ECONOMICS 331
Mathematical Economics
HOMEWORK

January 7, 1997

Instructions:

- Assignments are due in tutorial each week, starting in week three (2nd tutorial). There will be no assignment due the week of the midterm (week 7).
- Assignments will be marked primarily on effort. Therefore there is little return to copying. Students can be *expected to explain* their work in the tutorial.
- Students are encouraged to attempt to put their assignments on *MAPLE*. Therefore bonus marks will be given for each successful attempt.

1 Homework Assignment #1

[1] Two markets for two commodities interact with each other in the sense that the demand for each product depends not only on its own price, but also on the prices of other products. Suppose that the demand functions are as follows:

$$q_1^d = 20 - 2p_1 + p_2 \qquad q_2^d = 25 + p_1 - 3p_2$$

Suppliers are assumed willing to produce these two products according to the following supply functions

$$q_1^s = 2p_1 \qquad q_2^s = 2p_2$$

a) What relationship in consumption do these two product have?
b) Find an expression for their inverse demand functions.
(i.e. write as $p_i = f(q_i, q_j)$)
c) Find the market clearing prices and quantities for both goods. Graph your results

[2] Find the vector x satisfying the matrix equation $\mathbf{A}(\mathbf{x} + \mathbf{c}) = \mathbf{d}$
where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \qquad c = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

[3] Consider the following macroeconomic model:

	goods market		money market
1	$Y = C + I + G_0$	1	$M^d = kY - \beta r$
2	$C = C_0 + b(Y - T)$	2	$M^d = M_0^s$
3	$I = I_0 - \alpha r$		
4	$T = tY$		

a) Use equations 1 through 4 to find an expression for the IS curve (Y as a function of r). Use equations 5 and 6 to find an expression for the LM curve (Y as a function of r).

b) Graph the IS and the LM with r on the vertical axis and Y on the horizontal axis.(Hint: invert the two equations you derived in **a**)

c) Write the two equations in matrix form (i.e. $Ax = d$); where the x vector contains two elements, Y and r. (Hint: A is 2×2)

2 Homework Assignment #2

[**1**] In the interest of prudent diversification, an investor wishes to have his \$100 of wealth invested as \$50 in the Canadian economy, \$30 in the U.S. economy, and \$20 in the English economy. Although he can purchase the shares of firms that are *based* in Canada, the U.S. and England, it happens that each of these firms conducts some of its operations through foreign subsidiaries.

In particular, the Canadian based firm (C) has 75% of its operations in Canada but 25% in the U.S., the U.S. based firm (U) has 85% of its operations in the U.S. but 15% in England, and the England based firm (E) has 50% of its operations in England but 30% in Canada and 20% in the U.S.

- The problem for the investor is to determine the proper amounts to invest in each of these three firms to achieve his desired investment in the three economies. Write down a matrix equation that represents the problem he has to solve. Let the amounts invested in the three firms be represented by the column vector:

$$(C, U, E,)^T$$

- Use **matrix inversion** to solve for the amount that he should invest in each firm simultaneously.

3 Homework Assignment #3

From homework assignment 2 we had the following demand functions:

$$q_1^d = 20 - 2p_1 + p_2 \qquad q_2^d = 25 + p_1 - 3p_2$$

Use Cramer's Rule to find the inverse demand functions

$$P_i = f(q_i, q_j) \qquad i = 1, 2$$

[**3**] In homework [**2**] you derived an expression for the IS and LM curves using the following macroeconomic model:

	goods market		money market
1	$Y = C + I + G_0$	1	$M^d = kY - \beta r$
2	$C = C_0 + b(Y - T)$	2	$M^d = M_0^s$
3	$I = I_0 - \alpha r$		
4	$T = tY$		

a) Use Cramer's rule to solve for the equilibrium level of income and interest rates, Y^e and r^e.
b) From your solution for Y^e what is the coefficient in front of M_0^s? What is its sign? What is the economic interpretation of this coefficient?

4 Homework Assignment 4

$$Q_d = kP^{-\epsilon}(\epsilon > 0) \qquad Q_s = c(P - T_0)$$

are the demand and supply curves for a commodity whose price is P. T_0 is a sales tax imposed by the government on sales of this commodity.

(a) Compute the price elasticity of demand and the price elasticity of supply of this commodity.
(b) Find an expression for $\partial P/\partial T_0$ where P is the equilibrium price in this market.
(c) Show that

$$0 < \partial P/\partial T_0 < 1.$$

(d) Total tax revenue collected by the government is $R = T_0$Q. Find an expression for $\partial R/\partial T_0$.

[2] Consider the operator of a small newsprint production plant. He has fixed costs of $1600 per day for property taxes, interest on debt, etc., and has additional costs that vary with the amount Q of newsprint he produces. His total costs per day are given by

$$C = 1600 + 40Q + \frac{2}{3}Q^{3/2}$$

He sells his output to local newspaper publishers and the price he obtains depends on how much he tries to sell. Let the demand curve he faces be

$$P = 100 + 512Q^{-1/2}$$

(a) What are his average and marginal costs of production expressed as functions of Q?
(b) What are his total and marginal revenues expressed as functions of Q?
(c) How much output per day should he produce if he expands production just to the point where marginal revenue equals marginal cost? What profit if any will he make in $/day?
(hint: substitute $X = Q^{1/2}$ into the equation you have to solve, determine X, then Q)

5 Homework Assignment 5

[1] Consider the firm with a single factor of production defined implicitly by the relation

$$z = q^3 + 4q$$

where z is the variable input and q is output. The firm faces the following average revenue function:

$$p = 10 - 2q$$

a) Calculate the point elasticity of the firm's total sales revenue with respect to the amount of labour used when $q = 2$.

[2] The following three equations define x, y, and w as functions of z.

$$\begin{aligned} xy - w &= 0 \\ y &= w^3 + 3z \\ w^3 + z^3 &= 2wz \end{aligned}$$

Find an expression for $\partial x/\partial z$ and evaluate it at the point where

$$w = z = 1$$

[3] The equation

$$x^2 + y^2 + z^2 + xy + xz + yz + x + y + z - 1 = 0$$

has one solution $(x, y, z) = (1, -1, -1)$. Check that the equation does indeed define z as a function of x and y at this point. Calculate the partial derivatives of z with respect to x and y at this point.

6 Homework Assignment 6

[1] The following function has zero slope at the point $z = 1$. Determine whether or not this point is a relative extremum, and, if so, whether it is a maximum or minimum.

$$z^4 - 6z^3 + 12z^2 - 10z + 37$$

[2] You are an assembler of specialty computer terminals with a modest amount of monopoly power. Suppose that your average revenue per unit depends on how many terminals per day you wish to sell, and is given by

$$AR(y) = -y^3 + 12y^2 - 30y + 1000$$

where y is sales per day. Suppose further that your average cost of production is given by

$$AC(y) = 2y + 1000 - \frac{100}{y}$$

Notice that your total costs are negative if you choose to produce nothing. This is because you recieve a grant from the government for setting up in Surrey, B. C.

a) Write out an expression for your profits as a function of output, y.

b) Determine the most profitable level of output. Show that this output level does indeed lead to a maximum rather than a minimum by checking the second order conditions.

[3] Given the general function $y = f(x)$ and the following conditions

$x = 0$	$f(x) = 0$	$f' = 0$	$f'' > 0$
$0 < x < 0$		$f' > 0$	$f'' > 0$
$x = 1$		$f' > 0$	$f'' = 0$
$1 < x < 2$		$f' > 0$	$f'' < 0$
$x = 2$		$f' = 0$	$f'' = 0$
$2 < x < 3$		$f' > 0$	$f'' > 0$
$x = 3$		$f' > 0$	$f'' = 0$
$3 < x < 4$		$f' > 0$	$f'' < 0$
$x = 4$		$f' = 0$	$f'' < 0$
$4 < x \leq 5$		$f' < 0$	$f'' < 0$

Graph f(x) over the range 0 to 5. Label and identify all critical points.

7 Homework Assignment 7

Suppose that the output q of a firm depends on the quantities of z_1 and z_2 that it employs as inputs. Its output level is determined by the production function

1. $$q = 26z_1 + 24z_2 - 7z_1^2 - 12z_1z_2 - 6z_2^2$$

2. Write down the firm's profit function when the price of q is \$1 and the factor prices are w_1 and w_2(per unit) respectively.

3. Find the levels of z_1^* and z_2^* which maximize the firm's profits. Note that these **are** the firm's **demand** curves for the two inputs.

4. Verify that your solution to **[2]** satisfies the second order conditions for a maximum.

5. What will be the effect of an increase in w_1 on the firm's use of each input and on its output q? [hint: You do not have to explicitly determine the firm's supply curve of output to determine $\partial q/\partial w_1$. Instead write out the total derivative of q and make use of the very simple expressions for $\partial q/\partial z_1$ and $\partial q/\partial z_2$ at the optimum that can be obtained from the first order conditions.]

6. Is the firm's production function strictly concave? Explain.

8 Homework Assignment 8

Skip has the following utility function: $U(x, y) = x(y + 1)$, where x and y are quantities of two consumption goods whose prices are p_x and p_y respectively. Skip has a budget of B. Therefore the Skip's maximization problem is

$$x(y + 1) + \lambda(B - p_x x - p_y y)$$

a) From the first order conditions find expressions for the demand functions

$$x^* = x(p_x, p_y, B) \qquad y^* = y(p_x, p_y, B)$$

Carefully graph x* and y*. Graph Skip's indifference curves. What kind of good is y?

b) Verify that skip is at a maximum by checking the second order conditions.

c) By substituting x^* and y^* into the utility function find an expressions for the indirect utility function,

$$U = U(p_x, p_y, B)$$

d) By rearranging the indirect utility function, derive an expression for the expenditure function,

$$B^* = B(p_x, p_y, U_0)$$

Interpret this expression. Find $\partial \mathbf{B}/\partial \mathbf{p}_x$ and $\partial \mathbf{B}/\partial \mathbf{p}_y$.

Skip's maximization problem could be recast as the following minimization problem:

$$p_x x + p_y y \text{ s.t. } U_0 = x(y+1)$$

e) Write down the lagrangian for this problem.

f) Find the values of x and y that solve this minimization problem and show that the values of x and y are equal to the partial derivatives of the expenditure function, $\partial \mathbf{B}/\partial \mathbf{p}_x$ and $\partial \mathbf{B}/\partial \mathbf{p}_y$ respectively. (Hint: use the indirect utility function)

9 Homework Assignment 9

[1] Given the utility maximization problem

$$\text{Max } U = xy \text{ subject to } B = p_x x + p_y y$$

a) Derive an expression for the Slutsky equation for x and y when p_x changes. Identify the income and substitution effects. Can you sign the effects?

[2] Consider the following duopoly market where the market demand curve is given by

$$p = 120 - (q_1 + q_2)$$

where q_1 and q_2 are the outputs of firm 1 and firm 2 respectively.

Firm $1's$ cost function is

$$TC(q_1) = 75 + 35q_1$$

and firm $2's$ cost function is

$$TC(q_2) = 100 + 40q_2$$

Find the equilibrium prices, quantities, and profits when:
a) When firm 1 is a monopolist using the limit output stratagy to keep firm 2 out of the market.
b) When firm 1 and firm 2 are cournot duopolists.
c) When firm 1 and firm 2 are duopolists but firm 1 chooses his output first, taking into account the fact that firm $2's$ choice of depends on firm $1's$ choice of output.
Show all your work.
Graph your results from a,b, and c.

10 Homework Assignment 10

[1] Suppose a perfectly competitive, profit maximizing firm has only two inputs, capital and labour. The firm can buy as many units of capital and labour as it wants at constant factor prices r and w respectively.

If the firm's production function has constant returns to scale, use Euler's theorem to show that in long-run equilibrium the firm earns zero profits.
(hint: look at the first order conditions for profit maximization)

[2] Suppose that there are two types of electricity (peak and off-peak).Half the day is peak and half the day is off-peak. To produce a unit of electricity per half-day requires a unit of turbine capacity costing 8 cents per day (interest charges on a permanent loan). The cost of a unit of capacity is the same whether it is used at peak times only or off-peak also. In addition to the costs of turbine capacity, it costs 6 cents in operating costs (labour and fuel) to produce 1 unit per half day.

Suppose the demand for electricity per half day during peak hours is

$$p = 22 - 10^{-5}q$$

and during off-peak hours is

$$p = 18 - 10^{-5}q$$

where **q** is units of electricity per half-day and **p** is price in cents.

a) Write down the Kuhn-Tucker conditions for profit maximization.
b) What are the profit maximizing peak and off-peak prices?
c) If a unit of capacity cost only 3 cents per day, what would the profit maximizing peak and off-peak prices be?