Lecture Notes for ECON 402
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Maximum Value Functions and the Envelope Theorem

A maximum (or minimum) value function is an objective function where the choice variables have been assigned their optimal values.

What is the significance of the indirect objective function? Consider that in any optimization problem the direct objective function is maximized (or minimized) for a given set of parameters.

The indirect objective function gives all the maximum values of the objective function as these parameters vary. Hence the indirect objective function is an "envelope" of the set of optimized objective functions generated by varying the parameters of the model.

For most students of economics the first illustration of this notion of an "envelope" arises in the comparison of short-run and long-run cost curves.

To illustrate, consider the following maximization problem with two choice variables x and y, and one parameter, ϕ:

Maximize

$$U = f(x, y, \phi) \tag{1}$$

The first-order necessary condition are

$$f_x(x, y, \phi) = f_y(x, y, \phi) = 0 \tag{2}$$

if second-order conditions are met, these two equations implicitly define the solutions

$$x = x^*(\phi) \quad y = x^*(\phi) \tag{3}$$

If we substitute these solutions into the objective function, we obtain a new function

$$V(\phi) = f(x^*(\phi), y^*(\phi), \phi) \tag{4}$$

where this function is the value of f when the values of x and y are those that maximize $f(x, y, \phi)$. Therefore, $V(\phi)$ is *the maximum value function* (or indirect objective function). If we differentiate V with respect to ϕ

$$\frac{\partial V}{\partial \phi} = f_x \frac{\partial x^*}{\partial \phi} + f_y \frac{\partial y^*}{\partial \phi} + f_\phi \tag{5}$$

However, from the first-order conditions we know $f_x = f_y = 0$. Therefore, the first two terms disappear and the result becomes

$$\frac{\partial V}{\partial \phi} = f_\phi \tag{6}$$

This result says that, at the optimum, as ϕ varies, with x^* and y^* allowed to adjust optimally gives the same result as if x^* and y^* were held constant! Note that ϕ enters maximum value function (equation 4) in three places: one direct and two indirect (through x^* and y^*). Equations 5 and 6 show that, at the optimum, only the direct effect of ϕ on the objective function matters. This is the essence of the envelope theorem.

The envelope theorem says only the direct effects of a change in an exogenous variable need be considered, even though the exogenous variable may enter the maximum value function indirectly as part of the solution to the endogenous choice variables.

The Profit Function

Consider the case where a firm uses two inputs: capital, K, and labour, L. The profit function is

$$\pi = pf(K, L) - wL - rK \tag{7}$$

where p is the output price and w and r are the wage rate and rental rate respectively.

The first-order conditions are

$$\begin{aligned} \pi_L &= f_L(K, L) - w = 0 \\ \pi_K &= f_K(K, L) - r = 0 \end{aligned} \tag{8}$$

which respectively define the factor demand equations

$$\begin{aligned} L &= L^*(w, r, p) \\ K &= K^*(w, r, p) \end{aligned} \tag{9}$$

substituting the solutions K^* and L^* into the objective function gives us

$$\pi^*(w, r, p) = pf(K^*, L^*) - wL^* - rK^* \tag{10}$$

$\pi^*(w, r, p)$ is the *profit function* (or indirect objective function). The profit function gives the maximum profit as a function of the exogenous variables w, r, and p.

Now consider the effect of a change in w on the firm's profits. If we differentiate the original profit function (equation 7) with respect to w,

$$\frac{\partial \pi}{\partial w} = -L \tag{11}$$

However, this result does not take into account the profit maximizing firms ability to make a substitution of capital for labour and adjust the level of output in accordance with profit maximizing behavior.

Since $\pi^*(w, r, p)$ is the maximum value of profits for any values of w, r, and p, changes in π^* from a change in w takes all capital for labour substitutions into account. To evaluate a change in the maximum profit function from a change in w, we differentiate $\pi^*(w, r, p)$ with respect to w yielding

$$\frac{\partial \pi^*}{\partial w} = [pf_L - w]\frac{\partial L^*}{\partial w} + [pf_K - r]\frac{\partial K^*}{\partial w} - L^* \tag{12}$$

From the first-order conditions, the two bracketed terms are equal to zero. Therefore, the resulting equation becomes

$$\frac{\partial \pi^*}{\partial w} = -L^*(w, r, p) \tag{13}$$

This result says that, at the profit maximizing position, a change in profits with respect to a change in the wage is the same whether or not the factors are held constant or allowed to vary as the factor price changes.

The derivative of the profit function with respect to w is the negative of $L^*(w, r, p)$.

Following the above procedure, we can also show the additional comparative statics results

$$\frac{\partial \pi^*(w, r, p)}{\partial r} = -K^*(r, w, p) \tag{14}$$

and

$$\frac{\partial \pi^*(w, r, p)}{\partial p} = f(K^*, L^*) = q^* \tag{15}$$

This is known as "Hotelling's Lemma".

The Envelope Theorem and Constrained Optimization

Again we will have an objective function (U), two choice variables, (x and y) and one parameter (ϕ) except now we introduce the following constraint:

$$g(x, y; \phi) = 0$$

The problem then becomes
Maximize

$$U = f(x, y; \phi) \tag{16}$$

subject to

$$g(x, y; \phi) = 0 \tag{17}$$

The Lagrangian for this problem is

$$Z = f(x, y; \phi) + \lambda g(x, y; \phi) \tag{18}$$

The first-order conditions are

$$\begin{aligned} Z_x &= f_x + \lambda g_x = 0 \\ Z_y &= f_y + \lambda g_y = 0 \\ Z_\lambda &= g(x, y; \phi) = 0 \end{aligned} \tag{19}$$

Solving this system of equations gives us

$$x = x^*(\phi) \quad y = y^*(\phi) \quad \lambda = \lambda^*(\phi) \tag{20}$$

Substituting the solutions into the objective function, we get

$$U^* = f(x^*(\phi), y^*(\phi), \phi) = V(\phi) \tag{21}$$

where $V(\phi)$ is the indirect objective function, or maximum value function. This is the maximum value of y for any ϕ and x_i's that satisfy the constraint.

How does $V(\phi)$ change as ϕ changes? First, we differentiate V with respect to ϕ

$$\frac{\partial V}{\partial \phi} = f_x \frac{\partial x^*}{\partial \phi} + f_y \frac{\partial y^*}{\partial \phi} + f_\phi \tag{22}$$

In this case,equation 22 will not simplify to $\frac{\partial V}{\partial \phi} = f_\phi$ since $f_x \neq 0$ and $f_y \neq 0$. However, if we substitute the solutions to x and y into the constraint (producing an identity)

$$g(x^*(\phi), y^*(\phi), \phi) \equiv 0 \tag{23}$$

and differentiating with respect to ϕ yields

$$g_x \frac{\partial x^*}{\partial \phi} + g_x \frac{\partial x^*}{\partial \phi} + g_\phi \equiv 0 \tag{24}$$

If we multiply equation 24 by λ and combine the result with equation 22 and rearranging terms, we get

$$\frac{\partial V}{\partial \phi} = (f_x + \lambda g_x)\frac{\partial x^*}{\partial \phi} + (f_y + \lambda g_y)\frac{\partial y^*}{\partial \phi} + f_\phi + \lambda g_\phi = Z_\phi \tag{25}$$

Where Z_ϕ is the partial derivative of the Lagrangian function with respect to ϕ, ceterus parabus.[1]

Interpretation of the Lagrange Multiplier

In the consumer choice problem in chapter 12 we loosely interpreted λ as the marginal utility of income. Now let us derive a more general interpretation of the Lagrange multiplier.

Consider the problem

Maximize

$$U = f(x, y) \tag{26}$$

Subject to

$$c - g(x, y) = 0 \tag{27}$$

[1] While the results in equation 25 nicely parallel the unconstrained case, it is important to note that some of the comparative static results depend critically on whether the parameters enter only the objective function or whether they enter only the constraints, or enter both. If a parameter enters only in the objective function then the comparative static results are the same as for unconstrained case. However, if the parameter enters the constraint, the relation

$$V_{\phi\phi} \geq f_{\phi\phi}$$

will no longer hold.

where c is a constant. The Lagrangian for this problem is

$$Z = f(x, y) + \lambda(c - g(x, y)) \tag{28}$$

The first-order equations are

$$\begin{aligned} Z_x &= f_x(x, y) - \lambda g_x(x, y) = 0 \\ Z_y &= f_y(x, y) - \lambda g_y(x, y) = 0 \\ Z_\lambda &= c - g(x, y) = 0 \end{aligned} \tag{29}$$

From the firs two equations in (29), we get

$$\lambda = \frac{f_x}{g_x} = \frac{f_y}{g_y} \tag{30}$$

which gives us the condition that the slope of the level curve of the objective function must equal the slope of the constraint at the optimum.

Equations (29) implicitly define the solutions

$$x = x^*(c) \quad y = y(c) \quad \lambda = \lambda^*(c) \tag{31}$$

substituting (31) back into the Lagrangian yields the maximum value function

$$V(c) = Z^*(c) = f\left(x^*(c), y^*(c)\right) + \lambda^*(c)\left(c - g(x_1^*(c), y^*(c))\right) \tag{32}$$

differentiating with respect to c yields

$$\begin{aligned} \frac{\partial Z^*}{\partial c} = \quad & f_x \frac{\partial x^*}{\partial c} + f_y \frac{\partial y^*}{\partial c} + \left(c - g(x^*(c), y^*(c))\right) \frac{\partial \lambda^*}{\partial c} \\ & -\lambda^*(c) g_x \frac{\partial x^*}{\partial c} - \lambda^*(c) g_y \frac{\partial y^*}{\partial c} + \lambda^*(c) \frac{\partial c}{\partial c} \end{aligned} \tag{33}$$

by rearranging we get

$$\frac{\partial Z^*}{\partial c} = (f_x - \lambda^* g_x)\frac{\partial x^*}{\partial c} + (f_y - \lambda^* g_y)\frac{\partial y^*}{\partial c} + (c - g(x^*, y^*))\frac{\partial \lambda^*}{\partial c} + \lambda^* \tag{34}$$

Note that the three terms in brackets are nothing more than the first-order equations and, at the optimal values of x, y and λ, these terms are all equal to zero. Therefore this expression simplifies to

$$\frac{\partial V(c)}{\partial c} = \frac{\partial Z^*}{\partial c} = \lambda^* \tag{35}$$

Therefore equals the rate of change of the maximum value of the objective function when c changes (λ is sometimes referred to as the "shadow price" of c).Note that, in this case, c enters the problem only through the constraint; it is not an argument of the original objective function.

Duality and the Envelope Theorem

A consumer's expenditure function and his indirect utility function are the minimum and maximum value functions for dual problems[2]. An expenditure function specifies the minimum expenditure required to obtain a fixed level of utility given the utility function and the prices of consumption goods. An indirect utility function specifies the maximum utility that can be obtained given prices, income and the utility function.

Let $U(x, y)$ be a utility function in x and y are consumption goods. The consumer has a budget, B, and faces market prices P_x and P_y for goods x and y respectively.

Setting up the Lagrangian:

$$Z = U(x, y) + \lambda(B - P_x x - P_y y) \tag{36}$$

[2] Duality in economic theory is the relationship between two constrained optimization problems. If one of the problems requires constrained maximization, the other problem will require constrained minimization. The structure and solution of either problem can provide information about the structure and solution of the other problem.

The first-order conditions are

$$\begin{aligned} Z_x &= U_x - \lambda P_x = 0 \\ Z_y &= U_y - \lambda P_y = 0 \\ Z_\lambda &= B - P_x X - P_y Y = 0 \end{aligned} \tag{37}$$

This system of equations implicitly defines a solution for x^M, y^M and λ^M as a function of the exogenous variables B, P_x, P_y.

$$\begin{aligned} x^M &= x^M(P_x, P_y, B) \\ y^M &= y^M(P_x, P_y, B) \\ \lambda^M &= \lambda^M(P_x, P_y, B, \phi) \end{aligned} \tag{38}$$

The solutions to x^M and y^M are the consumer's ordinary demand functions, sometimes called the "Marshallian" demand functions.

Substituting the solutions to x^* and y^* into the utility function yields

$$U^* = U^*(x^M(B, P_x, P_y), y^M(B, P_x, P_y)) = V(B, P_x, P_y) \tag{39}$$

Where V is the maximum value function, or indirect utility function.

Now consider the alternative, or dual, problem.

The Lagranian for this problem is

$$Z = P_x x + P_y y + \lambda(U^* - U(x, y)) \tag{40}$$

The first-order conditions are

$$\begin{aligned} Z_x &= P_x - \lambda U_x = 0 \\ Z_y &= P_y - \lambda U_y = 0 \\ Z_\lambda &= U^* - U(x, y; \phi) = 0 \end{aligned} \tag{41}$$

This system of equations implicitly define the solutions to x^h, y^h and λ^h

$$\begin{aligned} x^h &= x^h(U^*, P_x, P_y) \\ y^h &= y^h(U^*, P_x, P_y) \\ \lambda^h &= \lambda^h(U^*, P_x, P_y) \end{aligned} \tag{42}$$

x^h and y^h are the compensated, or "real income" held constant demand functions.

If we compare the first two equations from the first-order conditions in both utility maximization problem and expenditure minimization problem, we see

$$\frac{Px}{Py} = \frac{Ux}{Uy} (= MRS) \tag{43}$$

for both

The tangency condition is identical for both problems. If the target level of utility in the minimization problem is set equal to the value of the utility obtained in the solution to the maximization problem, namely U^*, we obtain the following

$$\begin{aligned} x^M(B, P_x, P_y) &= x^h(U^*, P_x, P_y) \\ y^M(B, P_x, P_y) &= y^h(U^*, P_x, P_y) \end{aligned} \tag{44}$$

However, the solutions are functions of different exogenous variables so any comparative statics exercises will produce different results.

Substituting x^h and y^h into the objective function of the minimization problem yields

$$P_x x^h(P_x, P_y, U^*) + P_y y^h(P_x, P_y, U^*) = E(P_x, P_y, U^*) \tag{45}$$

where E is the minimum value function or expenditure function. The duality relationship in this case is

$$E(P_x, P_y, U^*, \phi) = B \tag{46}$$

where B is the exogenous budget from the maximization problem.

Finally, it can be shown from the first-order conditions of the two problems that

$$\lambda^M = \frac{1}{\lambda^h} \tag{47}$$

Roy's Identity

One application of the envelope theorem is the derivation of Roy's identity. Roy's identity states that the individual consumer's Marshallian demand function is equal to the ratio of partial derivatives of the maximum value function. Substituting the optimal values of x^M, y^M and λ^M into the Lagrangian gives us

$$V(B, P_x, P_y) = U(x^M, y^M) + \lambda^M(B - P_x x^M - P_y y^M) \tag{48}$$

First differentiate with respect to P_x

$$\begin{aligned} \frac{\partial V}{\partial P_x} = \quad & (U_x - \lambda^M P_x)\frac{\partial x^M}{\partial P_x} + (U_y - \lambda^M P_y)\frac{\partial y^M}{\partial P_x} \\ & +(B - P_x x^M - P_y y^M)\frac{\partial \lambda^M}{\partial P_x} - \lambda^M x^M \end{aligned} \tag{49}$$

$$\frac{\partial V}{\partial P_x} = (0)\frac{\partial x^M}{\partial P_x} + (0)\frac{\partial y^M}{\partial P_x} + (0)\frac{\partial \lambda^M}{\partial P_x} - \lambda^M x^M = -\lambda^M x^M \tag{50}$$

Next, differentiate the value function with respect to B

$$\begin{aligned} \frac{\partial V}{\partial B} = \quad & (U_x - \lambda^M P_x)\frac{\partial x^M}{\partial B} + (U_y - \lambda^M P_y)\frac{\partial y^M}{\partial B} \\ & +B - P_x x^M - P_y y^M)\frac{\partial \lambda^M}{\partial B} + \lambda^M \end{aligned} \tag{51}$$

$$\frac{\partial V}{\partial B} = (0)\frac{\partial x^M}{\partial B} + (0)\frac{\partial y^M}{\partial B} + (0)\frac{\partial \lambda^M}{\partial B} + \lambda^M = \lambda^M \tag{52}$$

Finally, taking the ratio of the two partial derivatives

$$\frac{\frac{\partial V}{\partial P_x}}{\frac{\partial V}{\partial B}} = \frac{-\lambda^M x^M}{\lambda^M} = x^M \tag{53}$$

which is Roy's identity.

Shephard's Lemma

Earlier in the chapter an application of the envelope theorem was the derivation of Hotelling's Lemma, which states that the partial derivatives of the maximum value of the profit function yields the firm's factory demand functions and the supply functions. A similar approach applied to the expenditure function yields Shephard's Lemma.

Consider the consumer's minimization problem. The Lagrangian is

$$Z = P_x x + P_y y + \lambda(U^* - U(x, y)) \tag{54}$$

From the first-order conditions, the solutions are implicitly defined

$$\begin{aligned} x^h &= x^h(P_x, P_y, U^*) \\ y^h &= y^h(P_x, P_y, U^*) \\ \lambda^h &= \lambda^h(P_x, P_y, U^*) \end{aligned} \tag{55}$$

Substituting these solutions into the Lagrangian yields the minimum value function

$$V(P_x, P_y, U^*) = P_x x^h + P_y y^h + \lambda^h(U^* - U(x^h, y^h)) \tag{56}$$

The partial derivatives of the value function with respect to P_x and P_y are the consumer's conditional, or Hicksian, demands:

$$\begin{aligned} \frac{\partial V}{\partial P_x} &= (P_x - \lambda^h U_x)\frac{\partial x^h}{\partial P_x} + (P_y - \lambda^h U_y)\frac{\partial y^h}{\partial P_x} + (U^* - U(x^h, y^h))\frac{\partial \lambda^h}{\partial P_x} + x^h \\ \frac{\partial V}{\partial P_x} &= (0)\frac{\partial x^h}{\partial P_x} + (0)\frac{\partial y^h}{\partial P_x} + (0)\frac{\partial \lambda^h}{\partial P_x} + x^h = x^h \end{aligned} \tag{57}$$

and

$$\begin{aligned}\frac{\partial V}{\partial P_y} &= (P_x-\lambda^h U_x)\frac{\partial x^h}{\partial P_y}+(P_y-\lambda^h U_y)\frac{\partial y^h}{\partial P_y}+(U^*-U(x^h,y^h))\frac{\partial \lambda^h}{\partial P_y}+y^h\\ \frac{\partial V}{\partial P_y} &= (0)\frac{\partial x^h}{\partial P_y}+(0)\frac{\partial y^h}{\partial P_y}+(0)\frac{\partial \lambda^h}{\partial P_y}+y^h=y^h\end{aligned} \tag{58}$$

Differentiating V with respect to the constraint U* yields λ^h, the marginal cost of the constraint

$$\begin{aligned}\frac{\partial V}{\partial U^*} = &\quad (P_x-\lambda^h U_x)\frac{\partial x^h}{\partial U^*}+(P_y-\lambda^h U_y)\frac{\partial y^h}{\partial P_y}\\ &+(U^*-U(x^h,y^h))\frac{\partial \lambda^h}{\partial U^*}+\lambda^h\\ \frac{\partial V}{\partial U^*} = &\quad (0)\frac{\partial x^h}{\partial U^*}+(0)\frac{\partial y^h}{\partial U^*}+(0)\frac{\partial \lambda^h}{\partial U^*}+y^h=\lambda^h\end{aligned}$$

Together, these three partial derivatives are Shephard's Lemma.

Example of duality for the consumer choice problem

Utility Maximization

Consider a consumer with the utility function $U = xy$, who faces a budget constraint of $B = P_x x P_y y$, where all variables are defined as before.

The choice problem is

Maximize

$$U = xy \tag{59}$$

Subject to

$$B = P_x x P_y y \tag{60}$$

The Lagrangian for this problem is

$$Z = xy + \lambda(B - P_x x P_y y) \tag{61}$$

The first-order conditions are

$$\begin{aligned}Z_x &= y-\lambda P_x = 0\\ Z_y &= x-\lambda P_y = 0\\ Z_\lambda &= B-P_x x-P_y y=0\end{aligned} \tag{62}$$

Solving the first-order conditions yield the following solutions

$$x^M = \tfrac{B}{2P_x} \quad y^M = \tfrac{B}{2P_y} \quad \lambda = \tfrac{B}{2P_x P_y} \tag{63}$$

where x^M and y^M are the consumer's Marshallian demand functions. Checking second order conditions, the bordered Hessian is

$$|\overline{H}| = \begin{vmatrix} 0 & 1 & -P_x \\ 1 & 0 & -P_y \\ -P_x & -P_y & 0 \end{vmatrix} = 2P_x P_y > 0 \tag{64}$$

Therefore the solution does represent a maximum . Substituting x^M and y^M into the utility function yields the indirect utility function

$$V(P_x,P_y,B) = \left(\frac{B}{2P_x}\right)\left(\frac{B}{2P_y}\right) = \frac{B^2}{4P_xP_y} \tag{65}$$

If we denote the maximum utility by U$_0$ and re-arrange the indirect utility function to isolate B

$$\frac{B^2}{4P_xP_y} = U_0 \tag{66}$$

$$B = (4P_xP_yU_0)^{\frac{1}{2}} = 2P_x^{\frac{1}{2}}P_y^{\frac{1}{2}}U_0^{\frac{1}{2}} = E(P_x,P_y,U_0) \tag{67}$$

We have the expenditure function

Roy's Identity Let's verify Roy's identity which states

$$x^M = -\frac{\frac{\partial V}{\partial P_x}}{\frac{\partial V}{\partial B}} \tag{68}$$

Taking the partial derivative of V

$$\frac{\partial V}{\partial P_x} = -\frac{B^2}{4P_x^2 P_y} \tag{69}$$

and

$$\frac{\partial V}{\partial B} = -\frac{B}{P_x P_y} \tag{70}$$

Taking the negative of the ratio of these two partials

$$-\frac{\frac{\partial V}{\partial P_x}}{\frac{\partial V}{\partial B}} = -\frac{\left(\frac{B^2}{4P_x^2 P_y}\right)}{\left(\frac{B}{P_x P_y}\right)} = \frac{B}{2P_x} = x^M \tag{71}$$

Thus we find that Roy's Identity does hold.

The dual and Shephard's Lemma

Now consider the dual problem of cost minimization given a fixed level of utility. Letting U_0 denote the target level of utility, the problem is

Minimize

$$P_x x + P_y y \tag{72}$$

Subject to

$$U_0 = xy \tag{73}$$

The Lagrangian for the problem is

$$Z = P_x x + P_y y + \lambda(U_0 - xy) \tag{74}$$

The first-order conditions are

$$\begin{aligned} Z_x &= P_x - \lambda y = 0 \\ Z_y &= P_y - \lambda x = 0 \\ Z_\lambda &= U_0 - xy = 0 \end{aligned} \tag{75}$$

Solving the system of equations for x, y and λ

$$\begin{aligned} x^h &= \left(\frac{P_y U_0}{P_x}\right)^{\frac{1}{2}} \\ y^h &= \left(\frac{P_x U_0}{P_y}\right)^{\frac{1}{2}} \\ \lambda^h &= \left(\frac{P_x P_y}{U_0}\right)^{\frac{1}{2}} \end{aligned} \tag{76}$$

where x^h and y^h are the consumer's compensated (Hicksian) demand functions. Checking the second order conditions for a minimum

$$|\overline{H}| = \begin{vmatrix} 0 & -\lambda & -y \\ -\lambda & 0 & -x \\ -y & -x & 0 \end{vmatrix} = -2xy\lambda < 0 \tag{77}$$

Thus the sufficient conditions for a minimum are satisfied.

Substituting x^h and y^h into the original objective function gives us the minimum value function, or expenditure function

$$
\begin{aligned}
&P_x x^h + P_y y^h = P_x \left(\tfrac{P_y U_0}{P_x}\right)^{\frac{1}{2}} + P_y \left(\tfrac{P_x U_0}{P_y}\right)^{\frac{1}{2}} \\
&= (P_x P_y U_0)^{\frac{1}{2}} + (P_x P_y U_0)^{\frac{1}{2}} \\
&= 2P_x^{\frac{1}{2}} P_y^{\frac{1}{2}} U_0^{\frac{1}{2}}
\end{aligned}
\tag{78}
$$

Note that the expenditure function derived here is identical to the expenditure function obtained by re-arranging the indirect utility function from the maximization problem.

Shephard's Lemma We can now test Shephard's Lemma by differentiating the expenditure function directly.

First, we derive the conditional demand functions

$$
\frac{\partial E(P_x, P_y, U_0)}{\partial P_x} = \frac{\partial}{\partial P_x}\left(2P_x^{\frac{1}{2}} P_y^{\frac{1}{2}} U_0^{\frac{1}{2}}\right) = \frac{P_y^{\frac{1}{2}} U_0^{\frac{1}{2}}}{P_x^{\frac{1}{2}}} = x^h \tag{79}
$$

and

$$
\frac{\partial E(P_x, P_y, U_0)}{\partial P_y} = \frac{\partial}{\partial P_y}\left(2P_x^{\frac{1}{2}} P_y^{\frac{1}{2}} U_0^{\frac{1}{2}}\right) = \frac{P_y^{\frac{1}{2}} U_0^{\frac{1}{2}}}{P_y^{\frac{1}{2}}} = y^h \tag{80}
$$

Next, we can find the marginal cost of utility (the Lagrange multiplier)

$$
\frac{\partial E(P_x, P_y, U_0)}{\partial U^0} = \frac{\partial}{\partial U^0}\left(2P_x^{\frac{1}{2}} P_y^{\frac{1}{2}} U_0^{\frac{1}{2}}\right) = \frac{P_x^{\frac{1}{2}} P_y^{\frac{1}{2}}}{U_0^{\frac{1}{2}}} = \lambda^h \tag{81}
$$

Thus, Shephard's Lemma holds in this example.

Duality and the Alternative Slutsky

In chapter 12 we derived the Slutsky equation by differentiating the first-order conditions and seperating the income and substitution effects by use of laplace expansion.

Consider first that from the utility maximum problem we derived solutions for x and y

$$
\begin{aligned}
x^M &= x^M(P_x, P_y, B) \\
y^M &= y^M(P_x, P_y, B)
\end{aligned}
\tag{82}
$$

Substituting these solutions into the utility function yielded the indirect utility function (or maximum value function)

$$
U^* = U(x^M(P_x, P_y, B), y^M(P_x, P_y, B)) = U^*(P_x, P_y, B) \tag{83}
$$

which could be rewritten as

$$
B^* = B(P_x, P_{y,} U^*) \tag{84}
$$

Second, from the budget minimization problem we derived the Hicksian, or compensated, demand function

$$
x^* = x^h(P_x, P_{y,} U^*) \tag{85}
$$

which, by Shephard's lemma, is equivalent to the partial derivative of the expenditure function with respect to P_x:

$$
\frac{\partial B(P_x, P_{y,} U^*)}{\partial P_x} = x^c(P_x, P_{y,} U^*) \tag{86}
$$

Thus we know that if the maximum value of utility obtained from

$$
Max \quad U(x, y) + \lambda(B - P_x x - P_y y)
$$

is the same value as the exogenous level of utility found in the constrained minimization problem

$$Min \quad P_x x + P_y y + \lambda(U_0 - U(x,y)) \tag{87}$$

the values of x and y that satisfy the first-order conditions of both problems will be identical, or

$$x^c(P_x, P_y, U_0) = x^m(P_x, P_y, B) \tag{88}$$

at the optimum we substitute the expenditure function into x^M in place of the budget, B, we get

$$x^c(P_x, P_y, U_0) = x^M(P_x, P_y, B^*(P_x, P_y, U_0)) \tag{89}$$

Differentiate both sides of equation 89 with respect to P_x

$$\begin{aligned} \frac{\partial x^c(P_x,P_y,U_0)}{\partial P_x} = & \frac{\partial x^M(P_x,P_y,B^*(P_x,P_y,U_0))}{\partial P_x} \\ & + \frac{\partial x^M(P_x,P_y,B^*(P_x,P_y,U_0))}{\partial B}\frac{\partial B(P_x,P_y,U_0)}{\partial P_x} \end{aligned} \tag{90}$$

But we know from Shephard's lemma that

$$\frac{\partial B(P_x, P_y, U_0)}{\partial P_x} = x_c \tag{91}$$

substituting equation 91 in to equation 90 we get

$$\frac{\partial x^c}{\partial P_x} = \frac{\partial x^M}{\partial P_x} + x^c \frac{\partial x^M}{\partial B} \tag{92}$$

Subtract $(x^c \frac{\partial x^M}{\partial B})$ from both sides gives us

$$\frac{\partial x^M}{\partial P_x} = \underbrace{-x^c \frac{\partial x^M}{\partial B}}_{\text{Income effect}} + \underbrace{\frac{\partial x^c}{\partial P_x}}_{\text{Substitution effect}} \tag{93}$$

If we compare equation (93) to equation 12.42' we see that we have arrived at the identical result. The method of deriving the Slutsky decomposition through the application of duality and the envelope theorem is sometimes referred to as the "instant Slutsky".