Econ 431: Bang-Bang Optimal Control Example

Example 1 *Find the optimal control that will*

$$Max \qquad V = \int_0^2 (2y - 3u)\,dt$$

subject to

$$y' = y + u$$
$$y(0) = 4 \qquad y(2) \quad free$$

and

$$u(t) \in U = [0, 2]$$

Since the problem is characterized by linearity in u and a closed control set, we can expect boundary solutions to occur.

1. **Step i** The Hamiltonian of the problem, namely

   $$H = 2y - 3u + \lambda(y + u) = (2 + \lambda)\,y + (\lambda - 3)\,u$$

   is linear in u with the slope

   $$\frac{\partial H}{\partial u} = \lambda - 3$$

   therefore

   $$u^*(t) = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix} \quad if \quad \lambda(t) \begin{Bmatrix} > \\ < \end{Bmatrix} 3$$

   Both $u^* = 2$ and $u^8 = 0$ are boundary solutions

2. **Step ii** Determine $\lambda(t)$. From the equation of motion for λ we have the differential equation

   $$\lambda' = -\frac{\partial H}{\partial y} = -2 - \lambda$$

   or

   $$\lambda' + \lambda = -2$$

   the general solution is

   $$\lambda(t) = ke^{-t} - 2$$

   The transversality condition for this problem is $\lambda(T) = \lambda(2) = 0$. Using this we can write the definite solution as

   $$\lambda^*(t) = 2e^2e^{-t} - 2 = 2e^{2-t} - 2$$

   Note that $\lambda^*(t)$ is a decreasing function of t. with

   $$\lambda^*(0) = 2e^2 - 2 = 12.778$$
   $$\lambda^*(2) = 0$$

   therefore λ starts off being greater than 3, but at some point will become less than 3. Remembering that

   $$\begin{aligned} u^*(t) &= 2 \quad if \quad \lambda(t) > 3 \\ u^*(t) &= 0 \quad if \quad \lambda(t) < 3 \end{aligned}$$

   there is a critical point where $\lambda^*(t) = 3$. Let denote the time when $\lambda^*(t) = 3$. Therefore

   $$\begin{aligned} 2e^{2-\tau} - 2 &= 3 \\ e^{2-\tau} &= 2.5 \\ 2 - \tau &= \ln 2.5 \\ \tau &= 2 - \ln 2.5 = 1.096 \end{aligned}$$

The optimal control can be restated in two phases:

$$\begin{aligned} \text{Phase I} &: \quad u_I^*(t) \equiv u^*\left[0, \tau\right] = 2 \\ \text{Phase II} &: \quad u_{II}^*(t) \equiv u^*\left[\tau, 2\right] = 0 \end{aligned}$$

3. **Step iii** We can also find the optimal state path, which is in two phases. In phase I, the equation of motion is

$$y' = y + u = y + 2$$

or

$$y' - y = 2$$

with $y(0) = 4$. the solution is

$$y_I^* \equiv y^*\left[0, \tau\right] = 2\left(3e^t - 1\right)$$

In phase II, the equation of motion for y is $y' = y + 0$ or

$$y' - y = 0$$

with the general solution

$$y_{II}^* \equiv y^*\left[\tau, 2\right] = ce^t$$

where c is arbitrary. To solve for c we need to use the fact that, at τ, y^* of phase I equals y^* of phase II, or

$$y_I^*(\tau) = 2\left(3e^\tau - 1\right)$$
$$y_{II}^*(\tau) = ce^\tau$$

by equating the two we find that

$$c = 2(3 - e^{-\tau}) = 5.324$$

$$y_{II}^* = 5.324e^t$$

and the vlue of y^* at the time of switching is approximately 15.739

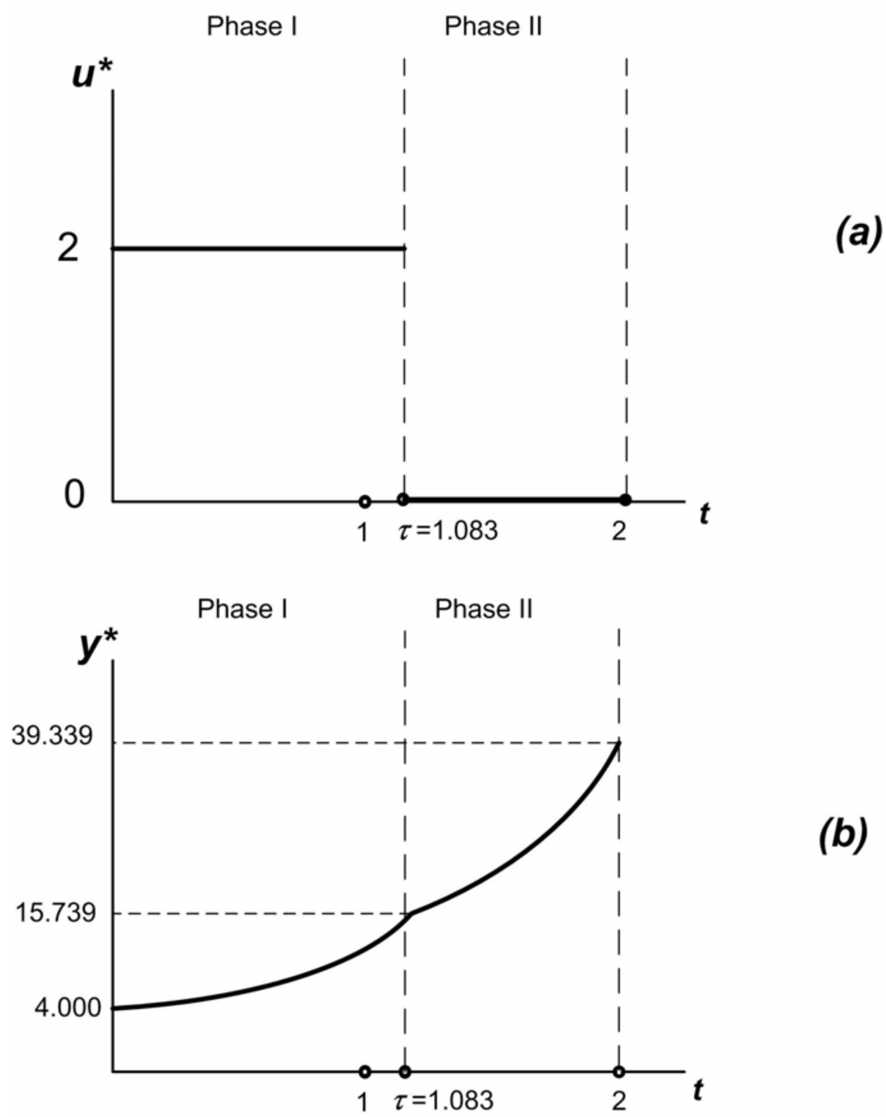
Phase I
Phase II
u*
2
0
1
τ=1.083
2
t
(a)
Phase I
Phase II
y*
39.339
15.739
4.000
1
τ=1.083
2
t
(b)

Figure 1: