# ECON 431

**Linear, First-Order Differential Equations**

## 1 Autonomous Equations

**Definition 1** *The general form of a linear, automomous, first-order differential equation is*

$$\frac{dy}{dt} + ay = b \tag{1}$$

*where $a, b$ are known constants*

### 1.1 Homogeneous solution

**Definition 2** *The homogeneous form of a linear, automomous, first-order differential equation is*

$$\frac{dy}{dt} + ay = 0 \qquad a \neq 0 \tag{2}$$

*where $a, b$ are known constants*

To solve the homogeneous case, re-write eq 2 as

$$\begin{aligned} \frac{dy}{dt} &= -ay \\ \frac{1}{y}\frac{dy}{dt} &= -a \end{aligned}$$

multiply by $dt$ and integrate both sides

$$\begin{aligned} \frac{1}{y}dy &= -adt \qquad (3) \\ \int \frac{1}{y}dy &= \int -adt \\ \ln y &= -at + c \end{aligned}$$

and take the anti-log

$$\begin{aligned} y(t) &= e^{-at+c} \qquad (4) \\ &= e^{-at}e^{c} \\ y(t) &= Ae^{-at} \qquad A \equiv e^{c} \end{aligned}$$

**Theorem 3** *The general solution to equation 2 is*

$$y(t) = Ae^{-at}$$

### 1.2 Particular Solution

**Definition 4** *A steady-state value of a differential equation is defined by the condition $\frac{dy}{dt} = 0$ thus making the value of $y$ stationary. Let $D$ be the stationary value of $y$.*

Letting $\frac{dy}{dt} = 0$, then equation 1 becomes

$$0 + aD = b \tag{5}$$

and solving for $D$ yields

$$D = \frac{b}{a} \qquad a \neq 0$$

For autonomous, linear, first-order differential equations, the steady state, $D$, will be the particular solution

## 1.3 The General Solution

The solution to The general form of a linear, automomous, first-order differential equation is made up of the sum of two parts: the complementary function $y_c$ (or $CF$) and the particular solution (or particular integral), $y_p$ (or $PS$) such that

$$y(t) = y_c + y_p = CF + PS \tag{6}$$

**Definition 5** *The complementary function is the solution to the homogeneous form*

$$y_c = CF = Ae^{-at}$$

**Definition 6** *The particular integral (solution) $y_p$ is ANY solution that completes equation 1. In the case of the linear, automomous, first-order differential equation, this will be the steady-state solution ($\frac{dy}{dt} = 0$).*

The general solution for linear, automomous, first-order differential equations is given by

$$y(t) = Ae^{-at} + \frac{b}{a} \tag{7}$$

To solve for $A$ we need the an initial condition, $y(0)$. In this case the solution can be expressed as

$$y(t) = \left[y(0) - \frac{b}{a}\right] e^{-at} + \frac{b}{a} \tag{8}$$

**Example 7** *Solve the differential equation*

$$\frac{dy}{dt} = 0.1y - 1$$

*with the initial condition $y(0) = 5$*

**Solution:**

$$\begin{aligned} y_c &= Ae^{0.1t} \\ y_p &= 10 \\ y(t) &= \left[5 - \frac{1}{.1}\right] e^{0.1t} + \frac{1}{.1} \\ y(t) &= -5e^{0.1t} + 10 \end{aligned}$$

**Example 8** *Let $K(t)$ be the capital stock at time $t$. Let $\delta$ be the rate of depreciation ($\delta > 0$) and $I_0$ is a constant level of investment.*

$$\frac{dK}{dt} = I_0 - \delta K$$

*and the initial condition $K(0) = K_0$. Find the solution to $K(t)$. Does $K(t)$ converge to a steady-state?*

**Solution:**

$$\begin{aligned} K(t) &= \left([K_0 - (I_0/\delta)]\, e^{-\delta t} + I_0/\delta\right) \\ K(t &\to \infty) = I_0/\delta \end{aligned}$$

**Example 9** *Dynamics of national debt accumulation.*
*Let $D(t)$ represent the dollar value of debt and let $Y(t)$ be the value of GDP at time $t$. Suppose that debt is a constant proportion of GDP, denoted by $.\theta$ such that*

$$\frac{dD}{dt} = \theta Y \qquad \theta > 0$$

*Further assume that national income grows according to the following differential equation*

$$\frac{dY}{dt} = \mu Y \qquad \mu > 0$$

*Finally, assume that, at time $t = 0$, the initial values of debt and income are $D_0$ and $Y_0$*
*Show the following*
*(a) the solution to $Y(t)$ is*

$$Y(t) = Y_0 e^{\mu t}$$

*(b) the general solution to $D(t)$ is*

$$D(t) = \theta Y_0 \frac{e^{\mu t}}{\mu} + c_0$$

*where $c_0$ is an arbitrary constant of integration.*
*(c) Using the initial condition, show that the specific solution for $D(t)$ is*

$$D(t) = D_0 + \frac{\theta}{\mu} Y_0 \left(e^{\mu t} - 1\right)$$

*Let the interest payments on the debt be $rD$, where $r$ is the rate of interest. The ratio*

$$z(t) = \frac{rD(t)}{Y(t)}$$

*is the share of national income used to service the interest on the national debt. By substituting in your previous solutions, show that*

$$\lim_{t\to\infty} z(t) = r\frac{\theta}{\mu}$$

# 2 Variable Coefficient and Variable Term

The general form of a linear, first-order differential equation is

$$\frac{dy}{dt} + u(t)y = w(t)$$

where $u(t), w(t)$ are the variable coefficient and variable term respectively.

## 2.1 Homogeneous case

For this case, where $w(t) = 0$, the solution is still easy

$$\begin{aligned} \frac{dy}{dt} + u(t)y &= 0 \\ \frac{1}{y}\frac{dy}{dt} &= -u(t) \end{aligned}$$

just integrate both sides; the left side

$$\int \frac{1}{y}\frac{dy}{dt}dt = \int \frac{dy}{y} = \ln y + c$$

the right side

$$\int -u(t)dt = -\int u(t)dt$$

equate both sides

$$\ln y = -c - \int u(t)dt \tag{9}$$

solving for $y(t)$

$$y(t) = e^{-c}e^{-\int u(t)dt} = Ae^{-\int u(t)dt} \tag{10}$$

Note that this is the general solution. When $u(t) = a$ then $\int adt = at + c$ which is our earlier solution

### 2.2 Nonhomogeneous Case

For the nonhomogeneous case, where $w(t) \neq 0$, the general solution is

$$y(t) = e^{-\int u(t)dt}\left(A + \int w e^{\int u(t)dt}\right) \tag{11}$$

Where $A$ is an arbitrary constant found by the appropriate initial conditions

Example

Find the general solution of the equation

$$\frac{dy}{dt} + 2ty = t$$

Here we have

$$u = 2t \quad w = t \quad and \quad \int udt = t^2 + k$$

thus by equation 11 we have

$$\begin{aligned}
y(t) &= e^{-(t^2+k)}\left(A + \int te^{(t^2+k)}dt\right) \\
&= e^{-t^2}e^{-k}\left(A + e^k \int te^{t^2}dt\right) \\
&= Ae^{-t^2}e^{-k} + e^{-t^2}\left(\tfrac{1}{2}e^{t^2} + c\right) \qquad (e^k e^{-k} = 1) \\
&= \left(Ae^{-k} + c\right)e^{-t^2} + \frac{1}{2} \\
&= Be^{-t^2} + \frac{1}{2}
\end{aligned}$$

## 3 Phase Diagrams

Given a differential equation (linear or nonlinear) of the form

$$\frac{dy}{dt} = f(y)$$

it is feasible to plot $\frac{dy}{dt}$ against $y$ whenever $dy/dt$ is a function of $y$ alone. Such a geometric representation is called a *phase diagram* and the line representing the function $f(y)$ is called a *phase line.*

Once a phase line is known, its configuration will impart significant information regarding the time path of $y(t)$.

Consider the case where

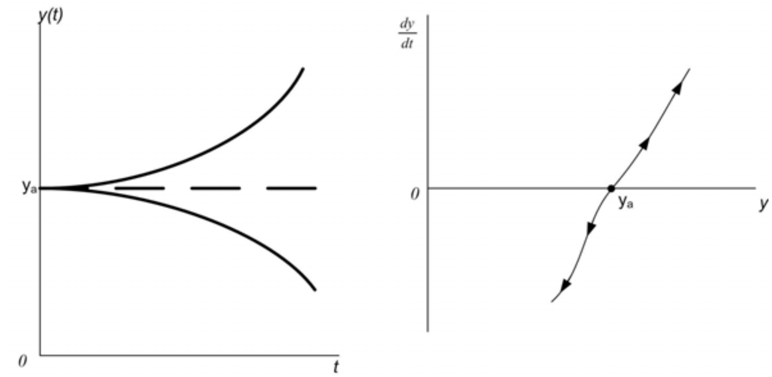
$$\frac{dy}{dt} + ay = b$$

or

$$\frac{dy}{dt} = -ay + b$$

where $-a$ is the slope of the phase diagram. We may infer that

$$a \gtrless \quad \Longleftrightarrow \quad y(t)\left\{\begin{array}{c}\text{converges to}\\ \text{diverges from}\end{array}\right\}\text{ equilibrium}$$

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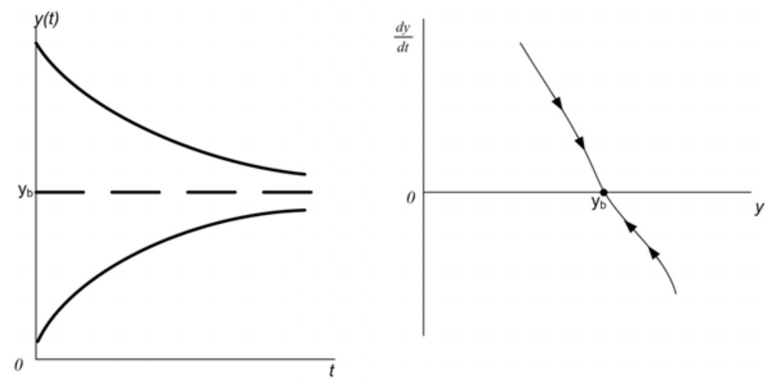


(a)

Figure 1:

# 4 Solow Growth Model

Domar's model had capital and labour in fixed proportions. Solow's model looks at the case where capital and labour can be combined in variable propotions.

Consider

$$Q = f(K, L)$$

where $f_L, f_K$ are positive and $f_{LL}, f_{KK}$ are negative. Furthermore, $f$ is linearly homogeneous. Therefore

$$Q = Lf(\frac{K}{L}, 1) = L\phi(k) \tag{12}$$

where $\phi'(k) > 0$, $\phi''(k) < 0$

Solow's assumption

$$\dot{K} = \frac{dK}{dt} = sQ \tag{13}$$

a constant proportion of output is invested (where $s$ is a constant MPS). and

$$\frac{\dot{L}}{L} = \frac{dL/dt}{L} = \lambda \tag{14}$$

Labour force grows exponentially

Combining equations 12,13 and 14 yields

$$\dot{K} = sL\phi(k) \tag{15}$$

Since $k = K/L$ and $K = kL$ we can obtain another expression for $\dot{K}$ by differentiating $K = kL$

$$\begin{aligned} \dot{K} &= L\dot{k} + k\dot{L} \\ \dot{K} &= L\dot{k} + k\lambda L \end{aligned} \tag{16}$$

equations 15 and 16 can combine to give us

$$\dot{k} = s\phi(k) - \lambda K \tag{17}$$

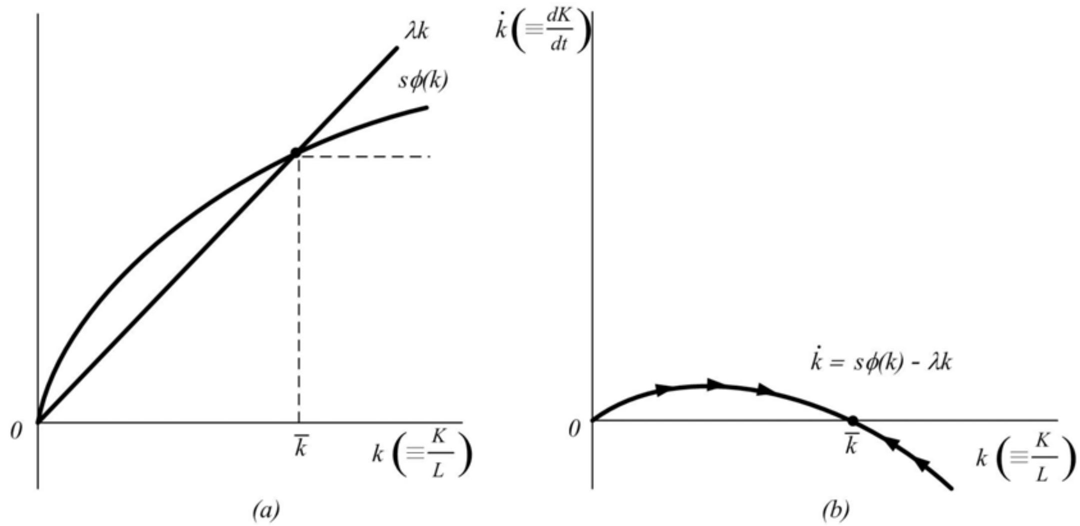


Figure 15.5

Figure 2: