Partial Derivatives

Single variable calculus is really just a "special case" of multivariable calculus. For the function \( y = f(x) \), we assumed that \( y \) was the endogenous variable, \( x \) was the exogenous variable and everything else was a parameter. For example, given the equations

\[
y = a + bx
\]

or

\[
y = ax^n
\]

we automatically treated \( a, b, \) and \( n \) as constants and took the derivative of \( y \) with respect to \( x \) (\( dy/dx \)). However, what if we decided to treat \( x \) as a constant and take the derivative with respect to one of the other variables? Nothing precludes us from doing this. Consider the equation

\[
y = ax
\]

where

\[
\frac{dy}{dx} = a
\]

Now suppose we find the derivative of \( y \) with respect to \( a \), but TREAT \( x \) as the constant. Then

\[
\frac{dy}{da} = x
\]

Here we just "reversed" the roles played by \( a \) and \( x \) in our equation.

Two Variable Case:

let \( z = f(x, y) \), which means "\( z \) is a function of \( x \) and \( y \)". In this case \( z \) is the endogenous (dependent) variable and both \( x \) and \( y \) are the exogenous (independent) variables. To measure the effect of a change in a single independent variable (\( x \) or \( y \)) on the dependent variable (\( z \)) we use what is known as the PARTIAL DERIVATIVE. The partial derivative of \( z \) with respect to \( x \) measures the instantaneous change in the function as \( x \) changes while HOLDING \( y \) constant. Similarly, we would hold \( x \) constant if we wanted to evaluate the effect of a change in \( y \) on \( z \). Formally:

- \( \frac{\partial z}{\partial x} \) is the "partial derivative" of \( z \) with respect to \( x \), treating \( y \) as a constant. Sometimes written as \( f_x \).

- \( \frac{\partial z}{\partial y} \) is the "partial derivative" of \( z \) with respect to \( y \), treating \( x \) as a constant. Sometimes written as \( f_y \).
The "∂" symbol ("bent over" lower case D) is called the "partial" symbol. It is interpreted in exactly the same way as \( \frac{dy}{dx} \) from single variable calculus. The \( \partial \) symbol simply serves to remind us that there are other variables in the equation, but for the purposes of the current exercise, these other variables are held constant.

**EXAMPLES:**

\[
\begin{align*}
  z &= x + y \quad \frac{\partial z}{\partial x} = 1 \quad \frac{\partial z}{\partial y} = 1 \\
  z &= xy \quad \frac{\partial z}{\partial x} = y \quad \frac{\partial z}{\partial y} = x \\
  z &= x^2 y^2 \quad \frac{\partial z}{\partial x} = 2(y^2)x \quad \frac{\partial z}{\partial y} = 2(x^2)y \\
  z &= x^2 y^3 + 2x + 4y \quad \frac{\partial z}{\partial x} = 2xy^3 + 2 \quad \frac{\partial z}{\partial y} = 3x^2 y^2 + 4
\end{align*}
\]

• **REMEMBER:** When you are taking a partial derivative you treat the other variables in the equation as constants!

**Rules of Partial Differentiation**

**Product Rule:** given \( z = g(x,y) \cdot h(x,y) \)

\[
\begin{align*}
  \frac{\partial z}{\partial x} &= g(x,y) \cdot \frac{\partial h}{\partial x} + h(x,y) \cdot \frac{\partial g}{\partial x} \\
  \frac{\partial z}{\partial y} &= g(x,y) \cdot \frac{\partial h}{\partial y} + h(x,y) \cdot \frac{\partial g}{\partial y}
\end{align*}
\]

**Quotient Rule:** given \( z = \frac{g(x,y)}{h(x,y)} \) and \( h(x,y) \neq 0 \)

\[
\begin{align*}
  \frac{\partial z}{\partial x} &= \frac{h(x,y) \cdot \frac{\partial g}{\partial x} - g(x,y) \cdot \frac{\partial h}{\partial x}}{[h(x,y)]^2} \\
  \frac{\partial z}{\partial y} &= \frac{h(x,y) \cdot \frac{\partial g}{\partial y} - g(x,y) \cdot \frac{\partial h}{\partial y}}{[h(x,y)]^2}
\end{align*}
\]

**Chain Rule:** given \( z = [g(x,y)]^n \)

\[
\begin{align*}
  \frac{\partial z}{\partial x} &= n[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial x} \\
  \frac{\partial z}{\partial y} &= n[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial y}
\end{align*}
\]

**Further Examples:**

For the function \( U = U(x,y) \) find the the partial derivates with respect to \( x \) and \( y \) for each of the following examples

**Example 1**

\[ U = -5x^3 - 12xy - 6y^5 \]

**Answer:**

\[
\begin{align*}
  \frac{\partial U}{\partial x} &= U_x = 15x^2 - 12y \\
  \frac{\partial U}{\partial y} &= U_y = -12x - 30y^4
\end{align*}
\]
Example 2

\[ U = 7x^2y^3 \]

Answer:

\[
\frac{\partial U}{\partial x} = U_x = 14xy^3 \\
\frac{\partial U}{\partial y} = U_y = 21x^2y^2
\]

Example 3

\[ U = 3x^2(8x - 7y) \]

Answer:

\[
\frac{\partial U}{\partial x} = U_x = 3x^2(8) + (8x - 7y)(6x) = 72x^2 - 42xy \\
\frac{\partial U}{\partial y} = U_y = 3x^2(-7) + (8x - 7y)(0) = -21x^2
\]

Example 4

\[ U = (5x^2 + 7y)(2x - 4y^3) \]

Answer:

\[
\frac{\partial U}{\partial x} = U_x = (5x^2 + 7y)(2) + (2x - 4y^3)(10x) \\
\frac{\partial U}{\partial y} = U_y = (5x^2 + 7y)(-12y^2) + (2x - 4y^3)(7)
\]

Example 5

\[ U = \frac{9y^3}{x - y} \]

Answer:

\[
\frac{\partial U}{\partial x} = U_x = \frac{(x - y)(0) - 9y^3(1)}{(x - y)^2} = \frac{-9y^3}{(x - y)^2} \\
\frac{\partial U}{\partial y} = U_y = \frac{(x - y)(27y^2) - 9y^3(-1)}{(x - y)^2} = \frac{27xy^2 - 18y^3}{(x - y)^2}
\]

Example 6

\[ U = (x - 3y)^3 \]

Answer:

\[
\frac{\partial U}{\partial x} = U_x = 3(x - 3y)^2(1) = 3(x - 3y)^2 \\
\frac{\partial U}{\partial y} = U_y = 3(x - 3y)^2(-3) = -9(x - 3y)^2
\]
A Special Function: Cobb-Douglas

The Cobb-douglas function is a mathematical function that is very popular in economic models. The general form is

\[ z = x^a y^b \]

and its partial derivatives are

\[ \frac{\partial z}{\partial x} = ax^{a-1} y^b \quad \text{and} \quad \frac{\partial z}{\partial y} = bx^a y^{b-1} \]

Furthermore, the slope of the level curve of a Cobb-douglas is given by

\[ \frac{\partial z/\partial x}{\partial z/\partial y} = MRS = \frac{a}{b} \frac{y}{x} \]

Differentials

Given the function

\[ y = f(x) \]

the derivative is

\[ \frac{dy}{dx} = f'(x) \]

However, we can treat \( dy/dx \) as a fraction and factor out the \( dx \)

\[ dy = f'(x)dx \]

where \( dy \) and \( dx \) are called differentials. If \( dy/dx \) can be interpreted as ”the slope of a function”, then \( dy \) is the ”rise” and \( dx \) is the ”run”. Another way of looking at it is as follows:

- \( dy \) = the change in \( y \)
- \( dx \) = the change in \( x \)
- \( f'(x) \) = how the change in \( x \) causes a change in \( y \)

Example 7  if

\[ y = x^2 \]

then

\[ dy = 2x dx \]

Let's suppose \( x = 2 \) and \( dx = 0.01 \). What is the change in \( y(dy) \)?

\[ dy = 2(2)(0.01) = 0.04 \]

Therefore, at \( x = 2 \), if \( x \) is increased by 0.01 then \( y \) will increase by 0.04.
The two variable case

If

\[ z = f(x, y) \]

then the change in \( z \) is

\[ dz = \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy \]

or

\[ dz = f_x \, dx + f_y \, dy \]

which is read as "the change in \( z \) (\( dz \)) is due partially to a change in \( x \) (\( dx \)) plus partially due to a change in \( y \) (\( dy \)). For example, if

\[ z = xy \]

then the total differential is

\[ dz = ydx + xdy \]

and, if

\[ z = x^2y^3 \]

then

\[ dz = 2xy^3dx + 3x^2y^2dy \]

**REMEMBER:** When you are taking the total differential, you are just taking all the partial derivatives and adding them up.

**Example 8** Find the total differential for the following utility functions

1. \( U(x_1, x_2) = ax_1 + bx_2 \quad (a, b > 0) \)
2. \( U(x_1, x_2) = x_1^2 + x_2^2 + x_1x_2 \)
3. \( U(x_1, x_2) = x_1^{a}x_2^{b} \)

Answers:

1. \[ \frac{\partial U}{\partial x_1} = U_1 = a \]
   \[ \frac{\partial U}{\partial x_2} = U_2 = b \]
   \[ dU = U_1 \, dx_1 + U_2 \, dx_2 = adx_1 + bdx_2 \]

2. \[ \frac{\partial U}{\partial x_1} = U_1 = 2x_1 + x_2 \]
   \[ \frac{\partial U}{\partial x_2} = U_2 = 3x_2^2 + x_1 \]
   \[ dU = U_1 \, dx_1 + U_2 \, dx_2 = (2x_1 + x_2) \, dx_1 + (3x_2^2 + x_1) \, dx_2 \]

3. \[ \frac{\partial U}{\partial x_1} = U_1 = ax_1^{a-1}x_2^{b} = \frac{ax_1^{a}x_2^{b}}{x_2} \]
   \[ \frac{\partial U}{\partial x_2} = U_2 = bx_1^a x_2^{b-1} = \frac{bx_1^a x_2^{b}}{x_1} \]
   \[ dU = \left( \frac{ax_1^{a}x_2^{b}}{x_1} \right) \, dx_1 + \left( \frac{bx_1^a x_2^{b}}{x_2} \right) \, dx_2 = \left( \frac{adx_1}{x_1} + \frac{b dx_2}{x_2} \right) x_1^{a}x_2^{b} \]
The Implicit Function Theorem

Suppose you have a function of the form

\[ F(y, x_1, x_2) = 0 \]

where the partial derivatives are \( \frac{\partial F}{\partial x_1} = F_{x_1}, \frac{\partial F}{\partial x_2} = F_{x_2} \) and \( \frac{\partial F}{\partial y} = F_y \). This class of functions are known as implicit functions where \( F(y, x_1, x_2) = 0 \) implicitly define \( y = y(x_1, x_2) \). What this means is that it is possible (theoretically) to rewrite to get \( y \) isolated and expressed as a function of \( x_1 \) and \( x_2 \). While it may not be possible to explicitly solve for \( y \) as a function of \( x \), we can still find the effect on \( y \) from a change in \( x_1 \) or \( x_2 \) by applying the implicit function theorem:

**Theorem 9** If a function

\[ F(y, x_1, x_2) = 0 \]

has well defined continuous partial derivatives

\[
\begin{align*}
\frac{\partial F}{\partial y} &= F_y \\
\frac{\partial F}{\partial x_1} &= F_{x_1} \\
\frac{\partial F}{\partial x_2} &= F_{x_2}
\end{align*}
\]

and if, at the values where \( F \) is being evaluated, the condition that

\[ \frac{\partial F}{\partial y} = F_y \neq 0 \]

holds, then \( y \) is implicitly defined as a function of \( x \). The partial derivatives of \( y \) with respect to \( x_1 \) and \( x_2 \), are given by the ratio of the partial derivatives of \( F \), or

\[
\frac{\partial y}{\partial x_i} = -\frac{F_{x_i}}{F_y}, \quad i = 1, 2
\]

To apply the implicit function theorem to find the partial derivative of \( y \) with respect to \( x_1 \) (for example), first take the total differential of \( F \)

\[ dF = F_y dy + F_{x_1} dx_1 + F_{x_2} dx_2 = 0 \]

then set all the differentials except the ones in question equal to zero (i.e. set \( dx_2 = 0 \)) which leaves

\[ F_y dy + F_{x_1} dx_1 = 0 \]

or

\[ F_y dy = -F_{x_1} dx_1 \]

dividing both sides by \( F_y \) and \( dx_1 \) yields

\[
\frac{dy}{dx_1} = -\frac{F_{x_1}}{F_y}
\]

which is equal to \( \frac{\partial y}{\partial x_1} \) from the implicit function theorem.
Example 10 For each \( f(x, y) = 0 \), find \( \frac{dy}{dx} \) for each of the following:

1. \( y - 6x + 7 = 0 \)
   
   Answer:
   \[
   \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(-6)}{1} = 6
   \]

2. \( 3y + 12x + 17 = 0 \)
   
   Answer:
   \[
   \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(-12)}{3} = 4
   \]

3. \( x^2 + 6x - 13 - y = 0 \)
   
   Answer:
   \[
   \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(2x + 6)}{-1} = 2x + 6
   \]

4. \( f(x, y) = 3x^2 + 2xy + 4y^3 \)
   
   Answer:
   \[
   \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{6x + 2y}{12y^2 + 2x}
   \]

5. \( f(x, y) = 12x^5 - 2y \)
   
   Answer:
   \[
   \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{60x^4}{-2} = 30x^4
   \]

6. \( f(x, y) = 7x^2 + 2xy^2 + 9y^4 \)
   
   Answer:
   \[
   \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{14x + 2y^2}{36y^3 + 4xy}
   \]

Example 11 For \( f(x, y, z) \) use the implicit function theorem to find \( \frac{dy}{dx} \) and \( \frac{dy}{dz} \):

1. \( f(x, y, z) = x^2y^3 + z^2 + xyz \)
   
   Answer:
   \[
   \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2xy^3 + yz}{3x^2y^2 + xz}
   \]
   \[
   \frac{dy}{dz} = -\frac{f_z}{f_y} = -\frac{2z + xy}{3x^2y^2 + xz}
   \]
2. \[ f(x, y, z) = x^3 z^2 + y^3 + 4xyz \]

Answer:
\[
\frac{dy}{dx} = -\frac{fx}{fy} = -\frac{3x^2 z^2 + 4yz}{3y^2 + 4xz}
\]
\[
\frac{dy}{dz} = -\frac{fz}{fy} = -\frac{2x^3 z + 4xy}{3y^2 + 4xz}
\]

3. \[ f(x, y, z) = 3x^2 y^3 + xz^2 y^2 + y^3 z x^4 + y^2 z \]

Answer:
\[
\frac{dy}{dx} = -\frac{fx}{fy} = -\frac{6xy^3 + x^2 y^2 + 4y^3 xx^3}{9x^2 y^4 + 2xz^2 x^2 + 3y^2 z x^4 + 2y^2 z}
\]
\[
\frac{dy}{dz} = -\frac{fz}{fy} = -\frac{2xyz y^2 + y^3 z^4 + y^2}{9x^2 y^4 + 2xz^2 x^2 + 3y^2 z x^4 + 2y^2 z}
\]