ECONOMICS 331
Mathematical Economics
Kevin Wainwright

Homework Assignment 10 ANSWER KEY

1. A new cellphone company "Yap.com" is setting up in Burnaby and it has to plan its capacity. The peak period demand is given by $p_1 = 200 - 0.25q_1$ and the off-peak is given by $p_2 = 190 - 0.25q_2$. Let K be the cell capacity which costs 15 per unit and is only paid once and is used in both periods. There are no other costs. The quantity of cell usage in either market (q_1, q_2) cannot excede K.

   (a) write down the Kuhn-Tucker conditions.

$$Z = 200q_1 - 0.25q_1^2 + 190q_2 - 0.25q_2^2 - 15K + \lambda_1 (K - q_1) + \lambda_2 (K - q_2)$$

$$\begin{array}{ll} Z_1 = 200 - 0.5q_1 - \lambda_1 \leq 0 & q_1 \geq 0 \\ Z_2 = 190 - 0.5q_2 - \lambda_2 \leq 0 & q_2 \geq 0 \\ Z_K = -15 + \lambda_1 + \lambda_2 \leq 0 & K \geq 0 \\ Z_{\lambda_1} = K - q_1 \geq 0 & \lambda_1 \geq 0 \\ Z_{\lambda_2} = K - q_2 \geq 0 & \lambda_2 \geq 0 \end{array}$$

   (b) find the optimal outputs and capacity for this problem.How much of the capacity price is paid for by each market?
   In this case, both constraints are binding

$$\begin{array}{l} Z_1 = 200 - 0.5q_1 = \lambda_1 \\ Z_2 = 190 - 0.5q_2 = \lambda_2 \\ Z_K = -15 + \lambda_1 + \lambda_2 = 0 \\ Z_{\lambda_1} = K - q_1 = 0 \\ Z_{\lambda_2} = K - q_2 = 0 \end{array}$$

   By substitution

$$\begin{array}{l} Z_1 = 200 - 0.5K = \lambda_1 \\ Z_2 = 190 - 0.5K = 15 - \lambda_1 \end{array}$$

$$\begin{aligned} K &= q_1 = q_2 = 375 \\ \lambda_1 &= 12.5, \lambda_2 = 2.5 \end{aligned}$$

   (c) Suppose price of capacity is now 5 per unit of capacity. Redo part (b)

$$\begin{array}{l} Z_1 = 200 - 0.5q_1 = 5 \\ Z_2 = 190 - 0.5q_2 = 0 \\ \lambda_1 = 5 \\ K = q_1 \\ K = q_1 = 390, \qquad q_2 = 380 \end{array}$$

2. Consider the following utility maximization problem with a Cobb-Douglas utility function: $U(x, y) = Ax^\alpha y^\beta$ subject to $M = p_x x + p_y y$
(where $A, \alpha, \beta > 0$)

(a) Show that the indirect utility function for this problem is

$$U^* = A\left(\frac{M}{\alpha+\beta}\right)^{\alpha+\beta}\left(\frac{\alpha}{p_x}\right)^{\alpha}\left(\frac{\beta}{p_y}\right)^{\beta}$$

$$\begin{aligned} L &= Ax^\alpha y^\beta + \lambda\left(M - p_x x - p_y y\right) \\ L_x &= \alpha A x^{\alpha-1} y^\beta - \lambda p_x = 0 \\ L_x &= \beta A x^\alpha y^{\beta-1} - \lambda p_y = 0 \\ L_\lambda &= M - p_x x - p_y y = 0 \end{aligned}$$

From (1) an (2) of the FOC's

$$\frac{\alpha}{\beta}\frac{y}{x} = \frac{p_x}{p_y}$$

Substitute into (3) to get x^* and y^*. Use your solution to x and FOC (1) to find λ

$$\begin{aligned} x^* &= x^M = \frac{\alpha}{\alpha+\beta}\frac{M}{p_x}, \qquad y^* = y^M = \frac{\beta}{\alpha+\beta}\frac{M}{p_y} \\ \lambda^* &= A\left(\frac{M}{\alpha+\beta}\right)^{\alpha+\beta-1}\left(\frac{\alpha}{p_x}\right)^{\alpha}\left(\frac{\beta}{p_y}\right)^{\beta} \\ U^* &= A\left(\frac{M}{\alpha+\beta}\right)^{\alpha+\beta}\left(\frac{\alpha}{p_x}\right)^{\alpha}\left(\frac{\beta}{p_y}\right)^{\beta} \end{aligned}$$

(b) Verify that Roy's identity holds for this problem

$$\begin{aligned} \frac{\partial U/\partial p_x}{\partial U/\partial M} &= \frac{A\left(\frac{M}{\alpha+\beta}\right)^{\alpha+\beta}\alpha\left(\frac{\alpha}{p_x}\right)^{\alpha-1}\left(\frac{-\alpha}{p_x^2}\right)\left(\frac{\beta}{p_y}\right)^{\beta}}{(\alpha+\beta)\,A\left(\frac{M}{\alpha+\beta}\right)^{\alpha+\beta-1}\left(\frac{1}{\alpha+\beta}\right)\left(\frac{\alpha}{p_x}\right)^{\alpha}\left(\frac{\beta}{p_y}\right)^{\beta}} \\ &= \frac{\alpha}{\alpha+\beta}\frac{M}{p_x} \end{aligned}$$

3. The dual for problem (3) can be expressed as minimize $p_x x + p_y y$ subject to $U_0 = Ax^\alpha y^\beta$, where $U_0 = U^*$ (from above).

(a) Find x^* and y^* that satisfies this minimization problem.

$$L = p_x x + p_y y + \lambda(U_0 - Ax^\alpha y^\beta)$$

The FOC's are

$$\begin{aligned} L_x &= p_x - \lambda\alpha A x^{\alpha-1} y^\beta = 0 \\ L_y &= p_y - \lambda\beta A x^\alpha y^{\beta-1} = 0 \\ L_\lambda &= U_0 - Ax^\alpha y^\beta = 0 \end{aligned}$$

From the first two equations, we get

$$\frac{p_x}{p_y} = \frac{\alpha}{\beta}\frac{y}{x}$$

Substituting into the third yields

$$\begin{aligned} x^* &= x^h = \left[\frac{U_0}{A}\left(\frac{\alpha p_y}{\beta p_x}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} \\ y^* &= y^h = \left[\frac{U_0}{A}\left(\frac{\beta p_x}{\alpha p_y}\right)^{\alpha}\right]^{\frac{1}{\alpha+\beta}} \end{aligned}$$

Find an expression for the expenditure function $M^*(p_x, p_y, U_0)$

$$M^*(p_x, p_y, U_0) = (\alpha+\beta)\left[\frac{U^*}{A}\left(\frac{p_x}{\alpha}\right)^{\alpha}\left(\frac{p_y}{\beta}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}}$$

(b) Use the expenditure function derived in (a) to verify that Shephard's Lemma holds for this minimization problem.

$$\begin{aligned} \frac{\partial M^*}{\partial p_x} &= \left[\frac{U_0}{A}\left(\frac{\alpha p_y}{\beta p_x}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} \\ \frac{\partial M^*}{\partial p_y} &= \left[\frac{U_0}{A}\left(\frac{\beta p_x}{\alpha p_y}\right)^{\alpha}\right]^{\frac{1}{\alpha+\beta}} \end{aligned}$$

(c) show that λ from the minimization problem equals $1/\lambda$ from the maximization problem in problem 3

First, we take L_x from the Max problem and solve for λ

$$\begin{aligned} L_x &= \alpha A x^{\alpha-1} y^{\beta} - \lambda p_x = 0 \\ \alpha A x^{\alpha-1} y^{\beta} &= \lambda p_x \\ \lambda &= \frac{\alpha A x^{\alpha-1} y^{\beta}}{p_x} \end{aligned}$$

Then we take L_x from the Min problem and solve for λ

$$\begin{aligned} L_x &= p_x - \lambda \alpha A x^{\alpha-1} y^{\beta} = 0 \\ p_x &= \lambda \alpha A x^{\alpha-1} y^{\beta} \\ \lambda &= \frac{p_x}{\alpha A x^{\alpha-1} y^{\beta}} \end{aligned}$$

Which verifies the duality condition that $\lambda^{MAX} = 1/\lambda^{MIN}$

4. Consider the following duopoly market where the market demand curve is given by

$$\begin{aligned} p &= 120 - 0.5Q \\ Q &= \sum_{i=1}^{N} q_i \end{aligned}$$

where Q is the market output, q_i is the output of firm i and N is the number of firms

(a) Suppse $N = 1$ and Firm $1's$ cost function is

$$C(q_1) = 75 + 35q_1$$

Find the monopoly price, quantity and profit

$$\begin{aligned} q &= 85 \\ p &= 77.5 \\ \pi &= (77.5 - 35)85 - 75 = 3537.5 \end{aligned}$$

:

(b) Suppose $N = 2$. Firm 1's cost function is the same as berfore but firm $2's$ cost function is

$$C(q_2) = 100 + 40q_2$$

Write down the profit function for each firm. Find each firm's "Best Response Function" from the first order conditions. Carefully graph and label each response function in a graph drawn in $q_1 \cdot q_2$ space

$$\begin{aligned} q_1 &= 85 - \frac{1}{2}q_2 \\ q_2 &= 80 - \frac{1}{2}q_1 \end{aligned}$$

(c) Find the Cournot duopoly equilibrium prices, quantities, and profits. Label your solution in your graph from (b)

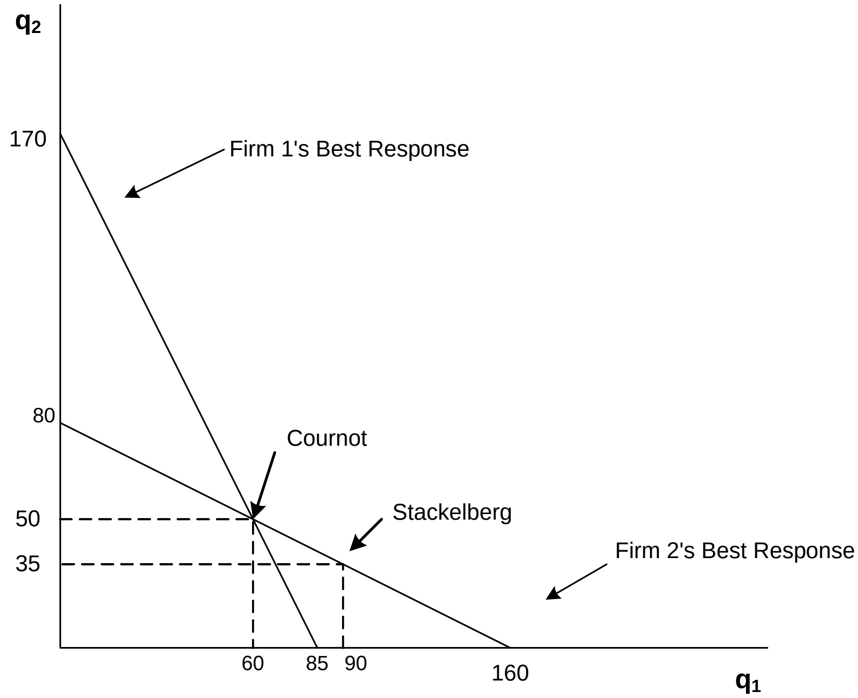
$$\begin{aligned} q_1 &= 60, q_2 = 50, p = 65 \\ \pi_1 &= 1725, \pi_2 = 1150 \end{aligned}$$

(d) First Mover Advantage: When firm 1 and firm 2 are duopolists but firm 1 chooses his output first, taking into account the fact that firm $2's$ choice of depends on firm $1's$ choice of output. This is done by (i) substituting firm 2's best response function into firm 1's profit function, (ii) maximizing firm 1's profit function by choice of q_1 and (c) using the solution to q_1 in firm 2's best response function to find q_2. LABEL this solution in your graph for (b).
First, set up Firm 1's profit function including firm 2's Best Response:

$$\begin{aligned} \pi_1 &= 120q_1 - 0.5q_1^2 - q_1 q_2 - 75 - 35q_1 \\ \pi_1 &= 85q_1 - 0.5q_1^2 - q_1 \left[80 - \frac{1}{2}q_1\right] - 75 \end{aligned}$$

Max firm 1's profit first to find q_1 and then find q_2 from 2's best response:

$$\begin{aligned} q_1 &= 90, q_2 = 35, p = 57.5 \\ \pi_1 &= 1950, \pi_2 = 512 \end{aligned}$$



(e) Now suppose $N = 3$ and all firms have the same cost function: $C(q_i) = 100 + 40q_i$. Using the same approach as in (b), find the Cournot quantities, price and profits for three firms.

The Best Response functions are:

$$\begin{aligned} q_1 &= 80 - \frac{1}{2}q_2 - \frac{1}{2}q_3 \\ q_2 &= 80 - \frac{1}{2}q_1 - \frac{1}{2}q_3 \\ q_3 &= 80 - \frac{1}{2}q_2 - \frac{1}{2}q_1 \end{aligned}$$

The solutions are:

$$\begin{aligned} q_1 &= q_2 = q_3 = 40, p = 60 \\ \pi_i &= 700 \end{aligned}$$