Agenda

Curve $y = f(x)$ becomes **Surface** $z = f(x,y)$

$\frac{df}{dx}$ becomes two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$\frac{d^2 f}{\partial x^2}$ becomes four second derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial y^2}$

Tangent Line $y - y_0 = f'(x_0)(x - x_0)$ becomes **Tangent Plane** $z - z_0 = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$

Differential $dy = f'(x)\, dx$ becomes **Total Differential** $dz = f_x(x,y)\, dx + f_y(x,y)\, dy$

$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ becomes the chain rule $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$\frac{df}{dx} = 0$ becomes two MAX/MIN equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

MATH TOOLS

FUNCTIONS – ONE VARIABLE CASE

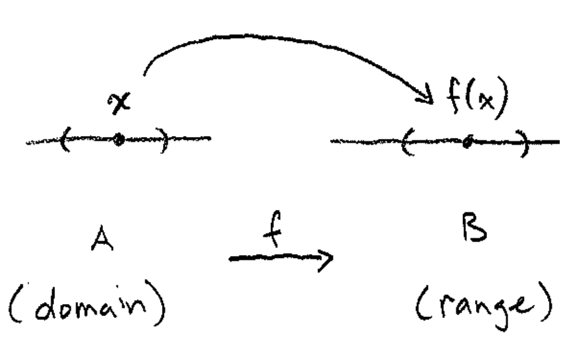
Definition: A function f is a rule that assigns to each element x in a set A exactly one element $f(x)$ in a set B.

FOUR POINTS OF VIEW	
(1) Verbal	(description in words)
(2) Numerical	(table of values)
(3) Algebraic	(explicit formula)
(4) Geometric	(graph of curve or surface)

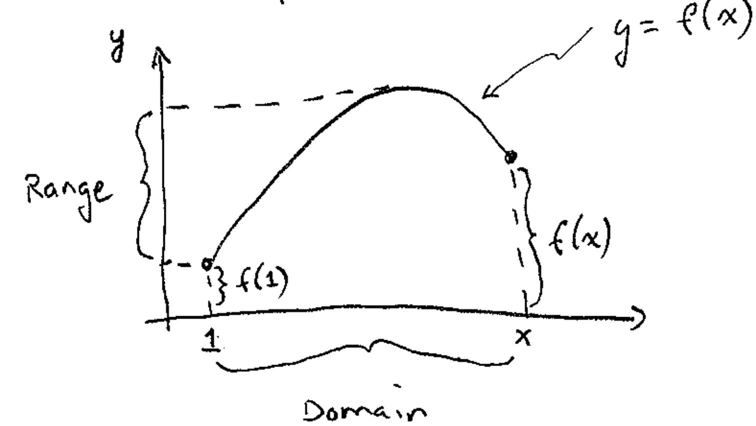
We focus on (3) & (4)!

VISUAL / Geometric

Arrow Diagram



Graph



A set of ordered pairs of the form $(x, f(x))$ such that $x \in A$ and $f(x) \in B$

↑ "element of"

Algebraic

A function $f: A \to B$ consists of

- Domain set A
- Range set B
- Rule that assigns each $x \in A$ a unique element $f(x) \in B$.

EX 1 Let $A = \{ x \in \mathbb{R} \mid 0 \leq x \leq 100 \}$

Let $B = \{ A^+, A, A^-, \ldots, F \}$

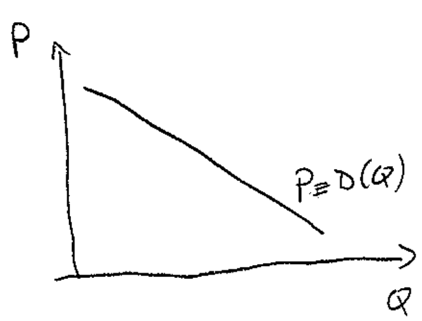
THEN, $G: A \to B$ is a function that sends numerical scores into letter grades.

EX 2 $g(x) = x^3 + x^2$

- assigns an output number $x^3 + x^2$ to each input number x.
- $g: x \to x^3 + x^2$ (g adds the cube & square for each x.

EX 3 Let P be Price / unit
Let Q be Sales / unit of time

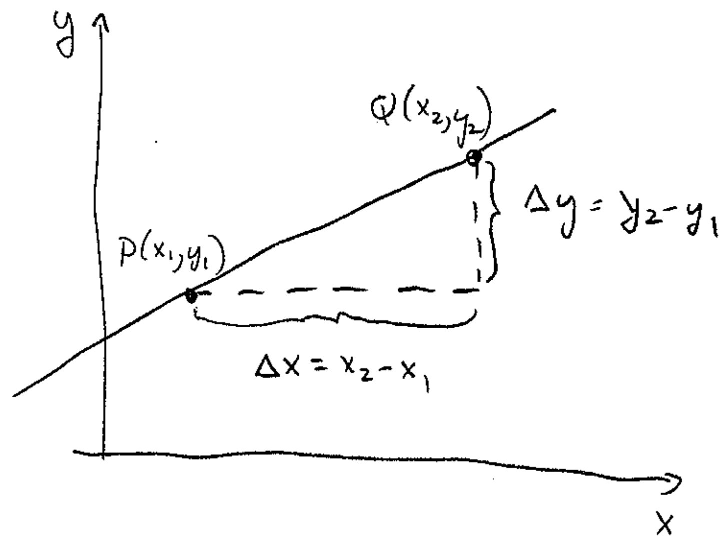
Then, $D: Q \to P$ is an inverse demand curve that sends each level of sales into a unique price.



$P = a - bQ$ consists of set of ordered pairs $(Q, D(Q))$ such that $Q \in \mathbb{R}^+$ and $D(Q) \in \mathbb{R}^+$

LINES

slope $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$



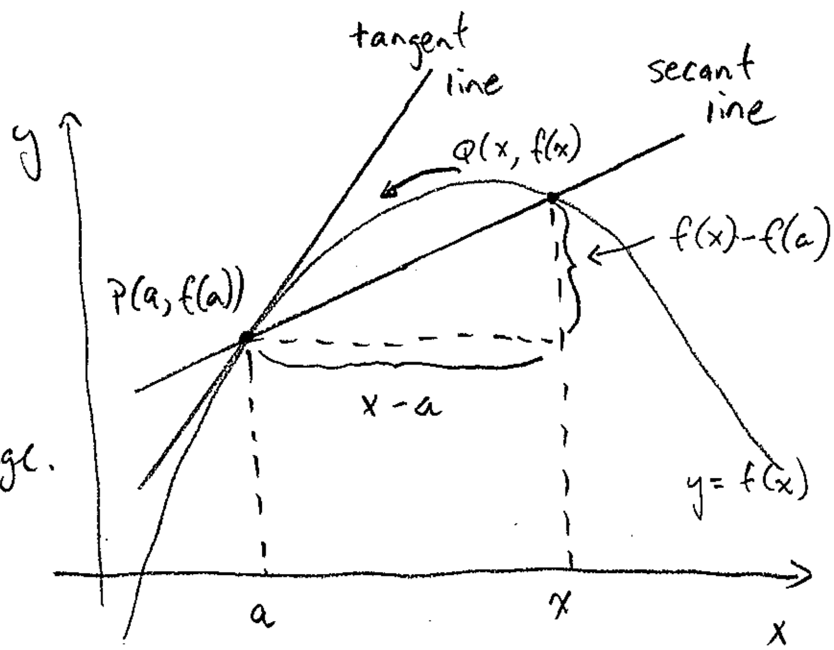
Given $P(x_1, y_1)$ and slope m, any other arbitrary point $Q(x, y)$ is found by:

$$\boxed{\text{Point-slope form} \qquad y - y_1 = m(x - x_1)}$$

or

$$\boxed{\text{Slope-Intercept form} \qquad y = mx + b \quad \text{if } (x_1, y_1) = (0, b)}$$

SECANT & TANGENT LINES



slope of secant line:

$$\boxed{m_{PQ} = \frac{f(x) - f(a)}{x - a}}$$

↳ interpretation: avg. rate of change.

slope of tangent line:

$$\boxed{m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$ (Newton Quotient)

↳ interpretation: instantaneous rate of change.

↳ Tangent line is the limiting position of the secant line as Point Q approaches Point P.

Derivative

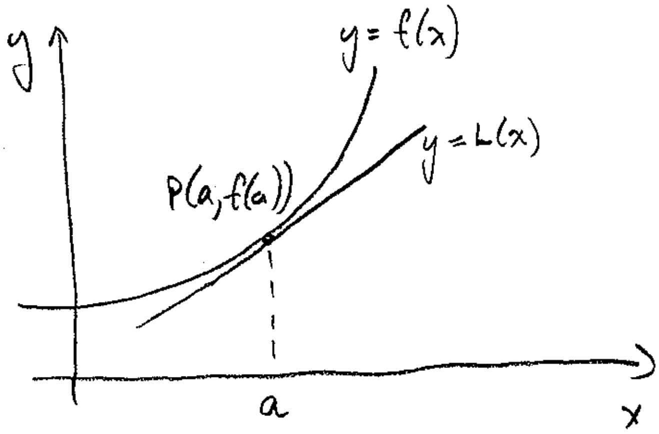
Derivative of function f at fixed number a is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{(if the limit exists)}$$

If $x = a+h \Rightarrow h = x - a$. and $h \to 0$ iff $x \to a$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Linear approximations



Tangent line at $P(a, f(a))$ approx. curve $y = f(x)$ when x is near a.

$$\boxed{\begin{aligned} &\underline{\text{Equation:}} \quad f(x) - f(a) = f'(a)(x-a) \\ &\text{or} \quad y - f(a) = f'(a)(x-a) \end{aligned}}$$

write tangent line as:

$$y = L(x) = f(a) + f'(a)(x-a)$$

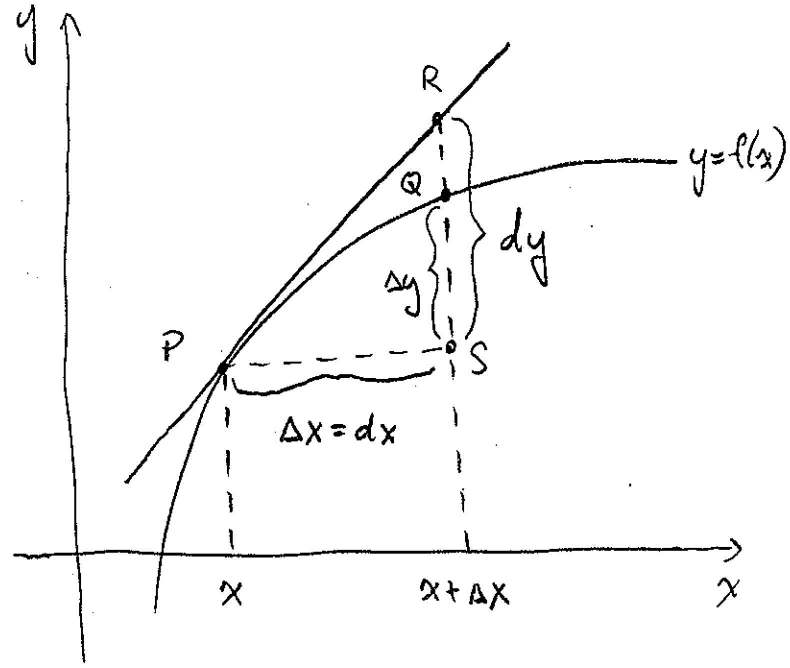
THUS, $f(x) \approx L(x)$ when $x \to a$.

KEY IDEA OF CALCULUS:

As we zoom in toward Point $P(a, f(a))$ the curve $y = f(x)$ looks more like a straight line $y = L(x)$.

Differentials

Ideas behind linear approx. are formulated as differentials.



Let $P = (x, f(x))$

$Q = (x+\Delta x, f(x+\Delta x))$

$\Delta x = dx$

Change in height of curve $y=f(x)$: $\Delta y = f(x+\Delta x) - f(x)$

Change in height of tangent line: $dy = f'(x)\,dx$

Approximation $\Delta y \approx dy = f'(x)\,dx$ gets better as $dx = \Delta x \to 0$.

NOTE: If $dx \neq 0$, we can divide both sides of the differential $dy = f'(x)\,dx$ to obtain

$$\boxed{\frac{dy}{dx} = f'(x)}$$

Derivative of a function f as a ratio of differentials.

Derivatives of Functions

For Constants α : $\frac{d}{dx}(\alpha) = 0$

For Sums : $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Power rule : $\frac{d}{dx}(\alpha x^n) = n \alpha n^{-1}$

Product rule : $\frac{d}{dx}[f(x) g(x)] = f'(x) g(x) + f(x) g'(x)$

Quotient rule : $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) g(x) - f(x) g'(x)}{[g(x)]^2}$

Chain Rule : $\frac{d}{dx}[f(g(x))] = f'[g(x)] g'(x)$

Logarithmic function: $\frac{d}{dx} \ln(x) = \frac{1}{x}$

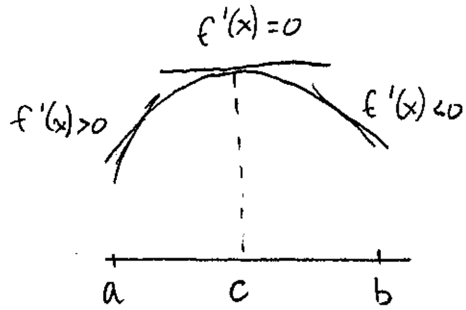
Exponential function : $\frac{d}{dx} e^x = e^x$

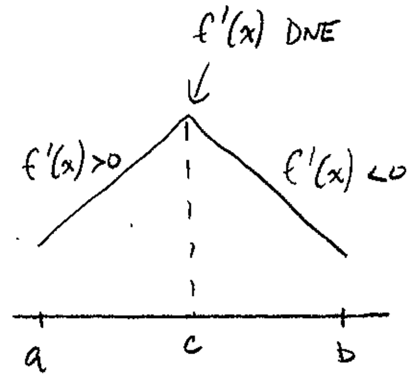
OPTIMIZATION – ONE VARIABLE

Def'n: A critical value for function f are those numbers c for which $f'(c) = 0$ or $f'(c)$ DNE. A critical point is $(c, f(c))$.

Local MAX

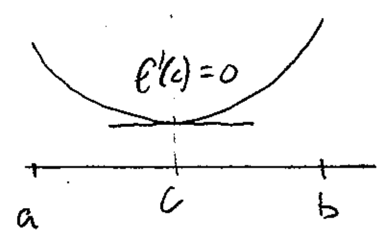
$f'(x) > 0$ in (a, c)
$f'(x) < 0$ in (c, b)

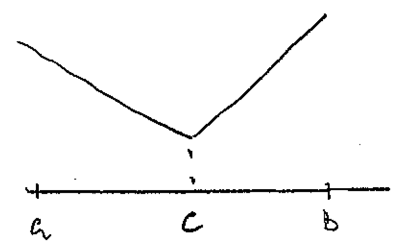




Local MIN

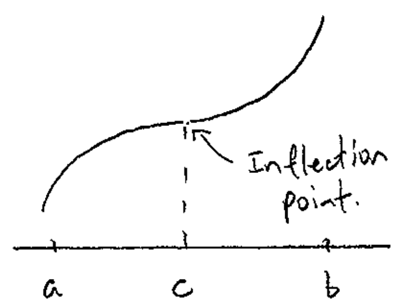
$f'(x) < 0$ in (a, c)
$f'(x) > 0$ in (c, b)

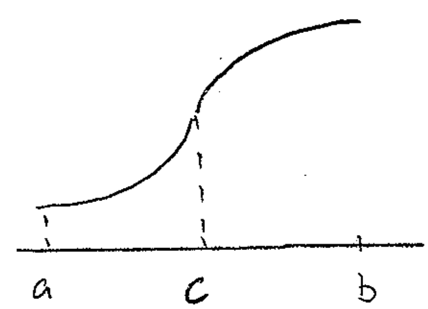




NO Local Extrema

$f'(x) > 0$ in (a, c)
$f'(x) > 0$ in (c, b)



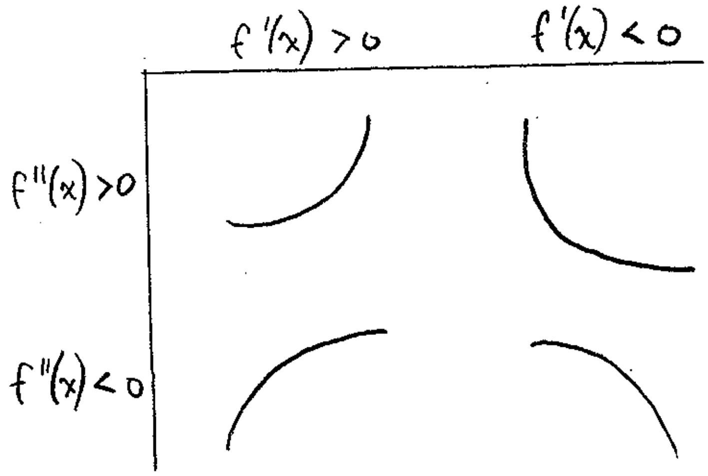


SECOND DERIVATIVE TEST

Let $f''(x)$ exist in (a, b) containing c and let $f'(c) = 0$.

1. If $f''(c) > 0$, then $f(c)$ is local MIN
2. If $f''(c) < 0$, then $f(c)$ is local MAX
3. If $f''(c) = 0$, no info. about extrema.

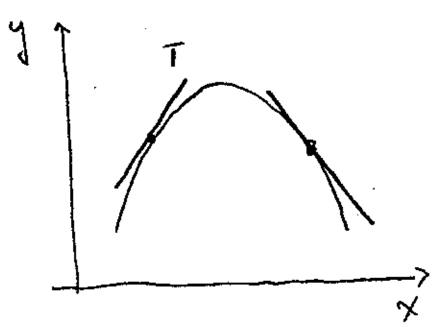
Concavity / Convexity



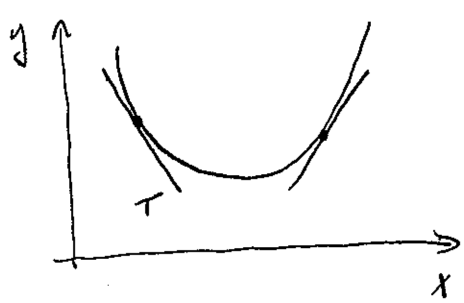
convex (i.e. concave up)
↳ slope is increasing

concave (i.e. concave down)
↳ slope is decreasing.

Another Method

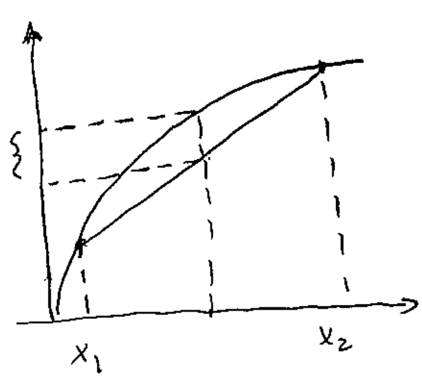


concave: slope of tangent line T is always <u>above</u> the curve $y = f(x)$



convex: slope of tangent line T is always <u>below</u> the curve $y = f(x)$.

Yet Another Method



Concave: For every pair of points, the <u>cord</u> joining them lies below the graph.

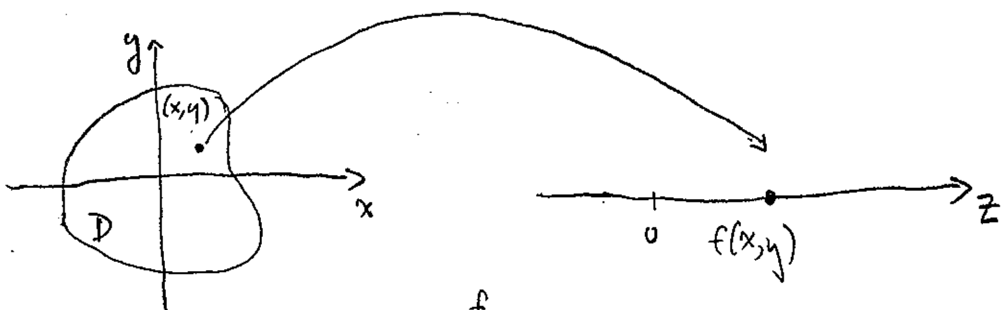
$$f(\alpha x_1 + (1-\alpha)x_2) > \alpha f(x_1) + (1-\alpha) f(x_2) \quad \forall \alpha \in [0,1]$$

MULTIVARIABLE CALCULUS

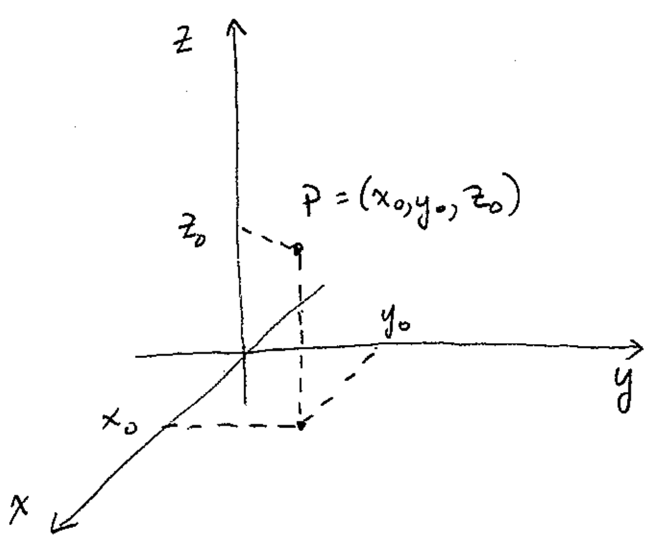
Functions - Two Variable case

Def'n: Function f of two variables is a rule that assigns to each ordered pair (x,y) in a set D a unique number $f(x,y)$.

Domain-Range Picture



$$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$



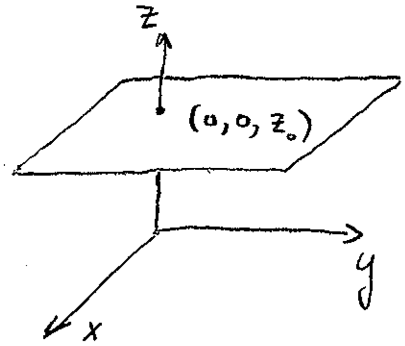
With $P=(x_0,y_0,z_0)$ in $\mathbb{R}^3$ displacement from the origin is:

x_0 in x-direction
y_0 in y-direction
z_0 in z-direction.

There are 8 octants in $\mathbb{R}^3$

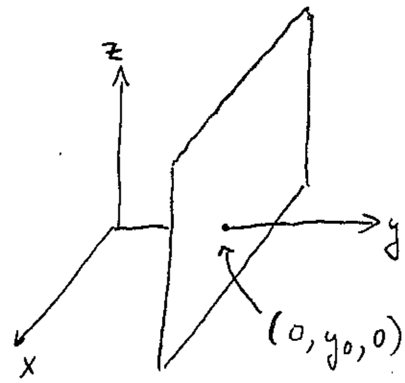
Most economic variables require $x, y, z \geq 0$. (1^{st} octant)

Coordinate Planes



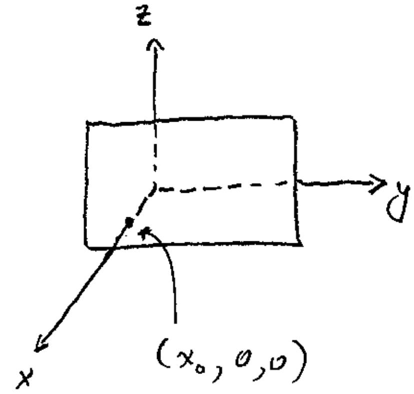
$z = z_0$

parallel to xy-plane



$y = y_0$

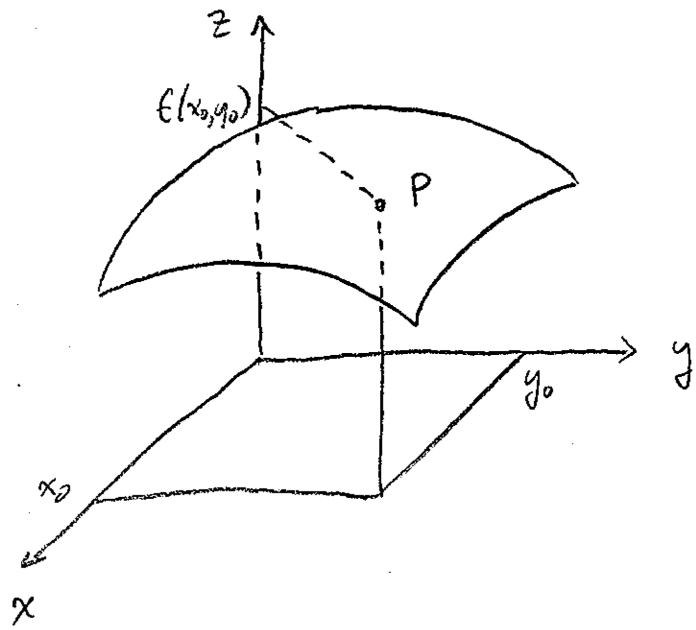
parallel to xz-plane



$x = x_0$

parallel to yz-plane.

Graph of function in $\mathbb{R}^3$

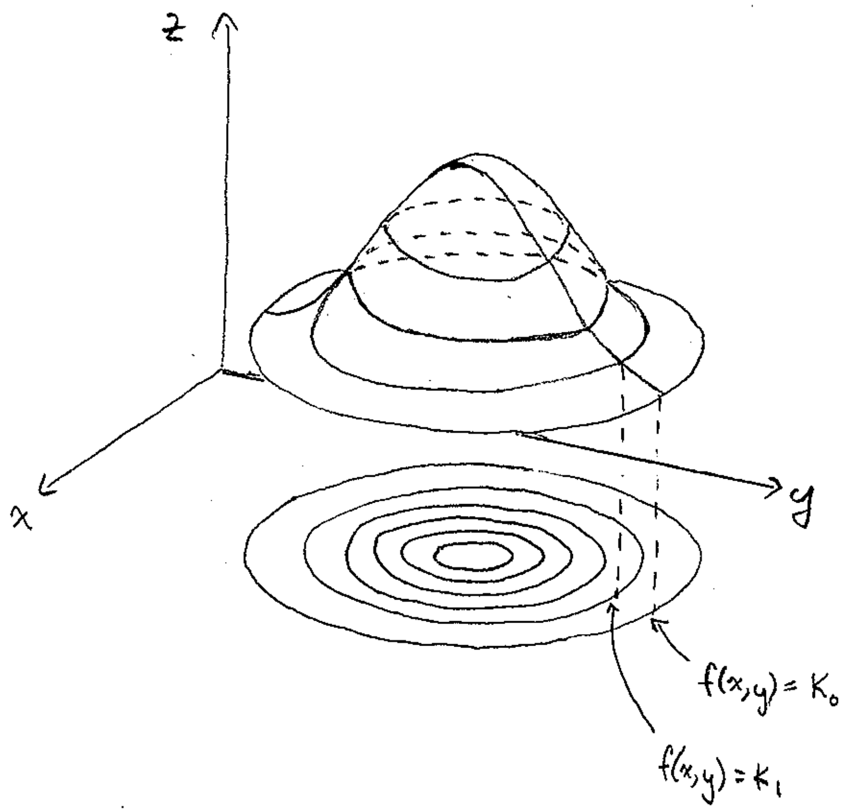


graph of $z = f(x, y)$ is a surface in $\mathbb{R}^3$. Point $P = (x_0, y_0, f(x_0, y_0))$ lies above (or below) point (x_0, y_0) in the xy-plane.

Level curves

Third way to visualize functions is a contour map. Points of constant "elevation" joined to form level curves.

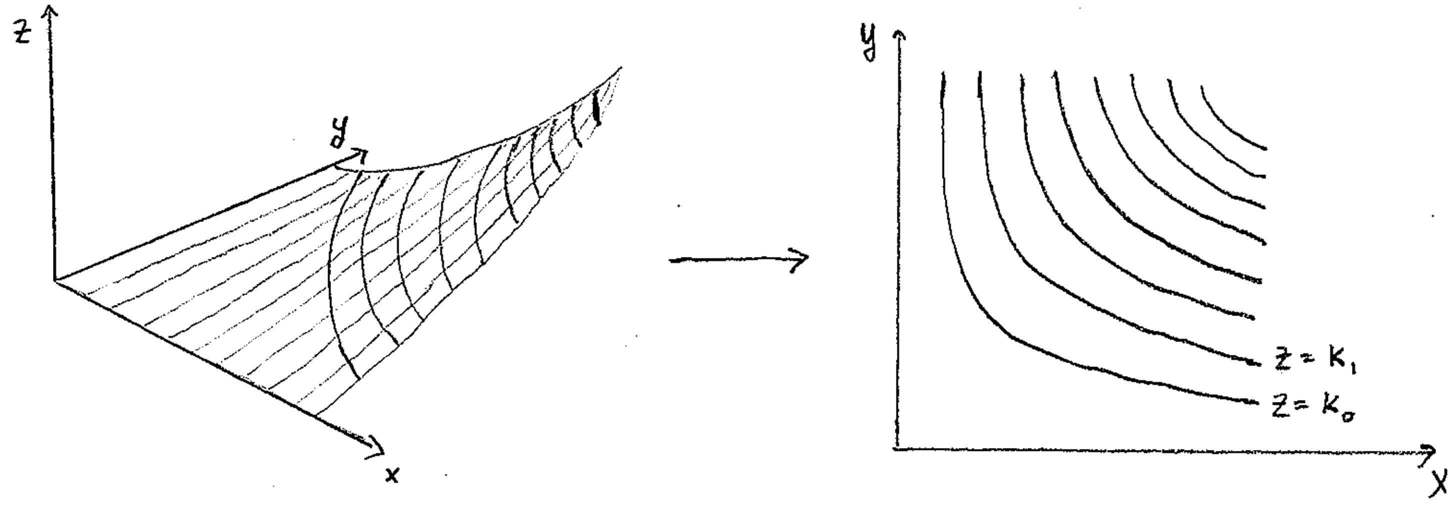
Def'n: Level curves of function f are curves with equations $f(x,y) = K$, where K is a constant.



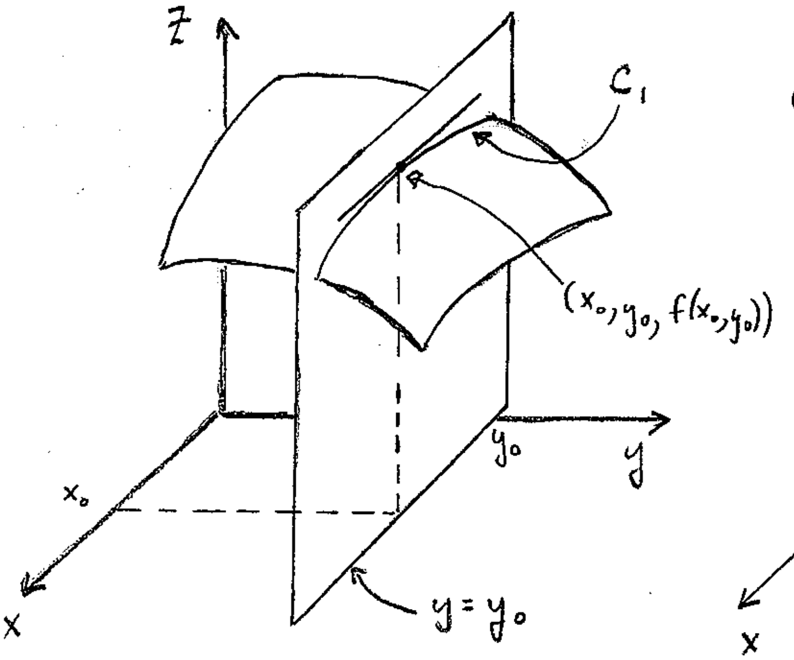
Horizontal Traces: (in $\mathbb{R}^3$)
Intersection of surface with plane parallel to xy-plane.

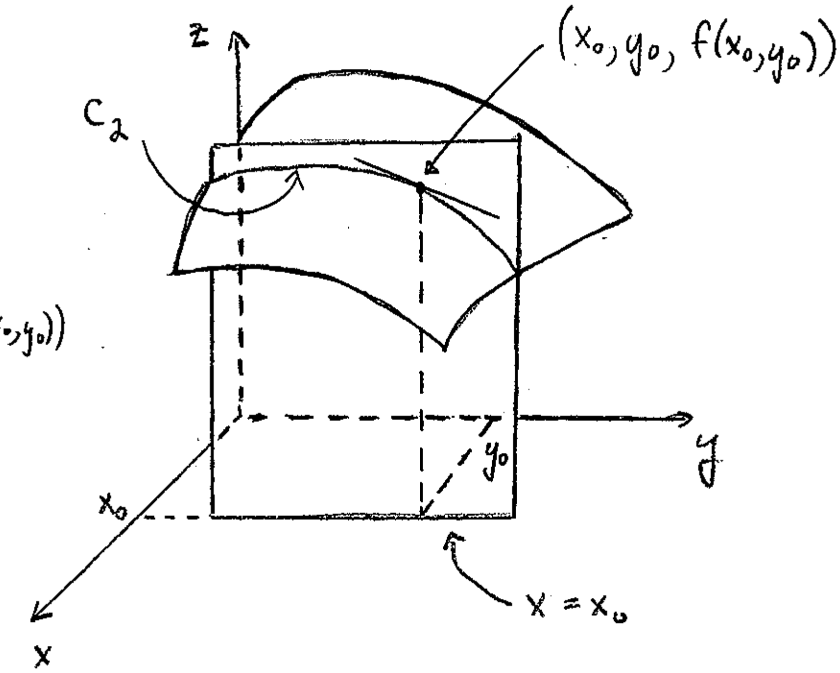
Level curves: (in $\mathbb{R}^2$)
are traces of the graph of f in the horizontal plane $z = K$ projected down on xy-plane.

Cobb-Douglas Function: $z = x \cdot y$



PARTIAL DERIVATIVES





- Curves C_1 & C_2 are <u>traces</u> from the intersection of the surface with vertical planes $y = y_0$ and $x = x_0$.

- For C_1, $y = y_0$. So C_1 is described by equation $z = f(x, y_0)$. Since y_0 is constant, z is a function of one variable, x.

- Derivative at x_0, we get the slope of tangent line to C_1 at $(x_0, y_0, f(x_0, y_0))$

Def$^{\text{n}}$ $\quad f_x(x,y) = \lim_{h \to 0} \dfrac{f(x+h, y) - f(x,y)}{h}$

- Rule for finding f_x : treat y as a constant, differentiate $f(x,y)$ w.r.t. x.

For C_2, $x = x_0$ and eq'n of C_2 is $z = f(x_0, y)$

↳ Derivative at y_0, we get slope of tangent line to C_2 at $(x_0, y_0, f(x_0, y_0))$

<u>Def'n</u>: $f_y(x,y) = \lim_{h \to 0} \dfrac{f(x, y+h) - f(x,y)}{h}$

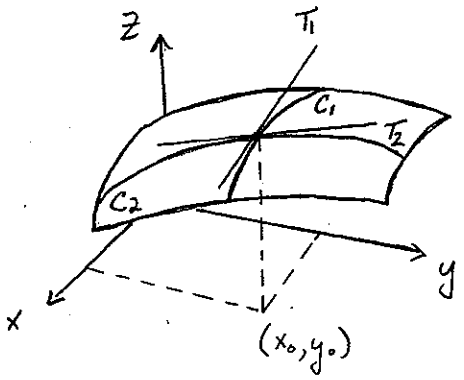
Rule for finding f_y: treat x as a constant, diff. $f(x,y)$ w.r.t. y.

** Partial Derivatives are Ordinary derivatives of a Partial Function!

<u>Notation</u>

Partial of f (or z) w.r.t. x	Partial of f (or z) w.r.t. y
$f_x(x,y)$	$f_y(x,y)$
$\partial/\partial x\,[f(x,y)]$	$\partial/\partial y\,[f(x,y)]$
$\partial f/\partial x$	$\partial f/\partial y$
$\partial z/\partial x$	$\partial z/\partial y$
f_1	f_2

Partial of f w.r.t x evaluated at (x_0, y_0)	Partial of f w.r.t. y evaluated at (x_0, y_0)
$f_x(x_0, y_0)$	$f_y(x_0, y_0)$
$\left.\partial f/\partial x\right\|_{(x_0,y_0)}$	$\left.\partial f/\partial y\right\|_{(x_0,y_0)}$



$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \text{slope of } T_1 \text{ (tangent line to } C_1)$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \text{slope of } T_2 \text{ (tangent line to } C_2)$$

Tangent Plane

Equation of tangent (in $\mathbb{R}^2$): $y - y_0 = m(x - x_0)$

Equation of tangent plane: $z - z_0 = a(x - x_0) + b(y - y_0)$

slope in x-direction

slope in y-direction

Tangent line T_1 : $z - z_0 = a(x - x_0)$ since $y = y_0$

Tangent line T_2 : $z - z_0 = b(y - y_0)$ since $x = x_0$

Since a = slope of T_1 and b = slope of T_2, rewrite Equation of Tangent Plane at $(x_0, y_0, f(x_0, y_0))$ as:

$$z - z_0 = \underbrace{f_x(x_0, y_0)}_{\text{slope in } x\text{-direction}}(x - x_0) + \underbrace{f_y(x_0, y_0)}_{\text{slope in } y\text{-direction}}(y - y_0)$$

Total Differential

For $y = f(x)$ we have $dy = f'(x)\,dx$

For $z = f(x,y)$ we have $dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

where $dx = \Delta x = (x - x_0)$

$dy = \Delta y = (y - y_0)$

$dz \approx \Delta z = (z - z_0)$

change in height of tangent plane

change in height of surface $z = f(x,y)$

NOTE $dz = \Delta z$ as $(x,y) \longrightarrow (x_0, y_0)$

SECOND DERIVATIVES

Two first Derivatives become Four second Derivatives!

$$f_{xx} = f_{11} = \partial/\partial x\left(\partial f/\partial x\right) = \partial^2 f/\partial x^2$$

$$f_{yy} = f_{22} = \partial/\partial y\left(\partial f/\partial y\right) = \partial^2 f/\partial y^2$$

$$f_{xy} = f_{12} = \partial/\partial y\left(\partial f/\partial x\right) = \partial^2 f/\partial y\,\partial x$$

$$f_{yx} = f_{21} = \partial/\partial x\left(\partial f/\partial y\right) = \partial^2 f/\partial x\,\partial y$$

YOUNG'S / CLAIRAUT'S THEOREM: $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$

Implicit Functions

1. $y = mx + b$ Explicit
2. $y - mx - b = 0$ Implicit
3. $f(y, x; m, b) = 0$ Implicit

In economics, we often get implicit functions where exogenous and endogenous variables are all mixed up.

More generally

- Suppose $F(x, y) = 0$ defines y implicitly as a function of x, so that $y = f(x)$.
- Rewrite $F(x, f(x)) = 0 \quad \forall \; x$ in domain of f.
- apply Chain Rule to get: $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$

↳ But $\frac{dx}{dx} = 1$, so if $\frac{\partial F}{\partial y} \neq 0$ we solve for $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}}$$

Implicit Function Theorem

Another Way

Let $F(x,y) = 0$

Now totally differentiate: $F_x\,dx + F_y\,dy = 0$

Solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = -\frac{F_x}{F_y}$

Ex $2x^2 + y^2 = 225$

Find dy/dx.

Method 1: Rewrite as $y = \sqrt{225 - 2x^2}$

$$\frac{dy}{dx} = \frac{1}{2}\left(225 - 2x^2\right)^{-\frac{1}{2}}(-4x)$$

$$= \frac{-4x}{2\sqrt{225 - 2x^2}} = -\frac{2x}{y}$$

Method 2: Rewrite as $2x^2 + y^2 - 225 = 0$

Find total differential: $4x\,dx + 2y\,dy = 0$

Solve for dy/dx: $dy/dx = -\frac{2x}{y}$

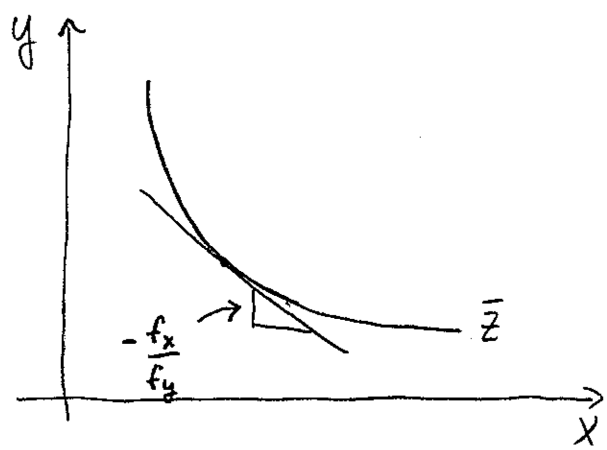
Application – Level curves

- Along a level curve, we have $f(x,y) = \bar{z}$ (fixed z)
- Implicit function $f(x, y(x)) = \bar{z}$ tells us how much y we'd give up for a little more x (holding z fixed)

$$f(x, y(x)) = \bar{z}$$

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$\left.\frac{dy}{dx}\right|_{z=\bar{z}} = -\frac{f_x}{f_y}$$

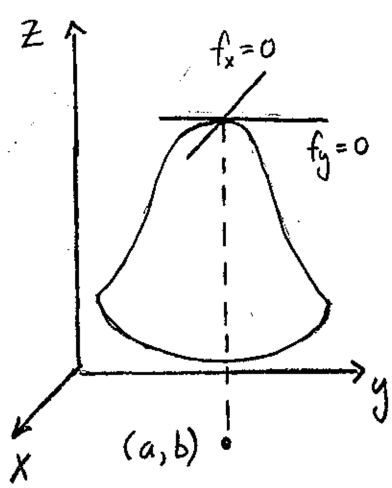


MAXIMA / MINIMA

Location of Extrema (critical Points): If $z = f(x,y)$ has local MAX/MIN at Point (a,b), then

$$\left.\begin{array}{l} f_x(a,b) = 0 \\ \text{and} \\ f_y(a,b) = 0 \end{array}\right\}$$

Tangent Plane $z - z_0 = f_x(x - x_0) + f_y(y - y_0)$ is horizontal



Test for relative Extrema (OPTIONAL)

For $z = f(x,y)$, let f_{xx}, f_{yy} and f_{xy} all exist in some region contained in xy-plane with center (a,b).

Further let $f_x(a,b) = 0$ and $f_y(a,b) = 0$

Define the number D by:

$$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

Then:

1. $f(a,b)$ is a local MAX if $D > 0$ and $f_{xx} < 0$
2. $f(a,b)$ is a local MIN if $D > 0$ and $f_{xx} > 0$
3. $f(a,b)$ is a saddle point (neither MAX nor MIN) if $D < 0$.
4. if $D = 0 \Rightarrow$ no information

Constrained Optimization

Most optimization problems in economics are subject to constraints:

(1) Maximize utility subject to a budget constraint.

(2) Minimize Expenditure subject to a utility constraint.

(3) Maximize social welfare subject to resource constraint.

(4) Maximize profit subject to a technological constraint.

(5) Minimize cost subject to a production constraint.

The tool for maximizing / minimizing constrained functions is the Lagrangian Method. This is a "trick" with very useful economic content.

Lagrange Method

Used for problems of the form:

Find the relative extrema for $z = f(x,y)$

subject to $g(x,y) = 0$.

NOTE: Applies ~~with~~ to any number of variables and constraints!

THEOREM

All relative extrema of $z = f(x,y)$ subject to a constraint $g(x,y) = 0$ will be found among those points (x,y) for which there exists a λ value such that

$$L_x(x,y,\lambda) = 0$$
$$L_y(x,y,\lambda) = 0$$
$$L_\lambda(x,y,\lambda) = 0,$$

where

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

and all partial derivatives exist.

SET UP

STEP 1: Write the constraint in the form $g(x,y) = 0$

STEP 2: Form the Lagrange function

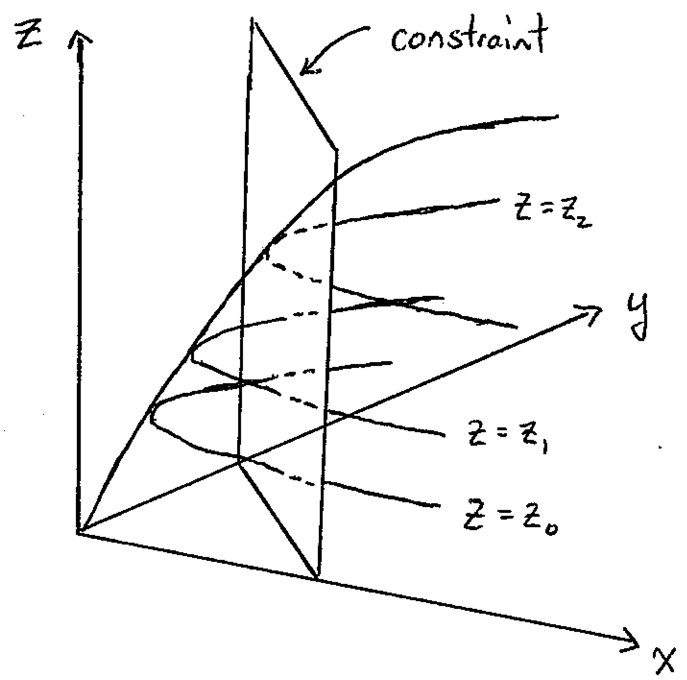
$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

STEP 3: Find $L_x(x,y,\lambda)$, $L_y(x,y,\lambda)$, and $L_\lambda(x,y,\lambda)$

STEP 4: Form system of equations (called First-order conditions)

$$L_x(x,y,\lambda) = 0, \quad L_y(x,y,\lambda) = 0, \quad L_\lambda(x,y,\lambda) = 0$$

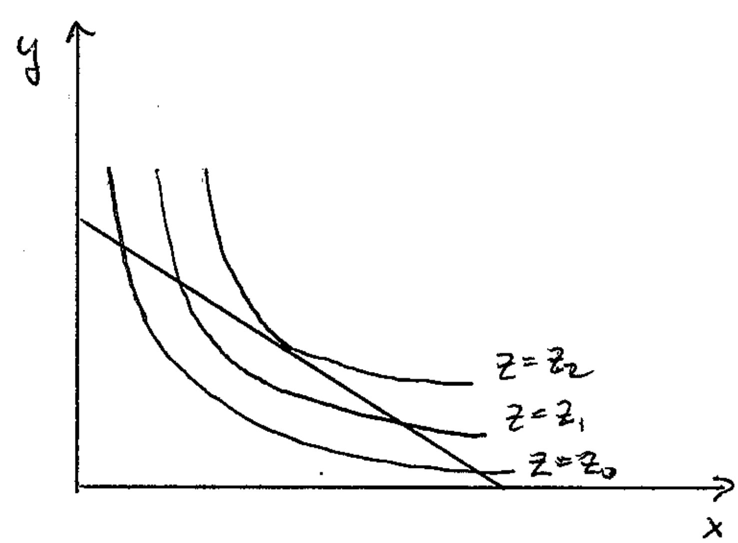
STEP 5: Solve system from STEP 4. Extrema for $f(x,y)$ are among the solutions of the system.



Convert 3D image to 2D by:

(1) Projecting horizontal traces $z = z_i$ for $i = 0, 1, 2$ onto the xy-plane to form Level curves

(2) Projecting plane onto the xy-plane to form a line.



Problem : (General form)

$$\max_{\{x,y\}} \; z = f(x,y) \qquad \text{s.t.} \qquad g(x,y) = 0$$

↑ "subject to"

SET UP: $L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$

FOC's ↑ "First-order conditions"

$$\frac{\partial L}{\partial x} = f_x + \lambda g_x = 0$$

$$\frac{\partial L}{\partial y} = f_y + \lambda g_y = 0$$

$$\frac{\partial L}{\partial \lambda} = g(x,y) = 0$$

NOTE: we obtain as many eq'ns as unknowns, since we introduced another unknown, λ, called the Lagragian Multiplier.

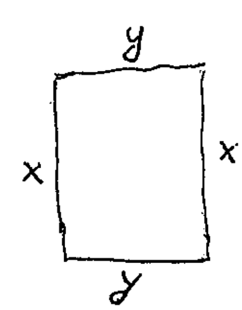
One then solves for x^*, y^*, and λ.

λ has a special interpretation that we will discuss,

Example : optimal Fence Dimensions

Given fencing perimeter of length P, max the fenced area provided P is of rectangular shape.

Problem: $\max_{\{x,y\}} \; x \cdot y \quad \text{s.t.} \quad 2x + 2y = P$



Form the Lagragian:

$$L(x,y,\lambda) = xy + \lambda(p - 2x - 2y)$$

<u>FOC's</u>: (1) $L_x = y - 2\lambda = 0$

(2) $L_y = x - 2\lambda = 0$

(3) $L_\lambda = p - 2x - 2y = 0$

<u>Solve</u>: solve (1) & (2) for λ. Then set (1) & (2) equal to each other (to get rid of λ)

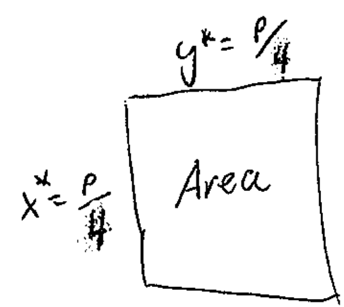
from (1): $y = 2\lambda \Rightarrow \lambda = y/2$

from (2): $x = 2\lambda \Rightarrow \lambda = x/2$

$\Rightarrow y/2 = x/2 \iff \boxed{x = y}$ perimeter must be <u>square</u> to maximize area.

– Now substitute $x = y$ into (3) to solve for x^* to get $p - 2x - 2x = 0$ or $\boxed{x^* = \frac{p}{4} = y^*}$

– To solve for λ, sub y^* into (1) or x^* into (2) to get $\boxed{\lambda = \frac{p}{8}}$



MAX AREA is $x^* \cdot y^* = \left(\frac{p}{4}\right)^2$