Econ 431: *Mathematical Economics* *Summer 2007*

Assignment 5 ANSWER KEY Kevin Wainwright

Definition 1 *The particular integral (solution) y_p is ANY solution that completes equation ??. In the case of the linear, automomous, first-order differential equation, this will be the steady-state solution ($\frac{dy}{dt} = 0$).*

The general solution for linear, automomous, first-order differential equations is given by

$$y(t) = Ae^{-at} + \frac{b}{a} \tag{1}$$

To solve for A we need the an initial condition, $y(0)$. In this case the solution can be expressed as

$$y(t) = \left[y(0) - \frac{b}{a}\right] e^{-at} + \frac{b}{a} \tag{2}$$

1. Solve the differential equation

$$\frac{dy}{dt} = 0.1y - 1$$

with the initial condition $y(0) = 5$

Solution:

$$\begin{aligned} y_c &= Ae^{0.1t} \\ y_p &= 10 \\ y(t) &= \left[5 - \frac{1}{.1}\right] e^{0.1t} + \frac{1}{.1} \\ y(t) &= -5e^{0.1t} + 10 \end{aligned}$$

2. Let $K(t)$ be the capital stock at time t. Let δ be the rate of depreciation ($\delta > 0$) and I_0 is a constant level of investment.

$$\frac{dK}{dt} = I_0 - \delta K$$

and the initial condition $K(0) = K_0$. Find the solution to $K(t)$. Does $K(t)$ converge to a steady-state?

Solution:

$$\begin{aligned} K(t) &= \left([K_0 - (I_0/\delta)]\, e^{-\delta t} + I_0/\delta\right) \\ K(t &\to \infty) = I_0/\delta \end{aligned}$$

3. Dynamics of national debt accumulation.

Let $D(t)$ represent the dollar value of debt and let $Y(t)$ be the value of GDP at time t. Suppose that debt is a constant proportion of GDP, denoted by .θ such that

$$\frac{dD}{dt} = \theta Y \qquad \theta > 0$$

Further assume that national income grows according to the following differential equation

$$\frac{dY}{dt} = \mu Y \qquad \mu > 0$$

Finally, assume that, at time $t = 0$, the initial values of debt and income are D_0 and Y_0

Show the following

(a) the solution to $Y(t)$ is

$$Y(t) = Y_0 e^{\mu t}$$

(b) the general solution to $D(t)$ is

$$D(t) = \theta Y_0 \frac{e^{\mu t}}{\mu} + c_0$$

where c_0 is an arbitrary constant of integration.
(c) Using the initial condition, show that the specific solution for $D(t)$ is

$$D(t) = D_0 + \frac{\theta}{\mu} Y_0 \left(e^{\mu t} - 1\right)$$

Let the interest payments on the debt be rD, where r is the rate of interest. The ratio

$$z(t) = \frac{rD(t)}{Y(t)}$$

is the share of national income used to service the interest on the national debt. By substituting in your previous solutions, show that

$$\lim_{t \to \infty} z(t) = r\frac{\theta}{\mu}$$