Econ 431: *Mathematical Economics* — *Summer 2007*

Assignment 6 — Kevin Wainwright

From Chapter 20, exercise 20.2 questions 2, 3, 4

2 The Hamiltonian is $H = 6y + \lambda y + \lambda u$ (linear in u). Thus to maximize H, we have $u = 2$ (if λ is positive) and $u = 0$ (if λ is negative)

From $\lambda' = -\partial H/\partial y = -6 - \lambda$, we find that $\lambda(t) = ke^{-t} - 6$, but since $\lambda(4) = 0$ from the transversality condition, we have $k = 6e^4$, and

$$\lambda^*(t) = 6e^{4-t} - 6$$

which is positive for all t in the interval $[0, 4]$. Hence the optimal control is $u^*(t) = 2$. From $y' = y + u = y + 2$, we obtain $y(t) = ce^t - 2$. Since $y(0) = 10$, then $c = 12$, and

$$y^*(t) = 12e^t - 2$$

The optimal terminal state is

$$y^*(4) = 12e^4 - 2$$

3 From the maximum principle, the system of differential equations are

$$\begin{aligned} \lambda' &= -\lambda \\ y' &= y + \frac{a + \lambda}{2b} \end{aligned}$$

solving first for λ, we get $\lambda(t) = c_0 e^{-t}$. Using $\lambda(T) = 0$ yields $c_0 = 0$. Therefore,

$$u(t) = \frac{-a}{2b}$$

and

$$y(t) = \left(y_0 + \frac{a}{2b}\right) e^t - \frac{a}{2b}$$

4 The maximum principle yields $u = (y + \lambda)/2$, and the following system of differential equations

$$\begin{aligned} \lambda' &= -(u - 2y) \\ y' &= u \end{aligned}$$

with the boundary conditions $y(0) = y_0$ and $\lambda(T) = 0$.
There are a couple of approaches one may use to arrive at the solutions for λ^*, y^*, and u^*. For the method described below, the reader may wish to review **Sec. 19.2.**

Substituting for u in the system of equations yields

$$\begin{bmatrix} \lambda' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \lambda \\ y \end{bmatrix}$$

The coefficient matrix has a determinant of -1, the roots are ± 1. For $r_1 = 1$, the eigenvector is

$$\begin{bmatrix} -\frac{1}{2} - 1 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} - 1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = 0$$

which yields $m = 1$ and $n = 1$.For $r_2 = -1$, the eigenvector is $m = 1$ and $n = -1/3$. The complete solutions are the homogeneous solutions,

$$\begin{bmatrix} \lambda(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} c_1 e^t + \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix} c_2 e^t$$

From the transversality conditions we get

$$\begin{aligned} c_1 &= \frac{y_0 e^{-2T}}{1/3 + e^{-2T}} \\ c_2 &= \frac{-y_0}{1/3 + e^{-2T}} \end{aligned}$$

The final solution is

$$\begin{aligned} \lambda^* &= \frac{y_0}{1/3 + e^{-2T}} \left(e^{t-2T} - e^{-t}\right) \\ y^* &= \frac{y_0}{1/3 + e^{-2T}} \left(e^{t-2T} + \frac{1}{3} e^{-t}\right) \\ u^* &= \frac{y_0}{1/3 + e^{-2T}} \left(e^{t-2T} - \frac{1}{3} e^{-t}\right) \end{aligned}$$