Cost Minimization

Cost Minimization

- **A firm is a cost-minimizer if it produces any given output level $q \geq 0$ at smallest possible total cost.**
- **c(q) denotes the firm's smallest possible total cost for producing q units of output.**
- **When the firm faces given input prices $w = (w_1, w_2, \ldots, w_n)$ the total cost function will be written as**

$$c(w_1, \ldots, w_n, q).$$

The Cost-Minimization Problem

- Consider a firm using two inputs to make one output.
- The production function is
$$q = f(x_1,x_2).$$
- Take the output level $q \geq 0$ as given.
- Given the input prices w_1 and w_2, the cost of an input bundle (x_1,x_2) is $w_1x_1 + w_2x_2$.

The Cost-Minimization Problem

- For given w_1, w_2 and q, the firm's cost-minimization problem is to solve $\min_{x_1, x_2 \geq 0} w_1 x_1 + w_2 x_2$

  subject to $f(x_1, x_2) = q.$

The Cost-Minimization Problem

- The levels x_1*(w_1,w_2,q) and x_1*(w_1,w_2,q) in the least-costly input bundle are the firm's **conditional demands for inputs 1 and 2**.
- The (smallest possible) total cost for producing q output units is therefore

$$c(w_1, w_2, q) = w_1 x_1^*(w_1, w_2, q)$$

$$+ w_2 x_2^*(w_1, w_2, q).$$

Conditional Input Demands

- Given w_1, w_2 and q, how is the least costly input bundle located?
- And how is the total cost function computed?

Iso-cost Lines

- A curve that contains all of the input bundles that cost the same amount is an iso-cost curve.
- E.g., given w_1 and w_2, the \$100 iso-cost line has the equation

$$w_1x_1 + w_2x_2 = 100.$$

Iso-cost Lines

- Generally, given w_1 and w_2, the equation of the \$c iso-cost line is

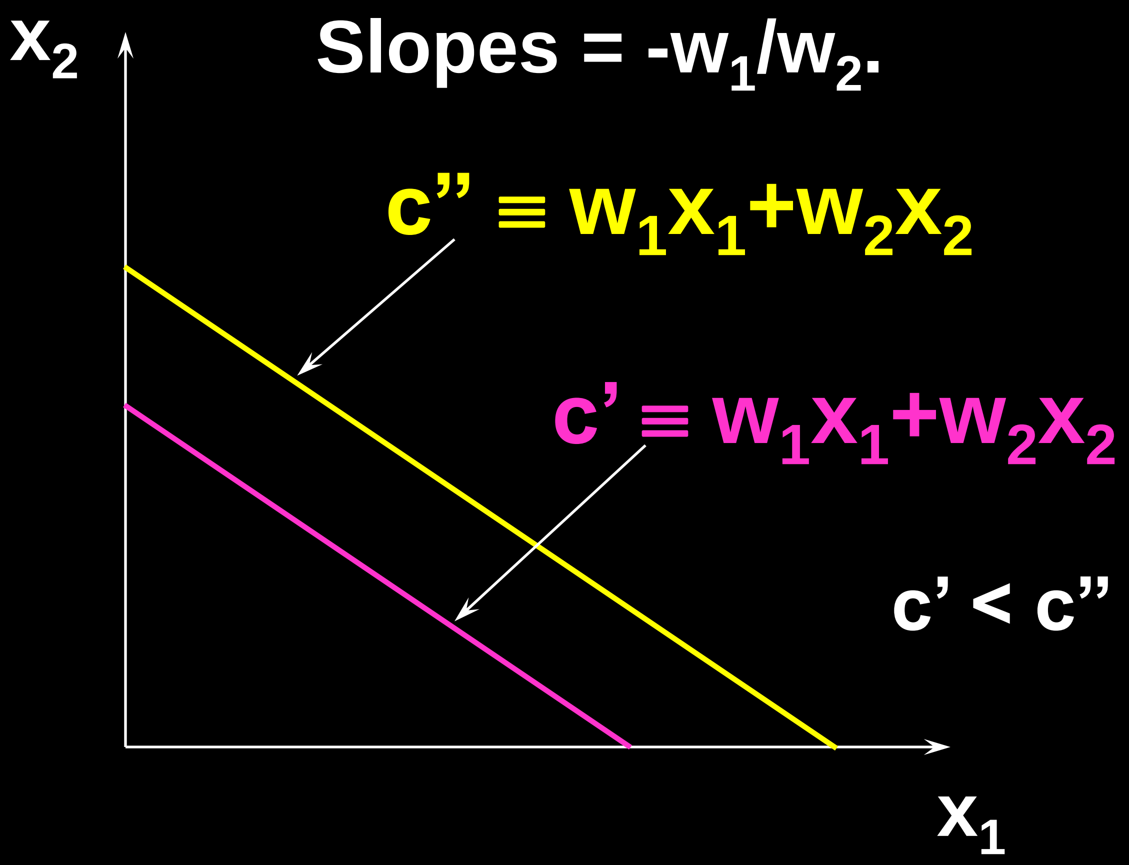
$$w_1 x_1 + w_2 x_2 = c$$

i.e.

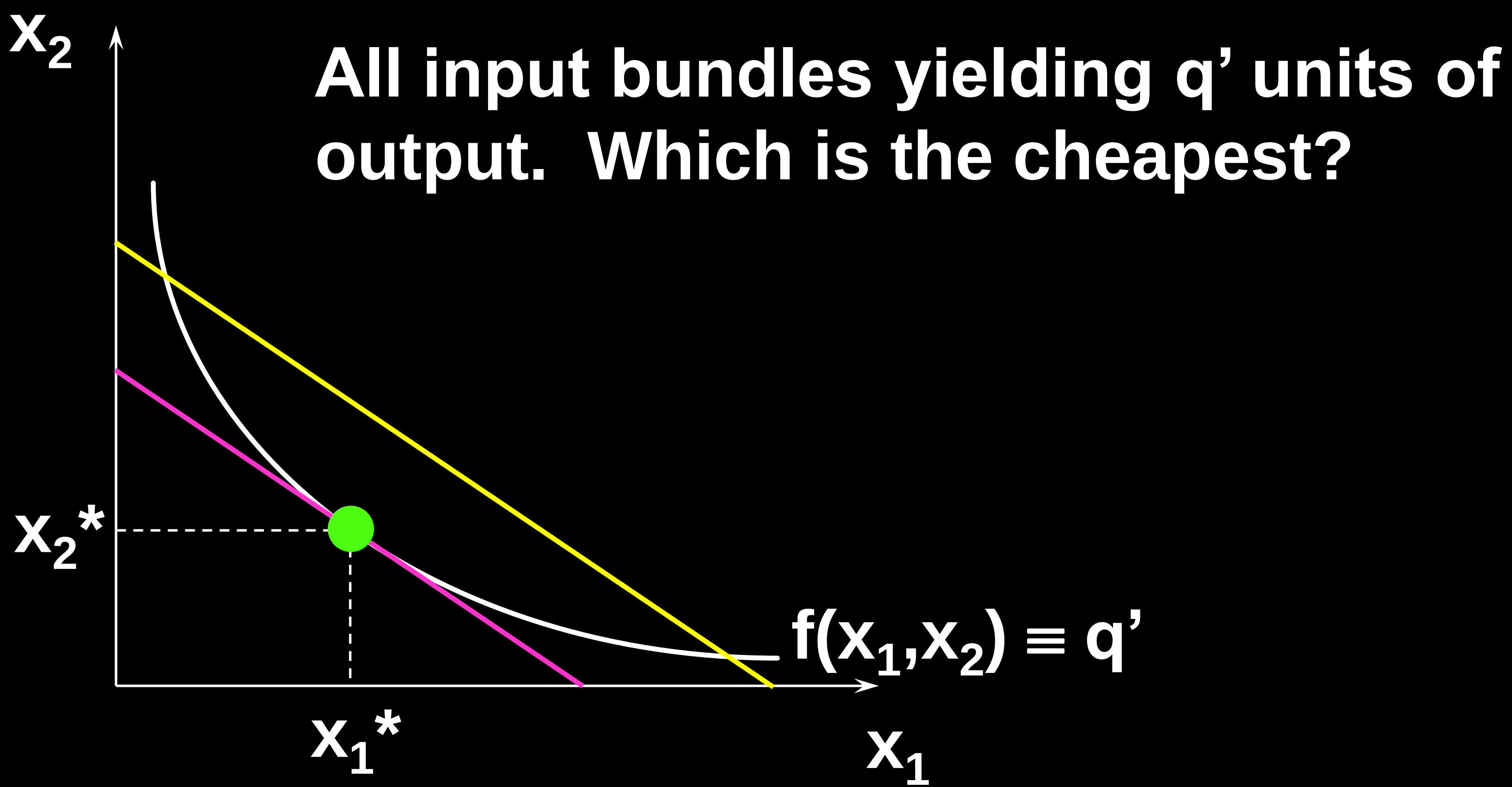
$$x_2 = -\frac{w_1}{w_2} x_1 + \frac{c}{w_2}.$$

- Slope is $- w_1/w_2$.

Iso-cost Lines



The Cost-Minimization Problem



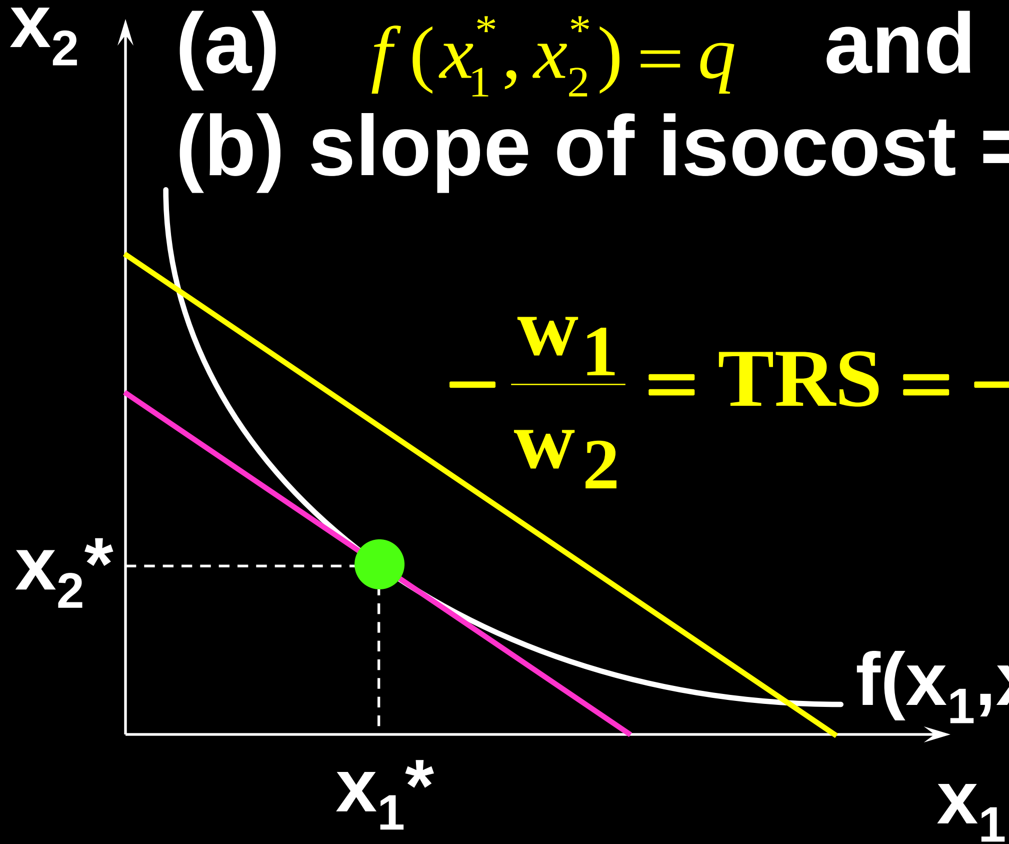
The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = q$ and

(b) slope of isocost = slope of isoquant; i.e.

$$-\frac{w_1}{w_2} = TRS = -\frac{MP_1}{MP_2} \text{ at } (x_1^*, x_2^*).$$



A Cobb-Douglas Example of Cost Minimization

- **A firm's Cobb-Douglas production function is**

$$q = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}.$$

- **Input prices are w_1 and w_2.**
- **What are the firm's conditional input demand functions?**

A Cobb-Douglas Example of Cost Minimization

At the input bundle (x_1^*,x_2^*) which minimizes the cost of producing q output units:

(a) $$q = (x_1^*)^{1/3}(x_2^*)^{2/3}$$ **and**

(b)

$$-\frac{w_1}{w_2} = -\frac{\partial q/\partial x_1}{\partial q/\partial x_2} = -\frac{(1/3)(x_1^*)^{-2/3}(x_2^*)^{2/3}}{(2/3)(x_1^*)^{1/3}(x_2^*)^{-1/3}}$$

$$= -\frac{x_2^*}{2x_1^*}.$$

A Cobb-Douglas Example of Cost Minimization

(a) $q = (x_1^*)^{1/3}(x_2^*)^{2/3}$ **(b)** $\frac{w_1}{w_2} = \frac{x_2^*}{2x_1^*}.$

From (b), $x_2^* = \frac{2w_1}{w_2} x_1^*.$

Now substitute into (a) to get

$$q = (x_1^*)^{1/3}\left(\frac{2w_1}{w_2}x_1^*\right)^{2/3} = \left(\frac{2w_1}{w_2}\right)^{2/3} x_1^*.$$

So $x_1^* = \left(\frac{w_2}{2w_1}\right)^{2/3} q$ **is the firm's conditional demand for input 1.**

A Cobb-Douglas Example of Cost Minimization

Since $\mathbf{x_2^* = \frac{2w_1}{w_2}x_1^*}$ **and** $x_1^* = \left(\frac{w_2}{2w_1}\right)^{2/3} q$

$$x_2^* = \frac{2w_1}{w_2}\left(\frac{w_2}{2w_1}\right)^{2/3} q = \left(\frac{2w_1}{w_2}\right)^{1/3} q$$

is the firm's conditional demand for input 2.

A Cobb-Douglas Example of Cost Minimization

For the production function

$$q = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$$

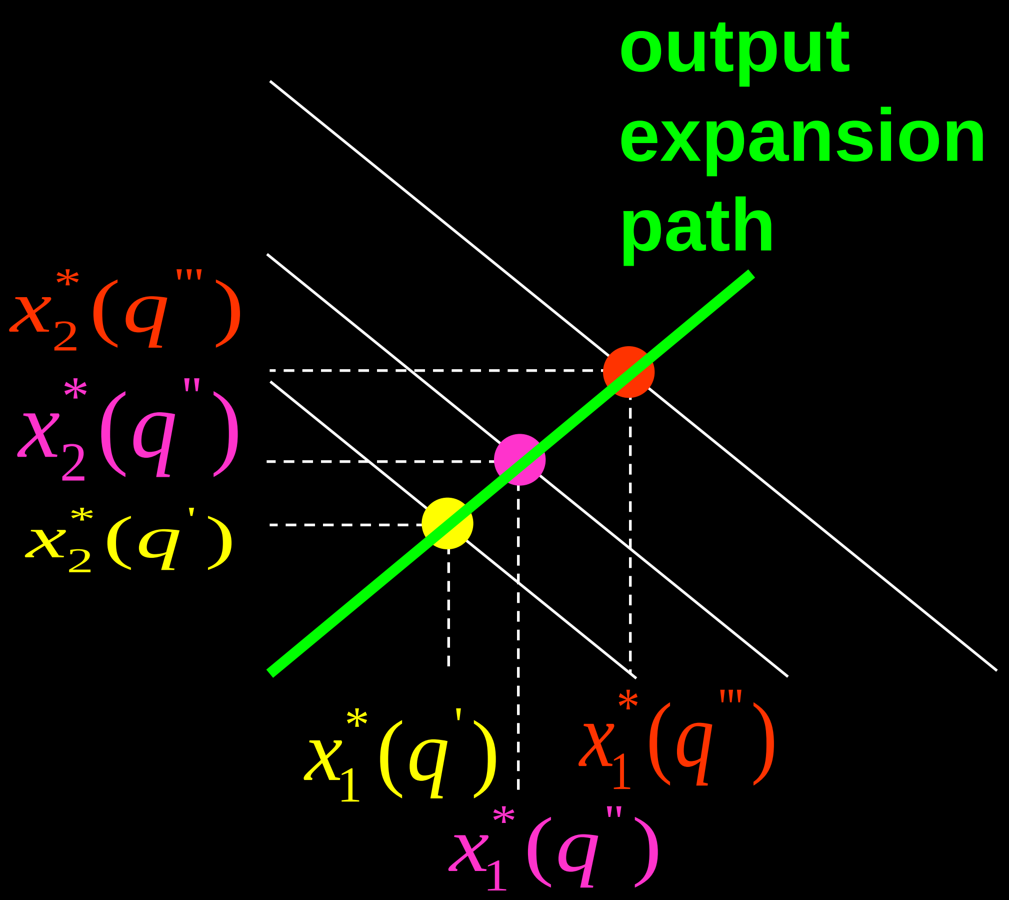
the cheapest input bundle yielding q output units is

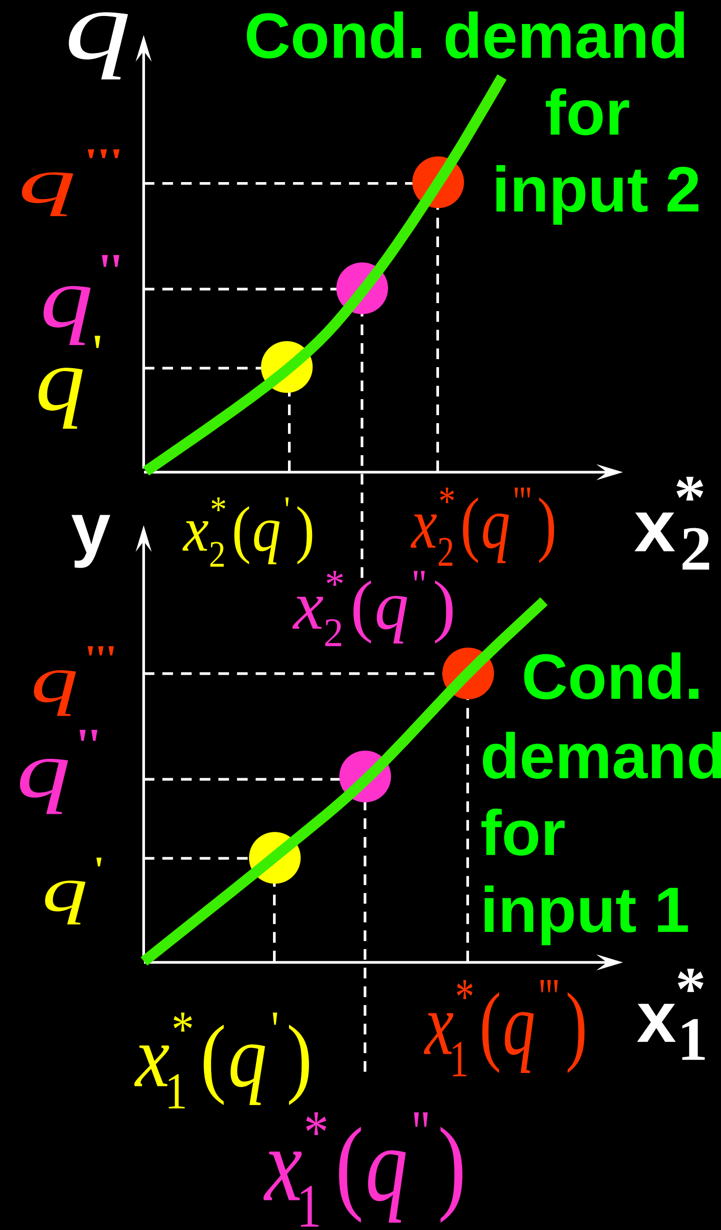
$$\left(x_1^*(w_1, w_2, q), x_2^*(w_1, w_2, q)\right)$$

$$= \left(\left(\frac{w_2}{2w_1}\right)^{2/3} q, \left(\frac{2w_1}{w_2}\right)^{1/3} q\right).$$

Conditional Input Demand Curves

Fixed w_1 and w_2.





A Cobb-Douglas Example of Cost Minimization

So the firm's total cost function is

$$c(w_1, w_2, q) = w_1 x_1^*(w_1, w_2, q) + w_2 x_2^*(w_1, w_2, q)$$

$$= w_1 \left(\frac{w_2}{2w_1} \right)^{2/3} q + w_2 \left(\frac{2w_1}{w_2} \right)^{1/3} q$$

$$= \left(\frac{1}{2} \right)^{2/3} w_1^{1/3} w_2^{2/3} q + 2^{1/3} w_1^{1/3} w_2^{2/3} q$$

A Fixed Proportion Example of Cost Minimization

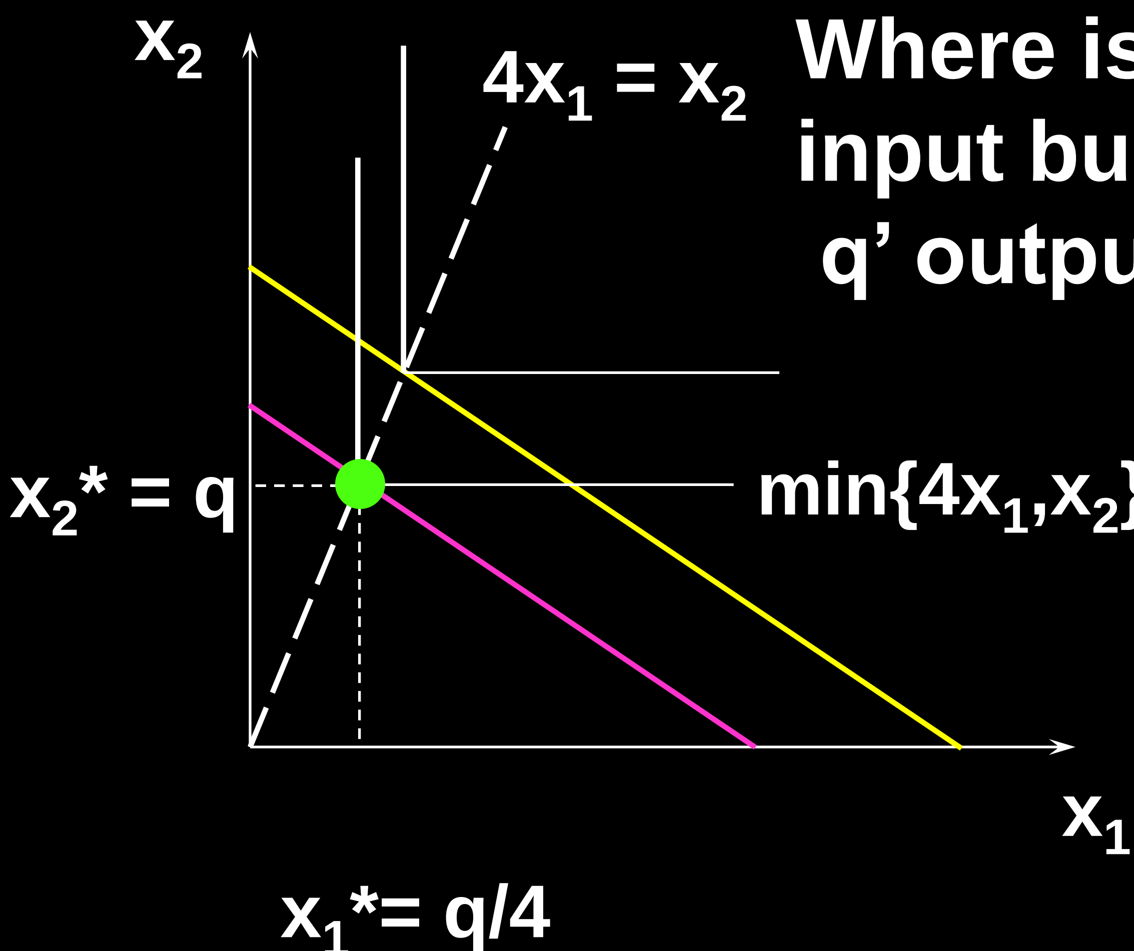
- The firm's production function is

$$q = \min\{4x_1, x_2\}.$$

- Input prices w_1 and w_2 are given.
- What are the firm's conditional demands for inputs 1 and 2?
- What is the firm's total cost function?

A Perfect Complements Example of Cost Minimization

Where is the least costly input bundle yielding q' output units?



A Perfect Complements Example of Cost Minimization

The firm's production function is

$$q = \min\{4x_1, x_2\}$$

and the conditional input demands are

$$x_1^*(w_1, w_2, q) = \frac{q}{4} \quad \text{and} \quad x_2^*(w_1, w_2, q) = q.$$

So the firm's total cost function is

$$c(w_1, w_2, q) = w_1 x_1^*(w_1, w_2, q) + w_2 x_2^*(w_1, w_2, q)$$

$$= w_1 \frac{q}{4} + w_2 q = \left(\frac{w_1}{4} + w_2\right) q.$$

Average Total Production Costs

- **For positive output levels q, a firm's average total cost of producing q units is**

$$AC(w_1, w_2, q) = \frac{c(w_1, w_2, q)}{q}.$$

Returns-to-Scale and Av. Total Costs

- The returns-to-scale properties of a firm's technology determine how average production costs change with output level.
- Our firm is presently producing q' output units.
- How does the firm's average production cost change if it instead produces 2q' units of output?

Constant Returns-to-Scale and Average Total Costs

- **If a firm's technology exhibits constant returns-to-scale then doubling its output level from q' to 2q' requires doubling all input levels.**
- **Total production cost doubles.**
- **Average production cost does not change.**

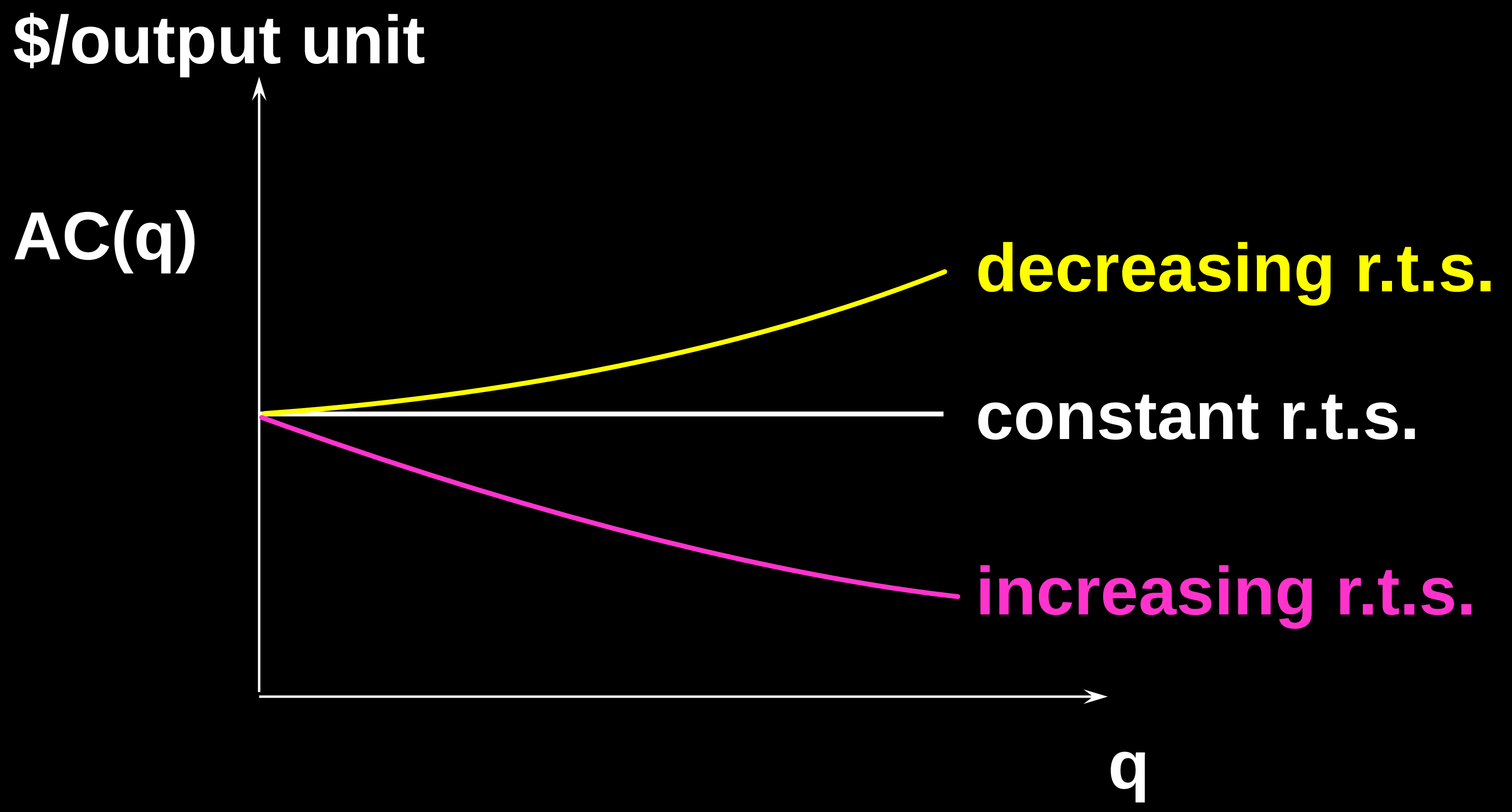
Decreasing Returns-to-Scale and Average Total Costs

- If a firm’s technology exhibits decreasing returns-to-scale then doubling its output level from q’ to 2q’ requires more than doubling all input levels.
- Total production cost more than doubles.
- Average production cost increases.

Increasing Returns-to-Scale and Average Total Costs

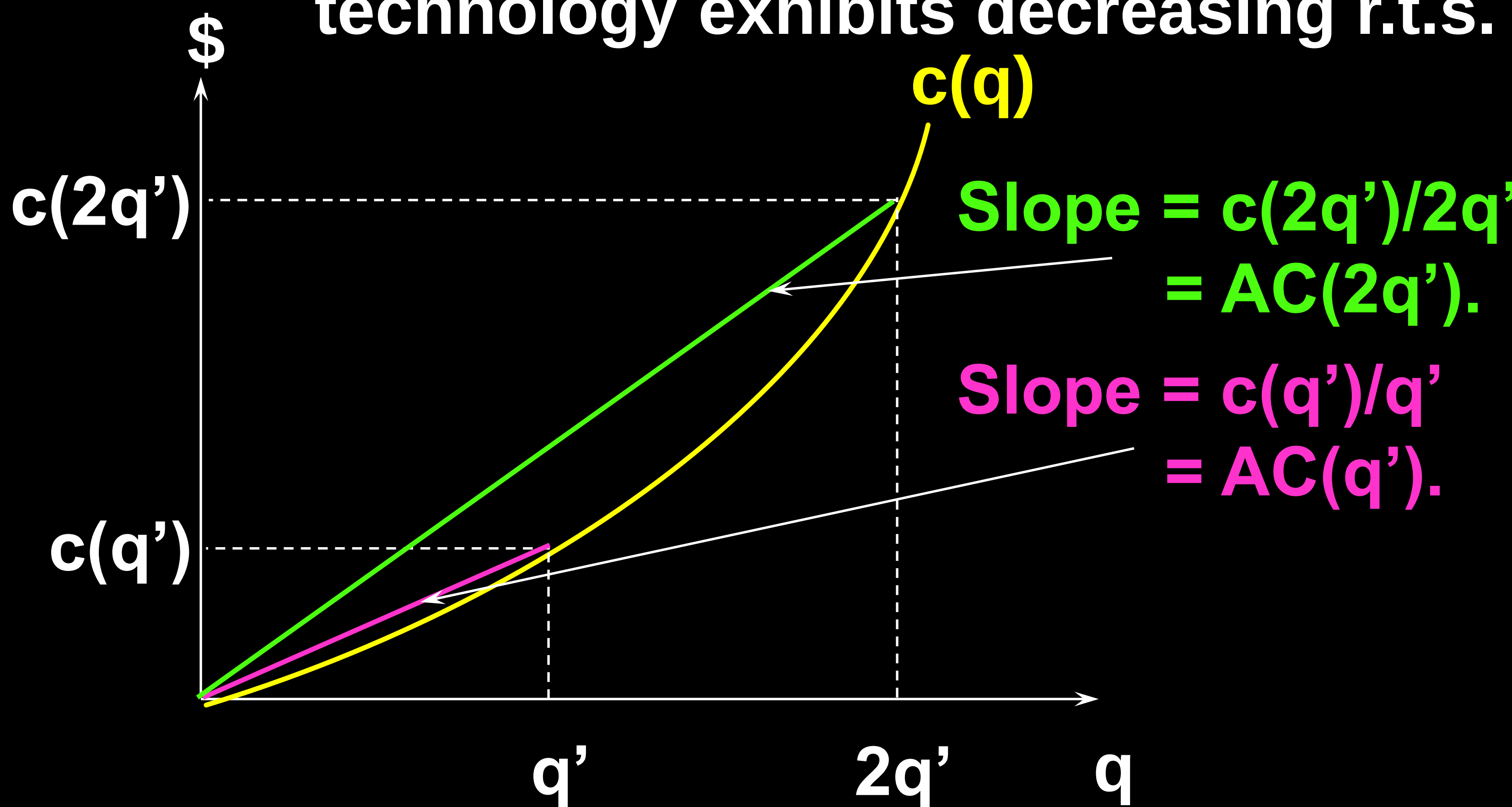
- If a firm's technology exhibits increasing returns-to-scale then doubling its output level from q' to 2q' requires less than doubling all input levels.
- Total production cost less than doubles.
- Average production cost decreases.

Returns-to-Scale and Av. Total Costs



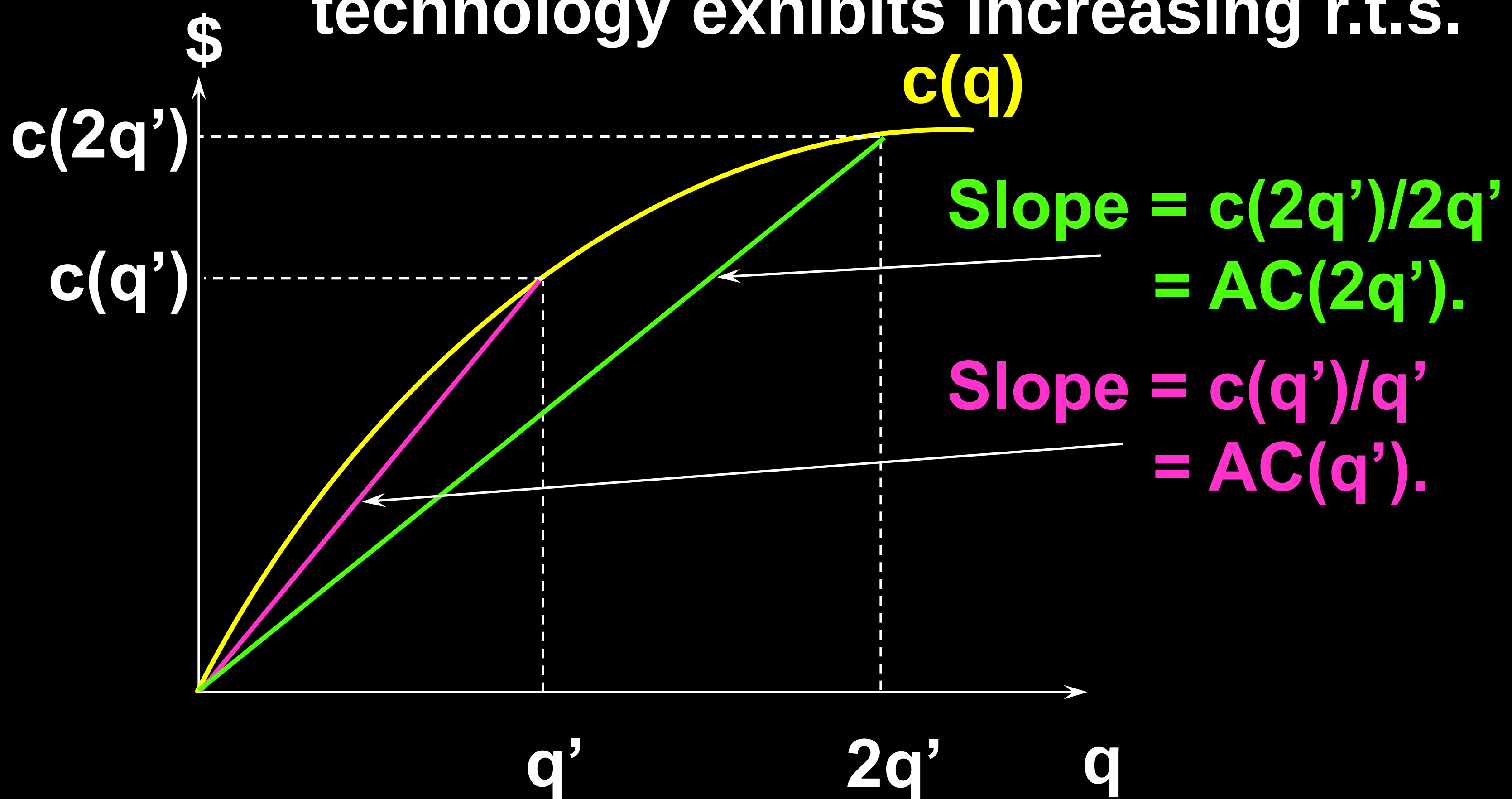
Returns-to-Scale and Total Costs

Av. cost increases with q if the firm's technology exhibits decreasing r.t.s.



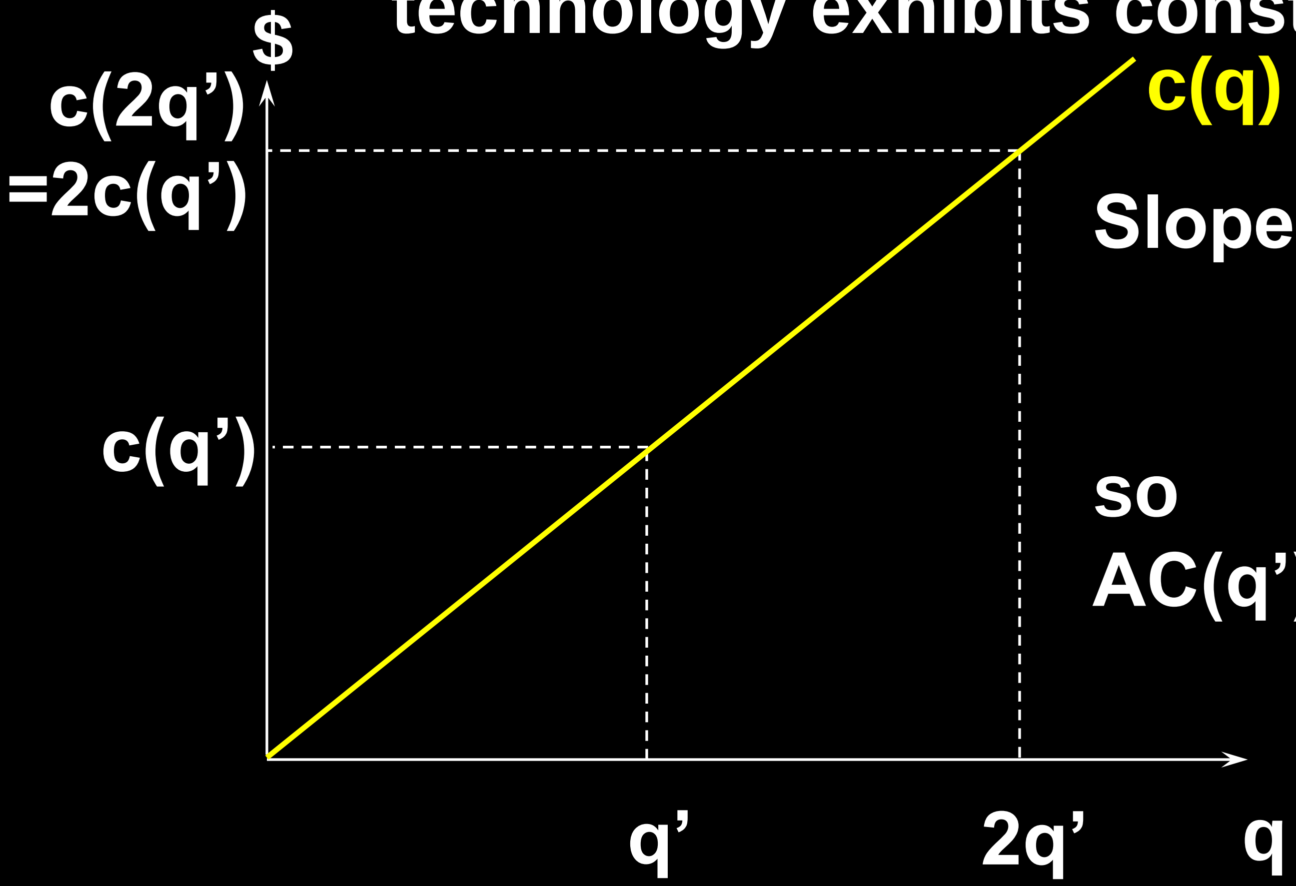
Returns-to-Scale and Total Costs

Av. cost decreases with q if the firm's technology exhibits increasing r.t.s.



Returns-to-Scale and Total Costs

Av. cost is constant when the firm's technology exhibits constant r.t.s.



Slope = c(2q')/2q'
= 2c(q')/2q'
= c(q')/q'
so
AC(q') = AC(2q').

Short-Run & Long-Run Total Costs

- **In the long-run a firm can vary all of its input levels.**
- **Consider a firm that cannot change its input 2 level from x_2' units.**
- **How does the short-run total cost of producing q output units compare to the long-run total cost of producing q units of output?**

Short-Run & Long-Run Total Costs

- The long-run cost-minimization problem is $\min_{x_1, x_2 \geq 0} w_1 x_1 + w_2 x_2$ subject to $f(x_1, x_2) = q.$
- The short-run cost-minimization problem is $\min_{x_1 \geq 0} w_1 x_1 + w_2 x_2'$ subject to $f(x_1, x_2') = q.$

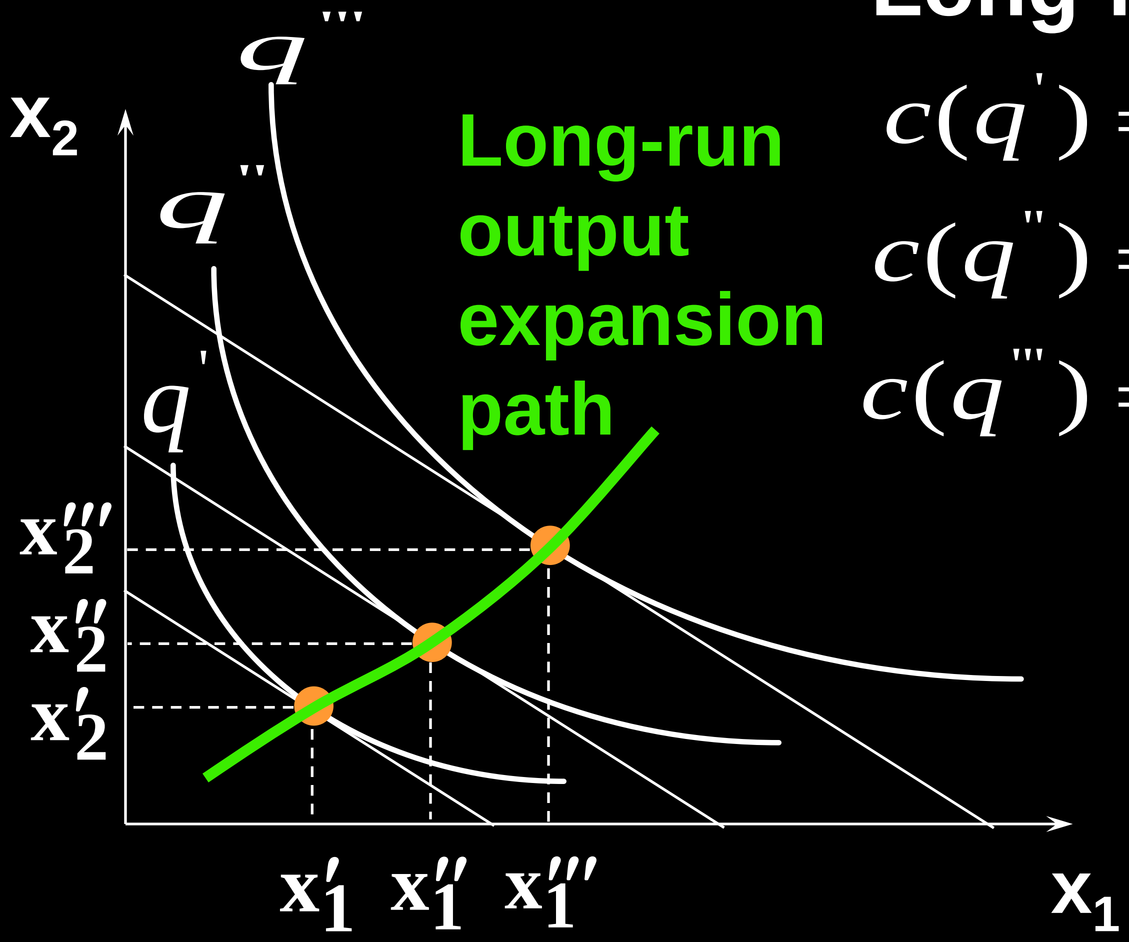
Short-Run & Long-Run Total Costs

- The short-run cost-min. problem is the long-run problem subject to the extra constraint that $x_2 = x_2'$.
- If the long-run choice for x_2 was x_2' then the extra constraint $x_2 = x_2'$ is not really a constraint at all and so the long-run and short-run total costs of producing q output units are the same.

Short-Run & Long-Run Total Costs

- The short-run cost-min. problem is therefore the long-run problem subject to the extra constraint that $x_2 = x_2''$.
- But, if the long-run choice for $x_2 \neq x_2''$ then the extra constraint $x_2 = x_2''$ prevents the firm in this short-run from achieving its long-run production cost, causing the short-run total cost to exceed the long-run total cost of producing q output units.

Short-Run & Long-Run Total Costs



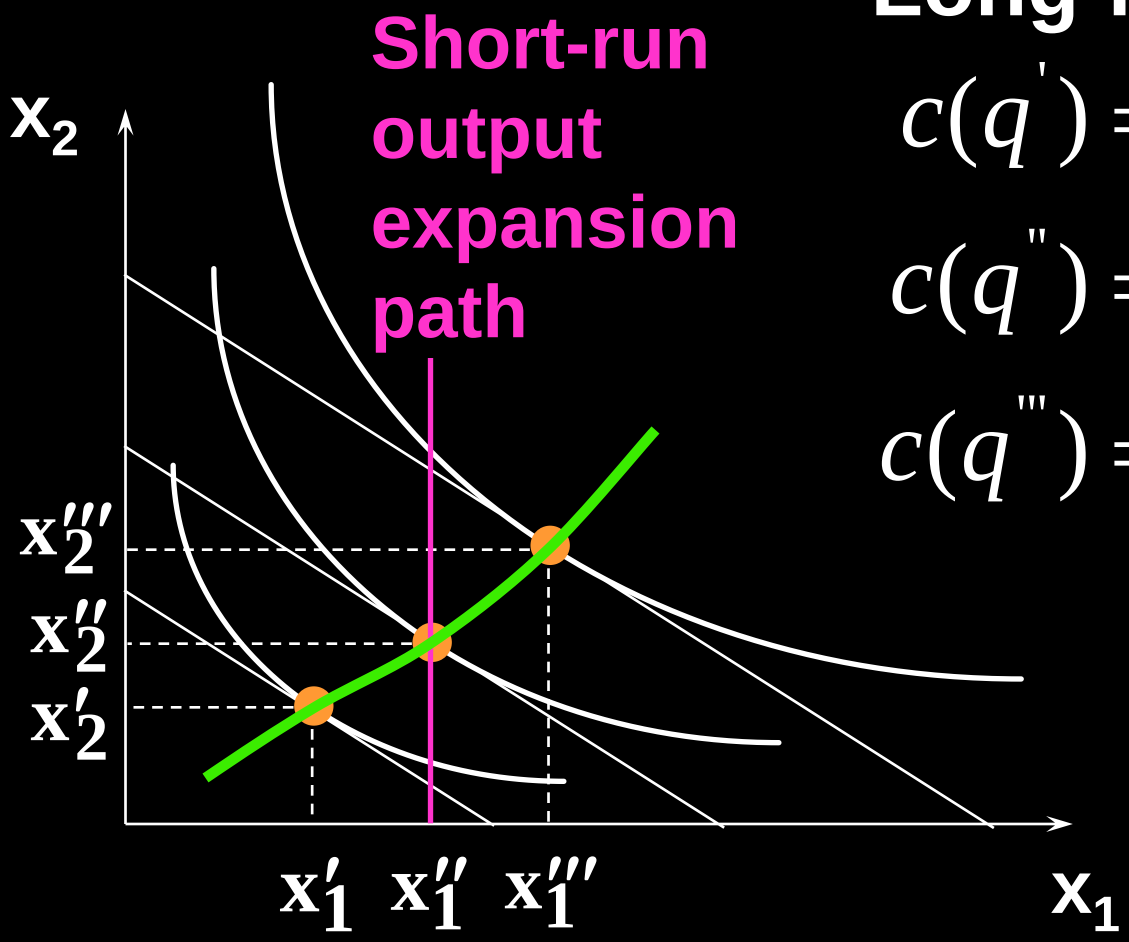
Long-run costs are:

$$c(q') = w_1 x_1' + w_2 x_2'$$

$$c(q'') = w_1 x_1'' + w_2 x_2'''$$

$$c(q''') = w_1 x_1''' + w_2 x_2'''$$

Short-Run & Long-Run Total Costs



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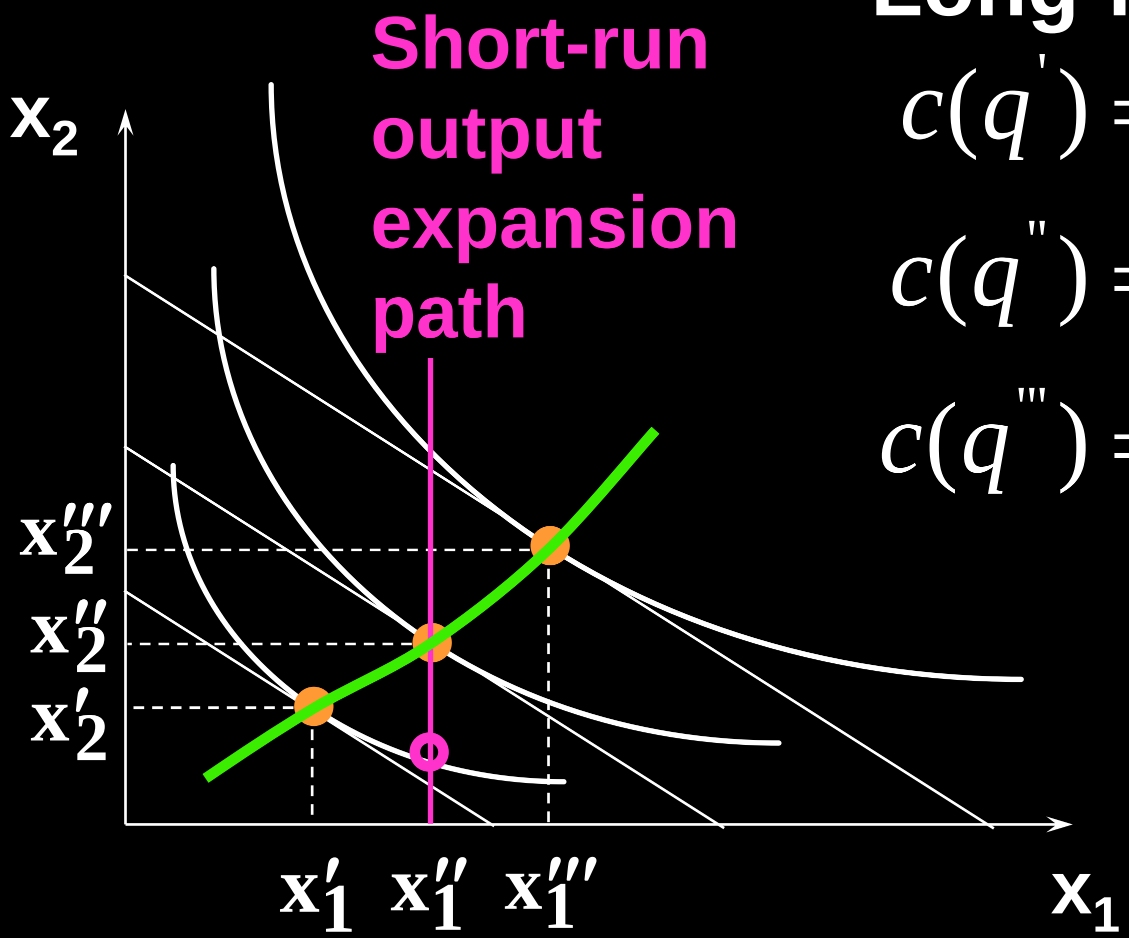
Short-Run & Long-Run Total Costs



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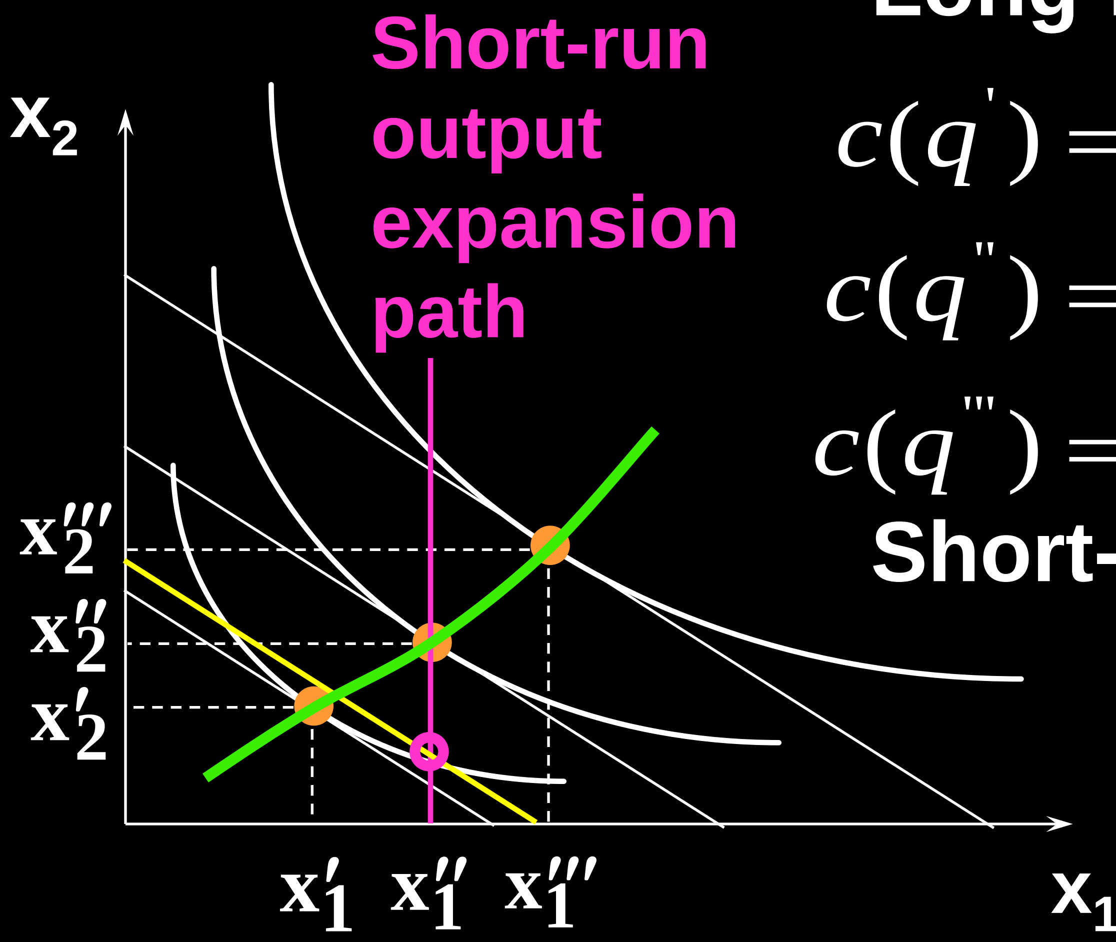
Short-Run & Long-Run Total Costs



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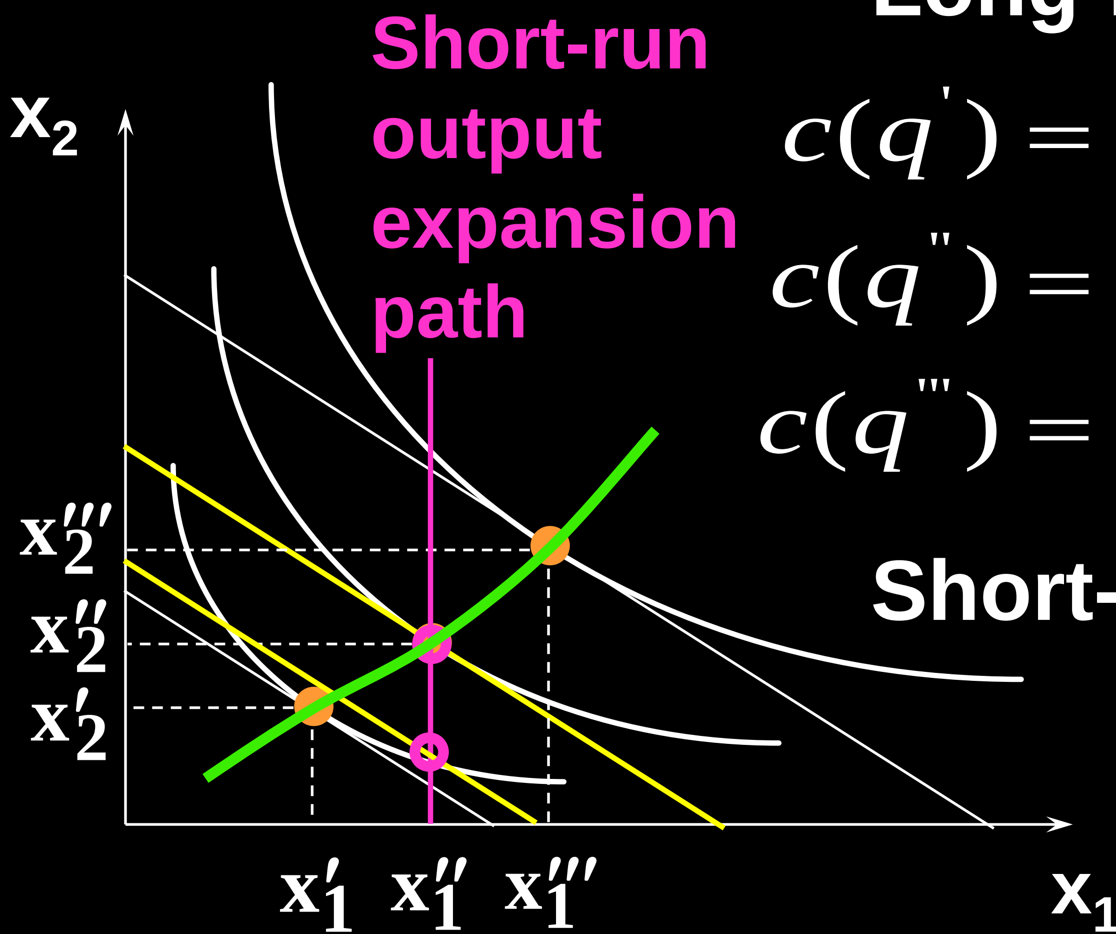
Short-run costs are:

$$c_s(q') > c(q')$$

Short-Run & Long-Run Total Costs



Long-run costs are:

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$$c(q'') = w_1 x_1'' + w_2 x_2'''$$

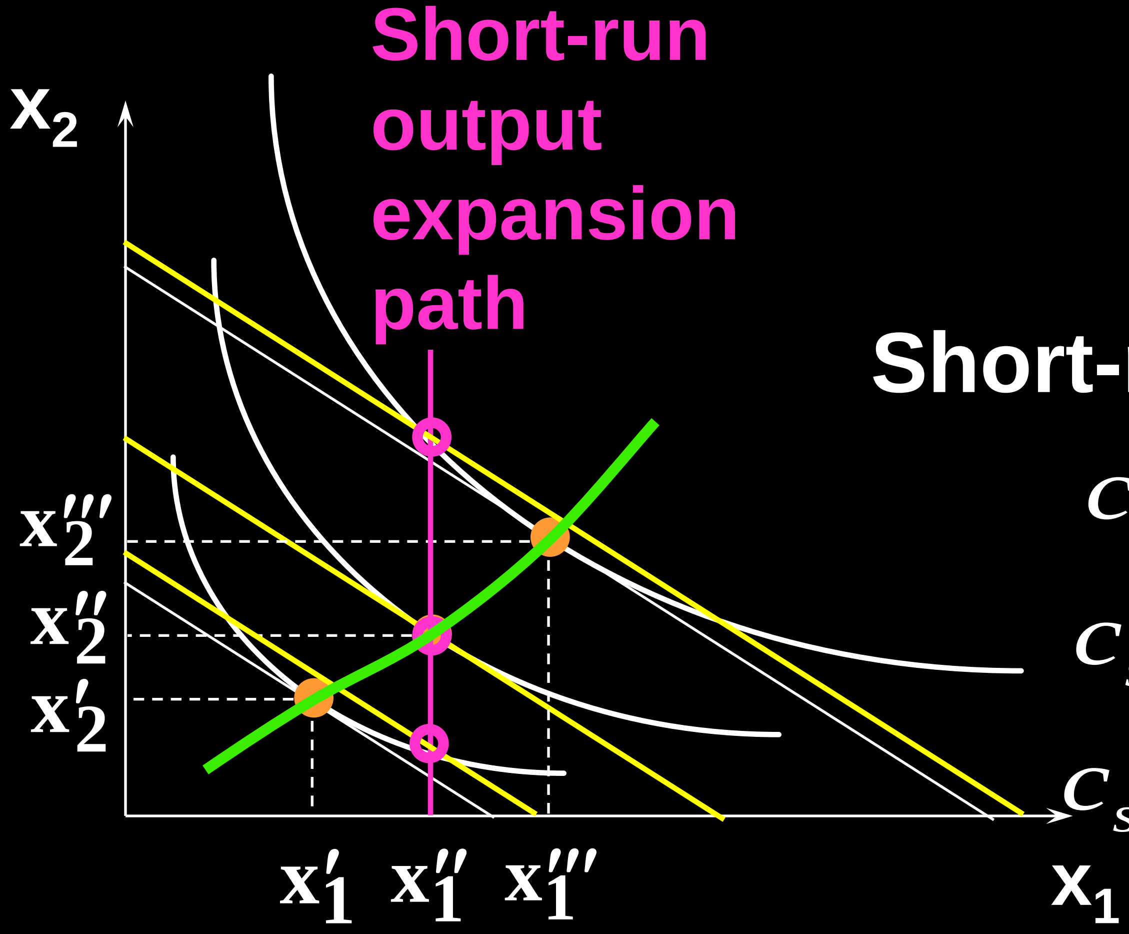
$$c(q''') = w_1 x_1''' + w_2 x_2'''$$

Short-run costs are:

$$c_s(q') > c(q')$$

$$c_s(q'') = c(q'')$$

Short-Run & Long-Run Total Costs



Short-run costs are:

$$c_s(q') > c(q')$$

$$c_s(q'') = c(q'')$$

$$c_s(q''') > c(q''')$$

Short-Run & Long-Run Total Costs

- Short-run total cost exceeds long-run total cost except for the output level where the short-run input level restriction is the long-run input level choice.
- This says that the long-run total cost curve always has one point in common with any particular short-run total cost curve.

Short-Run & Long-Run Total Costs

A short-run total cost curve always has one point in common with the long-run total cost curve, and is elsewhere higher than the long-run total cost curve.