ECON 6500
Utility Maximization Homework

Instructions:

For next week's class, read your notes on calculus and the illustrations found here. In the last section of this document you will find a set of utility maximization problems. Attempt each of them and bring your results to next class.

REVIEW:Partial Derivative Rules:

$U = xy$	$\partial U/\partial x = y$	$\partial U/\partial y = x$
$U = x^a y^b$	$\partial U/\partial x = ax^{a-1}y^b$	$\partial U/\partial y = bx^a y^{b-1}$
$U = x^a y^{-b} = \frac{x^a}{y^b}$	$\partial U/\partial x = ax^{a-1}y^{-b}$	$\partial U/\partial y = -bx^a y^{-b-1}$
$U = ax + by$	$\partial U/\partial x = a$	$\partial U/\partial y = b$
$U = ax^{1/2} + by^{1/2}$	$\partial U/\partial x = a\left(\frac{1}{2}\right)x^{-1/2}$	$\partial U/\partial y = b\left(\frac{1}{2}\right)y^{-1/2}$

The Lagrange Multiplier Method

Sometimes we need to to maximize a function that is subject to some sort of constraint. For example

$$\textit{Maximize} \quad z = f(x, y)$$

$$\textit{subject to the constraint} \quad x + y \leq 100$$

For this kind of problem there is a technique, or *trick,* developed for this kind of problem known as the *Lagrange Multiplier method.* This method involves adding an extra variable to the problem called the lagrange multiplier, or λ.

We then set up the problem as follows:

1. Create a new equation form the original information

$$L = f(x, y) + \lambda(100 - x - y)$$
$$or$$
$$L = f(x, y) + \lambda\,[Zero]$$

2. Then follow the same steps as used in a regular maximization problem

$$\frac{\partial L}{\partial x} = f_x - \lambda = 0$$
$$\frac{\partial L}{\partial y} = f_y - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 100 - x - y = 0$$

3. In most cases the λ will drop out with substitution. Solving these 3 equations will give you the constrained maximum solution

Example 1:

Suppose $z = f(x, y) = xy$. and the constraint is the one from above. The problem then becomes

$$L = xy + \lambda(100 - x - y)$$

Now take partial derivatives, one for each unknown, including λ

$$\frac{\partial L}{\partial x} = y - \lambda = 0$$
$$\frac{\partial L}{\partial y} = x - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 100 - x - y = 0$$

Starting with the first two equations, we see that $x = y$ and λ drops out. From the third equation we can easily find that $x = y = 50$ and the constrained maximum value for z is $z = xy = 2500$.

Example 2:

Maximize

$$u = x^2 y$$

subject to

$$x + y = 60$$

Set up the Lagrangian Equation:

$$L = x^2 y + \lambda(60 - x - y)$$

Take the first-order partials and set them to zero

$$\begin{aligned} L_x &= 2xy - \lambda = 0 \\ L_y &= x^2 - \lambda = 0 \\ L_\lambda &= 60 - x - y = 0 \end{aligned}$$

From the first two equations we get

$$\begin{aligned} 2xy &= x^2 \\ 2y &= x \end{aligned}$$

Substitute this result into the third equation

$$\begin{aligned} 60 - 2y - y &= 0 \\ 60 &= 3y \\ y &= 20 \end{aligned}$$

therefore

$$x = 2y = 40$$

Example 3: Utility Maximization

Consider a consumer with the utility function $U = xy$, who faces a budget constraint of $B = P_x x + P_y y$, where B, P_x and P_y are the budget and prices, which are given.

The choice problem is

Maximize

$$U = xy \tag{1}$$

Subject to

$$B = P_x x + P_y y \tag{2}$$

The Lagrangian for this problem is

$$Z = xy + \lambda(B - P_x x - P_y y) \tag{3}$$

The first order conditions are

$$\begin{aligned} Z_x &= y - \lambda P_x = 0 \\ Z_y &= x - \lambda P_y = 0 \\ Z_\lambda &= B - P_x x - P_y y = 0 \end{aligned} \tag{4}$$

Solving the first order conditions yield the following solutions

$$x = \tfrac{B}{2P_x} \quad y = \tfrac{B}{2P_y} \tag{5}$$

Problems:

1. Skippy lives on an island where she produces two goods, x *and* y, according the the production possibility frontier $200 = x + y$, and she consumes all the goods herself. Her utility function is

   $$u = x \cdot y^3$$

   Find her utility maximizing x and y.

2. Re-do problem 1 with $u = x^2 y^3$

3. Skippy has the following utility function: $u = x^{\frac{1}{2}} y^{\frac{1}{2}}$ and faces the budget constraint: $M = p_x x + p_y y$.

   (a) Suppose $M = 120$, $P_y = 1$ and $P_x = 4$. Find the optimal x and y

   (b) Suppose both prices change such that $P_x = 3$ and $P_y = 2$ and $M = 120$ (as before). Find the new optimal x and y

4. A consumer has the following utility function: $U(x, y) = x(y + 1)$, where x and y are quantities of two consumption goods whose prices are p_x and p_y respectively. The consumer also has a budget of B. Therefore the consumer's maximization problem is

   $$x(y + 1) + \lambda(B - p_x x - p_y y)$$

   find expressions for x^* and y^*. These are the consumer's demand functions. What kind of good is y? In particular what happens when $p_y > B/2$?