Partial Solutions to Practice Final #2

1. Minimize $z = 3x^2 - xy + 2y^2 - 4x - 7y + 12$ subject to the inequality constraint $x + y \geq 15$. Is the constraint binding or non-binding?ANSWER

$$L = 3x^2 - xy + 2y^2 - 4x - 7y + 12 + \lambda(x + y - 15)$$

kuhn-tucker

$$\begin{aligned} L_x &= 6x - y - 4 + \lambda \geq 0 \qquad x \geq 0 \\ L_y &= -x + 4y - 7 \geq 0 \qquad y \geq 0 \\ L_\lambda &= x + y - 15 \geq 0 \qquad \lambda \geq 0 \end{aligned}$$

Try non-binding $\lambda = 0$: Solution $x = 1$ and $y = 2$. Violates constraint

2. Given the utility function $u = x^a y^b$, show that both u and the monotonic transformation, $v = \ln(x^a y^b)$ have the same *MRS*. *ANSWER*

$$\begin{aligned} dU &= ax^{a-1}y^b dx + bx^a y^{b-1} dy = 0 \qquad \frac{dy}{dx} = -\frac{ay}{bx} \\ & \textit{then} \\ v &= \ln u = a \ln x + b \ln y \\ dv &= a\left(\frac{1}{x}\right)dx + b\left(\frac{1}{y}\right)dy = 0 \qquad \frac{dy}{dx} = -\frac{ay}{bx} \end{aligned}$$

Question 8 (second part only)

1. Myrtle's utility maximization problem could be recast as the following:

$$Minimize \quad p_x x + p_y y \quad s.t. \quad U_0 = x^{1/3}y^{2/3}$$

where U_0 is equivalent to the maximum utility

(a) *(4 marks)*Find the values of x and y that solve this minimization problem.

$$L = p_x x + p_y y + \lambda(U_0 - x^{1/3}y^{2/3})$$

FOC's

$$\begin{aligned} L_x &= p_x - \tfrac{1}{3}\lambda x^{-2/3}y^{2/3} = 0 \\ L_y &= p_y - \tfrac{2}{3}\lambda x^{1/3}y^{-1/3} = 0 \\ L_\lambda &= U_0 - x^{1/3}y^{2/3} = 0 \end{aligned}$$

divide the first equation by the second

$$\frac{p_x}{p_y} = \frac{-\frac{1}{3}\lambda x^{-2/3}y^{2/3}}{-\frac{2}{3}\lambda x^{1/3}y^{-1/3}} = \frac{y}{2x}$$

rewriting:

$$y = \frac{p_x}{p_y}(2x)$$

sub into the constraint

$$\begin{aligned} U_0 &= x^{1/3}y^{2/3} = x^{1/3}\left(\frac{p_x}{p_y}2x\right)^{2/3} = \left(\frac{2p_x}{p_y}\right)^{2/3} x \\ x^* &= \left(\frac{p_y}{2p_x}\right)^{2/3} U_0 \end{aligned}$$

similarly,

$$y^* = \left(\frac{2p_x}{p_y}\right)^{1/3} U_0$$

(b) Verify that this is a minimum by checking second order conditions

Find SOC's: Totally differentiate the FOC's

$$\begin{gathered}\left(\tfrac{2}{9}\lambda x^{-5/3}y^{2/3}\right)dx+\left(-\tfrac{2}{9}\lambda x^{-2/3}y^{-1/3}\right)dy+\left(-\tfrac{1}{3}x^{-2/3}y^{2/3}\right)d\lambda=0\\ \left(-\tfrac{2}{9}\lambda x^{-2/3}y^{-1/3}\right)dx+\left(\tfrac{2}{9}\lambda x^{1/3}y^{-2/3}\right)dy+\left(-\tfrac{2}{3}x^{1/3}y^{-1/3}\right)d\lambda=0\\ \left(-\tfrac{1}{3}x^{-2/3}y^{2/3}\right)dx+\left(-\tfrac{2}{3}x^{1/3}y^{-1/3}\right)dy+(0)\,d\lambda=0\end{gathered}$$

$$\left|\bar{H}\right|=\begin{vmatrix}\frac{2}{9}\lambda x^{-5/3}y^{2/3} & -\frac{2}{9}\lambda x^{-2/3}y^{-1/3} & -\frac{1}{3}x^{-2/3}y^{2/3}\\ -\frac{2}{9}\lambda x^{-2/3}y^{-1/3} & \frac{2}{9}\lambda x^{1/3}y^{-2/3} & -\frac{2}{3}x^{1/3}y^{-1/3}\\ -\frac{1}{3}x^{-2/3}y^{2/3} & -\frac{2}{3}x^{1/3}y^{-1/3} & 0\end{vmatrix}<0$$

(c) Find the expenditure function and show that your solution to x from the minimization problem is equal to $\frac{\partial B(p_x,p_y,U_0)}{\partial p_x}$. **ANSWER:**

$$\begin{aligned}B(p_x,p_y,U_0) &= p_x x^* + p_y y^*\\ &= p_x U_0 + p_y\left(\frac{2p_x}{p_y}\right)^{1/3}U_0\\ &= p_x^{1/3}\left(\frac{p_y}{2}\right)^{2/3}U_0 + p_y^{2/3}\left(2p_x\right)^{1/3}U_0\\ &\quad \text{differentiate w.r.t. } p_x\\ x^* &= \left(\frac{p_y}{2p_x}\right)^{2/3}\end{aligned}$$