Basic Calculus for Economics Students

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Fall, 1996

1 Introduction

Calculus is about finding slopes of curves. You know that the slope of a straight line is the "rise over the run". Calculus applies the same idea to curves. However, since the slope of a curve changes continuously, calculus allows us to find the slope at particular points along the curve.

2 One Variable Calculus

If $y = f(x)$ then the derivative, which is the formula for the slope of $f(x)$, is symbolized by

$$\frac{dy}{dx} \quad or \quad f'(x) \quad or \quad f_x(x) \quad \left(\text{which are the same as: } \frac{RISE}{RUN}\right)$$

There are a few basic rules which let you find the slope of any curve where you know the formula.

1.Constant Rule:

If $y = 4$ then f(x) is just a horizontal line at a height of 4 and a slope of zero. In general, if $y = f(x) = k$ then $\frac{dy}{dx} = f'(x) = 0$

2.Power function rule

$$y = f(x) = ax^n \qquad \text{then} \qquad \frac{dy}{dx} = f'(x) = anx^{n-1} \qquad \text{(where a, n are constants)}$$

Example:

$$y = x^2 \quad then \quad \frac{dy}{dx} = 2x$$

$$y = 4x^3 \quad then \quad \frac{dy}{dx} = (4)(3)x^{3-1} = 12x^2$$

$$y = x^{\frac{1}{2}} \quad then \quad \frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$$

3.Additive Rule:

$$y = f(x) + g(x) \quad then \quad \tfrac{dy}{dx} = f'(x) + g'(x)$$

Example:

$$\begin{array}{lll} y = 3x^2 + 2x^4 & then & \frac{dy}{dx} = 6x + 8x^3 \\ y = x + x^3 & then & \frac{dy}{dx} = 1 + 3x^2 \end{array}$$

4.Multiplicitive Rule:

$$y = f(x)g(x) \quad then \quad \tfrac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$$

Example: Let $f(x) = x^2$ and $g(x) = (x+1)^3$ then

$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x) = (2x)(x+1)^3 + (x^2)(3(x+1)^2)$$

5.Chain Rule: Sometime a function is "nested " inside a function; or, a variable will be a function of one variable, which in turn, is a function of a third variable. For example: *Sales depend on quantity of output produced, which, in turn, depends on the quantity of inputs used by the firm.*

For the next bit of calculus, it may be helpful to think of y as *sales,* x as *output,* and z as units of *labour* (or some other input).

$$y = f(x) \quad and \quad x = g(z) \quad then \quad \tfrac{dy}{dz} = f'(x)g'(z)$$

Example

$$\begin{array}{lll} y = (x+1)^3 \quad and \quad x = z^2 & then & \frac{dy}{dz} = [3(x+1)^2]\,[2z] \\ & and & \frac{dy}{dz} = [3(z^2+1)^2]\,[2z] \end{array}$$

3 Partial Derivatives

let $z = z(x, y)$, which means **"z is a function of x and y"**.

- $\frac{\partial z}{\partial x}$ is the **"partial derivative"** of z with respect to x, treating y as a constant. Sometimes written as f_x.
- $\frac{\partial z}{\partial y}$ is the **"partial derivative"** of z with respect to y, treating x as a constant. Sometimes written as f_y.

EXAMPLES:

$$\begin{array}{l} z = x + y \quad \partial z/\partial x = 1 \quad\quad \partial z/\partial y = 1 \\ z = xy \quad\quad \partial z/\partial x = y \quad \partial z/\partial y = x \\ z = x^2y^3 + 2x + 4y \quad \partial z/\partial x = 2xy^3 + 2 \quad \partial z/\partial y = 3x^2y^2 + 4 \end{array}$$

- REMEMBER: When you are taking a partial derivative you treat the other variables in the equation as constants!

Differentials

Given the function

$$y = f(x)$$

the derivative is

$$\frac{dy}{dx} = f'(x)$$

However, we can treat dy/dx as a fraction and factor out the dx

$$dy = f'(x)dx$$

where dy and dx are called *differentials*. If dy/dx can be interpreted as "the slope of a function", then dy is the "rise" and dx is the "run". Another way of looking at it is as follows:

- dy = the change in y
- dx = the change in x
- $f'(x)$ = how the change in x causes a change in y

Example:
if

$$y = x^2$$

then

$$dy = 2xdx$$

Lets suppose $x = 2$ and $dx = 0.01$. What is the change in $y(dy)$?

$$dy = 2(2)(0.01) = 0.04$$

Therefore, at $x = 2$, if x is increased by 0.01 then y will increase by 0.04.

The two variable case

If

$$z = f(x, y)$$

then the change in z is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad or \quad dz = f_x dx + f_y dy$$

which is read as "the change in z (dz) is due partially to a change in x (dx) plus partially due to a change in y (dy).

Examples:

$$if \quad z = xy \quad then \qquad dz = ydx + xdy$$

$$if z = x^2y^2 \quad then \quad dz = 2xy^2dx + 2x^2ydy$$

REMEMBER: When you are taking the total differential, you are just taking all the partial derivatives and adding them up.

4 Special Functions and rules

4.1 Cobb-Douglas

The Cobb-douglas function is a mathematical function that is very popular in economic models. The general form is

$$z = x^a y^b$$

and its partial derivatives are

$\partial z/\partial x = ax^{a-1}y^b$ and $\partial z/\partial y = bx^a y^{b-1}$

Furthermore, the slope of the level curve of a Cobb-douglas is given by

$$\frac{\partial z/\partial x}{\partial z/\partial y} = MRS = \frac{a}{b}\frac{y}{x}$$

4.2 e^x and Natural Log (ln(x))

- If $y = e^x$ Then its derivative is $\frac{dy}{dx} = e^x$
- If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$

5 Maximization

5.1 One Variable Case

If we have the following function

$$y = 10x - x^2$$

we have an example of a *dome shaped* function. To find the maximum of the dome, we simply need to find the point where the slope of the dome is zero, or

$$\begin{gathered} \tfrac{dy}{dx} = 10 - 2x = 0 \\ 10 = 2x \\ x = 5 \\ and \\ y = 25 \end{gathered}$$

5.2 Two Variable Case

Suppose we want to maximize the following function

$$z = 10x + 10y + xy - x^2 - y^2$$

Note that there are two unknowns that must be solved for, x and y. This function is an example of a *three-dimensional dome.* (i.e. the roof of *BC Place*)

To solve this maximization problem we use **partial derivatives.** We take a partial derivative for each of the unknown choice variables and set them equal to zero

$$\partial z/\partial x = 10 + y - 2x = 0$$

$$\partial z/\partial y = 10 + x - 2y = 0$$

This gives us a set of equations, one equation for each of the unknown variables. When you have the same number of independent equations as unknowns, you can solve for each of the unknowns.

rewrite each equation as

$$y = 2x - 10$$

$$x = 2y - 10$$

substitute one into the other

$$x = 2(2x - 10) - 10$$

$$x = 4x - 30$$

$$3x = 30$$

$$x = 10$$

similarly,

$$y = 10$$

REMEMBER: To maximize (minimize) a function of many variables you use the technique of partial differentiation. This produces a set of equations, one equation for each of the unknowns. You then solve the set of equations simulaneously to derive solutions for each of the unknowns.

5.3 Maximization with Constraints

Sometimes we need to to maximize (minimize) a function that is subject to some sort of constraint. For example

$$Maximize \quad z = f(x, y)$$

$$subject\ to\ the\ constraint \quad x + y \leq 100$$

For this kind of problem there is a technique, or *trick,* developed for this kind of problem known as the *Lagrange Multiplier method.* This method involves adding an extra variable to the problem called the lagrange multiplier, or λ.

We then set up the problem as follows:

1. Create a new equation form the original information

$$\begin{gathered} L = f(x, y) + \lambda(100 - x - y) \\ or \\ L = f(x, y) + \lambda\,[Zero] \end{gathered}$$

2. Then follow the same steps as used in a regular maximization problem

$$\begin{aligned} \tfrac{\partial L}{\partial x} &= f_x - \lambda = 0 \\ \tfrac{\partial L}{\partial y} &= f_y - \lambda = 0 \\ \tfrac{\partial L}{\partial \lambda} &= 100 - x - y = 0 \end{aligned}$$

3. In most cases the λ will drop out with substitution. Solving these 3 equations will give you the constrained maximum solution

An Example:

Suppose $z = f(x, y) = xy$. and the constraint is the one from above. The problem then becomes

$$L = xy + \lambda(100 - x - y)$$

Now take partial derivatives, one for each unknown, including λ

$$\frac{\partial L}{\partial x} = y - \lambda = 0$$
$$\frac{\partial L}{\partial y} = x - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 100 - x - y = 0$$

Starting with the first two equations, we see that $x = y$ and λ drops out. From the third equation we can easily find that $x = y = 50$ and the constrained maximum value for z is $z = xy = 2500$.