British Columbia Institute of Technology
Calculus for Business and Economics Lecture Notes

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Contents

1 Matrix Algebra

1. Gives us a shorthand way of writing a large system of equations.
2. Allows us to test for the existance of solutions to simultaneous systems.
3. Allows us to solve a simultaneous system.

DRAWBACK: Only works for linear systems. However, we can often covert non-linear to linear systems.

Example

$$y = ax^b$$
$$\ln y = \ln a + b \ln x$$

1.1 Matrices and Vectors

Given

$$y = 10 - x \quad \Rightarrow \quad x + y = 10$$
$$y = 2 + 3x \quad \Rightarrow \quad -3x + y = 2$$

In matrix form

$$\underset{\text{Coefficients}}{\overset{\text{Matrix of}}{\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}}} \underset{\text{Unknowns}}{\overset{\text{Vector of}}{\begin{bmatrix} x \\ y \end{bmatrix}}} = \underset{\text{Constants}}{\overset{\text{Vector of}}{\begin{bmatrix} 10 \\ 2 \end{bmatrix}}}$$

In general

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = d_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = d_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = d_m$$

n-unknowns $(x_1, x_2, \ldots x_n)$

Matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$$

Matrix shorthand

$$Ax = d$$

Where:
A= coefficient martrix or an array
x= vector of unknowns or an array
d= vector of constants or an array

Subscript notation

$$a_{ij}$$

is the coefficient found in the i-th row (i=1,. . . ,m) and the j-th column (j=1,. . . ,n)

1.2 Vectors as special matrices

The number of rows and the number of columns define the DIMENSION of a matrix.

A is m rows and n is columns or "mxn."

A matrix containing 1 column is called a "column VECTOR"

x is a n×1 column vector

d is a m×1 column vector

If x were arranged in a horizontal array we would have a row vector.

Row vectors are denoted by a prime

$$x' = [x_1, x_2, \ldots, x_n]$$

A 1×1 vector is known as a scalar.

$$x = [4] \text{ is a scalar}$$

1.2.1 Matrix Operators

If we have two matrices, A and B, then

$$A = B \quad iff \quad a_{ij} = b_{ij}$$

Addition and Subtraction of Matrices Suppose A is an m×n matrix and B is a p×q matrix then A and B is possible only if m=p and n=q. Matrices must have the same dimensions.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}+b_{11}) & (a_{12}+b_{12}) \\ (a_{21}+b_{21}) & (a_{22}+b_{22}) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Subtraction is identical to addition

$$\begin{bmatrix} 9 & 4 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} (9-7) & (4-2) \\ (3-1) & (1-6) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix}$$

Scalar Multiplication Suppose we want to multiply a matrix by a scalar

$$\underset{1\times 1}{k} \times \underset{m\times n}{A}$$

We multiply every element in A by the scalar k

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & & & \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

Example

Let k=[3] and A= $\begin{bmatrix} 6 & 2 \\ 4 & 5 \end{bmatrix}$

then kA=

$$kA = \begin{bmatrix} 3\times 6 & 3\times 2 \\ 3\times 4 & 3\times 5 \end{bmatrix} = \begin{bmatrix} 18 & 6 \\ 12 & 15 \end{bmatrix}$$

Multiplication of Matrices To multiply two matrices, A and B, together it must be true that for

$$\underset{m\times n}{A} \times \underset{n\times q}{B} = \underset{m\times q}{C}$$

That A must have the same number of columns (n) as B has rows (n).

The product matrix, C, will have the same number of rows as A and the same number of columns as B.

Example

$$\begin{matrix} A & \times & B & = & C \\ (1\times 3) & & (3\times 4) & & (1\times 4) \\ 1row & & 3rows & & 1row \\ 3cols & & 4cols & & 4cols \end{matrix}$$

In general

$$\begin{matrix} A & \times & B & \times & C & \times & D & = & E \\ (3\times 2) & & (2\times 5) & & (5\times 4) & & (4\times 1) & & (3\times 1) \end{matrix}$$

To multiply two matrices:

(1) Multiply each element in a given row by each element in a given column

(2) Sum up their products

Example 1

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Where:
c_{11}=$a_{11}b_{11} + a_{12}b_{21}$ (sum of row 1 times column 1)
c_{12}=$a_{11}b_{12} + a_{12}b_{22}$ (sum of row 1 times column 2)
c_{21}=$a_{21}b_{11} + a_{22}b_{21}$ (sum of row 2 times column 1)
c_{22}=$a_{21}b_{12} + a_{22}b_{22}$ (sum of row 2 times column 2)
Example 2

$$\begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (3 \times 1) & +(2 \times 3) & (3 \times 2) & +(2 \times 4) \end{bmatrix} = \begin{bmatrix} 9 & 14 \end{bmatrix}$$

Example 3

$$\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} (3 \times 2) & +(2 \times 1) & +(1 \times 4) \end{bmatrix} = [12]$$

12 is the inner product of two vectors.

Suppose

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{then} \quad x' = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

therefore

$$x'x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1^2 + x_2^2]$$

However

$$xx' = 2 \text{ by } 2 \text{ matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_2x_1 & x_2^2 \end{bmatrix}$$

Example 4

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$Ab = \begin{bmatrix} (1 \times 5) & + & (3 \times 9) \\ (2 \times 5) & + & (8 \times 9) \\ (4 \times 5) & + & (0 \times 9) \end{bmatrix} = \begin{bmatrix} 32 \\ 82 \\ 20 \end{bmatrix}$$

Example

$$Ax = d$$

$$\overset{A}{\begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix}}_{(3\times 3)} \quad \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{(3\times 1)} = \overset{d}{\begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}}_{(3\times 1)}$$

This produces

$$\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}$$

1.2.2 National Income Model

$$y = c + I_0 + G_0$$
$$C = a + bY$$

Arrange as

$$\begin{aligned} y - C &= I_0 + G_0 \\ -bY + C &= a \end{aligned}$$

Matrix form

$$\overset{A}{\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}} \quad \overset{x}{\begin{bmatrix} Y \\ C \end{bmatrix}} \overset{=d}{=\begin{bmatrix} I_o + G_o \\ a \end{bmatrix}}$$

1.2.3 Division in Matrix Algebra

In ordinary algebra

$$\frac{a}{b} = c$$

is well defined iff b$\neq$ 0.
Now $\frac{1}{b}$ can be rewritten as b^{-1}, therefore $ab^{-1} = c$, also b$^{-1}a = c$.

But in matrix algebra

$$\frac{A}{B} = C$$

is not defined. However,

$$AB^{-1} = C$$

is well defined. BUT

$$AB^{-1} \neq B^{-1}A$$

B^{-1} is called the inverse of B

$$B^{-1} \neq \frac{1}{B}$$

In some ways B^{-1} has the same properties as b^{-1} but in other ways it differs. We will explore these differences later.

1.3 Linear Dependance

Suppose we have two equations

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 6x_2 &= 3 \end{aligned}$$

To solve

$$\begin{aligned} &3\left[-2x_2 + 1\right] - 6x_2 = 3 \\ &6x_2 + 3 - 6x_2 = 3 \\ &3 = 3 \end{aligned}$$

There is no solution. These two equations are linearly dependent. Equation 2 is equal to two times equation one.

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$Ax = d$$

where A is a two column vectors

$$\begin{bmatrix} U_1 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} U_2 \\ 2 \\ 6 \end{bmatrix}$$

Or A is two row vector

$$\begin{aligned} V_1' = &\quad [1 \quad 2] \\ V_2' = &\quad [3 \quad 6] \end{aligned}$$

Where column two is twice column one and/or row two is three times row one

$$2U_1 = U_2 \text{ or } 3V_1' = V_2'$$

Linear Dependence Generally:

A set of vectors is said to be linearly dependent iff any one of them can be expressed as a linear combination of the remaining vectors.

Example:

Three vectors,

$$V_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad V_2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix} \quad V_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

are linearly dependent since

$$3V_1 - 2V_2 = V_3$$

$$\begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

or expressed as

$$3V_1 - 2V_2 - V_3 = 0$$

General Rule

A set of vectors, $\mathrm{V}_{1,}V_{2,...,}V_n$ are linearly dependent if there exsists a set of scalars (i=1,. . . ,n). Not all equal to zero, such that

$$\sum_{i=1}^{n} = k_i V_i = 0$$

Note

$$\sum_{i=1}^{n} k_i V_i = k_1 V_1 + k_2 V_2 + \ldots + k_n V_n$$

1.4 Commutative, Associative, and Distributive Laws

From Highschool algebra we know commutative law of addition,

$$a + b = b + a$$

commutative law of multiplication,

$$ab = ba$$

Associative law of addition,

$$(a + b) + c = a + (b + c)$$

associative law of multiplication,

$$(ab)c = a(bc)$$

Distributive law

$$a(b + c) = ab + ac$$

In matrix algebra most, but not all, of these laws are true.

1.4.1 Communicative Law of Addition

$$A + B = B + A$$

Since we are adding individual elements and $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ for all i and j.

1.4.2 Similarly Associative Law of Addition

$$A + (B + C) = (A + B) + C$$

for the same reasons.

1.4.3 Matrix Multiplication

Matrix multiplication in not communtative

$$AB \neq BA$$

Example 1
Let A be 2×3 and B be 3×2

$$\underset{(2\times 3)}{A} \times \underset{(3\times 2)}{B} = \underset{(2\times 2)}{C} \quad whereas \quad \underset{(3\times 2)}{B} \times \underset{(2\times 3)}{A} = \underset{(3\times 3)}{C}$$

Example 2
Let A=$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B=$\begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$

$$AB = \begin{bmatrix} (1\times 10) + (2\times 6) & (1\times -1) + (2\times 7) \\ (3\times 0) + (4\times 6) & (3\times -1) + (4\times 7) \end{bmatrix} = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}$$

But

$$BA = \begin{bmatrix} (0)(1) - (1)(3) & (0)(2) - (1)(4) \\ (6)(1) + (7)(3) & (6)(2) + (7)(4) \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}$$

Therefore, we realize the distinction of post multiply and pre multiply. In the case

$$AB = C$$

B is pre multiplied by A, A is post multiplied by B.

1.4.4 Associative Law

Matrix multiplication is associative

$$(AB)C = A(BC) = ABC$$

as long as their dimensions conform to our earlier rules of multiplication.

$$\underset{(m\times n)}{A} \times \underset{(n\times p)}{B} \times \underset{(p\times q)}{C}$$

1.4.5 Distributive Law

Matrix multiplication is distributive

$$\begin{array}{ll} A(B+C) = AB + AC & \text{Pre multiplication} \\ (B+C)A = BA + CA & \text{Post multiplication} \end{array}$$

1.5 Identity Matrices and Null Matrices

1.5.1 Identity matrix:

is a square matrix with ones on its principal diagonals and zeros everywhere else.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_n = \begin{bmatrix} 1 & 0 & \dots & n \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

Identity Matrix in scalar algebra we know

$$1 \times a = a \times 1 = a$$

In matrix algebra the identity matrix plays the same role

$$IA = AI = A$$

Example 1
Let A=$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} (1\times 1)+(0\times 2) & (1\times 3)+(0\times 4) \\ (0\times 1)+(1\times 2) & (0\times 3)+(1\times 4) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Example 2
Let A=$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix}$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A\,\{I_2 Case\}$$

$$AI = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A\,\{I_3 Case\}$$

Furthermore,

$$\underset{(m\times n)(n\times p)}{AIB} = (AI)B = A(IB) = \underset{(m\times n)(n\times p)}{AB}$$

1.5.2 Null Matrices

A null matrix is simply a matrix where all elements equal zero.

$$0 = \underset{(2\times 2)}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \quad 0 = \underset{(2\times 3)}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

The rules of scalar algebra apply to matrix algebra in this case.

Example

$$a + 0 = a \Rightarrow \{scalar\}$$

$$A + 0 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A \qquad \{matrix\}$$

$$A \times 0 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

1.6 Idiosyncracies of matrix algebra

1) We know AB≠BA

2)ab=0 implies a or b=0

In matrix

$$AB = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

1.7 Transposes and Inverses

1) Transpose: is when the rows and columns are interchanged.

Transpose of $A = A\prime$or A^T

Example

If A=$\begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix}$ and B=$\begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix}$

A'=$\begin{bmatrix} 3 & 1 \\ 8 & 0 \\ -9 & 4 \end{bmatrix}$ and B'=$\begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix}$

Symmetrix Matrix

If A=$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$ then A'=$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$

A is a symmetric matrix.

Properties of Transposes
1) $(A\prime)\prime = A$
2) $(A+B)\prime = A\prime + B\prime$
3) $(AB)\prime = B\prime A\prime$

Inverses and their Properties

In scalar algebra if

$$ax = b$$

then

$$x = \frac{b}{a} \text{ or } ba^{-1}$$

In matrix algebra
if

$$Ax = d$$

then

$$x = A^{-1}d$$

where A^{-1} is the inverse of A.

Properties of Inverses 1) Not all matrices have inverses
non-singular: if there is an inverse
singular: if there is no inverse
2) A matrix must be square in order to have an inverse. (Necessary but not sifficient)
3) In scalar algebra $\frac{a}{a} = 1$, in matrix algebra $AA^{-1} = A^{-1}A = I$
4) If an inverse exists then it must be unique.

Example
Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{1}{2} \end{bmatrix}$

$A^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$ by factoring $\left\{\frac{1}{6} \text{ is a scalar}\right\}$
Post Multiplication

$$AA^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \frac{1}{6} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \frac{1}{6} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Pre Multiplication

$$A^{-1}A = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Further properties If A and B are square and non-singular then:
1) $(A^{-1})^{-1} = A$
2) $(AB)^{-1} = B^{-1}A^{-1}$
3) $(A\prime)^{-1} = (A^{-1})^{1}$

Solving a linear system

Suppose

$$\underset{(3\times 3)}{A}\ \underset{(3\times 1)}{x} = \underset{(3\times 1)}{d}$$

then

$$\underset{(3\times 3)}{A^{-1}}\ \underset{(3\times 3)}{A}\ \underset{(3\times 1)}{x} = \underset{(3\times 3)}{A^{-1}}\ \underset{(3\times 1)}{d}$$

$$\underset{(3\times 3)}{I}\ \underset{(3\times 1)}{x} = \underset{(3\times 3)}{A^{-1}}\ \underset{(3\times 1)}{d}$$

$$x = A^{-1}d$$

Example

$$Ax = d$$

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} \quad A^{-1} = \frac{1}{52}\begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}$$

then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{52}\begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}\begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$x_1^* = 2 \quad x_2^* = 3 \quad x_3^* = 1$$

2 Linear Dependence and Determinants

Suppose we have the following

$$\begin{array}{ll} 1. & x_1 + 2x_2 = 1 \\ 2. & 2x_1 + 4x_2 = 2 \end{array}$$

where equation two is twice equation one. Therefore, there is no solution for x_1, x_2.

In matrix form:

$$Ax = d$$

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{d}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

The determinant of the coefficient matrix is

$$|A| = (1)(4) - (2)(2) = 0$$

a determinant of zero tells us that the equations are linearly dependent. Sometimes called a "vanishing determinant."

In general, the determinant of a square matrix, A is written as $|A|$ or $detA$.

For two by two case

$$|A| = \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} = a_{11}a_{22} - a_{12}a_{21} = k$$

where k is unique
any k$\neq$ 0 implies linear independence

Example 1
A=$\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$

$$|A| = (3 \times 5) - (1 \times 2) = 13 \qquad \{\text{Non-singular}\}$$

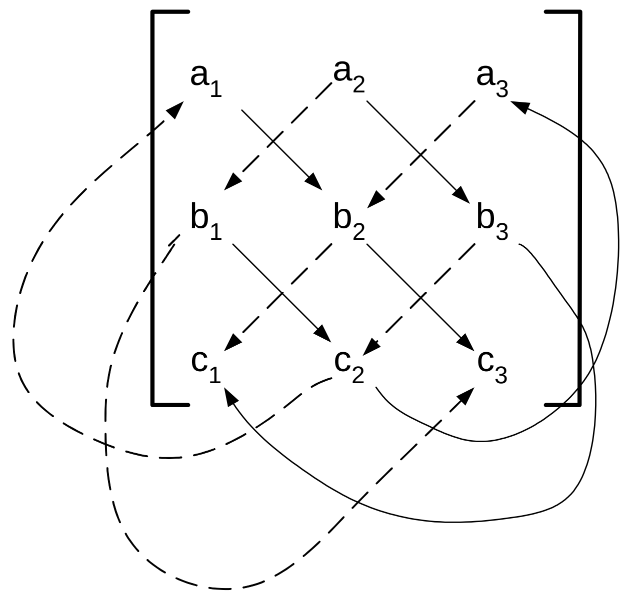
Example 2
$B = \begin{bmatrix} 2 & 6 \\ 8 & 24 \end{bmatrix}$

$$|B| = (2 \times 24) - (6 \times 8) = 0 \qquad \{\text{Singular}\}$$

Three by three case
Given $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$
then

$$|A| = (a_1b_2c_3) + (a_2b_3c_1) + (b_1c_2a_3) - (a_3b_2c_1) - (a_2b_1c_3) - (b_3c_2a_1)$$

Cross-diagonals



Multiple along the diagonals and add up their products
⇒The product along the solid lines are given a positive sign
⇒The product of the dashed lines are negative.

2.1 Using Laplace expansion

⇒ The cross diagonal method does not work for matrices greater than three by three

⇒ Laplace expansion evaluates the determinant of a matrix, A, by means of subdeterminants of A.

Subdeterminants or Minors

Given $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

By deleting the first row and first column, we get

$$|M_{11}| = \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix}$$

The determinant of this matrix is the minor element a_1.

$|M_{ij}| \equiv$ is the subdeterminant from deleting the i-th row and the j-th column.

Given A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

then

$$M_{21} \equiv \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} \quad M_{31} \equiv \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

2.2 Cofactors

A cofactor is a minor with a specific algebraic sign.

$$C_{ij} = (-1)^{i+j} |M_{ij}|$$

therefore

$$C_{11} = (-1)^2 |M_{11}| = |M_{11}|$$

$$C_{21} = (-1)^3 |M_{21}| = -|M_{21}|$$

The determinant by Laplace
Expanding down the first column

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11} |C_{11}| + a_{21} |C_{21}| + a_{31} |C_{31}| = \sum_{i=1}^{3} a_{i1} |C_{i1}|$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Note: minus sign $(-1)^{(1+2)}$

$$|A| = a_{11} [a_{22}a_{33} - a_{23}a_{32}] - a_{21} [a_{12}a_{33} - a_{13}a_{32}] + a_{31} [a_{12}a_{23} - a_{13}a_{22}]$$

Laplace expansion can be used to expand along any row or any column.

Example
Third row

$$|A| = a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example
$A = \begin{bmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}$
(1) Expand the first column

$$|A| = 8 \begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} - 4 \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} + 6 \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$$

$$|A| = (8 \times 0) - (4 \times 3) + (6 \times 1) = -6$$

(2) Expand the second column

$$|A| = -1 \begin{vmatrix} 4 & 1 \\ 6 & 3 \end{vmatrix} + 0 \begin{vmatrix} 8 & 3 \\ 6 & 3 \end{vmatrix} - 0 \begin{vmatrix} 8 & 3 \\ 4 & 1 \end{vmatrix}$$

$$|A| = (-1 \times 6) + (0) - (0) = -6$$

Suggestion: Try to choose an easy row or column to expand. (i.e. the ones with zero's in it.)

2.3 Rank of a Matrix

Definition

The rank of a matrix is the maximum number linearly independent rows in the matrix.

If A is an m×n matrix, then the rank of A is

$$r(A) \leq \min{[m,n]}$$

Read as: the rank of A is less than or equal to the minimum of m or n.

Using Determinants to Find the Rank

(1) If A is n×m and $|A|$=0

(2) Then delete one row and one column, and find the determinant of this new (n-1)×(n-1) matrix.

(3) Continue this process until you have a non-zero determinant.

3 Matrix Inversion

Given an n×n matrix, A, the inverse of A is

$$A^{-1} = \frac{1}{|A|} \bullet AdjA$$

where $AdjA$ is the adjoint matrix of A. $AdjA$ is the transpose of matrix A's cofactor matrix. It is also the adjoint, which is an n×n matrix

Cofactor Matrix (denoted C)

The cofactor matrix of A is a matrix who's elements are the cofactors of the elements of A

$$If\ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ then } C = \begin{bmatrix} |C_{11}| & |C_{12}| \\ |C_{21}| & |C_{22}| \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

Example
Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \Rightarrow |A|$=-2
Step 1: Find the cofactor matrix

$$C = \begin{bmatrix} |C_{11}| & |C_{12}| \\ |C_{21}| & |C_{22}| \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$$

Step 2: Transpose the cofactor matrix

$$C^T = Adj A = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$$

Step 3: Multiply all the elements of AdjA by $\frac{1}{|A|}$ to find A^{-1}

$$A^{-1} = \frac{1}{|A|} \bullet Adj A = \left(-\frac{1}{2}\right) \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Step 4: Check by $AA^{-1} = I$

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} (3)(0) + (2)(\frac{1}{2}) & (3)(1) + (2)(-\frac{3}{2}) \\ (1)(0) + (0)(\frac{1}{2}) & (1)(1) + (0)(-\frac{3}{2}) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4 Cramer's Rule

Suppose:

$$\text{Equation 1} \quad a_1x_1 + a_2x_2 = d_1$$

$$\text{Equation 2} \quad b_1x_1 + b_2x_2 = d_2$$

or

$$\begin{matrix} A & x & = & d \\ \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \end{matrix}$$

where

$$A = a_1b_2 - a_2b_1 \neq 0$$

Solve for x_1 by substitution
From equation 1

$$x_2 = \frac{d_1 - a_1x_1}{a_2}$$

and equation 2

$$x_2 = \frac{d_2 - b_1 x_1}{b_2}$$

therefore:

$$\frac{d_1 - a_1 x_1}{a_2} = \frac{d_2 - b_1 x_1}{b_2}$$

Cross multiply

$$d_1 b_2 - a_1 b_2 x_1 = d_2 a_2 - b_1 a_2 x_1$$

Collect terms

$$d_1 b_2 - d_2 a_2 = (a_1 b_2 - b_1 a_2) x_1$$

$$x_1 = \frac{d_1 b_2 - d_2 a_2}{a_1 b_2 - b_1 a_2}$$

The denominator is the determinant of $|A|$
The numerator is the same as the denominator except $d_1 d_2$ replaces $a_1 b_1$.

Cramer's Rule

$$x_1 = \frac{\begin{vmatrix} d_1 & a_2 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{d_1 b_2 - d_2 a_2}{a_1 b_2 - b_1 a_2}$$

Where the d vector replaces column 1 in the A matrix

To find x_2 replace column 2 with the d vector

$$x_2 = \frac{\begin{vmatrix} a_1 & d_1 \\ b_1 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{a_1 d_2 - d_1 b_1}{a_1 b_2 - b_1 a_2}$$

Generally: to find x_i,replace column i with vector d; find the determinant.
x_i= the ratio of two determinants
x_i= $\frac{|A_i|}{|A|}$

4.1 Example: The Market Model

$$\text{Equation 1} \quad Q^d = 10 - P \quad \text{Or} \quad Q + P = 10$$

$$\text{Equation 2} \quad Q^s = P - 2 \quad \text{Or} \quad -Q + P = 2$$

Matrix form

$$\begin{matrix} A & x & = & d \\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} Q \\ P \end{bmatrix} & = & \begin{bmatrix} 10 \\ 2 \end{bmatrix} \end{matrix}$$

$$|A| = (1)(1) - (-1)(1) = 2$$

Find Q^e

$$Q^e = \frac{\begin{vmatrix} 10 & 1 \\ 2 & 1 \end{vmatrix}}{2} = \frac{10-2}{2} = 4$$

Find P^e

$$P^e = \frac{\begin{vmatrix} 1 & 10 \\ -1 & 2 \end{vmatrix}}{2} = \frac{2-(-10)}{2} = 6$$

Substitute P and Q into either equation 1 or equation 2 to verify

$$Q^d = 10 - P$$
$$10 - 6 = 4$$

4.2 Example: National Income Model

$$Y = C + I_0 + G_0 \quad \text{Or} \quad Y - C = I_0 + G_0$$

$$C = a + bY \quad \text{Or} \quad -bY + c = a$$

In matrix form $\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$

Solve for Y^e

$$Y^e = \frac{\begin{vmatrix} I_0 + G_0 & -1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{I_0 + G_0 + a}{1-b}$$

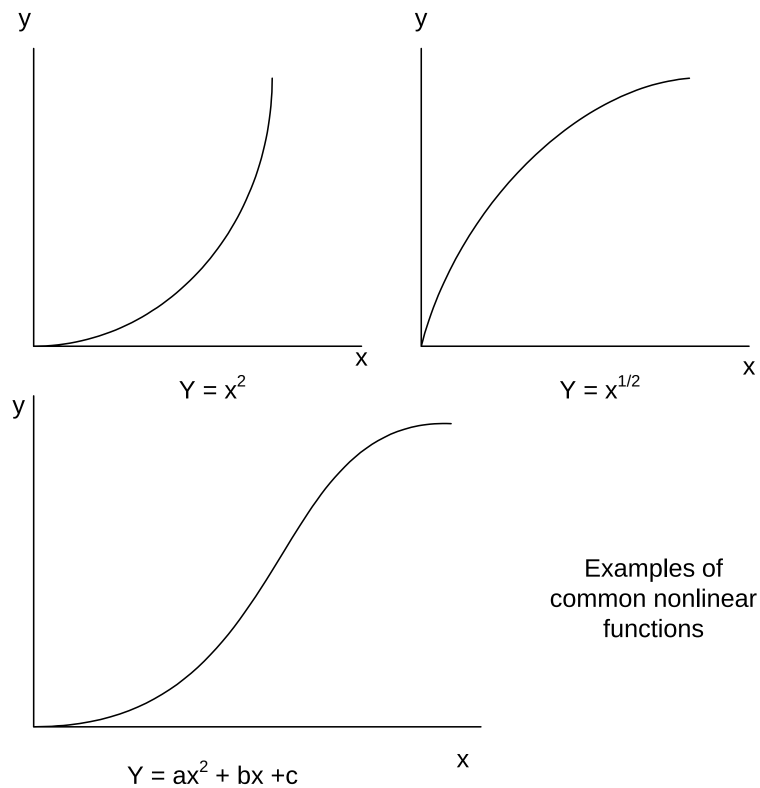
Solve for C^e

$$C^e = \frac{\begin{vmatrix} 1 & I_0 + G_0 \\ -b & a \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{a + b(I_0 + G_0)}{1-b}$$

5 Derivatives: The Five Basic Rules

5.1 Nonlinear Functions

The term derivative means "slope" or rate of change. The five rules we are about to learn allow us to find the slope of about 90% of functions used in economics, business, and social sciences.



Examples of common nonlinear functions

Suppose we have a function

$$y = f(x) \tag{1}$$

where $f(x)$ is a non linear function. For example:

$$\begin{array}{ll} 1 & y = x^2 \\ 2 & y = 3\sqrt{x} = 3x^{1/2} \\ 3 & y = ax + bx^2 + c \end{array} \tag{2}$$

Each equation is illustrated in Figure 1.

5.2 The Derivative

Given the general function

$$y = f(x)$$

the derivative of y is denoted as

$$\frac{dy}{dx} = f'(x) \quad (= y')$$

The symbol $\frac{dy}{dx}$ is an abbreviation for "the change in y (dy) FROM a change in x (dx)"; or the "rise over the run". In other words, the slope.

5.3 The Five Rules

5.3.1 The Constant Rule

Given $y = f(x) = c$, where c is an arbitrary constant, then

$$\frac{dy}{dx} = f'(x) = 0 \tag{3}$$

5.3.2 Power Function Rule

Suppose

$$y = ax^n \tag{4}$$

where a and n are any two constants. The power function rule states that the slope of the function is given by

$$\frac{dy}{dx} = f'(x) = anx^{n-1} \tag{5}$$

This is probably the most commonly used rule in an introductory calculus course. Examples

$$\begin{array}{ll} y = x^2 & \frac{dy}{dx} = y' = 2x \\ y = 4x^3 & \frac{dy}{dx} = y' = 12x^2 \\ y = 5x^{1/3} & \frac{dy}{dx} = y' = (5)(1/3)x^{1/3-1} = \frac{5}{3}x^{-2/3} \\ y = x & \frac{dy}{dx} = y' = (1)x^{1-1} = (1)x^0 = 1 \end{array}$$

Some functions that do not appear to be "power functions" can be manipulated to take the form of equation 4. For example, if

$$y = \frac{1}{x}$$

then it can also be written as

$$y = x^{-1}$$

thus

$$\frac{dy}{dx} = (-1)x^{-2}$$

Another example,

$$y = \sqrt{x}$$

which can also be written as

$$y = x^{1/2}$$

therefore, by equation 5,

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

5.3.3 Sum-difference Rule

If y is a function created by adding or subtracting multiple functions such written as

$$y = f(x) \pm g(x)$$

where $f(x)$ and $g(x)$ are each functions similar to equation 4, then the derivative of y (y') is given by

$$y' = f'(x) \pm g'(x)$$

Example 1:

$$y = 4x^3 + 5x^2$$

the derivative is

$$y' = 12x^2 + 10x$$

Example 2:

$$\begin{aligned} y &= x^5 + 3x^{1/2} - 4x + 7 \\ y' &= 5x^4 + \frac{3}{2}x^{-1/2} - 4 \end{aligned}$$

In each case we apply the power function rule (or constant rule) term-by-term

5.3.4 Product Rule

Suppose y is a composite function created by multiplying two functions together

$$y = f(x)g(x)$$

the derivative is given by

$$\frac{dy}{dx} = f'g + fg' \tag{6}$$

Example:

$$y = (3x^2 + 4)(x^3 - 5x)$$

First, break this up into $f(x)$ and $g(x)$:

$$f(x) = 3x^2 + 4 \qquad g(x) = x^3 - 5x$$

then find the derivative for each

$$f'(x) = 6x \qquad g'(x) = 3x^2 - 5$$

Now re-combine the parts according to equation 6

$$\frac{dy}{dx} = f'g + fg' = [6x]\left[x^3 - 5x\right] + \left[3x^2 + 4\right]\left[3x^2 - 5\right]$$

then you simply collect terms and simplify

5.3.5 Quotient Rule

Suppose

$$y = \frac{f(x)}{g(x)}$$

then

$$\frac{dy}{dx} = \frac{f'g - fg'}{[g]^2} \tag{7}$$

Example

$$y = \frac{x^2 + 3}{2x - 1}$$

therefore $f = x^2 + 3$ and $g = 2x - 1$. The derivatives are

$$f' = 2x$$
$$g' = 2$$

Subsitute the componants into equation 7

$$\frac{dy}{dx} = \frac{f'g - fg'}{[g]^2} = \frac{(2x)(2x-1) - \left(x^2+3\right)(2)}{(2x-1)^2}$$

which (of course) can be further simplified

OPMT 5701 Lecture Notes

6 The Chain Rule

Of all the basic rules of derivatives, the most challenging one is the chain rule. However, like the other rules, if you break it down to simple steps, it too is quite manageble. There are a couple of approaches to learning the chain rule. Both are equally good, it just comes down to preference.

The good news is that once you have mastered the chain rule – combined with the first five – you are ready to tackle about 90% of the calculus problems found in business courses (including graduate programs like the MBA!)

6.1 The Rule:

Suppose y is a "nested" function of x, where nested mean "a function inside a function".

$$y = f[g(x)]$$

where $g(x)$ is nested inside f. then the derivative is

$$\frac{dy}{dx} = y' = f'[g(x)] \times g'(x) \tag{8}$$

The process is to start with the outside function, taking the derivative of f but leaving $g(x)$ inside unchanged. Then find $g'$and multiply it by f'.

For example, if

$$y = \left(x^2 + 1\right)^3$$

then $f = (\quad)^3$ and $g = x^2 + 1$. From the power function rule, we know that the derivative of x^3 is $3x^2$. This is true for anything cubed! So

$$f' = 3\,(\quad)^2$$

(and $g' = 2x$)Using equation 8, we get

$$y' = f'\left[g(x)\right] \times g'(x) = 3(x^2 + 1)^2 \cdot (2x)$$

6.2 Some Examples

1. **Example**: if

$$y = (x^3 + x)^{1/2}$$

where $f = (\quad)^{1/2}$, then

$$y' = f'\left[g(x)\right] \times g'(x) = \frac{1}{2}\left(x^3 + x\right)^{-1/2}\left(3x^2 + 1\right)$$

2. **Example**: suppose

$$y = \frac{1}{\sqrt{3x + 7}}$$

For this one, we need to re-write this to look more like a "power-function" problem. (Remember the basics: $\sqrt{x} = x^{1/2}$ and $\frac{1}{x} = x^{-1}$) therefore

$$y = (3x + 7)^{-1/2}$$

Here $f = (\quad)^{-1/2}$. apply equation 8

$$y' = f'\left[g(x)\right] \times g'(x) = -\frac{1}{2}\left(3x + 7\right)^{-3/2}(3)$$

6.3 Combining the Chain Rule with Product and Quotient Rule

The chain rule, or "function in a function" rule: if $y = f(g(x))$ then $y' = f'(g(x)) \times g'(x)$

It is usually in the form like this

$$y = \overbrace{\left[g(x)\right]^n}^{f(x)}$$

where the derivative is

$$y' = \frac{dy}{dx} = \overbrace{n\left[g(x)\right]^{n-1}}^{f'(x)} \times g'(x)$$

Sometimes the inside function can be a bit more complex. For example

$$y = \left[h(x)j(x)\right]^n$$

by breaking it down, we see that

$$y = \overbrace{\left[\underbrace{h(x) \cdot j(x)}_{g(x)}\right]^n}^{f(x)}$$

In this case the inside function, $g(x)$ is two functions, $h(x)$ and $j(x)$. The steps are the same as before except, in this case, we need to use the product rule on $g(x)$ ($g' = h'j + hj'$)

1. **Example**: let y be

$$y = \left[(2x+1)(x^{1/2}-4)\right]^3$$

First, Identify the individual parts:

$$y = \overbrace{\left[\underbrace{(\overset{h}{2x+1})(\overset{j}{x^{1/2}-4})}_{g}\right]^3}^{f}$$

The derivative of f (power-function rule) is

$$f' = 3\left[(2x+1)(x^{1/2}-4)\right]^2$$

and the derivative of g (product rule) is

$$\begin{aligned} g' &= h'j + hj' = (2)(x^{1/2}-4) + (2x+1)\left(\tfrac{1}{2}x^{-1/2}\right) \\ \text{(simplify)} &= 2x^{1/2} - 8 + x^{1/2} + \tfrac{1}{2}x^{-1/2} \\ &= x^{1/2} + \tfrac{1}{2}x^{-1/2} - 8 \end{aligned}$$

Now put it all together

$$y' = f' \cdot g' = 3\left[(2x+1)(x^{1/2}-4)\right]^2 \left(x^{1/2} + \tfrac{1}{2}x^{-1/2} - 8\right)$$

2. **Example**: suppose

$$y = \left(\frac{6x+1}{x^3}\right)^2$$

Here this one is a little more complicated. The outside function (f) is

$$f = (\quad)^2$$

but the nested function (g) is

$$g(x) = \frac{6x+1}{x^3}$$

.which will require the quotient rule rule. This one should be done in parts seperately then subsitute into equation 8. First

$$\begin{aligned} g' &= \frac{(6)(x^3) - (2x+1)(3x^2)}{(x^3)^2} \\ &= \frac{6x^3 - 6x^3 - 3x^2}{x^6} \\ &= \frac{-3x^2}{x^6} = -3x^{-4} \end{aligned}$$

remember that $f = (\quad)^2$ and $f' = 2(\quad)$, we can use equation 8

$$y' = f'[g(x)] \times g'(x) = 2\overbrace{\left(\frac{6x+1}{x^3}\right)}^{f'} \cdot \overbrace{(-3x^{-4})}^{g'}$$

7 Natural Logarithm and the Exponential e

1. The Number e

$$\begin{aligned} \text{if } y &= e^x \text{ then } \frac{dy}{dx} = e^x \\ \text{if } y &= e^{f(x)} \text{ then } \frac{dy}{dx} = e^{f(x)} \cdot f'(x) \end{aligned}$$

2. Examples

1. (a)

$$\begin{aligned} y &= e^{3x} \\ \frac{dy}{dx} &= e^{3x}(3) \end{aligned}$$

(b)

$$\begin{aligned} y &= e^{7x^3} \\ \frac{dy}{dx} &= e^{7x^3}(21x^2) \end{aligned}$$

(c)

$$\begin{aligned} y &= e^{rt} \\ \frac{dy}{dt} &= re^{rt} \end{aligned}$$

2. Logarithm (Natural log) $\ln x$

(a) Rules of natural log

If	$Then$
$y = AB$	$\ln y = \ln(AB) = \ln A + \ln B$
$y = A/B$	$\ln y = \ln A - \ln B$
$y = A^b$	$\ln y = \ln(A^b) = b \ln A$

NOTE: $\ln(A + B) \neq \ln A + \ln B$ EVER!!!

(b) derivatives

IF	$THEN$
$y = \ln x$	$\frac{dy}{dx} = \frac{1}{x}$
$y = \ln(f(x))$	$\frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x)$

(c) Examples

i.

$$\begin{aligned} y &= \ln(x^2 - 2x) \\ dy/dx &= \frac{1}{(x^2 - 2x)}(2x - 2) \end{aligned}$$

ii.

$$\begin{aligned} y &= \ln(x^{1/2}) = \frac{1}{2} \ln x \\ dy/dx &= \left(\frac{1}{2}\right)\left(\frac{1}{x}\right) = \frac{1}{2x} \end{aligned}$$

7.1 Differentials

Given the function

$$y = f(x)$$

the derivative is

$$\frac{dy}{dx} = f'(x)$$

However, we can treat dy/dx as a fraction and factor out the dx

$$dy = f'(x)dx$$

where dy and dx are called *differentials*. If dy/dx can be interpreted as "the slope of a function", then dy is the "rise" and dx is the "run". Another way of looking at it is as follows:

- dy = the change in y
- dx = the change in x
- $f'(x)$ = how the change in x causes a change in y

Example 1 *if*

$$y = x^2$$

then

$$dy = 2xdx$$

Lets suppose $x = 2$ *and* $dx = 0.01$*. What is the change in* $y(dy)$*?*

$$dy = 2(2)(0.01) = 0.04$$

Therefore, at $x = 2$*, if* x *is increased by 0.01 then* y *will increase by 0.04.*

8 Implicit Differentiation

Suppose we have the following:

$$2y + 3x = 12$$

we can rewrite it as

$$\begin{aligned} 2y &= 12 - 3x \\ y &= 6 - \frac{3}{2}x \end{aligned}$$

Now we have $y = f(x)$ and we can take the derivative

$$\frac{dy}{dx} = -\frac{3}{2}$$

Lets consider an alternative. We know that y is a function of x or, $y = y(x)$ and the derivative of y is $\frac{dy}{dx}$. If we return to our original equation, $2y+3x = 12$, we can differentiate it IMPLICITLY in the following manner

$$\begin{aligned}
2y + 3x &= 12 \\
2dy + 3dx &= 0 \qquad \left(\frac{d(12)}{dx} = 0\right) \\
2\frac{dy}{dx} + 3\frac{dx}{dx} &= 0 \\
2\frac{dy}{dx} + 3 &= 0 \qquad \left(\frac{dx}{dx} = 1\right)
\end{aligned}$$

rearrange to get $\frac{dy}{dx}$ by itself

$$\begin{aligned}
2\frac{dy}{dx} &= -3 \\
\frac{dy}{dx} &= -\frac{3}{2}
\end{aligned}$$

which is what we got before!

Here is a few more examples:

1.

$$\begin{aligned}
y^2 + x^2 &= 36 \\
2ydy + 2xdx &= d(36) \\
2y\frac{dy}{dx} + 2x\frac{dx}{dx} &= 0 \qquad \left(\text{remember } \frac{d(36)}{dx} = 0\right) \\
2y\frac{dy}{dx} + 2x &= 0 \\
\frac{dy}{dx} &= -\frac{2x}{2y} = -\frac{x}{y}
\end{aligned}$$

2.

$$\begin{aligned}
5y^3 + 4x^5 &= 250 \\
15y^2\frac{dy}{dx} + 20x^4 &= 0 \\
15y^2\frac{dy}{dx} + 20x^4 &= 0 \\
\frac{dy}{dx} &= -\frac{20x^4}{15y^2} = -\frac{4x^4}{3y^2}
\end{aligned}$$

3.

$$\begin{aligned} y^{1/2} - 2x^2 + 5y &= 15 \\ \frac{1}{2}y^{-1/2}dy - 4xdx + 5dy &= 0 \\ \left(\frac{1}{2}y^{-1/2} + 5\right)\frac{dy}{dx} - 4x &= 0 \qquad (\div\ by\ dx) \\ \frac{dy}{dx} &= \frac{4x}{\left(\frac{1}{2}y^{-1/2} + 5\right)} \end{aligned}$$

When you are using implicit differentiation, there are two things to remember:

- First: All the rules apply as before
- Second: you are ASSUMING that you can rewrite the equation in the form $y = f(x)$

Example: Special application of the product rule.
Suppose you want to implicitly differentiate

$$xy = 24$$

what do we do here?

In this case we treat x and y as seperate functions and apply the product rule

$$\begin{aligned} x\frac{dy}{dx} + y\frac{dx}{dx} &= 0 \\ x\frac{dy}{dx} + y &= 0 \\ \frac{dy}{dx} &= -\frac{y}{x} \end{aligned}$$

Alternatively, we could first solve for y, then take the derivative

$$\begin{aligned} xy &= 24 \\ y &= \frac{24}{x} = 24x^{-1} \\ \frac{dy}{dx} &= (-1)24x^{-2} = -\frac{24}{x^2} \end{aligned}$$

which does not look like what we got with implicit differentiation, but, if we use a substitution trick, remembering that originally $xy = 24$, we will get

$$\begin{aligned} \frac{dy}{dx} &= -\frac{24}{x^2} = -\frac{xy}{x^2} \\ \frac{dy}{dx} &= -\frac{y}{x} \end{aligned}$$

Lets try it again

$$
\begin{aligned}
48 &= x^2y^3 \\
0 &= 3x^2y^2\frac{dy}{dx} + 2xy^3\frac{dx}{dx} \qquad \text{(Product rule and power-function rule)} \\
3x^2y^2\frac{dy}{dx} &= -2xy^3 \qquad \left(\text{again } \frac{dx}{dx} = 1\right) \\
\frac{dy}{dx} &= -\frac{2xy^3}{3x^2y^2} = -\frac{2y}{3x}
\end{aligned}
$$

Level Curves

If we have a function like $z = xy$ or $u = \ln x + \ln y$, then z and u are both functions of x and y. IF we fix z and u to be some particular values such as

$$z = \bar{z} \quad and \quad u = \bar{u}$$

then $\bar{z}$ and $\bar{u}$ are now treated as constants and we are evaluating the functions $\bar{z} = xy$ and $\bar{u} = \ln x + \ln y$ at a particular level. In other words, we are looking for values of x and y that keep z or u constant. This allows us to assume that y is an implicit function of x, i.e.

$$\begin{aligned} yx &= \bar{z} \\ y &= \frac{\bar{z}}{x} \end{aligned}$$

using implicit differentiation, we can find the slope of the level curve

$$\begin{aligned} yx &= \bar{z} \\ x\frac{dy}{dx} + y\frac{dx}{dx} &= \frac{d(\bar{z})}{dx} = 0 \\ \frac{dy}{dx} &= -\frac{y}{x} \end{aligned}$$

The level curve is illustrated in figure 1

In figure 1 we have graphed y as a function of x and a constant, $\bar{z}$. This curve plots all combinations of x and y that keep z at a constant level. Common examples of level curves in economics are *"indifference curves"* (constant utility) and *"isoquants"* (constant levels of output).

Lets look at the utility function example

$$u = \ln x + \ln y$$

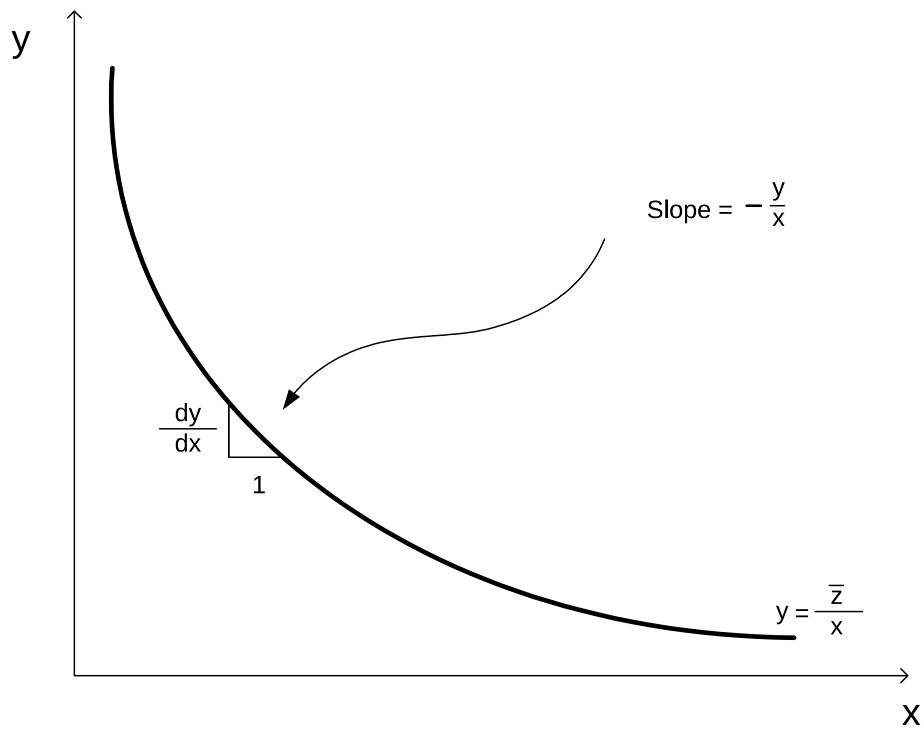
where $u = \bar{u}$. using implicit differentiation and the rule of logarithm derivatives

$$\begin{aligned} \frac{d(\bar{u})}{dx} &= \left(\frac{1}{x}\right) + \left(\frac{1}{y}\right)\frac{dy}{dx} = 0 \\ \frac{dy}{dx} &= -\frac{\frac{1}{x}}{\frac{1}{y}} = -\frac{y}{x} \end{aligned}$$

Alternatively, we could try to first write this function such that we explicitly have y as a function of x. However, this would require us to "unlog" the function, i.e.

$$\begin{aligned} \bar{u} &= \ln x + \ln y \\ \bar{u} &= \ln(xy) \\ e^{\bar{u}} &= xy \qquad \text{(unlogged)} \\ y &= \frac{e^{\bar{u}}}{x} \end{aligned}$$

The result does not look easier to work with than when we used implicit differentiation. This is an example of where implicit differentiation would be preferred.


y
Slope = − y/x
dy/dx
1
y = z̄/x
x

8.1 The difference between dy and Δy

dy is an approximation found by moving along the tangency. Δy is the difference between two points on the actual function $y = f(x)$. Given the function

$$y = x^2$$

the differential is

$$dy = 2xdx$$

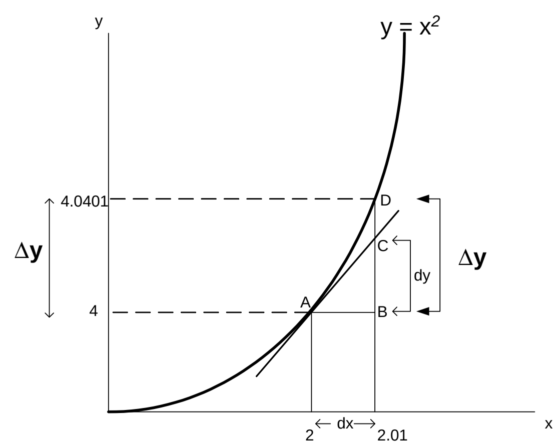
suppose $x = 2$ and $dx = .01$ then the differential, dy is

$$dy = 2xdx = 2(2)(.01) = .04$$

The other change Δy is given by

$$\begin{aligned}\Delta y &= (x + dx)^2 - x^2 \\ \Delta y &= (2.01)^2 - (2)^2 = 0.0401\end{aligned}$$

See the Graph for the difference



9 Partial Derivatives

Single variable calculus is really just a "special case" of multivariable calculus. For the function $y = f(x)$, we assumed that y was the endogenous variable, x was the exogenous variable and everything else was a parameter. For example, given the equations

$$y = a + bx$$

or

$$y = ax^n$$

we automatically treated a, b, and n as constants and took the derivative of y with respect to x (dy/dx). However, what if we decided to treat x as a constant and take the derivative with respect to one of the other variables? Nothing precludes us from doing this. Consider the equation

$$y = ax$$

where

$$\frac{dy}{dx} = a$$

Now suppose we find the derivative of y with respect to a, *but TREAT x as the constant.* Then

$$\frac{dy}{da} = x$$

Here we just "reversed" the roles played by a and x in our equation.

9.1 Two Variable Case:

let $z = f(x, y)$, which means **"z is a function of x and y"**. In this case z is the endogenous (dependent) variable and both x and y are the exogenous (independent) variables.

To measure the the effect of a change in a single independent variable (x or y) on the dependent variable (z) we use what is known as the *PARTIAL DERIVATIVE*.

The partial derivative of z with respect to x measures the instantaneous change in the function as x changes while *HOLDING y constant*. Similarly, we would hold x constant if we wanted to evaluate the effect of a change in y on z. Formally:

- $\frac{\partial z}{\partial x}$ is the **"partial derivative"** of z with respect to x, treating y as a constant. Sometimes written as f_x.
- $\frac{\partial z}{\partial y}$ is the **"partial derivative"** of z with respect to y, treating x as a constant. Sometimes written as f_y.

The "∂" symbol ("bent over" lower case D) is called the "partial" symbol. It is interpreted in exactly the same way as $\frac{dy}{dx}$ from single variable calculus. The ∂ symbol simply serves to remind us that there are other variables in the equation, but for the purposes of the current exercise, these other variables are held constant.

EXAMPLES:

$$
\begin{array}{lll}
z = x + y & \partial z/\partial x = 1 & \partial z/\partial y = 1 \\
z = xy & \partial z/\partial x = y & \partial z/\partial y = x \\
z = x^2y^2 & \partial z/\partial x = 2(y^2)x & \partial z/\partial y = 2(x^2)y \\
z = x^2y^3 + 2x + 4y & \partial z/\partial x = 2xy^3 + 2 & \partial z/\partial y = 3x^2y^2 + 4
\end{array}
$$

- **REMEMBER:** When you are taking a partial derivative you treat the other variables in the equation as constants!

9.2 Rules of Partial Differentiation

Product Rule: given $z = g(x,y) \cdot h(x,y)$

$$
\begin{array}{l}
\frac{\partial z}{\partial x} = g(x,y) \cdot \frac{\partial h}{\partial x} + h(x,y) \cdot \frac{\partial g}{\partial x} \\
\frac{\partial z}{\partial y} = g(x,y) \cdot \frac{\partial h}{\partial y} + h(x,y) \cdot \frac{\partial g}{\partial y}
\end{array}
$$

Quotient Rule: given $z = \frac{g(x,y)}{h(x,y)}$ and $h(x,y) \neq 0$

$$
\begin{array}{l}
\frac{\partial z}{\partial x} = \frac{h(x,y) \cdot \frac{\partial g}{\partial x} - g(x,y) \cdot \frac{\partial h}{\partial x}}{[h(x,y)]^2} \\
\frac{\partial z}{\partial y} = \frac{h(x,y) \cdot \frac{\partial g}{\partial y} - g(x,y) \cdot \frac{\partial h}{\partial y}}{[h(x,y)]^2}
\end{array}
$$

Chain Rule: given $z = [g(x,y)]^n$

$$
\begin{array}{l}
\frac{\partial z}{\partial x} = n\,[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial x} \\
\frac{\partial z}{\partial y} = n\,[g(x,y)]^{n-1} \cdot \frac{\partial g}{\partial y}
\end{array}
$$

9.3 Further Examples:

For the function $U = U(x,y)$ find the the partial derivates with respect to x and y

for each of the following examples

Example 2

$$U = -5x^3 - 12xy - 6y^5$$

Answer:

$$
\begin{aligned}
\frac{\partial U}{\partial x} &= U_x = 15x^2 - 12y \\
\frac{\partial U}{\partial y} &= U_y = -12x - 30y^4
\end{aligned}
$$

Example 3

$$U = 7x^2y^3$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = 14xy^3\\ \frac{\partial U}{\partial y} &= U_y = 21x^2y^2\end{aligned}$$

Example 4

$$U = 3x^2(8x - 7y)$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = 3x^2(8) + (8x - 7y)(6x) = 72x^2 - 42xy\\ \frac{\partial U}{\partial y} &= U_y = 3x^2(-7) + (8x - 7y)(0) = -21x^2\end{aligned}$$

Example 5

$$U = (5x^2 + 7y)(2x - 4y^3)$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = (5x^2 + 7y)(2) + (2x - 4y^3)(10x)\\ \frac{\partial U}{\partial y} &= U_y = (5x^2 + 7y)(-12y^2) + (2x - 4y^3)(7)\end{aligned}$$

Example 6

$$U = \frac{9y^3}{x - y}$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = \frac{(x-y)(0) - 9y^3(1)}{(x-y)^2} = \frac{-9y^3}{(x-y)^2}\\ \frac{\partial U}{\partial y} &= U_y = \frac{(x-y)(27y^2) - 9y^3(-1)}{(x-y)^2} = \frac{27xy^2 - 18y^3}{(x-y)^2}\end{aligned}$$

Example 7

$$U = (x - 3y)^3$$

Answer:

$$\begin{aligned}\frac{\partial U}{\partial x} &= U_x = 3(x - 3y)^2(1) = 3(x - 3y)^2\\ \frac{\partial U}{\partial y} &= U_y = 3(x - 3y)^2(-3) = -9(x - 3y)^2\end{aligned}$$

9.4 A Special Function: Cobb-Douglas

The Cobb-douglas function is a mathematical function that is very popular in economic models. The general form is

$$z = x^a y^b$$

and its partial derivatives are

$\partial z/\partial x = ax^{a-1}y^b$ and $\partial z/\partial y = bx^a y^{b-1}$

Furthermore, the absolute value of the slope of the level curve of a Cobb-douglas is given by

$$\frac{\partial z/\partial x}{\partial z/\partial y} = MRS = \frac{a}{b}\frac{y}{x}$$

9.5 Differentials

Given the function

$$y = f(x)$$

the derivative is

$$\frac{dy}{dx} = f'(x)$$

However, we can treat dy/dx as a fraction and factor out the dx

$$dy = f'(x)dx$$

where dy and dx are called *differentials*. If dy/dx can be interpreted as "the slope of a function", then dy is the "rise" and dx is the "run". Another way of looking at it is as follows:

- dy = the change in y
- dx = the change in x
- $f'(x)$ = how the change in x causes a change in y

Example 8 *if*

$$y = x^2$$

then

$$dy = 2xdx$$

Lets suppose $x = 2$ *and* $dx = 0.01$. *What is the change in* $y(dy)$*?*

$$dy = 2(2)(0.01) = 0.04$$

Therefore, at $x = 2$, *if* x *is increased by 0.01 then* y *will increase by 0.04.*

9.6 The two variable case

If

$$z = f(x, y)$$

then the change in z is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad or \quad dz = f_x dx + f_y dy$$

which is read as "the change in z (dz) is due partially to a change in x (dx) plus partially due to a change in y (dy). For example, if

$$z = xy$$

then the total differential is

$$dz = ydx + xdy$$

and, if

$$z = x^2 y^3$$

then

$$dz = 2xy^3 dx + 3x^2 y^2 dy$$

REMEMBER: When you are taking the total differential, you are just taking all the partial derivatives and adding them up.

Example 9 *Find the total differential for the following utility functions*

1. $U(x_1, x_2) = ax_1 + bx_2 \qquad (a, b > 0)$
2. $U(x_1, x_2) = x_1^2 + x_2^3 + x_1 x_2$
3. $U(x_1, x_2) = x_1^a x_2^b$

Answers:

1. $\frac{\partial U}{\partial x_1} = U_1 = a$
   $\frac{\partial U}{\partial x_2} = U_2 = b$
   $dU = U_1 dx_1 + U_2 dx_2 = adx_1 + bdx_2$

2. $\frac{\partial U}{\partial x_1} = U_1 = 2x_1 + x_2$
   $\frac{\partial U}{\partial x_2} = U_2 = 3x_2^2 + x_1$
   $dU = U_1 dx_1 + U_2 dx_2 = (2x_1 + x_2)dx_1 + (3x_2^2 + x_1)dx_2$

3. $\frac{\partial U}{\partial x_1} = U_1 = ax_1^{a-1}x_2^b = \frac{ax_1^a x_2^b}{x_1}$
   $\frac{\partial U}{\partial x_2} = U_2 = bx_1^a x_2^{b-1} = \frac{bx_1^a x_2^b}{x_2}$
   $dU = \left(\frac{ax_1^a x_2^b}{x_1}\right)dx_1 + \left(\frac{bx_1^a x_2^b}{x_2}\right)dx_2 = \left[\frac{adx_1}{x_1} + \frac{bdx_2}{x_2}\right]x_1^a x_2^b$

10 The Implicit Function Theorem

Suppose you have a function of the form

$$F(y, x_1, x_2) = 0$$

where the partial derivatives are $\partial F/\partial x_1 = F_{x_1}$, $\partial F/\partial x_2 = F_{x_2}$and $\partial F/\partial y = F_y$. This class of functions are known as implicit functions where $F(y, x_1, x_2) = 0$ implicity define $y = y(x_1, x_2)$. What this means is that it is possible (theoretically) to rewrite to get y isolated and expressed as a function of x_1 and x_2. While it may not be possible to explicitly solve for y as a function of x, we can still find the effect on y from a change in x_1 or x_2 by applying the implicit function theorem:

Theorem 10 *If a function*

$$F(y, x_1, x_2) = 0$$

has well defined continuous partial derivatives

$$\begin{aligned}
\frac{\partial F}{\partial y} &= F_y \\
\frac{\partial F}{\partial x_1} &= F_{x_1} \\
\frac{\partial F}{\partial x_2} &= F_{x_2}
\end{aligned}$$

and if, at the values where F is being evaluated, the condition that

$$\frac{\partial F}{\partial y} = F_y \neq 0$$

holds, then y is implicitly defined as a function of x. The partial derivatives of y with respect to x_1 and x_2, are given by the ratio of the partial derivatives of F, or

$$\frac{\partial y}{\partial x_i} = -\frac{F_{x_i}}{F_y} \qquad i = 1, 2$$

To apply the implicit function theorem to find the partial derivative of y with respect to x_1 (for example), first take the total differential of F

$$dF = F_y dy + F_{x_1} dx_1 + F_{x_2} dx_2 = 0$$

then set all the differentials except the ones in question equal to zero (i.e. set $dx_2 = 0$) which leaves

$$F_y dy + F_{x_1} dx_1 = 0$$

or

$$F_y dy = -F_{x_1} dx_1$$

dividing both sides by F_y and dx_1 yields

$$\frac{dy}{dx_1} = -\frac{F_{x_1}}{F_y}$$

which is equal to $\frac{\partial y}{\partial x_1}$ from the implicit function theorem.

Example 11 *For each $f(x,y)=0$, find dy/dx for each of the following:*

1. $$y - 6x + 7 = 0$$

   Answer:
   $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(-6)}{1} = 6$$

2. $$3y + 12x + 17 = 0$$

   Answer:
   $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(-12)}{3} = 4$$

3. $$x^2 + 6x - 13 - y = 0$$

   Answer:
   $$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{-(2x+6)}{-1} = 2x + 6$$

4. $$f(x,y) = 3x^2 + 2xy + 4y^3$$

   Answer:
   $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{6x+2y}{12y^2+2x}$$

5. $$f(x,y) = 12x^5 - 2y$$

   Answer:
   $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{60x^4}{-2} = 30x^4$$

6. $$f(x,y) = 7x^2 + 2xy^2 + 9y^4$$

   Answer:
   $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{14x+2y^2}{36y^3+4xy}$$

Example 12 *For $f(x,y,z)$ use the implicit function theorem to find dy/dx and dy/dz :*

1.

$$f(x, y, z) = x^2y^3 + z^2 + xyz$$

Answer:

$$\frac{dy}{dx} = -\frac{fx}{fy} = -\frac{2xy^3+yz}{3x^2y^2+xz}$$

$$\frac{dy}{dz} = -\frac{fz}{fy} = -\frac{2z+xy}{3x^2y^2+xz}$$

2.

$$f(x, y, z) = x^3z^2 + y^3 + 4xyz$$

Answer:

$$\frac{dy}{dx} = -\frac{fx}{fy} = -\frac{3x^2z^2+4yz}{3y^2+4xz}$$

$$\frac{dy}{dz} = -\frac{fz}{fy} = -\frac{2x^3z+4xy}{3y^2+4xz}$$

3.

$$f(x, y, z) = 3x^2y^3 + xz^2y^2 + y^3zx^4 + y^2z$$

Answer:

$$\frac{dy}{dx} = -\frac{fx}{fy} = -\frac{6xy^3+z^2y^2+4y^3zx^3}{9x^2y^2+2xz^2y+3y^2zx^4+2yz}$$

$$\frac{dy}{dz} = -\frac{fz}{fy} = -\frac{2xzy^2+y^3x^4+y^2}{9x^2y^2+2xz^2y+3y^2zx^4+2yz}$$

11 Using Calculus For Maximization Problems

11.1 One Variable Case

If we have the following function

$$y = 10x - x^2$$

we have an example of a *dome shaped* function. To find the maximum of the dome, we simply need to find the point where the slope of the dome is zero, or

$$\begin{gathered}\frac{dy}{dx} = 10 - 2x = 0 \\ 10 = 2x \\ x = 5 \\ \textit{and} \\ y = 25\end{gathered}$$

11.2 Two Variable Case

Suppose we want to maximize the following function

$$z = f(x, y) = 10x + 10y + xy - x^2 - y^2$$

Note that there are two unknowns that must be solved for: x and y. This function is an example of a *three-dimensional dome.* (i.e. the roof of *BC Place*)

To solve this maximization problem we use **partial derivatives.** We take a partial derivative for each of the unknown choice variables and set them equal to zero

$\frac{\partial z}{\partial x} = f_x = 10 + y - 2x = 0$ *The slope in the "x" direction = 0*
$\frac{\partial z}{\partial y} = f_y = 10 + x - 2y = 0$ *The slope in the "y" direction = 0*

This gives us a set of equations, one equation for each of the unknown variables. When you have the same number of independent equations as unknowns, you can solve for each of the unknowns.

rewrite each equation as

$$y = 2x - 10$$
$$x = 2y - 10$$

substitute one into the other

$$x = 2(2x - 10) - 10$$
$$x = 4x - 30$$
$$3x = 30$$
$$x = 10$$

similarly,

$$y = 10$$

REMEMBER: To maximize (minimize) a function of many variables you use the technique of partial differentiation. This produces a set of equations, one equation for each of the unknowns. You then solve the set of equations simulaneously to derive solutions for each of the unknowns.

11.2.1 Second order Conditions (second derivative Test)

To test for a maximum or minimum we need to check the second partial derivatives. Since we have two first partial derivative equations (f_x,f_y) and two variable in each equation, we will get four *second partials* ($f_{xx}, f_{yy}, f_{xy}, f_{yx}$)

Using our original first order equations and taking the partial derivatives for each of them (a second time) yields:

$$f_x = 10 + y - 2x = 0 \quad f_y = 10 + x - 2y = 0$$

$$f_{xx} = -2 \qquad f_{yy} = -2$$
$$f_{xy} = 1 \qquad f_{yx} = 1$$

The two partials,f_{xx}, and f_{yy} are the direct effects of of a small change in x and y on the respective slopes in in the x and y direction. The partials, f_{xy} and f_{yx} are the indirect effects, or the cross effects of one variable on the slope in the other variable's direction. For both *Maximums and Minimums*, the direct effects must outweigh the cross effects

11.3 Rules for two variable Maximums and Minimums

1. Maximum

$$\begin{aligned} f_{xx} &< 0 \\ f_{yy} &< 0 \\ f_{yy}f_{xx} - f_{xy}f_{yx} &> 0 \end{aligned}$$

2. Minimum

$$\begin{aligned} f_{xx} &> 0 \\ f_{yy} &> 0 \\ f_{yy}f_{xx} - f_{xy}f_{yx} &> 0 \end{aligned}$$

3. Otherwise, we have a *Saddle Point*

From our second order conditions, above,

$$\begin{array}{ll} f_{xx} = -2 < 0 & f_{yy} = -2 < 0 \\ f_{xy} = 1 & f_{yx} = 1 \end{array}$$

and

$$f_{yy}f_{xx} - f_{xy}f_{yx} = (-2)(-2) - (1)(1) = 3 > 0$$

therefore we have a maximum.

11.4 Hessian Matrix of Second Partials:

Sometimes the Second Order Conditions are checked in matrix form, using a Hession Matrix. The Hessian is written as

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

where the determinant of the Hessian is

$$|H| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{yy}f_{xx} - f_{xy}f_{yx}$$

which is the measure of the direct versus indirect strengths of the second partials. This is a common setup for checking maximums and minimums, but it is not necessary to use the Hessian.

11.5 Example: Two Market Monopoly with Joint Costs

A monopolist offers two different products, each having the following market demand functions

$$\begin{aligned} q_1 &= 14 - \tfrac{1}{4}p_1 \\ q_2 &= 24 - \tfrac{1}{2}p_2 \end{aligned}$$

The monopolist's joint cost function is

$$C(q_1, q_2) = q_1^2 + 5q_1q_2 + q_2^2$$

The monopolist's profit function can be written as

$$\pi = p_1q_1 + p_2q_2 - C(q_1, q_2) = p_1q_1 + p_2q_2 - q_1^2 - 5q_1q_2 - q_2^2$$

which is the function of four variables: p_1, p_2, q_1,and q_2. Using the market demand functions, we can eliminate $p_1$and p_2 leaving us with a two variable maximization problem. First, rewrite the demand functions to get the inverse functions

$$p_1 = 56 - 4q_1$$
$$p_2 = 48 - 2q_2$$

Substitute the inverse functions into the profit function

$$\pi = (56 - 4q_1)q_1 + (48 - 2q_2)q_2 - q_1^2 - 5q_1q_2 - q_2^2$$

The first order conditions for profit maximization are

$$\frac{\partial\pi}{\partial q_1} = 56 - 10q_1 - 5q_2 = 0$$
$$\frac{\partial\pi}{\partial q_2} = 48 - 6q_2 - 5q_1 = 0$$

Solve the first order conditions using Cramer's rule. First, rewrite in matrix form

$$\begin{bmatrix} 10 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 56 \\ 48 \end{bmatrix}$$

where $|A| = 35$

$$q_1^* = \frac{\begin{vmatrix} 56 & 5 \\ 48 & 6 \end{vmatrix}}{35} = 2.75$$

$$q_2^* = \frac{\begin{vmatrix} 10 & 56 \\ 5 & 48 \end{vmatrix}}{35} = 5.7$$

Using the inverse demand functions to find the respective prices, we get

$$p_1^* = 56 - 4(2.75) = 45$$
$$p_2^* = 48 - 2(5.7) = 36.6$$

From the profit function, the maximum profit is

$$\pi = 213.94$$

Next, check the second order conditions to verify that the profit is at a maximum. The various second derivatives can be set up in a matrix called a *Hessian* The Hessian for this problem is

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -10 & -5 \\ -5 & -6 \end{bmatrix}$$

The sufficient conditions are

$|H_1| = \pi_{11} = -10 < 0 \qquad$ (First Principle Minor of Hessian)
$|H_2| = \pi_{11}\pi_{22} - \pi_{12}\pi_{21} = (-10)(-6) - (-5)^2 = 35 > 0 \qquad$ (determinant)

Therefore the function is at a maximum. Further, since the signs of $|H_1|$ and $|H_2|$ are invariant to the values of $q_1$and q_2, we know that the profit function is strictly concave.

11.6 Example: Profit Max Capital and Labour

Suppose we have the following production function

$$q = f(K, L) = L^{\frac{1}{2}} + K^{\frac{1}{2}} \qquad \begin{array}{l} q = \text{ Output} \\ L = \text{ Labour} \\ K = \text{ Capital} \end{array}$$

Then the profit function for a competitive firm is

$$\begin{array}{ll} \pi = Pq - wL - rK & P = \text{ Market Price} \\ or & w = \text{ Wage Rate} \\ \pi = PL^{\frac{1}{2}} + PK^{\frac{1}{2}} - wL - rK & r = \text{ Rental Rate} \end{array}$$

First order conditions

$$\begin{array}{lll} & & \text{General Form} \\ 1. & \frac{\partial \pi}{\partial L} = \frac{P}{2}L^{\frac{-1}{2}} - w = 0 & Pf_L - w = 0 \\ 2. & \frac{\partial \pi}{\partial k} = \frac{P}{2}K^{\frac{-1}{2}} - r = 0 & Pf_K - r = 0 \end{array}$$

Solving (1) and (2), we get

$$L^* = (\tfrac{2w}{P})^{-2} \qquad K^* = (\tfrac{2r}{P})^{-2}$$

Second order conditions (Hessian)

$$\begin{array}{l} \pi_{LL} = Pf_{LL} = \frac{-P}{4}L^{\frac{-3}{2}} < 0 \\ \pi_{KK} = Pf_{KK} = \frac{-P}{4}K^{\frac{-3}{2}} < 0 \\ \pi_{LK} = \pi_{KL} = Pf_{LK} = Pf_{KL} = 0 \\ \text{or, in matrix form} \end{array}$$

$$H = \begin{vmatrix} \pi_{LL} & \pi_{LK} \\ \pi_{KL} & \pi_{KK} \end{vmatrix} = \begin{vmatrix} \frac{-P}{4}L^{\frac{-3}{2}} & 0 \\ 0 & \frac{-P}{4}K^{\frac{-3}{2}} \end{vmatrix}$$

$$P\left[f_{LL}f_{KK} - (f_{LK})^2\right] = \left(\frac{-P}{4}L^{\frac{-3}{2}}\right)\left(\frac{-P}{4}K^{\frac{-3}{2}}\right) - 0 > 0$$

Differentiate first order of conditions with respect to capital (K) and labour (L)

$\Longrightarrow$Therefore profit maximization

Example: If $P = 1000$, $w = 20$, and $r = 10$

1. Find the optimal K, L, and π
2. Check second order conditions

11.7 Example: Cobb-Douglas production function and a competitive firm

Consider a competitive firm with the following profit function

$$\pi = TR - TC = PQ - wL - rK$$

where P is price, Q is output, L is labour and K is capital, and w and r are the input prices for L and K respectively. Since the firm operates in a competitive market, the exogenous variables are P,w and r. There are three endogenous variables, K, L and Q. However output, Q, is in turn a function of K and L via the production function

$$Q = f(K, L)$$

which in this case, is the Cobb-Douglas function

$$Q = L^a K^b$$

where a and b are positive parameters. If we further assume decreasing returns to scale, then a + b < 1. For simplicity, let's consider the symmetric case where $a = b = \frac{1}{4}$

$$Q = L^{\frac{1}{4}} K^{\frac{1}{4}}$$

Substituting Equation 3 into Equation 1 gives us

$$\pi(K, L) = PL^{\frac{1}{4}} K^{\frac{1}{4}} - wL - rK$$

The first order conditions are

$$\begin{aligned} \frac{\partial \pi}{\partial L} &= P\left(\frac{1}{4}\right) L^{-\frac{3}{4}} K^{\frac{1}{4}} - w = 0 \\ \frac{\partial \pi}{\partial K} &= P\left(\frac{1}{4}\right) L^{\frac{1}{4}} K^{-\frac{3}{4}} - r = 0 \end{aligned}$$

This system of equations define the optimal L and K for profit maximization. But first, we need to check the second order conditions to verify that we have a maximum.

The Hessian for this problem is

$$H = \begin{bmatrix} \pi_{LL} & \pi_{LK} \\ \pi_{KL} & \pi_{KK} \end{bmatrix} = \begin{bmatrix} P(-\frac{3}{16})L^{-\frac{7}{4}}K^{\frac{1}{4}} & P\left(\frac{1}{4}\right)^2 L^{-\frac{3}{4}}K^{-\frac{3}{4}} \\ P\left(\frac{1}{4}\right)^2 L^{-\frac{3}{4}}K^{-\frac{3}{4}} & P\left(-\frac{3}{16}\right)L^{\frac{1}{4}}K^{\frac{7}{4}} \end{bmatrix}$$

The sufficient conditions for a maximum are that $|H_1| < 0$ and $|H| > 0$. Therefore, the second order conditions are satisfied.

We can now return to the first order conditions to solve for the optimal K and L. Rewriting the first equation in Equation 5 to isolate K

$$P\left(\frac{1}{4}\right)L^{-\frac{3}{4}}K^{\frac{1}{4}} = w$$
$$K = (\frac{4w}{p}L^{\frac{3}{4}})^4$$

Substituting into the second equation of Equation 5

$$\frac{P}{4}L^{\frac{1}{4}}K^{-\frac{3}{4}} = \left(\frac{P}{4}\right)L^{\frac{1}{4}}\left[\left(\frac{4w}{p}L^{\frac{3}{4}}\right)^4\right]^{-\frac{3}{4}} = r$$
$$= P^4\left(\frac{1}{4}\right)^4 w^{-3}L^{-2} = r$$

Re-arranging to get L by itself gives us

$$L^* = (\frac{P}{4}w^{-\frac{3}{4}}r^{-\frac{1}{4}})^2$$

Taking advantage of the symmetry of the model, we can quickly find the optimal K

$$K^* = (\frac{P}{4}r^{-\frac{3}{4}}w^{-\frac{1}{4}})^2$$

L* and K^* are the firm's factor demand equations.

11.8 Cournot Doupoly (Game Theory)

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Assumption: Each firm takes the other firms output as exogenous and chooses output to maximize its own profits.

Market Demand:

$$P = a - bq$$
or
$$P = a - b(q_1 + q_2) \quad (q_1 + q_2 = q)$$

Where q_i is firm i's output

$$i = 1, 2$$

Each firm faces the same cost function

$$TC = K + cq_i$$
$$(i = 1, 2)$$

Each firm's profit function is

$$\pi_i = Pq_i - cq_i - K$$

Firm 1

$$\pi_1 = Pq_1 - cq_1 - K$$
$$\pi_1 = (a - bq_1 - bq_2)q_1 - cq_1 - K$$

Max $\pi_{1,}$ treating q_2 as constant

$$\frac{\partial\pi}{\partial q_1} = a - bq_2 - 2bq_1 - c = 0$$
$$2bq_1 = a - c - bq_2$$
$$q_1 = \frac{a-c}{2b} - \frac{q_2}{2} \qquad \Rightarrow \text{"Best Response Function"}$$

Best Response Function tells Firm 1 the profit maximizing q_1 for any level of q_2.

For Firm 2

$$\pi_2 = (a - bq_1 - bq_2)q_2 - cq_2 - K$$

Max π_2 — Treating q_1 as constant

$q_2 = \frac{a-c}{2b} - \frac{q_1}{2}$ — Firm 2's "Best Response Function"

The two "Best Response Functions"

Firm 1	$q_1 = \frac{a-c}{2b} - \frac{q_2}{2}$
Firm 2	$q_2 = \frac{a-c}{2b} - \frac{q_1}{2}$

gives us two equations and two unknowns.
The solution to this system of equations is the equilibrium to the "Cournot Duopoly Game."

Using Cramers Rule

1.	$q_1^* = \frac{a-c}{3b}$
2.	$q_2^* = \frac{a-c}{3b}$
Market Output	$q_1^* + q_2^* = \frac{2(a-c)}{3b}$

11.9 Review of Some Derivative Rules

1. Partial Derivative Rules:

$U = xy$	$\partial U/\partial x = y$	$\partial U/\partial y = x$
$U = x^a y^b$	$\partial U/\partial x = ax^{a-1}y^b$	$\partial U/\partial y = bx^a y^{b-1}$
$U = x^a y^{-b} = \frac{x^a}{y^b}$	$\partial U/\partial x = ax^{a-1}y^{-b}$	$\partial U/\partial y = -bx^a y^{-b-1}$
$U = ax + by$	$\partial U/\partial x = a$	$\partial U/\partial y = b$
$U = ax^{1/2} + by^{1/2}$	$\partial U/\partial x = a\left(\frac{1}{2}\right)x^{-1/2}$	$\partial U/\partial y = b\left(\frac{1}{2}\right)y^{-1/2}$

2. Logarithm (Natural log) $\ln x$

(a) Rules of natural log

If	*Then*
$y = AB$	$\ln y = \ln(AB) = \ln A + \ln B$
$y = A/B$	$\ln y = \ln A - \ln B$
$y = A^b$	$\ln y = \ln(A^b) = b \ln A$

NOTE: $\ln(A + B) \neq \ln A + \ln B$

(b) derivatives

IF	*THEN*
$y = \ln x$	$\frac{dy}{dx} = \frac{1}{x}$
$y = \ln(f(x))$	$\frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x)$

(c) Examples

If	*Then*
$y = \ln(x^2 - 2x)$	$dy/dx = \frac{1}{(x^2-2x)}(2x - 2)$
$y = \ln(x^{1/2}) = \frac{1}{2} \ln x$	$dy/dx = \left(\frac{1}{2}\right)\left(\frac{1}{x}\right) = \frac{1}{2x}$

3. The Number e

$$\text{if } y = e^x \text{ then } \frac{dy}{dx} = e^x$$
$$\text{if } y = e^{f(x)} \text{ then } \frac{dy}{dx} = e^{f(x)} \cdot f'(x)$$

(a) Examples

$y = e^{3x}$	$\frac{dy}{dx} = e^{3x}(3)$
$y = e^{7x^3}$	$\frac{dy}{dx} = e^{7x^3}(21x^2)$
$y = e^{rt}$	$\frac{dy}{dt} = re^{rt}$

11.10 Finding the MRS from Utility functions

EXAMPLE: Find the total differential for the following utility functions

1. $U(x_1, x_2) = ax_1 + bx_2$ where $(a, b > 0)$
2. $U(x_1, x_2) = x_1^2 + x_2^3 + x_1x_2$
3. $U(x_1, x_2) = x_1^a x_2^b$ where $(a, b > 0)$
4. $U(x_1, x_2) = \alpha \ln c_1 + \beta \ln c_2$ where $(\alpha, \beta > 0)$

Answers:
1. $\frac{\partial U}{\partial x_1} = U_1 = a \quad \frac{\partial U}{\partial x_2} = U_2 = b$
and

$$dU = U_1 dx_1 + U_2 dx_2 = a dx_1 + b dx_2 = 0$$

If we rearrange to get dx_2/dx_1

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = -\frac{U_1}{U_2} = -\frac{a}{b}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{a}{b}$$

2. $\frac{\partial U}{\partial x_1} = U_1 = 2x_1 + x_2 \quad \frac{\partial U}{\partial x_2} = U_2 = 3x_2^2 + x_1$
and

$$dU = U_1 dx_1 + U_2 dx_2 = (2x_1 + x_2)dx_1 + (3x_2^2 + x_1)dx_2 = 0$$

Find dx_2/dx_1

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2} = -\frac{(2x_1 + x_2)}{(3x_2^2 + x_1)}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{(2x_1 + x_2)}{(3x_2^2 + x_1)}$$

iii) $\frac{\partial U}{\partial x_1} = U_1 = a x_1^{a-1} x_2^b \qquad \frac{\partial U}{\partial x_2} = U_2 = b x_1^a x_2^{b-1}$
and

$$dU = \left(a x_1^{a-1} x_2^b\right) dx_1 + \left(b x_1^a x_2^{b-1}\right) dx_2 = 0$$

Rearrange to get

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2} = -\frac{a x_1^{a-1} x_2^b}{b x_1^a x_2^{b-1}} = -\frac{a x_2}{b x_1}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{a x_2}{b x_1}$$

iv) $\frac{\partial U}{\partial c_1} = U_1 = \alpha \left(\frac{1}{c_1}\right) dc_1 = \left(\frac{\alpha}{c_1}\right) dc_1 \qquad \frac{\partial U}{\partial x_2} = U_2 = \beta \left(\frac{1}{c_2}\right) dc_2 = \left(\frac{\beta}{c_2}\right) dc_2$
and

$$dU = \left(\frac{\alpha}{c_1}\right) dc_1 + \left(\frac{\beta}{c_2}\right) dc_2 = 0$$

Rearrange to get

$$\frac{dc_2}{dc_1} = -\frac{U_1}{U_2} = \frac{\left(\frac{\alpha}{c_1}\right)}{\left(\frac{\beta}{c_2}\right)} = -\frac{\alpha c_2}{\beta c_1}$$

The MRS is the Absolute value of $\frac{dc_2}{dc_1}$:

$$MRS = \frac{\alpha c_2}{\beta c_1}$$

12 The Lagrange Multiplier Method

Sometimes we need to to maximize (minimize) a function that is subject to some sort of constraint. For example

$$\textit{Maximize} \quad z = f(x, y)$$

$$\textit{subject to the constraint} \quad x + y \leq 100$$

For this kind of problem there is a technique, or *trick,* developed for this kind of problem known as the *Lagrange Multiplier method.* This method involves adding an extra variable to the problem called the lagrange multiplier, or λ.

We then set up the problem as follows:

1. Create a new equation form the original information

$$L = f(x, y) + \lambda(100 - x - y)$$
$$or$$
$$L = f(x, y) + \lambda\,[Zero]$$

2. Then follow the same steps as used in a regular maximization problem

$$\frac{\partial L}{\partial x} = f_x - \lambda = 0$$
$$\frac{\partial L}{\partial y} = f_y - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 100 - x - y = 0$$

3. In most cases the λ will drop out with substitution. Solving these 3 equations will give you the constrained maximum solution

12.0.1 Example 1: max xy subject to linear constraint

Suppose $z = f(x, y) = xy$. and the constraint is the one from above. The problem then becomes

$$L = xy + \lambda(100 - x - y)$$

Now take partial derivatives, one for each unknown, including λ

$$\frac{\partial L}{\partial x} = y - \lambda = 0$$
$$\frac{\partial L}{\partial y} = x - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 100 - x - y = 0$$

Starting with the first two equations, we see that $x = y$ and λ drops out. From the third equation we can easily find that $x = y = 50$ and the constrained maximum value for z is $z = xy = 2500$.

12.0.2 Example 2: Utility Maximization

Maximize

$$u = 4x^2 + 3xy + 6y^2$$

subject to

$$x + y = 56$$

Set up the Lagrangian Equation:

$$L = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$$

Take the first-order partials and set them to zero

$$\begin{aligned} L_x &= 8x + 3y - \lambda = 0 \\ L_y &= 3x + 12y - \lambda = 0 \\ L_\lambda &= 56 - x - y = 0 \end{aligned}$$

From the first two equations we get

$$\begin{aligned} 8x + 3y &= 3x + 12y \\ x &= 1.8y \end{aligned}$$

Substitute this result into the third equation

$$\begin{aligned} 56 - 1.8y - y &= 0 \\ y &= 20 \end{aligned}$$

therefore

$$x = 36 \qquad \lambda = 348$$

12.0.3 Example 3: Cost minimization

A firm produces two goods, x and y. Due to a government quota, the firm must produce subject to the constraint $x + y = 42$. The firm's cost functions is

$$c(x, y) = 8x^2 - xy + 12y^2$$

The Lagrangian is

$$L = 8x^2 - xy + 12y^2 + \lambda(42 - x - y)$$

The first order conditions are

$$\begin{aligned} L_x &= 16x - y - \lambda = 0 \\ L_y &= -x + 24y - \lambda = 0 \\ L_\lambda &= 42 - x - y = 0 \end{aligned} \tag{9}$$

Solving these three equations simultaneously yields

$$x = 25 \qquad y = 17 \qquad \lambda = 383$$

12.1 Duality of the consumer choice problem

12.1.1 Example 4: Utility Maximization

Consider a consumer with the utility function $U = xy$, who faces a budget constraint of $B = P_x x + P_y y$, where B, P_x and P_y are the budget and prices, which are given.

The choice problem is

Maximize

$$U = xy \tag{10}$$

Subject to

$$B = P_x x + P_y y \tag{11}$$

The Lagrangian for this problem is

$$Z = xy + \lambda(B - P_x x - P_y y) \tag{12}$$

The first order conditions are

$$\begin{array}{l} Z_x = y - \lambda P_x = 0 \\ Z_y = x - \lambda P_y = 0 \\ Z_\lambda = B - P_x x - P_y y = 0 \end{array} \tag{13}$$

Solving the first order conditions yield the following solutions

$$x^M = \tfrac{B}{2P_x} \quad y^M = \tfrac{B}{2P_y} \quad \lambda = \tfrac{B}{2P_x P_y} \tag{14}$$

where x^M and y^M are the consumer's Marshallian demand functions.

12.1.2 Example 5: Minimization Problem

Minimize

$$P_x x + P_y y \tag{15}$$

Subject to

$$U_0 = xy \tag{16}$$

The Lagrangian for the problem is

$$Z = P_x x + P_y y + \lambda(U_0 - xy) \tag{17}$$

The first order conditions are

$$\begin{aligned} Z_x &= P_x - \lambda y = 0 \\ Z_y &= P_y - \lambda x = 0 \\ Z_\lambda &= U_0 - xy = 0 \end{aligned} \tag{18}$$

Solving the system of equations for x, y and λ

$$\begin{aligned} x^h &= \left(\tfrac{P_y U_0}{P_x}\right)^{\frac{1}{2}} \\ y^h &= \left(\tfrac{P_x U_0}{P_y}\right)^{\frac{1}{2}} \\ \lambda^h &= \left(\tfrac{P_x P_y}{U_0}\right)^{\frac{1}{2}} \end{aligned} \tag{19}$$

12.2 Application: Intertemporal Utility Maximization

Consider a simple two period model where a consumer's utility is a function of consumption in both periods. Let the consumer's utility function be

$$U(c_1, c_2) = \ln c_1 + \beta \ln c_2$$

where c_1 is consumption in period one and c_2 is consumption in period two. The consumer is also endowments of y_1 in period one and y_2 in period two.

Let r denote a market interest rate with the consumer can choose to borrow or lend across the two periods. The consumer's intertemporal budget constraint is

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

12.2.1 Method One:Find MRS and Substitute

Differentiate the Utility function

$$dU = \left(\frac{1}{c_1}\right) dc_1 + \left(\frac{\beta}{c_2}\right) dc_2 = 0$$

Rearrange to get

$$\frac{dc_2}{dc_1} = -\frac{c_2}{\beta c_1}$$

The MRS is the Absolute value of $\frac{dc_2}{dc_1}$:

$$MRS = \frac{c_2}{\beta c_1}$$

substitute into the budget constraint

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= c_1 + \frac{\beta c_1(1+r)}{1+r} = (1+\beta)c_1 \\ c_1^* &= \frac{y_1 + \frac{y_2}{1+r}}{(1+\beta)} \end{aligned}$$

Similarly, solving for c_2^* using the first order conditions

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= \frac{c_2}{\beta(1+r)} + \frac{c_2}{1+r} \\ (1+r)y_1 + y_2 &= \left(\frac{1}{\beta} + 1\right) c_2 \\ c_2^* &= \frac{(1+r)y_1 + y_2}{\frac{1}{\beta} + 1} \end{aligned}$$

12.2.2 Method Two: Use the Lagrange Multiplier Method

The Lagrangian for this utility maximization problem is

$$L = \ln c_1 + \beta \ln c_2 + \lambda \left(y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right)$$

The first order conditions are

$$\begin{aligned} &\tfrac{\partial L}{\partial \lambda} = y_1 + \tfrac{y_2}{1+r} - c_1 - \tfrac{c_2}{1+r} = 0 \\ &\tfrac{\partial L}{\partial C_1} = \tfrac{1}{c_1} - \lambda = 0 \\ &\tfrac{\partial L}{\partial C_1} = \tfrac{\beta}{c_2} - \tfrac{\lambda}{1+r} = 0 \end{aligned}$$

Combining the last two first order equations to eliminate λ gives us

$$\begin{aligned} \frac{1/c_1}{\beta/c_2} &= \frac{c_2}{\beta c_1} = \frac{\lambda}{\frac{\lambda}{1+r}} = 1+r \\ c_2 &= \beta c_1(1+r) \qquad \textit{and} \qquad c_1 = \frac{c_2}{\beta(1+r)} \end{aligned}$$

sub into the Budget constraint

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= c_1 + \frac{\beta c_1(1+r)}{1+r} = (1+\beta)c_1 \\ c_1^* &= \frac{y_1 + \frac{y_2}{1+r}}{(1+\beta)} \end{aligned}$$

Similarly, solving for c_2^* using the first order conditions

$$\begin{aligned} y_1 + \frac{y_2}{1+r} &= \frac{c_2}{\beta(1+r)} + \frac{c_2}{1+r} \\ (1+r)y_1 + y_2 &= \left(\frac{1}{\beta} + 1\right) c_2 \\ c_2^* &= \frac{(1+r)y_1 + y_2}{\frac{1}{\beta} + 1} \end{aligned}$$

12.3 Problems:

1. Skippy lives on an island where she produces two goods, *x and y,* according the the production possibility frontier $200 = x^2 + y^2$, and she consumes all the goods herself. Her utility function is

$$u = x \cdot y^3$$

   FInd her utility maximizing x and y as well as the value of λ

2. A consumer has the following utility function: $U(x, y) = x(y + 1)$, where x and y are quantities of two consumption goods whose prices are p_x and p_y respectively. The consumer also has a budget of B. Therefore the consumer's maximization problem is

$$x(y + 1) + \lambda(B - p_x x - p_y y)$$

   (a) From the first order conditions find expressions for x^* and y^*. These are the consumer's demand functions. What kind of good is y? In particular what happens when $p_y > B/2$?

3. This problem could be recast as the following dual problem

$$Minimize \quad p_x x + p_y y \text{ subject to } U^* = x(y + 1)$$

   Find the values of x and y that solve this minimization problem.

4. Skippy has the following utility function: $u = x^{\frac{1}{2}} y^{\frac{1}{2}}$ and faces the budget constraint: $M = p_x x + p_y y$.

   (a) Suppose $M = 120$, $P_y = 1$ and $P_x = 4$. Find the optimal x and y

13 Kuhn-Tucker: Lagrange with Inequality Constraints

13.1 Utility Maximization with a simple rationing constraint

Consider a familiar problem of utility maximization with a budget constraint:

$$\begin{array}{ll} \text{Maximize} & U = U(x, y) \\ & \\ \text{subject to} & B = P_x x + P_y y \\ \text{and} & \overline{x} \geqslant x \end{array}$$

But where a ration on x has been imposed equal to $\overline{x}$. We now have two constraints. The Lagrange method easily allows us to set up this problem by adding the second constraint in the same manner as the first. The Lagrange becomes

$$\underset{x,y}{Max} \qquad U(x, y) + \lambda_1(B - P_x x - P_y y) + \lambda_2(\overline{x} - x)$$

However, in the case of more than one constraint, it is possible that one of the constraints is nonbinding. In the example we are using here, we know that the budget constraint will be binding but it is not clear if the ration constraint will be binding. It depends on the size of $\overline{x}$.

The two possibilities are illustrated in figure one. In the top graph, we see the standard utility maximization result with the solution at point E. In this case the ration constraint, $\overline{x}$, is larger than the optimum value x^*. In this case the second constraint could have been ignored.

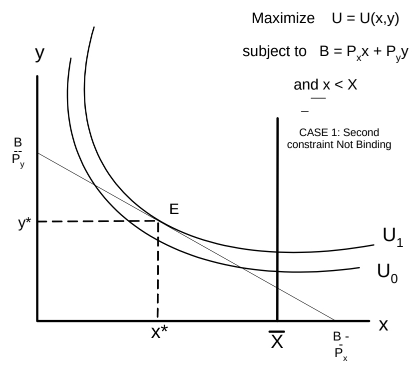
In the bottom graph the ration constraint is binding. Without the constraint, the solution to the maximization problem would again be at point E. However, the solution for x violates the second constraint. Therefore the solution is determined by the intersection of the two constraints at point *E'*

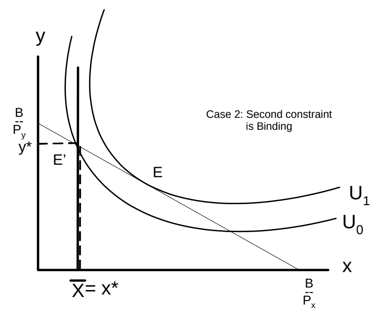
13.1.1 Procedure:

This type of problem requires us to vary the first order conditions slightly. Cases where constraints may or not be binding are often referred to as *Kuhn-Tucker* conditions.

The Kuhn-Tucker conditions are

$$\begin{array}{ll} L_x = U_x - P_x\lambda_1 - \lambda_2 = 0 & x \geq 0 \\ L_y = U_y - P_y\lambda_1 = 0 & y \geq 0 \\ and & \\ L_{\lambda_1} = B - P_x x - P_y y \geq 0 & \lambda_1 \geq 0 \\ L_{\lambda_2} = \overline{x} - x \geq 0 & \lambda_2 \geq 0 \end{array}$$





Now let us interpret the Kuhn-Tucker conditions for this particular problem. Looking at the Lagrange

$$U(x, y) + \lambda_1(B - P_x x - P_y y) + \lambda_2(\overline{x} - x)$$

We require that

$$\lambda_1(B - P_x x - P_y y) = 0$$

therefore either

$$\lambda_1 = 0$$
$$or$$
$$B - P_x x - P_y y = 0$$

If we interpret $\lambda_1$as the marginal utility of the budget (Income), then if the budget constraint is not met the marginal utility of additional B is zero ($\lambda_1 = 0$).

(2) Similarly for the ration constraint, either

$$\overline{x} - x = 0$$
$$or$$
$$\lambda_2 = 0$$

λ_2 can be interpreted as the marginal utility of relaxing the ration constraint.

13.1.2 Solving by Trial and Error

Solving these types of problems is a bit like detective work. Since there are more than one possible outcomes, we need to try them all. But before you start, it is important to think about the problem and try to make an educated guess as to which constraint is more likely to be nonbinding. In this example we can be sure that the budget constraint will always be binding, therefore we only need to worry about the effects of the ration constraint.

Step one: Assume $\lambda_2 = 0, \lambda_1 > 0$ (simply ignore the second constraint) the first order conditions become

$$\begin{aligned} &L_x = U_x - P_x\lambda_1 - \lambda_2 = 0 \\ &L_y = U_y - P_y\lambda_1 = 0 \\ &L_{\lambda_1} = B - P_x x - P_y y = 0 \end{aligned}$$

Find a solution for x^* and y^* then check if you have violated the constraint you ignored. If you have, go to step two.

Step two: Assume $\lambda_2 > 0, \lambda_1 > 0$ (use both constraints, assume they are binding)

The first order conditions become

$$\begin{aligned} &L_x = U_x - P_x\lambda_1 - \lambda_2 = 0 \\ &L_y = U_y - P_y\lambda_1 = 0 \\ &L_{\lambda_1} = B - P_x x - P_y y = 0 \\ &L_{\lambda_2} = \overline{x} - x = 0 \end{aligned}$$

In this case, the solution will simply be where the two constraints intersect.

Step three: Assume $\lambda_2 > 0, \lambda_1 = 0$ (use the second constraint, but ignore the first constraint)

Numerical example

$$Maximize \quad U = xy$$

subject to:

$$\begin{gathered} 100 \geq x + y \\ and \\ x \leq 40 \end{gathered}$$

The Lagrange is

$$xy + \lambda_1(100 - x - y) + \lambda_2(40 - x)$$

and the Kuhn-Tucker conditions become

$$\begin{array}{ll} L_x = y - \lambda_1 - \lambda_2 = 0 & x \geq 0 \\ L_y = x - \lambda_1 = 0 & y \geq 0 \\ L_{\lambda_1} = 100 - x - y \geq 0 & \lambda_1 \geq 0 \\ L_{\lambda_2} = 40 - x \geq 0 & \lambda_2 \geq 0 \end{array}$$

Which gives us four equations and four unknowns: x, y, λ_1 and λ_2.

To solve, we typically approach the problem in a stepwise manner. First, ask if any λ_i could be zero Try $\lambda_2 = 0$ ($\lambda_1 = 0$ does not make sense, given the form of the utility function), then

$$x - \lambda_1 = y - \lambda_1 \quad or \quad x = y$$

from the constraint $100 - x - y$ we get $x^* = y^* = 50$ which violates our constraint $x \leq 40$. Therefore $x^* = 40$ and $y^* = 60$, also $\lambda_1^* = 40$ and $\lambda_2^* = 20$

13.2 War-Time Rationing

Typically during times of war the civilian population is subject to some form of rationing of basic consumer goods. Usually, the method of rationing is through the use of redeemable coupons used by the government. The government will supply each consumer with an allotment of coupons each month. In turn, the consumer will have to redeem a certain number of coupons at the time of purchase of a rationed good. This effectively means the consumer "pays" two "prices" at the time of the purchase. He or she pays both the coupon price and the monetary price of the rationed good. This requires the consumer to have both sufficient funds and sufficient coupons in order to buy a unit of the rationed good.

Consider the case of a two-good world where both goods, x and y. are rationed. Let the consumer's utility function be $U = U(x, y)$. The consumer has a fixed money budget of B and faces the money prices P_x and P_y. Further, the consumer has an allotment of coupons, denoted C, which can be used to purchase both x or y at a coupon price of c_x and c_y. Therefore the consumer's maximization problem is

Maximize

$$U = U(x, y)$$

Subject to

$$B \geq P_x x + P_y y$$

and

$$C \geq c_x x + c_y y$$

in addition, the non-negativity constraint $x \geq 0$ and $y \geq 0$.

The Lagrangian for the problem is

$$Z = U(x, y) + \lambda(B - P_x x - P_y y) + \lambda_2(C - c_x x + c_y y)$$

where λ, λ_2 are the Lagrange multiplier on the budget and coupon constraints respectively. The Kuhn-Tucker conditions are

$$\begin{aligned}
&Z_x = U_x - \lambda_1 P_x - \lambda_2 c_x = 0 \\
&Z_y = U_y - \lambda_1 P_y - \lambda_2 c_y = 0 \\
&Z_{\lambda_1} = B - P_x x - P_y y \geq 0 \qquad \lambda_1 \geq 0 \\
&Z_{\lambda_2} = C - c_x x - c_y y \geq 0 \qquad \lambda_2 \geq 0
\end{aligned}$$

Numerical Example

Let's suppose the utility function is of the form $U = x \cdot y^2$. Further, let $B = 100, P_x = P_y = 1$ while $C = 120$ and $c_x = 2, c_y = 1$.

The Lagrangian becomes

$$Z = xy^2 + \lambda_1(100 - x - y) + \lambda_2(120 - 2x - y)$$

The Kuhn-Tucker conditions are now

$$\begin{array}{lll} Z_x = y^2 - \lambda_1 - 2\lambda_2 \leq 0 & x \geq 0 & x \cdot Z_x = 0 \\ Z_y = 2xy - \lambda_1 - \lambda_2 \leq 0 & y \geq 0 & y \cdot Z_y = 0 \\ Z_{\lambda_1} = 100 - x - y \geq 0 & \lambda_1 \geq 0 & \lambda_1 \cdot Z_{\lambda_1} = 0 \\ Z_{\lambda_2} = 120 - 2x - y \geq 0 & \lambda_2 \geq 0 & \lambda_2 \cdot Z_{\lambda_2} = 0 \end{array}$$

Solving the problem:

Typically the solution involves a certain amount of trial and error. We first choose one of the constraints to be non-binding and solve for the x and y. Once found, use these values to test if the constraint chosen to be non-binding is violated. If it is, then redo the procedure choosing another constraint to be non-binding. If violation of the non-binding constraint occurs again, then we can assume both constraints bind and the solution is determined only by the constraints.

Step one: Assume $\lambda_2 = 0, \lambda_1 > 0$

By ignoring the coupon constraint, the first order conditions become

$$\begin{array}{l} Z_x = y^2 - \lambda_1 = 0 \\ Z_y = 2xy - \lambda_1 = 0 \\ Z_{\lambda_1} = 100 - x - y = 0 \end{array}$$

Solving for x and y yields

$$x^* = 33.33 \quad y^* = 66.67$$

However, when we substitute these solutions into the coupon constraint we find that

$$2(33.33) + 66.67 = 133.67 > 120$$

The solution violates the coupon constraints.

Step two: Assume $\lambda_1 = 0, \lambda_2 > 0$

Now the first order conditions become

$$\begin{array}{l} Z_x = y^2 - 2\lambda_2 = 0 \\ Z_y = 2xy - \lambda_2 = 0 \\ Z_{\lambda_1} = 120 - 2x - y = 0 \end{array}$$

Solving this system of equations yields

$$x^* = 20 \quad y^* = 80$$

When we check our solution against the budget constraint, we find that the budget constraint is just met. In this case, we have the unusual result that the budget constraint is met but is not binding due to the particular location of the coupon constraint. The student is encouraged to carefully graph the solution, paying careful attention to the indifference curve, to understand how this result arose.

13.3 Peak Load Pricing

Peak and off-peak pricing and planning problems are common place for firms with capacity constrained production processes. Usually the firm has invested in capacity in order to target a primary market. However there may exist a secondary market in which the firm can often sell its product. Once the capital has been purchased to service the firm's primary market, the capital is freely available (up to capacity) to be used in the secondary market. Typical examples include: schools and universities who build to meet day-time needs (peak), but may offer night-school classes (off-peak); theatres who offer shows in the evening (peak) and matinees (off-peak); or trucking companies who have dedicated routes but may choose to enter "back-haul" markets. Since the capacity price is a factor in the profit maximizing decision for the peak market and is already paid, it normally, should not be a factor in calculating optimal price and quantity for the smaller, off-peak market. However, if the secondary market's demand is close to the same size as the primary market, capacity constraints may be an issue, especially given that it is common practice to price discriminate and charge lower prices in off-peak periods. Even though the secondary market is smaller than the primary, it is possible at the lower (profit maximizing) price that off-peak demand exceeds capacity. In such cases capacity choices maust be made taking both markets into account, makeing the problem a classic application of Kuhn-Tucker.

Consider a profit maximizing Company who faces two demand curves

$$P_1 = D^1(Q_1) \quad \text{in the day time (peak period)}$$
$$P_2 = D^2(Q_2) \quad \text{in the night time (off-peak period)}$$

to operate the firm must pay b per unit of output, whether it is day or night. Furthermore, the firm must purchase capacity at a cost of c per unit of output. Let K denote total capacity measured in units of Q. The firm must pay for capacity, regardless if it operates in the off peak period. Question: Who should be charged for the capacity costs? Peak, off-peak, or both sets of customers? The firm's maximization problem becomes

$$\underset{Q_1,Q_2,K}{Maximize} \quad P_1Q_1 + P_2Q_2 - b(Q_1 - Q_2) - cK$$

Subject to

$$K \geq Q_1$$
$$K \geq Q_2$$

Where

$$P_1 = D^1(Q_1)$$
$$P_2 = D^2(Q_2)$$

The Lagrangian for this problem is:

$$Z = D^1(Q_1)Q_1 + D^2(Q_2)Q_2 - b(Q_1 + Q_2) - cK + \lambda_1(K - Q_1) + \lambda_2(K - Q_2)$$

The Kuhn-Tucker conditions are

$$\begin{array}{lll}
Z_1 = D^1 + Q_1\frac{\partial D^1}{\partial Q_1} - b - \lambda_1 = 0 & & (MR_1 - b - \lambda_1 = 0) \\
Z_2 = D^2 + Q_2\frac{\partial D^2}{\partial Q_2} - b - \lambda_2 = 0 & & (MR_2 - b - \lambda_2 = 0) \\
Z_K = -c + \lambda_1 + \lambda_2 = 0 & & (c = \lambda_1 + \lambda_2) \\
Z_{\lambda_1} = K - Q_1 \geq 0 & \lambda_1 \geq 0 & \\
Z_{\lambda_2} = K - Q_2 \geq 0 & \lambda_2 \geq 0 &
\end{array}$$

Assuming that $Q_1, Q_2, K > 0$ the first-order conditions become

$$\begin{array}{ll}
MR_1 = b + \lambda_1 = b + c - \lambda_2 & (\lambda_1 = c - \lambda_2) \\
MR_2 = b + \lambda_2 &
\end{array}$$

Finding a solution:
Step One: Since $D^2(Q_2)$ is smaller than $D^1(Q_1)$ try $\lambda_2 = 0$
Therefore from the Kuhn-Tucker conditions

$$MR_1 = b + c - \lambda_2 = b + c$$
$$MR_2 = b + \lambda_2 = b$$

which implies that $K = Q_1$. Then we check to see if $Q_2^* \leq K$. If true, then we have a valid solution. Otherwise the second constraint is violated and the assumption that $\lambda_2 = 0$ was false. Therefore we proceed to the next step.

Step Two: if $Q_2^* > K$ then $Q_1^* = Q_2^* = K$ and

$$MR_1 = b + \lambda_1$$
$$MR_2 = b + \lambda_2$$

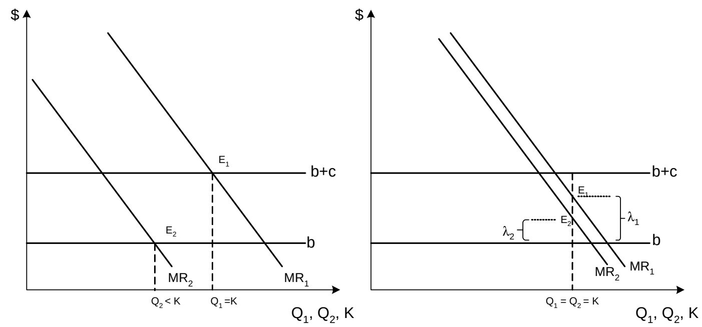
Since $c = \lambda_1 + \lambda_2$ then λ_1 and λ_2 represent the share of c each group pays. Both cases are illustrated in figure 13.3

Numerical Example Suppose the demand during peak hours is

$$P_1 = 22 - 10^{-5}Q_1$$

and during off-peak hours is

$$P_2 = 18 - 10^{-5}Q_2$$



Case 1: Off-peak constraint non-binding

Case 2: Off-peak constraint binding

To produce a unit of output per half-day requires a unit of capacity costing 8 cents per day. The cost of a unit of capacity is the same whether it is used at peak times only or off-peak also. In addition to the costs of capacity, it costs 6 cents in operating costs (labour and fuel) to produce 1 unit per half day (both day and evening)

If we assume that the capacity constraint is binding ($\lambda_2 = 0$), then the Kuhn-Tucker conditions (above) become

$$\begin{array}{lll} \lambda_1 = c = 8 & & \\ \overbrace{22 - 2 \times 10^{-5} Q_1}^{MR} & \overbrace{= b + c}^{MC} & = 14 \\ 18 - 2 \times 10^{-5} Q_2 & = b & = 6 \end{array}$$

Solving this system gives us

$$\begin{aligned} Q_1 &= 40000 \\ Q_2 &= 60000 \end{aligned}$$

which violates the assumption that the second constraint is non-binding ($Q_2 > Q_1 = K$).

Therefore, assuming that both constraints are binding, then $Q_1 = Q_2 = Q$ and the Kuhn-Tucker conditions become

$$\begin{aligned} \lambda_1 + \lambda_2 &= 8 \\ 22 - 2 \times 10^{-5} Q &= 6 + \lambda_1 \\ 18 - 2 \times 10^{-5} Q &= 6 + \lambda_2 \end{aligned}$$

which yields the following solutions

$$\begin{aligned} Q &= K = 50000 \\ \lambda_1 &= 6 \qquad \lambda_2 = 2 \\ P_1 &= 17 \qquad P_2 = 13 \end{aligned}$$

Since the capacity constraint is binding in both markets, market one pays $\lambda_1 = 6$ of the capacity cost and market two pays $\lambda_2 = 2$.

13.4 Problems

1. Suppose in the above example a unit of capacity cost only 3 cents per day.

   (a) What would be the profit maximizing peak and off-peak prices and quantitites?

   (b) What would be the values of the Lagrange multipliers? What interpretation do you put on their values?

2. Skippy lives on an island where she produces two goods, *x and y,* according the the production possibility frontier $200 \geq x^2 + y^2$, and she consumes all the goods herself. Her utility function is

$$u = x \cdot y^3$$

   Skippy also faces and environmental constraint on her total output of both goods. The environmental constraint is given by $x + y \leq 20$

   (a) Write down the Kuhn Tucker first order conditions.

   (b) Find Skippy's optimal *x and y.* Identify which constaints are binding.

3. An electric company is setting up a power plant in a foreign country and it has to plan its capacity. The peak period demand for power is given by $p_1 = 400 - q_1$ and the off-peak is given by $p_2 = 380 - q_2$. The variable cost to is 20 per unit (paid in both markets) and capacity costs 10 per unit which is only paid once and is used in both periods.

   (a) write down the lagrangian and Kuhn-Tucker conditions for this problem

   (b) Find the optimal outputs and capacity for this problem.

   (c) How much of the capacity is paid for by each market (i.e. what are the values of λ_1 and λ_2)?

   (d) Now suppose capacity cost is 30 per unit (paid only once). Find quantities, capacity and how much of the capacity is paid for by each market (i.e. λ_1 and λ_2)?