Utility Maximization Steps

MPP 801 Fall, 2007

The MRS and the Cobb-Douglas

Consider a two-good world, x and y. Our consumer, Skippy, wishes to maximize utility, denoted $U(x, y)$. Her problem is then to Maximize:

$$U = U(x, y)$$

subject to the constraint

$$B = p_x x + p_y y$$

Unless there is a *Corner Solution*, the solution will occur where the highest indifference curve is tangent to the budget constraint. Equivalent to that is the statement: *The Marginal Rate of Substitution equals the price ratio*, or

$$MRS = \frac{p_x}{p_y}$$

This rule, combined with the budget constraint, give us a two-step procedure for finding the solution to the utility maximization problem.

First, in order to solve the problem, we need more information about the MRS. As it turns out, every utility function has its own MRS, which can easily be found using calculus. However, if we restrict ourselves to some of the more common utility functions, we can adopt some short-cuts to arrive at the MRS without calculus.

For example, if the utility function is

$$U = xy$$

then

$$MRS = \frac{y}{x}$$

This is a special case of the "Cobb-Douglas" utility function, which has the form:

$$U = x^a y^b$$

where a and b are two constants. In this case the marginal rate of substitution for the Cobb-Douglas utility function is

$$MRS = \left(\frac{a}{b}\right)\left(\frac{y}{x}\right)$$

regardless of the values of a and b.

Solving the utility max problem

Consider our earlier example of "Skippy" where

$$\begin{aligned} U &= xy \\ MRS &= \frac{y}{x} \end{aligned}$$

Suppose Skippy's budget information is as follows: $B = 100, p_x = 1, p_y = 1$. Her budget constraint is

$$\begin{aligned} B &= p_x x + p_y y \\ 100 &= x + y \end{aligned}$$

Step 1 Set MRS equal to price ratio

$$\begin{aligned} MRS &= \frac{p_x}{p_y} \\ \frac{y}{x} &= \frac{1}{1} \\ y &= x \end{aligned}$$

this relationship must hold at the utility maximizing point.

Step 2 Substitute step 1 into budget constraint

Since $y = x$, the budget constraint becomes

$$\begin{aligned} 100 &= x + y \\ &= x + x \\ &= 2x \end{aligned}$$

Solving for x yields

$$x = \frac{100}{2} = 50$$

Therefore

$$y = 50$$

and

$$u = (50)(50) = 2500$$

Change the price of x

Now suppose the price of x falls to 0.5 or 1/2. Re-do steps 1 and 2,

$$\begin{aligned} MRS &= \frac{p_x}{p_y} \\ \frac{y}{x} &= \frac{0.5}{1} = \frac{1}{2} \\ y &= \frac{1}{2}x \end{aligned}$$

Substitute this new relationship into the budget constraint

$$\begin{aligned} 100 &= x + y \\ 100 &= x + \frac{1}{2}x \\ 100 &= 1.5x \\ x &= \frac{100}{1.5} = 66.7 \\ y &= 33.3 \end{aligned}$$

General Solution to Cobb-Douglas Utility

Using the general form of the Cobb-Douglas

$$U = x^a y^b$$

where

$$MRS = \frac{ay}{bx}$$

and the budget constraint in the form

$$B = p_x x + p_y y$$

where the price ratio is p_x/p_y, the first rule of utility maximization yields

$$\begin{aligned} MRS &= \frac{p_x}{p_y} \\ \frac{ay}{bx} &= \frac{p_x}{p_y} \\ y &= \frac{b}{a}\frac{p_x}{p_y}x \end{aligned}$$

Substituting into the budget constraint yields

$$\begin{aligned} B &= p_x x + p_y\left(\frac{b}{a}\frac{p_x}{p_y}x\right) \\ B &= p_x x + \frac{b}{a}p_x x \\ B &= \left(\tfrac{a+b}{a}\right)p_x x \qquad \textit{(see footnote for algebra)} \\ x^* &= \left(\tfrac{a}{a+b}\right)\frac{B}{p_x} \end{aligned}$$

Similarly, we can find y by the same method, which gives us

$$y^* = \left(\frac{b}{a+b}\right)\frac{B}{p_y}$$

The solutions for x and y are called the consumer's DEMAND FUNCTIONS.

Note that in our first example where $U = xy$, the values of a and b are $a = b = 1$ substituting into x^* and y^* we get

$$\begin{aligned} x^* &= \left(\tfrac{1}{1+1}\right)\frac{B}{p_x} \\ x^* &= \frac{B}{2p_x} \end{aligned}$$

and

$$y^* = \frac{B}{2p_y}$$

Use the values of p_x, p_y, and B to test to see if these equations give you the solutions in example One.

If we substitute the answers back into the utility function, we get

$$\begin{aligned} U &= xy = \left(\frac{B}{2p_x}\right)\left(\frac{B}{2p_y}\right) \\ U &= \frac{B^2}{4p_x p_y} \end{aligned}$$

This gives you the utility number directly from the budget and prices. If you re-arrange this expression to get B by itself, you get

$$B = \sqrt{4p_x p_y U}$$

You can use this equation to calculate the amount of budget is needed if you know prices AND the desired utility number (Helpful for CV and EV)

[0] The trick used here is as follows:

$$\begin{aligned} x + \frac{b}{a}x &= \frac{a}{a}x + \frac{b}{a}x \\ &= \left(\frac{a}{a} + \frac{b}{a}\right)x \\ &= \frac{a+b}{a}x \end{aligned}$$