > 0$$

$$\left.\frac{dq_1}{dp_1}\right|_{u=u_0} < 0$$



* Δy is the compensating variation in money income that makes consumer just as well off after $p_1 \uparrow$ to p_1'.

(i) $(q_1^B - q_1^A)$ is the SE (i.e. Δq_1 due solely to Δ in relative prices with real income (utility) constant)

(ii) $(q_1^C - q_1^B)$ is the IE (i.e. Δq_1 due solely to Δ in money income with relative prices constant)

(iii) $(q_1^B - q_1^A) + (q_1^c - q_1^B) = (q_1^c - q_1^A) \rightarrow$ Total Price Effect

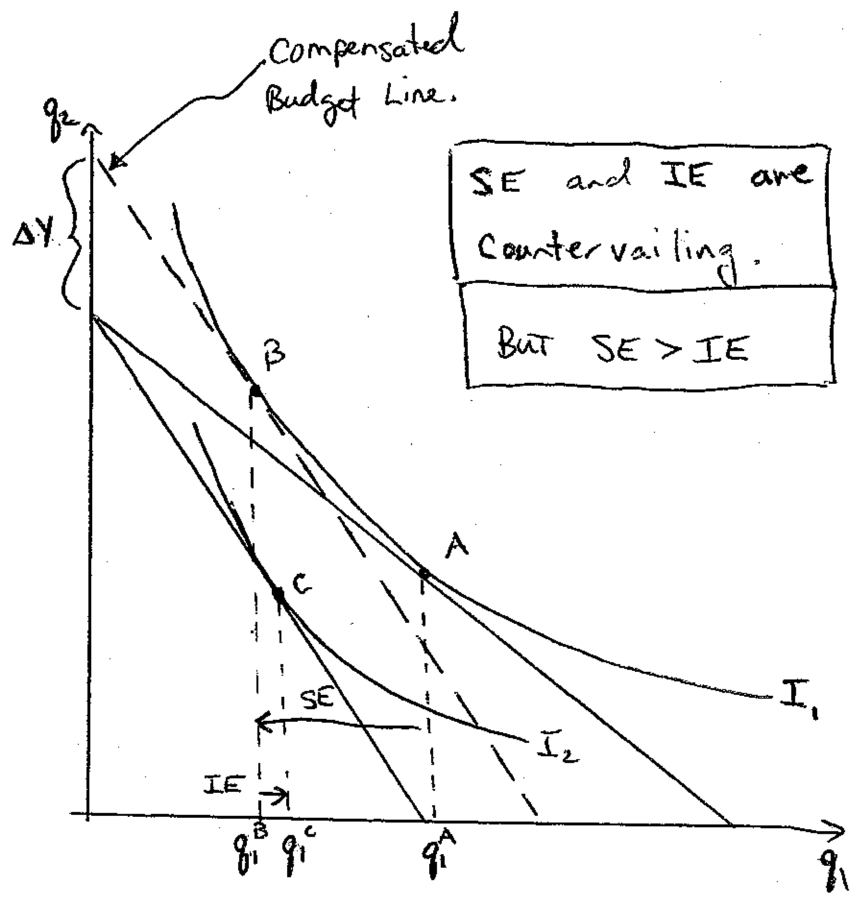
** REMEMBER: We can only ever observe the total price effect, not the isolated effects!

(2) Inferior Goods

$$\frac{dq_1}{dy} < 0$$

$$\left.\frac{dq_1}{dp_1}\right|_{u=u_0} < 0$$

* ΔY is compensating variation to make consumer "whole" after $P_1 \uparrow$ to new price.



(i) $(q_1^B - q_1^A)$ is the SE

(ii) $(q_1^C - q_1^B)$ is the IE

(iii) $(q_1^B - q_1^A) + (q_1^C - q_1^B) = (q_1^C - q_1^A)$ Total price effect.

NOTICE: $\underbrace{(q_1^C - q_1^A)}_{\text{Total Effect}} < \underbrace{(q_1^B - q_1^A)}_{\text{SE}}$

REASON: $\uparrow P_1$ reduces real income, thereby increasing consumption through the IE for Inferior Goods.

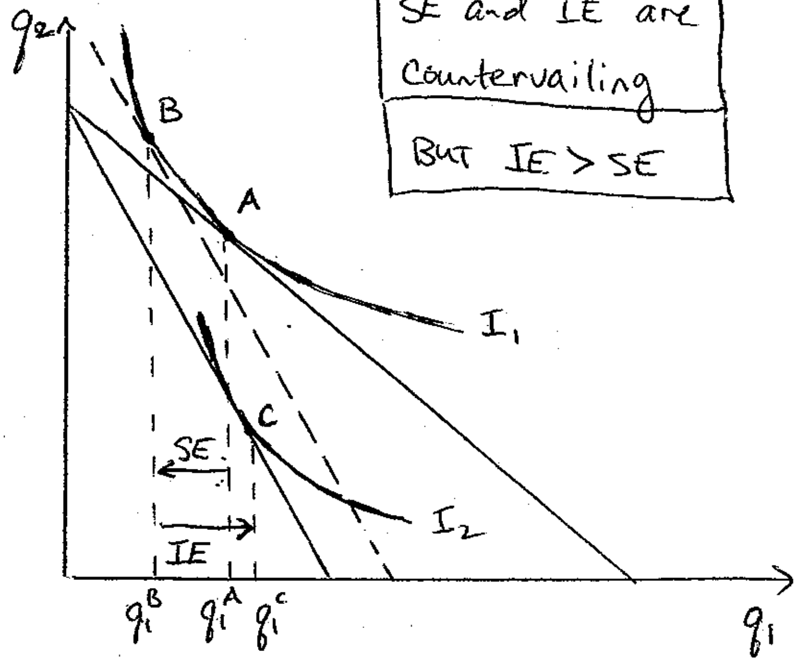
(3) Giffen Goods (Strongly Inferior)

$$\frac{dq_1}{dY} < 0$$

$$\left.\frac{dq_1}{dp_1}\right|_{u=u_0} < 0$$

SE and IE are Countervailing

BUT IE > SE



when $p_1 \uparrow$ the total price effect $(q_1^C - q_1^A) > 0$.
In other words, $p_1 \uparrow \Rightarrow q_1 \uparrow$ (demand curve for good 1 is upward sloping)

What is going on here?

Even though $p_1 \uparrow$ reduces qty demanded holding real income (utility) constant via the SE, the consumer is effectively so much poorer due to the loss in real income that the qty demanded for the strongly inferior good rises.

REMARK: Candidates for Giffen Goods must be goods which take up a relatively large share of the consumer's budget.

Duality

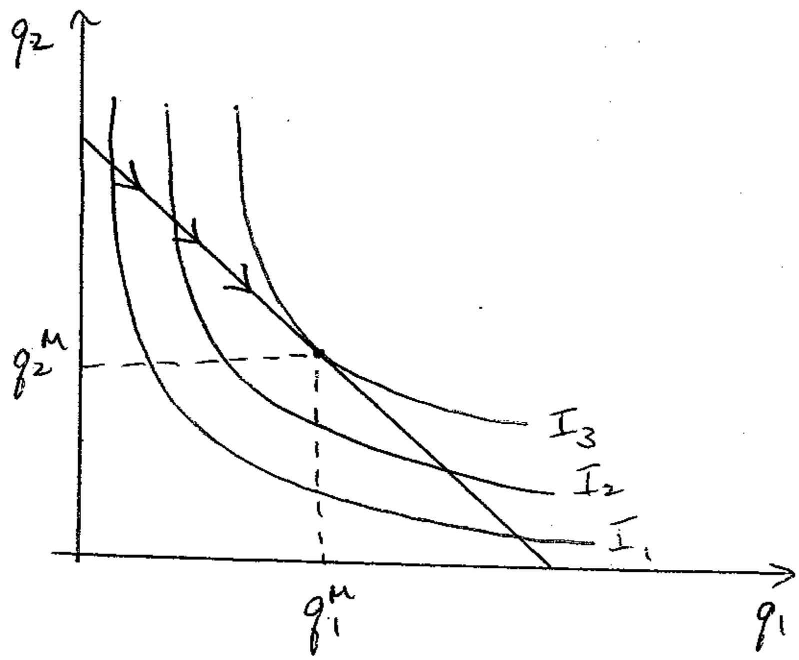
Primal Problem:

$$\max_{\{q_1,q_2\}} u(q_1,q_2) \quad \text{s.t.} \quad p_1 q_1 + p_2 q_2 \le y$$

Solutions

$$\left.\begin{aligned} q_1^M &= d_1(p_1, p_2, y) \\ q_2^M &= d_2(p_1, p_2, y) \end{aligned}\right\} \text{Ordinary or Marshallian Demand Functions}$$

Take p_1, p_2 and y as given. Solution is found by moving along the budget line to find (q_1, q_2) that yields maximal utility.



Maximal utility is given by:

$$u(q_1^*, q_2^*) = v^*$$

where v^* is the Indirect Utility Function

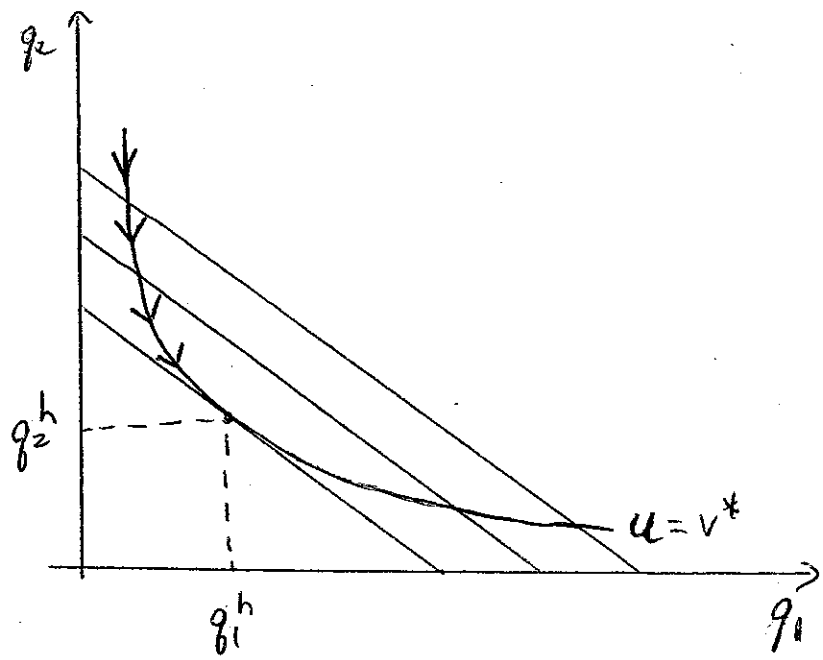
Dual Problem:

$$\underset{\{q_1,q_2\}}{\text{MIN}} \quad E = p_1 q_1 + p_2 q_2 \quad \text{s.t.} \quad u(q_1,q_2) \geq V^*$$

Solution:

$$\left.\begin{aligned} q_1^h &= h_1(p_1,p_2,u) \\ q_2^h &= h_2(p_1,p_2,u) \end{aligned}\right\} \quad \text{Compensated or Hicksian Demand Functions.}$$

Take p_1, p_2 and $u = V^*$ as given. Solution is found by moving along a given indifference curve (utility level) to find (q_1, q_2) that minimizes expenditures.



Minimized Expenditures $E^* = p_1 q_1^h + p_2 q_2^h$ is given by:

$$E(p_1, p_2, V^*) = E^*$$

where E^* is the Expenditure Function.

EX

$$\underset{\{q_1,q_2\}}{\text{MIN}}\ E = p_1 q_1 + p_2 q_2 \quad \text{s.t.} \quad u = q_1^{\frac{1}{2}} q_2^{\frac{1}{2}} \geq u_p$$

where u_p comes from the primal problem ($u_p = V^*$).

Setup $\quad L(q_1, q_2, \lambda) = p_1 q_1 + p_2 q_2 + \lambda\left(u_p - q_1^{\frac{1}{2}} q_2^{\frac{1}{2}}\right)$

$$\left.\begin{array}{ll} \text{(i)} & L_{q_1} = p_1 - \lambda \frac{1}{2} q_1^{-\frac{1}{2}} q_2^{\frac{1}{2}} = 0 \\ \text{(ii)} & L_{q_2} = p_2 - \lambda \frac{1}{2} q_1^{\frac{1}{2}} q_2^{-\frac{1}{2}} = 0 \\ \text{(iii)} & L_\lambda = u_p - q_1^{\frac{1}{2}} q_2^{\frac{1}{2}} = 0 \end{array}\right\} \quad \text{FOC's.}$$

(i) and (ii) yield (iv) $\boxed{q_1 = \dfrac{p_2 q_2}{p_1}}$

Substitute (iv) into the constraint (iii) to get:

$$u_p = \left(\frac{p_2 q_2}{p_1}\right)^{\frac{1}{2}} q_2^{\frac{1}{2}}$$

$\Rightarrow \quad q_2^* = \left(\dfrac{p_1}{p_2}\right)^{\frac{1}{2}} u_p \quad$ (sub q_2^* into (iv) to get q_1^*).

Solutions $\quad \boxed{q_1^h = \left(\dfrac{p_2}{p_1}\right)^{\frac{1}{2}} u_p, \quad q_2^h = \left(\dfrac{p_1}{p_2}\right)^{\frac{1}{2}} u_p.}$

$$E^* = p_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{2}} u_p + p_2\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}} u_p = 2 p_1^{\frac{1}{2}} p_2^{\frac{1}{2}} u_p$$

Expenditure Function: What's it good for?

Essential tool for making consumer theory operational for public policy analysis!

↳ Using $E(p_1, p_2, u)$, we can "monetize" tradeoffs to evaluate costs and benefits.

↳ Why the Exp. Function? We don't know what 'utils' are! This is a problem if we want to determine how much harm or benefit a certain policy imposes on people.

* Though we don't know how to measure utility, we do know money increases utility. (through Indirect Utility Function V by relaxing the constraint).

↳ Using the Exp. Function, we can figure out how much money to give or take away to leave a consumer equally well off after a policy is implemented.

* So, Exp. Function permits a "money metric" calculation.

EX

Consider a policy of banning SUV's b/c:

(1) they cause a disproportionate share of air pollution;

(2) they increase oil dependence from foreign sources.

Question: How much harm does this policy do to potential buyers of SUV's?

NOTE: We can't answer this question in "utils"!

* However, we may be able to determine how much money we would need to give these potential buyers to leave them equally well off as before the ban.

↳ This calculation depends on the Expenditure Function.

Suppose consumer utility prior to the ban is $\bar{u}$ and exp are:

$$\boxed{E_{pre} = E(P_{suv}, P_a, \bar{u})}$$

To attain same utility after the ban, would-be buyers would need:

$$\boxed{E_{post} = E(P_{suv} = \infty, P_a, \bar{u})}$$

The difference $E_{post} - E_{pre}$ is the amount of \$ we need to compensate buyers to leave utility unaffected by the ban.

NOTE: We don't always know the Expenditure Function. But if we have an estimate of Compensated Elasticity of Demand, its enough for rough guess.

Relationship Btw Exp. Function and Indirect Utility Function

Question: How do solutions to Primal and Dual problems compare?

Indirect Utility Function: $V(p_1, p_2, y_0) = U_0$

Expenditure Function: $E(p_1, p_2, U_0) = y_0$

* Now substitute either y_0 into V or U_0 into E to get:

$$V(p_1, p_2, E(p_1, p_2, U_0)) = U_0$$

and

$$E(p_1, p_2, V(p_1, p_2, y_0)) = y_0$$

* Expenditure Function and Indirect Utility Function are INVERSES of one another.

Let's verify for example given above: $U(q_1, q_2) = q_1^{\frac{1}{2}} q_2^{\frac{1}{2}}$.

Recall (1) primal solution gave Marshallian demand q_1^M, q_2^M as function of prices and income (not utility).

(2) dual solution gave Hicksian demand q_1^H, q_2^H as function of prices and utility (not income).

For $U(q_1, q_2) = q_1^{\frac{1}{2}} q_2^{\frac{1}{2}}$, solutions are:

$$q_1^M = \frac{Y}{2p_1} \qquad q_2^M = \frac{Y}{2p_2}$$

and

$$q_1^H = \left(\frac{p_2}{p_1}\right)^{\frac{1}{2}} U_p \qquad q_2^H = \left(\frac{p_1}{p_2}\right)^{\frac{1}{2}} U_p$$

From Primal Problem:

$$u = (q_1^M)^{\frac{1}{2}} (q_2^M)^{\frac{1}{2}} = \left(\frac{Y}{2p_1}\right)^{\frac{1}{2}} \left(\frac{Y}{2p_2}\right)^{\frac{1}{2}} = V^*$$

From Dual Problem:

$$E = p_1 q_1^h + p_2 q_2^h = 2 p_1^{\frac{1}{2}} p_2^{\frac{1}{2}} V^* = E^*$$

Now plug V^* into E^*:

$$\boxed{E^* = 2 p_1^{\frac{1}{2}} p_2^{\frac{1}{2}} \left(\frac{Y}{2p_1}\right)^{\frac{1}{2}} \left(\frac{Y}{2p_2}\right)^{\frac{1}{2}} = Y}$$

** Interpretation: The min. expenditure to reach utility $u = V^*$ is Y.

Now plug $E^* = Y$ into u:

$$\boxed{u = \left(\frac{2 p_1^{\frac{1}{2}} p_2^{\frac{1}{2}} V^*}{2p_1}\right)^{\frac{1}{2}} \left(\frac{2 p_1^{\frac{1}{2}} p_2^{\frac{1}{2}} V^*}{2p_2}\right)^{\frac{1}{2}} = V^*}$$

** Interpretation: The MAX utility from expenditure $E^* = Y$ is V^*

NOTE: The Multiplier in the Dual is the same as the multiplier in the primal.

Primal: $\lambda_P = U_1/p_1 = U_2/p_2$

Dual: $\lambda_D = p_1/U_1 = p_2/U_2$

} Interpretation is the inverse of one another.

DEMAND CURVES (AGAIN)

"Marshallian Cross" is staple tool of blackboard economics. They are the conventional individual or market demand curves in principles texts.

They answer the Question:

Holding Y and p_j constant, how does the quantity demanded of q_i change with p_i?

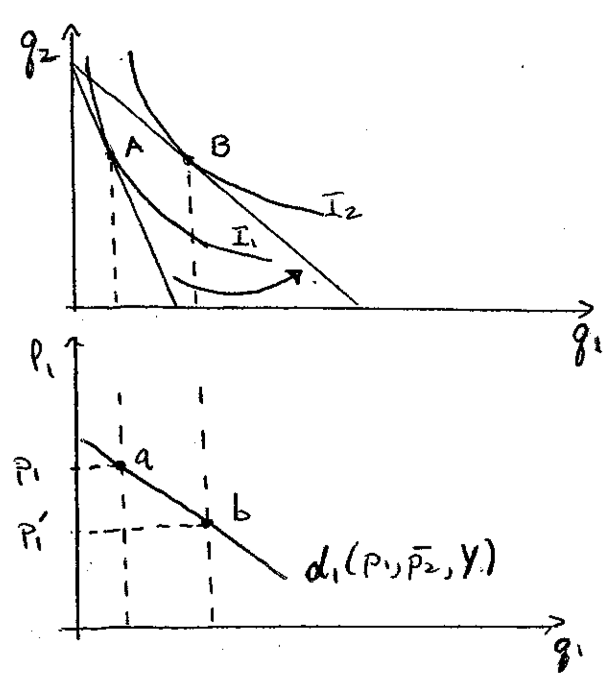
We notate this demand function as:

$$d_i(p_i, p_j, Y_0) \quad \text{for } i \neq j$$

This is our solutions to primal problem (q_1^M, q_2^M).

* Ordinary or Marshallian demand curves implicitly combine IE and SE. They are "net" demands that sum over these 2 conceptually distinct behavioral responses.

- Fix p_2 and Y.
- Lower p_1 to p_1'.
- optimal bundles: $A \to B$
  - ↳ Thus $q_1 \uparrow$ as $p_1 \downarrow$
  - ↳ Normal Good*.
  - ↳ Δq_1 is net change or total price effect!



From the Dual problem, we also have demand curves composed solely of substitution effects. we called these compensated or Hicksian demand curves.

They answer the Question:

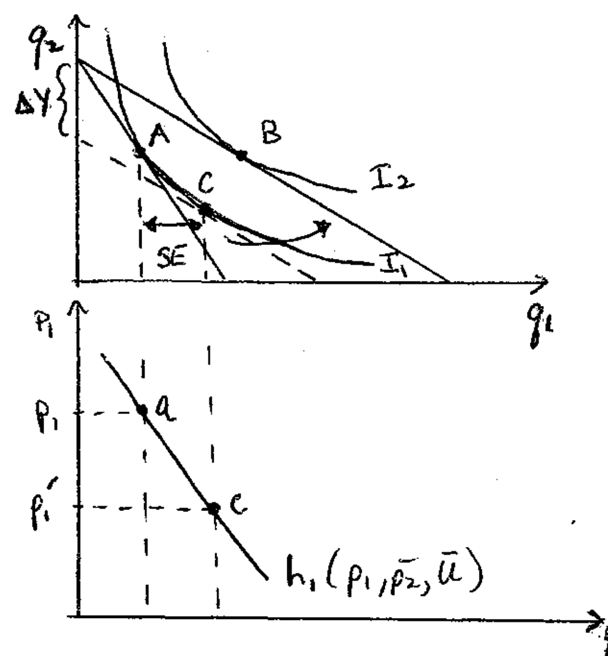
Holding utility u and p_j constant, how does the quantity demanded of q_i change with p_i?

We notate this demand function as:

$$\boxed{h_i(p_i, p_j, u_0) \quad \text{for } i \neq j}$$

This is our solution to dual problem (q_1^h, q_2^h).

The Presence of u as a parameter tells we hold utility constant (i.e. on same indifference curve) as price changes.
Its a compensated demand curve b/c to keep u constant as price changes we need to adjust money income.



- Fix p_2 and u.
- Lower p_1 to p_1', and then take away ΔY to leave consumer indifferent (equally well off).
- optimal Bundles: $A \to C$
  ↳ Δq_1 is an isolated SE.

Relationship Btw $d_i(\cdot)$ and $h_i(\cdot)$

$d_i(p_i, p_j, Y)$ is total price effect
$h_i(p_i, p_j, u)$ is the isolated SE

These demand curves are related, but not identical unless the income effect is zero.

Recall the Exp. Function: $E(p_1, p_2, \bar{u})$

↳ gives the MIN expenditure necessary to obtain utility $\bar{u}$ given prices p_1 and p_2.

For any chosen utility level $\bar{u}$, the following identity holds:

$$h_1(p_1, p_2, \bar{u}) = d_1(p_1, p_2, \underbrace{E(p_1, p_2, \bar{u})}_{Y})$$

** In words, for any chosen level of utility, compensated and "uncompensated" (i.e. Marshallian) demand must equal.

↳ Fix prices at p_1 and p_2. Fix utility $u = \bar{u}$.
Use Exp. Function to determine the income $\bar{Y}$ necessary to attain utility $\bar{u}$ given p_1 and p_2.
It must be the case that $h_1(p_1, p_2, \bar{u}) = d_1(p_1, p_2, \bar{Y})$

↳ By construction, the demand curves cross at chosen point!

Although $h(\cdot)$ and $d(\cdot)$ cross at the initial chosen bundle, they <u>do not respond identically to a price change.</u>

Differentiate the prior identity w.r.t. p_1 to get:

$$\boxed{\frac{\partial h_1}{\partial p_1} = \frac{\partial d_1}{\partial p_1} + \frac{\partial d_1}{\partial y}\frac{\partial E}{\partial p_1}}$$

Rearranging yields:

$$\underbrace{\frac{\partial d_1}{\partial p_1}} = \underbrace{\frac{\partial h_1}{\partial p_1}} - \underbrace{\frac{\partial d_1}{\partial y}\frac{\partial E}{\partial p_1}}$$

- Total Price Effect (i.e. SE + IE)
- Substitution Effect.
- Income Effect $\frac{\partial d_1}{\partial y}$ <u>scaled</u> by the change in expenditures due to price change.

So what exactly is $\frac{\partial E}{\partial p_1}$?

From the Dual Problem: $L = p_1 q_1 + p_2 q_2 + \lambda(\bar{u} - u(q_1, q_2))$

Solution to the Problem is: $h_1(\cdot)$ and $h_2(\cdot)$

Plug into Exp. Function: $E = p_1 h_1 + p_2 h_2$.

$\Rightarrow \boxed{\frac{\partial E}{\partial p_1} = h_1(\cdot)}$ This is called "<u>Shephard's Lemma</u>"

Recall $d_1(\cdot) = h_1(\cdot)$ at initial bundle (q_1, q_2).

Applying Shephard's Lemma $\partial E / \partial p_1 = h_1(\cdot)$ and letting $h_1(\cdot) = q_1$, re-write our expression for the total price effect as:

$$\boxed{\frac{\partial d_1}{\partial p_1} = \frac{\partial h_1}{\partial p_1} - \frac{\partial d_1}{\partial y} q_1}$$

This is called the "<u>SLUTSKY EQUATION</u>".

<u>Intuition</u>: You buy 10 bags of chips/day and price rises by ¢1 cent/bag. How much do we need to compensate you to keep utility constant?

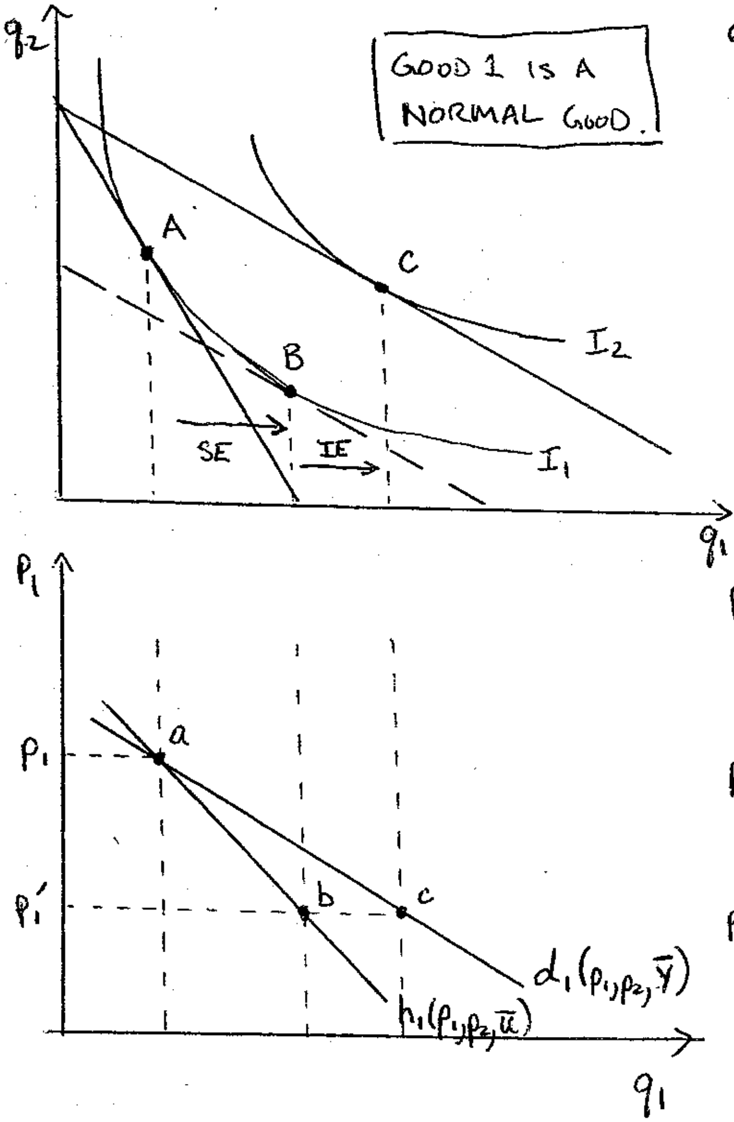
<u>Answer</u>: No more than ¢10 cents. To keep utility constant given the price increase, your exp. must rise by the price change Δp_1 times the initial level of consumption, q_1.

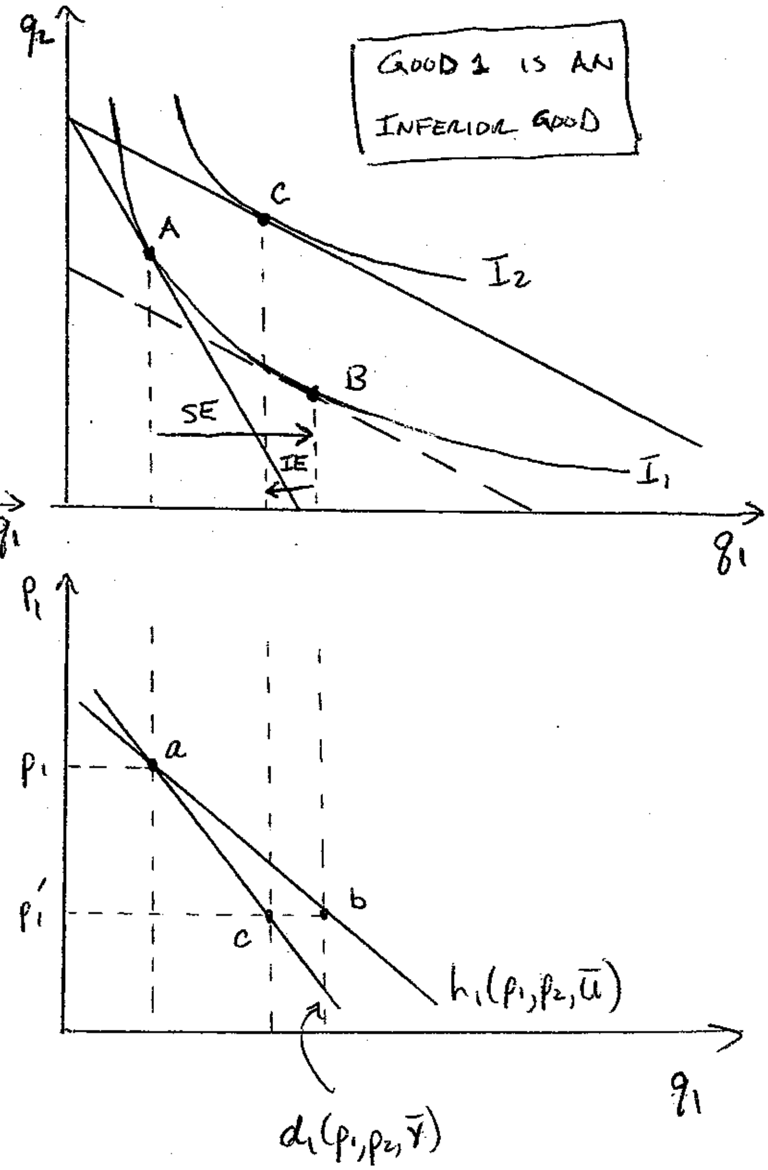
$\Rightarrow$ $\frac{\partial d_1}{\partial y} q_1$ means the IE <u>scaled</u> by the initial level of good 1 consumed.

$\Rightarrow$ If we buy a <u>large</u> qty of q_1 to begin with, an $\uparrow p_1$ has a potentially large IE.

$\Rightarrow$ If initially $q_1 = 0$, then IE $= 0$ and $d_1(\cdot) = h_1(\cdot)$

Slutsky equation gives a precise statement about the conclusions of our graphical analysis.





$$\underbrace{\frac{\partial d_1}{\partial p_1}}_{(-)} = \underbrace{\frac{\partial h_1}{\partial p_1}}_{SE(-)} \underbrace{- \underbrace{\frac{\partial d_1}{\partial Y}}_{IE(+)} q_1}_{(-)}$$

- Law of Demand Holds!
- Slope of Marshallian demand curve is downward sloping

↳ SE and IE are reinforcing

$$\underbrace{\frac{\partial d_1}{\partial p_1}}_{(?)} = \underbrace{\frac{\partial h_1}{\partial p_1}}_{SE(-)} \underbrace{- \underbrace{\frac{\partial d_1}{\partial Y}}_{IE(-)} q_1}_{(+)}$$

Law of Demand holds if $\frac{\partial h_1}{\partial p_1} > \frac{\partial d_1}{\partial Y} q_1$

If so ⇒ Inferior Good (as shown here)

↳ otherwise ⇒ Giffen good $\left(\frac{\partial d}{\partial p_1} > 0\right)$