Kuhn-Tucker conditions
(nonlinear programming)

Marketable Permits with spatial constraints

Utility Max problem

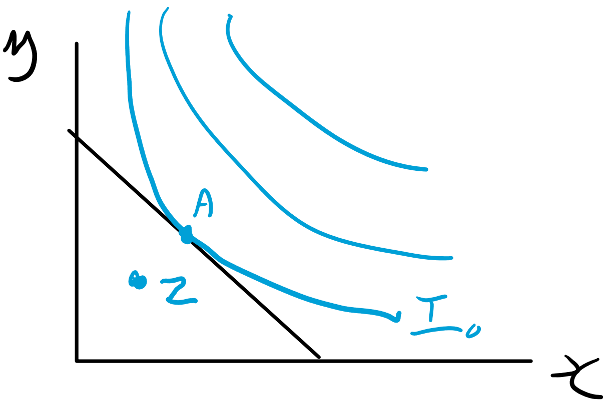
$\max\ U(x,y)$ s.t. $B = P_x X + P_y Y$

Lagrange

$$\mathcal{L} = U(x,y) + \lambda\big(\underbrace{B - P_x X - P_y Y}_{=0}\big)$$

$MRS = \frac{P_x}{P_y}$

$\frac{\Delta U}{\Delta B} = \lambda.$



$\max\ U$ s.t. $P_x X + P_y Y \leq B$

More than 1 constraint?

$\max\ U$ s.t. $P_x X + P_y Y \leq B$

and $x \le \bar{X}$

Lagrange

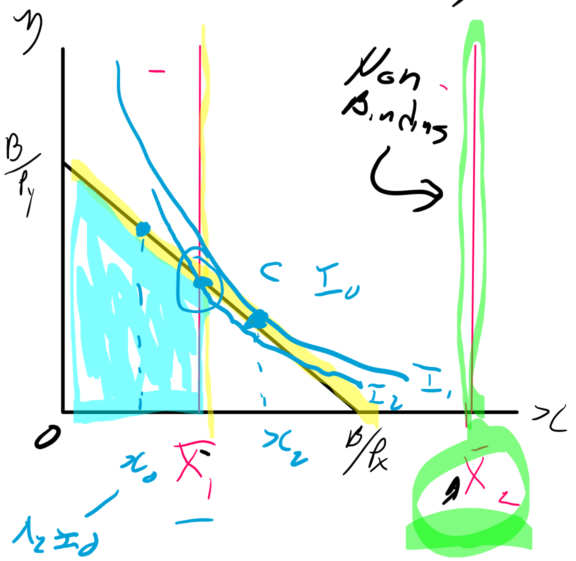
$$\mathcal{L} = \underbrace{u(x,y) + \lambda_1 (B - P_x X - P_y Y)}_{} + \lambda_2 (\bar{X} - x)$$

$x, y \ge 0$

either

$\bar{x} - x = 0$ or $\lambda_2 = 0$.

$\lambda_2 > 0$



$u = xy$ $\quad x + y \le 10$

$x \le \bar{X}$

$$\mathcal{L} = xy + \lambda_1 (10 - x - y) + \lambda_2 (\bar{X} - x)$$

stepwise approach

step 1 assume $\lambda_2 = 0$

$\mathcal{L} = xy + \lambda_1 (10 - x - y)$

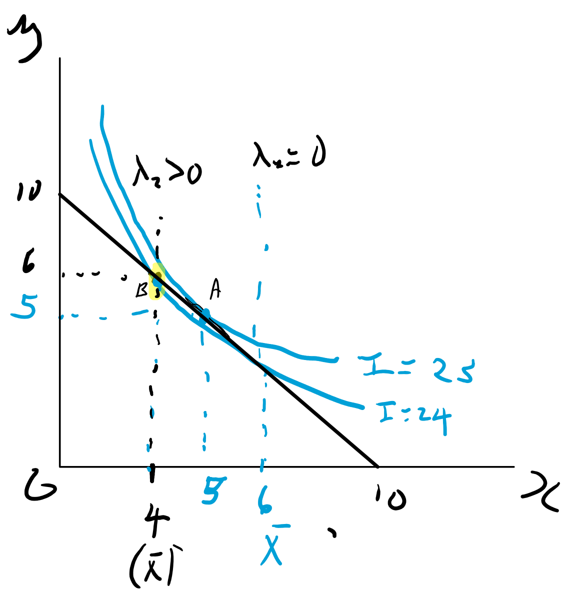
$\left.\begin{array}{l} \mathcal{L}_x = y - \lambda_1 = 0 \\ \mathcal{L}_y = x - \lambda_1 = 0 \end{array}\right\} x = y$

$\mathcal{L}_{\lambda_1} = 10 - x - y = 0$

$10 - 2x = 0$

$x = 5$

$y = 5$



step 2: ...

$(\bar{x})$ $\bar{X}$

step 2: check to see if constraint 2 is violated

$x \leq \bar{X}$

or $5 \leq \bar{X}$

Suppose
$$\begin{cases} \bar{X} = 4 \\ \bar{X} = 6 \checkmark \end{cases}$$

2nd EXAMPLE: Wartime Rationing

Budget

B = money

$\left.\begin{array}{l} P_x = \text{price of } X \\ P_y = \text{Price of } Y \end{array}\right\} \$$

Govt coupon

C = Total coupons

c_x = Coupon price of X

c_y = coupon price of Y

To buy a unit of x: you need P_x dollars
AND c_x in coupons

suppose

$B = 100$

$P_x = 1$

$P_y = 1$

and

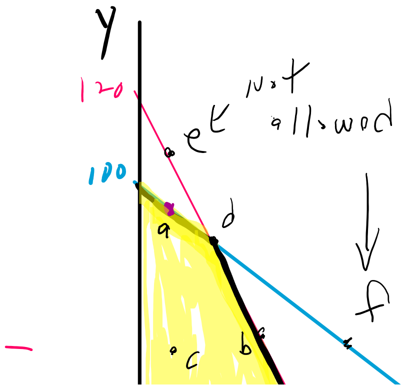
$C = 120$

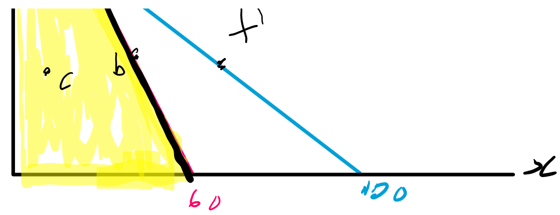
$c_x = 2$

$c_y = 1$

\$100 → 100 units of X but 120 coupons → 60 units of X

\$100 → 100 Y 120 Coupons → 120 Y





$U = xy$ $\qquad x + y \le 100$

$\qquad\qquad\quad 2x + y \le 120$

$$\mathcal{L} = xy + \lambda_1(100 - x - y) + \lambda_2(120 - 2x - y)$$

steps:

assume one constraint is nonbinding (either $\lambda_1 = 0$ or $\lambda_2 = 0$)

choose 1

Let $\lambda_2 = 0$

$$\mathcal{L} = xy + \lambda_1(100 - x - y)$$

$\mathcal{L}_x = y - \lambda_1 = 0$

$\mathcal{L}_y = x - \lambda_1 = 0$ $\Big\}$ $y = x$

$\mathcal{L}_{\lambda_1} = 100 - x - y = 0$

$100 - 2x = 0$

$x = 50$

$y = 50$

Then we test our assumption

constraint 2: $2x + y \le 120$?

$2(50) + 50 \le 120$?

$150 > 120$ ✗ failed

$150 > 120$ X failed

Try $\lambda_1 = 0$

$$\mathcal{L} = xy + \lambda_2(120 - 2x - y)$$

$\mathcal{L}_x = y - 2\lambda_2 = 0$ $\quad y = 2\lambda_2$

$\mathcal{L}_y = x - \lambda_2 = 0$ $\quad x = \lambda_2$ $\quad \} \; y = 2x$

$\mathcal{L}_{\lambda_2} = 120 - 2x - y = 0$

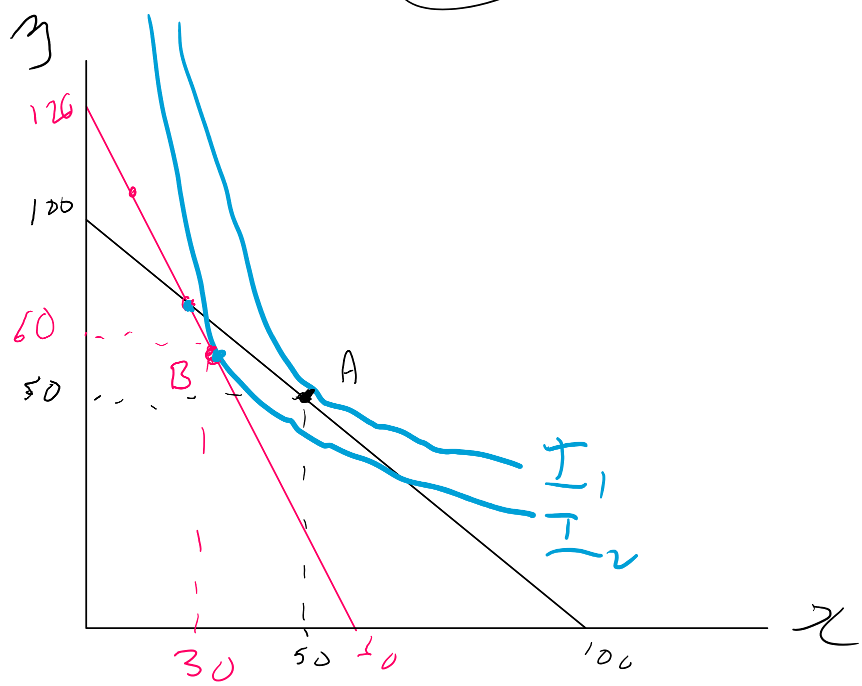
$120 - 4x = 0$

$x = 30, \; y = 60$

Test our assumption that $\lambda_1 = 0$

$x + y \leq 100$

$30 + 60 = 90 < 100$ ✓



New utility $U = xy^3$

$$\mathcal{L} = xy^3 + \lambda_1(100-x-y) + \lambda_2(120-2x-y)$$

(1) $\lambda_2 = 0$

$\mathcal{L} = xy^3 + \lambda_1(100-x-y)$

$\mathcal{L}_x = y^3 - \lambda_1 = 0$

$\mathcal{L}_y = 3xy^2 - \lambda_1 = 0$

$\mathcal{L}_{\lambda_1} = 100 - x - y = 0$

$y^3 = 3xy^2$

$y = 3x$

$100 - 4x = 0$

$x = 25$ $y = 75$

Test

$2x + y \le 120$

$50 + 75 = 125 > 120$

fail

(2) $\lambda_1 = 0$

$\mathcal{L} = xy^3 + \lambda_2(120 - 2x - y)$

$\mathcal{L}_x = y^3 - 2\lambda_2 = 0$

$\mathcal{L}_y = 3xy^2 - \lambda_2 = 0$

$\mathcal{L}_{\lambda_2} = 120 - 2x - y = 0$

$$\frac{y^{\cancel{3}} = 2\cancel{\lambda_2}}{3x\cancel{y^2} = \cancel{\lambda_2}}$$

$\frac{y}{3x} = 2$

$y = 6x$

$120 - 8x = 0$

$x = 15$ $y = 90$

Test

$x + y \le 100$

$15 + 90 = 105 > 100$ fail

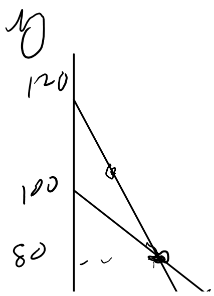
Both assumptions failed

then solution

$x + y = 100$

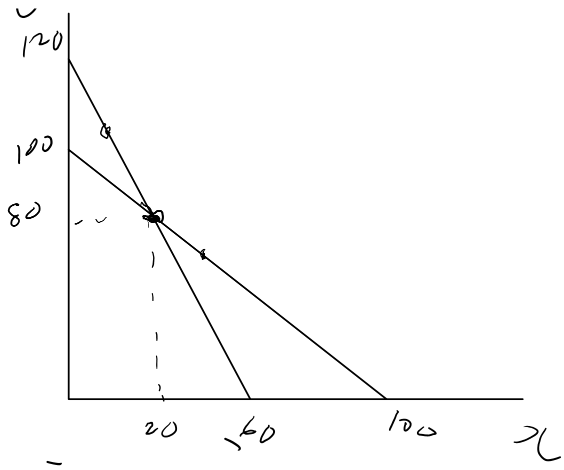
$2x + y = 120$

$x = 20$, $y = 80$



$2x + y = 120$

$x = 20 \;, y = 80$



this is known as
Kuhn - Tucker conditions

chapter 14 Kolstad:

Basic Model

$i = 1, \ldots, I$ emission Sources

$j = 1, \ldots, J$ Receptors (zones)

e_i = emissions from source i

a_{ij} = Transfer Coefficient

Damage 1 unit from i that occurs at j

P_j = Pollution (air Quality) at zone j

$$P_j = a_{1j} e_1 + a_{2j} e_2 + \ldots + a_{Ij} e_I + B_j$$

where β_j = background pollution

$$P_j = \sum_{i=1}^{I} a_{ij} e_i + \beta_j$$

assume $\beta_j = 0$

2 x 2 case

2 zones P_1, P_2

2 sources e_1, e_2

$$\begin{cases} P_1 = a_{11} e_1 + a_{21} e_2 \\ P_2 = a_{12} e_1 + a_{22} e_2 \end{cases}$$

matrix form

$$\underbrace{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}}_{2\times 2} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$$A \cdot E = P$$

$$P_1 = a_{11} e_1 + a_{21} e_2$$

$$\frac{\partial P_1}{\partial e_1} = a_{11} \qquad \frac{\partial P_1}{\partial e_2} = a_{21}$$

$$\Delta P_1 = a_{11} \Delta e_1 + a_{21} \Delta e_2$$

or

$$dP_1 = a_{11} de_1 + a_{21} de_2$$

change e_1, e_2 but keep P_1 fixed

$\therefore$ $\Delta P_1 = 0$ (or $dP_1 = 0$)

$$a_{11} de_1 + a_{21} de_2 = 0$$

$$a_{21} de_2 = - a_{11} de_1$$

$$\frac{de_2}{de_1} = - \frac{a_{11}}{a_{21}}$$ } Ratio of trade between e_1, e_2 that keeps P_1 = constant

Marginal Damage and Marginal Savings

Define

$MDE_i \equiv$ Marginal Damage of emissions from i

$MD(P_j) \equiv$ Marginal damage of Pollution in zone j

$$MDE_i = a_{ij}\, MD(P_j)$$

i.e.

$$P_1 = a_{11} e_1 + a_{21} e_2$$

$a_{11} = 3$ $\quad a_{21} = 2$

$$P_1 = 3e_1 + 2e_2$$

1 unit of $e_1 \longrightarrow$ 3 units of P_1

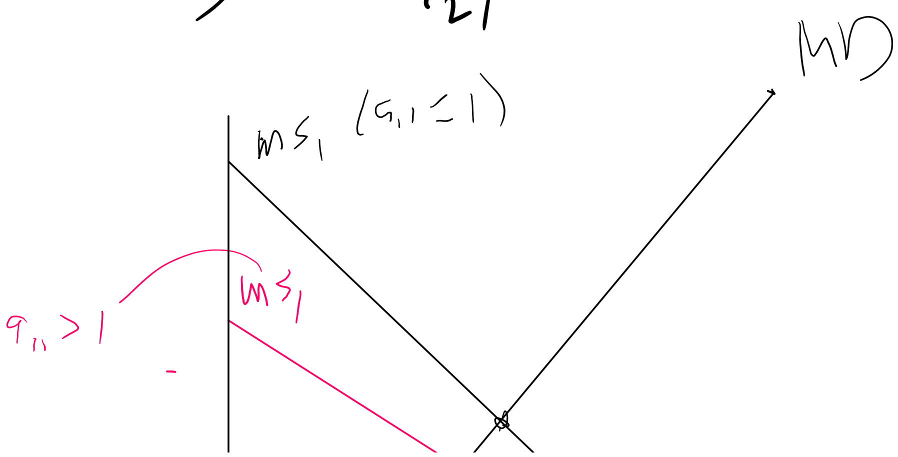
$MDE_1 = 3\, MD(P_1)$ $\quad MDE_2 = 2\, MD(P_1)$

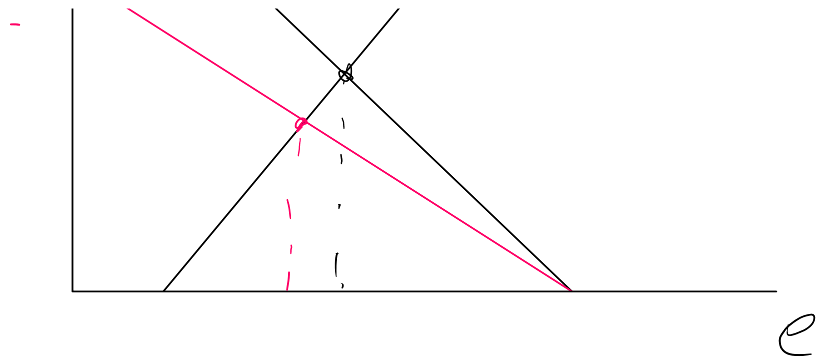
Efficiency

$MS_1 = MDE_1$ $\quad MS_2 = MDE_2$

$MS_1 = a_{11}\, MD(P_1)$ $\quad MS_2 = a_{21}\, MD(P_1)$

$$\frac{MS_1}{a_{11}} = MD(P_1) = \frac{MS_2}{a_{21}}$$





$$MS_1 = MS_2 \quad (\text{only if } a_{ij} = 1)$$

$$\frac{MS_1}{a_{11}} = \frac{MS_2}{a_{21}}$$

$$\frac{MS_1}{3} = \frac{MS_2}{2}$$

Basic framework

$$\boxed{P_1 = a_{11} e_1 + a_{21} e_2}$$

$$\frac{de_2}{de_1} = -\frac{a_{11}}{a_{21}}$$

Pollution constraints for just one zone