1 Selected answers to practice final 1

Spring 1998 (A)

Question 1.
The profit function:

$$\pi = pq - wL - rK = p(60K + 34L - 4KL - 6K^2 - 3L^2) - wL - rK$$

The FOC's give:

$$\begin{array}{c} 34 - 4K - 6L - \frac{w}{p} = 0 \\ 60 - 4L - 12K - \frac{r}{p} = 0 \end{array}$$

which can be solved for K and L. i.e. by Cramer's rule

$$\begin{bmatrix} 6 & 4 \\ 4 & 12 \end{bmatrix} \begin{bmatrix} L \\ K \end{bmatrix} = \begin{bmatrix} 34 - w/p \\ 60 - r/p \end{bmatrix}$$

where $|A| = 60$

$$\begin{aligned} L^* &= \frac{\begin{vmatrix} 34 - w/p & 4 \\ 60 - r/p & 12 \end{vmatrix}}{60} \\ K^* &= \frac{\begin{vmatrix} 6 & 34 - w/p \\ 4 & 60 - r/p \end{vmatrix}}{60} \end{aligned}$$

The Hessian is

$$\begin{aligned} |H| &= \begin{vmatrix} -6 & -4 \\ -4 & -12 \end{vmatrix} \\ |H_1| &= -6 < 0 \\ |H| &= 60 > 0 \end{aligned}$$

Question Two:
The Lagrangian is

$$Z = xy^2 + \lambda_1(100 - x - y) + \lambda_2(120 - 2x - y)$$

and the KT conditions:

$$\begin{aligned} Z_x &= y^2 - \lambda_1 - 2\lambda_2 \leq 0 \\ Z_y &= 2xy - \lambda_1 - \lambda_2 \leq 0 \\ Z_1 &= 100 - x - y \geq 0 \\ Z_2 &= 120 - 2x - y \geq 0 \end{aligned}$$

Solution: $x^* = 20$ and $y^* = 80$. The second constraint is Binding.

Question Three
The lagrange equation is

$$Z = 200Q_1 - 0.5Q_1^2 + 190Q_2 - 0.5Q_2^2 - 10(Q_1 + Q_2) - 20K + \lambda_1(K - Q_1) + \lambda_2(K - Q_2)$$

The Kuhn-Tucker conditions are

$$\begin{array}{ll} Z_1 = 200 - Q_1 - 10 - \lambda_1 \leq 0 & Q_1 \geq 0 \\ Z_2 \ = 190 - Q_2 - 10 - \lambda_2 \leq 0 & Q_2 \geq 0 \\ Z_K = -20 + \lambda_1 + \lambda_2 \leq 0 & K \geq 0 \\ Z_{\lambda_1} = K - Q_1 \geq 0 & \lambda_1 \geq 0 \\ Z_{\lambda_2} = K - Q_2 \geq 0 & \lambda_2 \geq 0 \end{array}$$

Assuming that $Q_1, Q_2, K > 0$ the first-order conditions become

$$\begin{array}{ll} 200 - Q_1 = 10 + \lambda_1 = 30 - \lambda_2 & (\lambda_1 = 20 - \lambda_2) \\ 190 - Q_2 = 10 + \lambda_2 & \end{array}$$

both constraints are binding and $Q_1 = Q_2 = K = 175$ and $\lambda_1 = 15$, $\lambda_2 = 5$

Question 4: Not applicable for this term's final.

Question five:

$$Z = x^2y + \lambda(B - p_x x - p_y y)$$

the FOC's

$$\begin{aligned} 2xy - \lambda p_x &= 0 \\ x^2 - \lambda p_y &= 0 \\ B - p_x x - p_y y &= 0 \end{aligned}$$

Where $x^* = 2B/3p_x$ and $y^* = B/3p_y$.
The Bordered Hessian is:

$$\begin{bmatrix} 2y & 2x & -p_x \\ 2x & 0 & -p_y \\ -p_x & -p_y & 0 \end{bmatrix}$$

$$\left|\bar{H}\right| = 2Bp_y > 0$$

Hint: substitute optimal x and y into the bordered hessian

$$\frac{\partial x}{\partial p_x} = \frac{\begin{vmatrix} \lambda & 2x & -p_x \\ 0 & 0 & -p_y \\ x & -p_y & 0 \end{vmatrix}}{\left|\bar{H}\right|} = \lambda \frac{\overset{S.E.}{\begin{vmatrix} 0 & -p_y \\ -p_y & 0 \end{vmatrix}}}{\left|\bar{H}\right|} + x \frac{\overset{I.E.}{\begin{vmatrix} 2x & -p_x \\ 0 & -p_y \end{vmatrix}}}{\left|\bar{H}\right|} = \frac{\overset{S.E.}{-\lambda p_y^2}}{\left|\bar{H}\right|} + \frac{\overset{I.E.}{-2x^2 p_y}}{\left|\bar{H}\right|} < 0$$

This answer can be simplified further