LOCAL WELL-POSEDNESS OF A DISPERSIVE NAVIER-STOKES SYSTEM

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Abstract. We establish local well-posedness and smoothing results for the Cauchy problem of a degenerate dispersive Navier-Stokes system that arises from kinetic theory. Under assumptions that the initial data have enough regularity and satisfy asymptotic flatness and nontrapping conditions, we show there exists a unique smooth solution for a finite time. Due to degeneracies in both dissipation and dispersion for the system, different components of the solution gain different regularity. The full system is decomposed accordingly into its strictly dispersive part and non-dispersive part. We apply the strategy of Kenig, Ponce, and Vega [13] to treat the strict dispersive part of the DNS system. Couplings of these two parts are then analyzed using normal form reductions and regularization from dispersion and dissipation.

1. Introduction

In this paper we establish the local well-posedness of the Cauchy problem for a dispersive Navier-Stokes (DNS) system that has the form

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0\,, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x(\rho\theta) &= \nabla_x \cdot \Sigma + \nabla_x \cdot \tilde{\Sigma}\,, \\ \partial_t(\rho e) + \nabla_x \cdot (\rho e u + \rho\theta u) &= \nabla_x \cdot (\Sigma u + q) + \nabla_x \cdot (\tilde{\Sigma} u + \tilde{q})\,, \\ (\rho, u, \theta)(x, 0) &= (\rho^{in}, u^{in}, \theta^{in})(x)\,, \end{aligned} \tag{1.1}$$

where $\rho(x,t)$ is the mass density, $u(x,t)$ is the bulk velocity, and $\theta(x,t)$ is the temperature at a position $x \in \mathbb{R}^d$ and time $t \geq 0$. We assume that $d \geq 2$. Here the total energy density ρe is given by

$$\rho e = \tfrac{1}{2}\rho|u|^2 + \tfrac{d}{2}\rho\theta\,,$$

while the classical Navier-Stokes stress tensor Σ and heat flux $-q$ are given by

$$\Sigma = \mu(\theta)\, D_x u\,, \qquad q = \kappa(\theta)\, \nabla_x \theta\,, \tag{1.2}$$

where $D_x u = \nabla_x u + (\nabla_x u)^T - \frac{2}{d}\nabla_x \cdot u I$ is the strain-rate tensor while $\mu(\theta) \geq 0$ and $\kappa(\theta) \geq 0$ are the coefficients of shear viscosity and heat conductivity. Dispersive corrections to the stress tensor $\tilde{\Sigma}$ and the heat flux $\tilde{q}$ are given by

$$\begin{aligned} \tilde{\Sigma} &= \tau_1(\rho,\theta)\left(\nabla_x^2\theta - \tfrac{1}{d}\Delta_x\theta I\right) + \tau_2(\rho,\theta)\left(\nabla_x\theta \otimes \nabla_x\theta - \tfrac{1}{d}|\nabla_x\theta|^2 I\right) \\ &\quad + \tau_3(\rho,\theta)\left(\nabla_x\rho \otimes \nabla_x\theta + \nabla_x\theta \otimes \nabla_x\rho - \tfrac{2}{d}\nabla_x\rho \cdot \nabla_x\theta I\right)\,, \\ \tilde{q} &= \tau_4(\rho,\theta)\left(\Delta_x u + \tfrac{d-2}{d}\nabla_x\nabla_x \cdot u\right) + \tau_5(\rho,\theta)\, D_x u \cdot \nabla_x\theta + \tau_6(\rho,\theta)\, D_x u \cdot \nabla_x\rho \\ &\quad + \tau_7(\rho,\theta)\left(\nabla_x u - (\nabla_x u)^T\right) \cdot \nabla_x\theta\,, \end{aligned} \tag{1.3}$$

where $\nabla_x^2\theta$ denotes the Hessian matrix of θ and $\tau_1(\rho,\theta), \tau_2(\rho,\theta), \cdots, \tau_7(\rho,\theta)$ are additional transport coefficients.

It is well known that there are regimes when classical fluid equations such as Navier-Stokes are not accurate [19, 20]. In these regimes higher-order corrections need to be added to the governing equation. These corrections can be formally derived from classical kinetic equations such as the Boltzmann equation in small mean-free-path regimes. For example, one classical way is to apply the Chapman-Enskog expansion in terms of the mean free path and truncate at various higher orders. The resulting equations are the so-called Burnett and super-Burnett equations. However, one essential problem related to these equations is that these systems are linearly unstable. It has been shown in [2] that linearized Burnett and super-Burnett equations amplify small wavelength perturbations. Therefore they are of limited use.

In [16] a balance argument is employed to derive a family of gas dynamical systems in a systematic manner. This family includes both classical fluid equations and equations that are beyond Navier-Stokes with various orders. One salient feature of these systems is that they all obey an entropy structure which shows their consistency with the second law of thermodynamics. This structure also guarantees that all the derived systems are at least linearly stable. Among them (1.1) is of the lowest order which provides a first correction to the compressible Navier-Stokes. One consistency of (1.1) with kinetic equations is verified in [17] where it is shown that certain non-classical fluid systems derivable from kinetic equations [19] can be recovered from (1.1).

In (1.1) the transport coefficients $\mu(\theta)$, $\kappa(\theta)$, and $\tau_i(\rho,\theta)$ for $i = 1, \cdots, 7$ have forms that depend on details of the underlying kinetic equation. In particular, the transport coefficients $\tau_i(\rho,\theta)$ for $i = 1, \cdots, 6$ will satisfy the relations

$$\tau_4 = \frac{\theta}{2}\tau_1 \,, \qquad \frac{\tau_2}{\theta} + \frac{2\tau_5}{\theta^2} = \partial_\theta\left(\frac{\tau_4}{\theta^2}\right) \,, \qquad \theta\tau_3 + \tau_6 = 2\partial_\rho\tau_4 \,. \tag{1.4}$$

The resulting DNS system (1.1) then inherits an entropy structure from the kinetic equation in which the mathematical entropy density η is given by

$$\eta = \rho \log\left(\frac{\rho}{\theta^{d/2}}\right) \,.$$

Direct calculation from system (1.1) shows that η satisfies

$$\partial_t\eta + \nabla_x\cdot\left(\eta\, u + \frac{q}{\theta} + \frac{\tilde{q}}{\theta}\right) = -\left(\frac{\Sigma}{\theta} : \nabla_x u + \frac{q}{\theta^2}\cdot\nabla_x\theta\right) - \left(\frac{\tilde{\Sigma}}{\theta} : \nabla_x u + \frac{\tilde{q}}{\theta^2}\cdot\nabla_x\theta\right) \,. \tag{1.5}$$

It follows from the constitutive relations (1.2) that

$$\frac{\Sigma}{\theta} : \nabla_x u + \frac{q}{\theta^2}\cdot\nabla_x\theta = \frac{\mu}{2\theta}|D_x u|^2 + \frac{\kappa}{\theta^2}|\nabla_x\theta|^2 \geq 0 \,,$$

while (1.3) and (1.4) gives

$$\tilde{\Sigma} : \frac{\nabla_x u}{\theta} + \tilde{q}\cdot\frac{\nabla_x\theta}{\theta^2} = \nabla_x\cdot\left(\frac{\tau_1}{2\theta}D_x u\cdot\nabla_x\theta\right) \,.$$

One thereby sees that the dispersion terms containing $\tilde{\Sigma}$ and $\tilde{q}$ contribute only to the entropy flux in the entropy equation (1.5). DNS systems (1.1) derived from kinetic equations therefore formally dissipate the entropy in the same way as the compressible Navier-Stokes system, but transport it differently.

The above calculation indicates that the DNS system is formally well-posed over domains without boundary. The main goal of our paper is to establish the local well-posedness of the DNS system rigorously. Because our theory is local in time, we will not need the entropy structure of the system, and so will not assume that (1.4) holds. We will however assume that $\mu(\theta)$, $\kappa(\theta)$, and $\tau_i(\rho,\theta)$ for $i = 1, \cdots, 7$ are smooth functions of ρ and θ with $\mu(\theta)$, $\kappa(\theta)$, and $\tau_1(\rho,\theta)\tau_4(\rho,\theta)$ being strictly positive whenever ρ and θ are bounded away from zero. In this paper we do not try to optimize the regularity needed for the initial data. Instead, we work with data that are smooth enough so that various Sobolev embeddings and estimates for pseudo-differential operators can be carried out.

Remark 1.1. We believe that the entropy structure would play an important role in any global well-posedness result for the DNS system with large initial data.

In our proof of local well-posedness, dispersive regularization plays a crucial role. We use the fact that solutions of dispersive equations gain spatial differentiability provided the initial data satisfy certain asymptotic flatness conditions at infinity. This type of smoothing was noticed by Kato when he showed in [10] that solutions of the 1D KdV equation gain 1/2 spatial derivative compared to its initial data. This kind of smoothing has since been generalized by various authors to more general dispersive equations and systems [5, 11]. In general, solutions of dispersive equations with order m gain $\frac{m-1}{2}$ derivatives locally for positive times [5].

Based on Kato's smoothing effect, various well-posedness results have been established for semilinear or quasi-linear dispersive equations and systems with strict or uniform dispersive effects [13, 6]. However, these existing results do not apply directly to the DNS system because its dispersion is degenerate. To see this, one first observes that the mass equation has no terms that are dissipative or dispersive. Another degeneracy occurs in the energy equation where the dispersive term $\nabla_x \cdot \tilde{q}$ is

$$\nabla_x \cdot \tilde{q} = \tfrac{2(d-1)}{d}\tau_4\, \Delta_x \nabla_x \cdot u + \text{lower order terms}\,.$$

The leading order term in $\nabla_x \cdot \tilde{q}$ gives the dispersive effect for the velocity field. It is clear that the incompressible part, *i.e.*, the divergence free part of the velocity field u vanishes in this term. Therefore, if u is decomposed into the divergence free part and the gradient part as done in the Hodge decomposition, then only the gradient component of u has a dispersive effect. These degeneracies suggest to decompose the DNS system into a strictly dispersive subsystem and a nondispersive subsystem. We apply the technique by Kenig, Ponce, and Vega in [13] to treat the principle part of the strictly dispersive subsystem. The coupling of the strictly dispersive and nondispersive parts will be treated using both dissipative and dispersive regularization. Our main theorem will imply the following.

Theorem 1.1. Well-Posedness Theorem. *In dimension $d \geq 2$, there exists $N = N(d)$ such that if s_1, $s \in \mathbb{R}_+$, $s_1 > d/2 + 6$, $s = \max\{s_1 + 6, N + d/2 + 5\}$, and the initial data ρ^{in}, u^{in}, and θ^{in} satisfy the following conditions: there exist two constants $\bar{\rho} > 0$ and $\bar{\theta} > 0$ such that*

- *the boundedness and asymptotic flatness condition*

$$\begin{aligned} &\|\rho^{in} - \bar{\rho}\|_{H^{s+1}} + \|(u^{in}, \theta^{in} - \bar{\theta})\|_{H^s} + \sum_{1 \leq |\alpha| \leq s_1} \left(\left\| \langle x \rangle^2 \partial_x^\alpha \rho^{in} \right\|_{H^1} + \left\| \langle x \rangle^2 \partial_x^\alpha (u^{in}, \theta^{in}) \right\|_{L^2} \right) \\ &\leq C^{in} < \infty\,, \end{aligned} \tag{1.6}$$

where $\alpha \in \mathbb{N}^d$ denote multi-indices with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and we define $\langle x \rangle^2 \triangleq 1 + |x|^2$;

- *there exists a constant $\alpha^{in} > 0$ such that for every $x \in \mathbb{R}^d$*

$$\begin{aligned} \alpha^{in} \leq \rho^{in}(x)\,, \qquad \alpha^{in} \leq \theta^{in}(x)\,, \qquad \alpha^{in} \leq \mu(\theta^{in})\,, \qquad \alpha^{in} \leq \kappa(\theta^{in})\,, \\ \alpha^{in} \leq \tfrac{4(d-1)^2}{d^3} \cdot \frac{\tau_1(\rho^{in}(x), \theta^{in}(x))\tau_4(\rho^{in}(x), \theta^{in}(x))}{\rho^{in}(x)^2}\,; \end{aligned} \tag{1.7}$$

- *the Hamiltonian defined by*

$$h^{in}(\xi, x) = \tfrac{2(d-1)}{d^{3/2}} \left(\frac{\tau_1(\rho^{in}(x), \theta^{in}(x))\tau_4(\rho^{in}(x), \theta^{in}(x))}{\rho^{in}(x)^2} \right)^{\frac{1}{2}} |\xi|^3 \tag{1.8}$$

  generates a flow that is nontrapping,

then for some $T_0 > 0$ depending only on C^{in}, α^{in}, and d there exist unique functions ρ, u, and θ with

$$\rho - \bar{\rho} \in C([0, T_0]; H^{s+1})\,, \quad (u,\, \theta - \bar{\theta}) \ \in C([0, T_0]; H^s) \cap L^2(0, T_0; H^{s+1})\,, \tag{1.9}$$

such that (ρ, u, θ) solves the DNS initial-value problem (1.1).

Remark 1.2. Here N depends only on the dimension d. We choose it to be large enough so that various Sobolev embeddings and symbolic calculus of pseudo-differential operators can carry out. See Remark (2.4) for a sufficient condition on N.

Here L^2 denotes the Lebesgue space $L^2(\mathbb{R}^d; \mathbb{R}^m)$ where $\mathbb{R}^m$ is the Euclidian space implied by the context, and $\| \cdot \|_{L^2}$ denotes its norm. Similarly, H^s denotes the Sobolev space $H^s(\mathbb{R}^d; \mathbb{R}^m)$ where $\mathbb{R}^m$ is the Euclidian space implied by the context, and $\| \cdot \|_{H^s}$ denotes its norm.

To prove the above theorem, we construct an approximating sequence of solutions by adding an artificial hyperviscosity term to the DNS system (1.1). An *a priori* estimate is established that is independent of the artificial hyperviscosity. Then using this a priori estimate and letting the artificial hyperviscosity term vanish, we show that the approximating sequence converges to a solution of the original system. Uniqueness is also shown by the a priori estimate.

This paper is laid out as follows. In Section 2 we establish an estimate for a linear system that we will later use to construct our approximating sequence of solutions to the DNS system (1.1) plus an artificial hyperviscosity. In Section 3 we establish the a priori estimate for this regularized DNS system. In section 4 we show the existence of the approximating sequence and the convergence of this sequence to the unique solution to the original DNS system.

2. Estimate for an Associated Linear System

In this section we establish the key estimate for a linear system associated with a regularization of the DNS system (1.1). One can see from the proof that the same estimate holds for the analogous linear system associated with the original DNS system (1.1).

2.1. **Regularized DNS System.** Our regularized system is obtained by first expressing the DNS system (1.1) as a system for the evolution of the fluid variables (ρ, u, θ) and then adding a fourth-order artificial hyperviscosity term to each dynamical equation. We also add a forcing term to each equation which will emerge in the higher-order estimate. The result is the regularized DNS system

$$
\begin{aligned}
&\partial_t\rho = -\epsilon\,\Delta_x^2\rho - \rho\nabla_x\cdot u - u\cdot\nabla_x\rho + f_1\,,\\
&\partial_t u = -\epsilon\,\Delta_x^2 u + \frac{1}{\rho}\nabla_x\cdot\Sigma + \frac{1}{\rho}\nabla_x\cdot\tilde{\Sigma} - \frac{1}{\rho}\nabla_x(\rho\theta) - u\cdot\nabla_x u + f_2\,,\\
&\partial_t\theta = -\epsilon\,\Delta_x^2\theta + \tfrac{2}{d}\frac{1}{\rho}\nabla_x\cdot q + \tfrac{2}{d}\frac{1}{\rho}\nabla_x\cdot\tilde{q} + \tfrac{2}{d}\frac{\tilde{\Sigma}:\nabla_x u}{\rho} + \tfrac{2}{d}\frac{\Sigma:\nabla_x u}{\rho} - \tfrac{2}{d}\theta\nabla_x\cdot u - u\cdot\nabla_x\theta + f_3\,,\\
&(\rho,u,\theta)(x,0) = (\rho^{in},u^{in},\theta^{in})(x)\,,
\end{aligned}
\tag{2.1}
$$

where Σ and q are given by (1.2) while $\tilde{\Sigma}$ and $\tilde{q}$ are given by (1.3). The structure of this system becomes explicit if we express Σ, $\tilde{\Sigma}$, q, and $\tilde{q}$ in terms of the fluid variables (ρ,u,θ). It follows from (1.3) that $\nabla_x\cdot\tilde{\Sigma}$ and $\nabla_x\cdot\tilde{q}$ have the forms

$$
\begin{aligned}
\nabla_x\cdot\tilde{\Sigma} &= \tfrac{d-1}{d}\tau_1(\rho,\theta)\Delta_x\nabla_x\theta + A^\rho(\rho,\theta,\nabla_x\rho,\nabla_x\theta):\nabla_x^2\rho + A^\theta(\rho,\theta,\nabla_x\rho,\nabla_x\theta):\nabla_x^2\theta\\
&\quad + B^\rho(\rho,\theta,\nabla_x\rho,\nabla_x\theta)\cdot\nabla_x\rho + B^\theta(\rho,\theta,\nabla_x\rho,\nabla_x\theta)\cdot\nabla_x\theta\,,\\
\nabla_x\cdot\tilde{q} &= \tfrac{2(d-1)}{d}\tau_4(\rho,\theta)\Delta_x\nabla_x\cdot u + A^u(\rho,\theta,\nabla_x\rho,\nabla_x\theta)\,\vdots\,\nabla_x^2 u + \tau_5(\rho,\theta)D_x u:\nabla_x^2\theta\\
&\quad + \tau_6(\rho,\theta)D_x u:\nabla_x^2\rho + B^u(\rho,\theta,\nabla_x\rho,\nabla_x\theta):\nabla_x u\,,
\end{aligned}
\tag{2.2}
$$

where

$$
\begin{aligned}
A^\rho(\rho,\theta,\nabla_x\rho,\nabla_x\theta):\nabla_x^2\rho &= \tau_3(\rho,\theta)\nabla_x\theta\Delta_x\rho + \tfrac{d-2}{d}\tau_3(\rho,\theta)\nabla_x\theta:\nabla_x^2\rho\,,\\
A^\theta(\rho,\theta,\nabla_x\rho,\nabla_x\theta):\nabla_x^2\theta &= \left[\left(\partial_\rho\tau_1 + \tfrac{d-2}{d}\tau_3\right)\nabla_x\rho + \left(\partial_\theta\tau_1 + \tfrac{d-2}{d}\tau_2\right)\nabla_x\theta\right]\cdot\nabla_x^2\theta\\
&\quad + \left[\left(\tfrac{1}{d}\partial_\rho\tau_1 + \tau_3\right)\nabla_x\rho + \left(\tfrac{1}{d}\partial_\theta\tau_1 + \tau_2\right)\nabla_x\theta\right]\Delta_x\theta\,,\\
A^u(\rho,\theta,\nabla_x\rho,\nabla_x\theta)\,\vdots\,\nabla_x^2 u &= \left[\left(\partial_\rho\tau_4 + \tau_6\right)\nabla_x\rho + \left(\partial_\rho\tau_4 + \tau_5 + \tau_7\right)\nabla_x\theta\right]\cdot\Delta_x u\\
&\quad + \left[\left(\tfrac{d-2}{d}\partial_\rho\tau_4 + \tfrac{d-2}{d}\tau_6\right)\nabla_x\rho + \left(\tfrac{d-2}{d}\partial_\theta\tau_4 + \tfrac{d-2}{d}\tau_5 - \tau_7\right)\nabla_x\theta\right]\cdot(\nabla_x\nabla_x\cdot u)\,,
\end{aligned}
$$

while B^ρ, B^θ have the form

$$
\begin{aligned}
&a_1(\rho,\theta)(\nabla_x\rho\cdot\nabla_x\theta)I + a_2(\rho,\theta)|\nabla_x\theta|^2 I + a_3(\rho,\theta)\nabla_x\rho\otimes\nabla_x\theta\\
&\quad + a_4(\rho,\theta)\nabla_x\theta\otimes\nabla_x\rho + a_5(\rho,\theta)\nabla_x\theta\otimes\nabla_x\theta\,,
\end{aligned}
$$

with $a_1,\cdots,a_5$ being given by the functional forms of τ_1,τ_2,τ_3 and B^u is of the form

$$
b_1(\rho,\theta)\nabla_x\rho\otimes\nabla_x\theta + b_2(\rho,\theta)\nabla_x\theta\otimes\nabla_x\rho + b_3(\rho,\theta)\nabla_x\theta\otimes\nabla_x\theta\,,
$$

where b_1,b_2,b_3 are determined by the functional forms of τ_5,τ_6, and τ_7. Notice that the forms of B^ρ and B^θ are not uniquely specified above, but the specific choice of B^ρ and B^θ does not affect our subsequent arguments. The main structure of B^θ, B^ρ, and B^u is that they are $d\times d$ tensors of linear combinations of quadratic forms of $\nabla_x\rho$, $\nabla_x\theta$.

The regularized DNS system (2.1) thereby has the form

$$
\begin{aligned}
\partial_t \rho &= -\epsilon\,\Delta_x^2 \rho - \rho \nabla_x \cdot u - u \cdot \nabla_x \rho + f_1 \,,\\
\partial_t u &= -\epsilon\,\Delta_x^2 u + \frac{1}{\rho}\nabla_x \cdot [\mu D_x u] + \tfrac{d-1}{d}\frac{\tau_1}{\rho}\Delta_x \nabla_x \theta + \frac{A^\rho}{\rho} : \nabla_x^2 \rho + \frac{A^\theta}{\rho} : \nabla_x^2 \theta\\
&\quad + \frac{B^\rho}{\rho}\cdot \nabla_x \rho + \frac{B^\theta}{\rho}\cdot \nabla_x \theta - \frac{1}{\rho}\nabla_x(\rho\theta) - u\cdot\nabla_x u + f_2 \,,\\
\partial_t \theta &= -\epsilon\,\Delta_x^2 \theta + \tfrac{2}{d}\frac{1}{\rho}\nabla_x \cdot [\kappa \nabla_x \theta] + \tfrac{4(d-1)}{d^2}\frac{\tau_4}{\rho}\Delta_x \nabla_x \cdot u\\
&\quad + \tfrac{2}{d}\frac{A^u}{\rho} \vdots \nabla_x^2 u + \tfrac{1}{d}\frac{\tau_1 + 2\tau_5}{\rho} D_x u : \nabla_x^2 \theta + \tfrac{2}{d}\frac{\tau_6}{\rho} D_x u : \nabla_x^2 \rho\\
&\quad + \tfrac{2}{d}\frac{B^u}{\rho} : \nabla_x u + \tfrac{1}{d}\frac{\tau_2}{\rho}\nabla_x \theta \cdot D_x u \cdot \nabla_x \theta + \tfrac{1}{d}\frac{\tau_3}{\rho}\nabla_x \rho \cdot D_x u \cdot \nabla_x \theta\\
&\quad + \tfrac{1}{d}\frac{\mu}{\rho}|D_x u|^2 - \tfrac{2}{d}\theta \nabla_x \cdot u - u \cdot \nabla_x \theta + f_3 \,.\\
&\qquad (\rho, u, \theta)(x,0) = (\rho^{in}, u^{in}, \theta^{in})(x)\,,
\end{aligned}
\tag{2.3}
$$

2.2. **Associated Linear System.** The linear system associated with the regularized DNS system (2.3) is obtained by replacing certain (ρ, u, θ) by a given state $(\hat\rho, \hat u, \hat\theta)$ which satisfies that for the constants $\bar\rho, \bar\theta > 0$ and $T, s > 0$ in Theorem 1.1,

$$
(\hat\rho - \bar\rho,\ \hat u,\ \hat\theta - \bar\theta) \in L^\infty(0,T; H^s(\mathbb{R}^d))\,. \tag{2.4}
$$

Specifically, it is the linear system for $(\tilde\rho, \tilde u, \tilde\theta)$ given by

$$
\begin{aligned}
\partial_t \tilde\rho &= -\epsilon\,\Delta_x^2 \tilde\rho - \hat\rho \nabla_x \cdot \tilde u - \hat u \cdot \nabla_x \tilde\rho + f_1 \,,\\
\partial_t \tilde u &= -\epsilon\,\Delta_x^2 \tilde u + \tfrac{1}{\hat\rho}\nabla_x \cdot \Big[\mu(\hat\theta) D_x \tilde u\Big] + \hat\tau_1 \Delta_x \nabla_x \tilde\theta + \hat A^\rho : \nabla_x^2 \tilde\rho + \hat A^\theta : \nabla_x^2 \tilde\theta\\
&\quad + \hat B^\rho \cdot \nabla_x \tilde\rho + \hat B^\theta \cdot \nabla_x \tilde\theta - \nabla_x \tilde\theta - \tfrac{\hat\theta}{\hat\rho}\nabla_x \tilde\rho - \hat u \cdot \nabla_x \tilde u + f_2 \,,\\
\partial_t \tilde\theta &= -\epsilon\,\Delta_x^2 \tilde\theta + \tfrac{2}{d}\tfrac{1}{\hat\rho}\nabla_x \cdot \Big[\kappa(\hat\theta)\nabla_x \tilde\theta\Big] + \hat\tau_4 \Delta_x \nabla_x \cdot \tilde u + \hat A^u \vdots \nabla_x^2 \tilde u + \hat\tau_5 D_x \hat u : \nabla_x^2 \tilde\theta + \hat\tau_6 D_x \hat u : \nabla_x^2 \tilde\rho\\
&\quad + \hat B^u : \nabla_x \tilde u + \hat\tau_2 \nabla_x \hat\theta \cdot D_x \tilde u \cdot \nabla_x \hat\theta + \hat\tau_3 \nabla_x \hat\rho \cdot D_x \tilde u \cdot \nabla_x \hat\theta\\
&\quad + \tfrac{1}{d}\tfrac{\mu(\hat\theta)}{\hat\rho} D_x \hat u : D_x \tilde u - \tfrac{2}{d}\hat\theta \nabla_x \cdot \tilde u - \hat u \cdot \nabla_x \tilde\theta + f_3 \,.\\
&\qquad (\tilde\rho, \tilde u, \tilde\theta)(x,0) = (\rho^{in} - \bar\rho, u^{in}, \theta^{in} - \bar\theta)(x)\,,
\end{aligned}
\tag{2.5}
$$

where

$$
\begin{aligned}
\hat\tau_1 &= \tfrac{d-1}{d}\frac{\tau_1(\hat\rho, \hat\theta)}{\hat\rho}\,, & \hat\tau_2 &= \tfrac{1}{d}\frac{\tau_2(\hat\rho, \hat\theta)}{\hat\rho}\,, & \hat\tau_3 &= \tfrac{1}{d}\frac{\tau_3(\hat\rho, \hat\theta)}{\hat\rho}\,,\\
\hat\tau_4 &= \tfrac{4(d-1)}{d^2}\frac{\tau_4(\hat\rho, \hat\theta)}{\hat\rho}\,, & \hat\tau_5 &= \tfrac{1}{d}\frac{\tau_1(\hat\rho, \hat\theta) + 2\tau_5(\hat\rho, \hat\theta)}{\hat\rho}\,, & \hat\tau_6 &= \tfrac{2}{d}\frac{\tau_6(\hat\rho, \hat\theta)}{\hat\rho}\,,\\
\hat A^\rho &= \frac{A^\rho(\hat\rho, \hat\theta, \nabla_x \hat\rho, \nabla_x \hat\theta)}{\hat\rho}\,, & \hat A^\theta &= \frac{A^\theta(\hat\rho, \hat\theta, \nabla_x \hat\rho, \nabla_x \hat\theta)}{\hat\rho}\,, & \hat A^u &= \tfrac{2}{d}\frac{A^u(\hat\rho, \hat\theta, \nabla_x \hat\rho, \nabla_x \hat\theta)}{\hat\rho}\,,\\
\hat B^\rho &= \frac{B^\rho(\hat\rho, \hat\theta, \nabla_x \hat\rho, \nabla_x \hat\theta)}{\hat\rho}\,, & \hat B^\theta &= \frac{B^\theta(\hat\rho, \hat\theta, \nabla_x \hat\rho, \nabla_x \hat\theta)}{\hat\rho}\,, & \hat B^u &= \tfrac{2}{d}\frac{B^u(\hat\rho, \hat\theta, \nabla_x \hat\rho, \nabla_x \hat\theta)}{\hat\rho}\,.
\end{aligned}
\tag{2.6}
$$

System (2.5) is satisfied by $(\tilde{\rho}, \tilde{u}, \tilde{\theta}) = (\rho - \bar{\rho}, u, \theta - \bar{\theta})$ when $(\hat{\rho}, \hat{u}, \hat{\theta}) = (\rho, u, \theta)$ solves (2.3). Note that is is the deviation $(\rho - \bar{\rho}, u, \theta - \bar{\theta})$ instead of the solution (ρ, u, θ) that is in H^s. This is because we need ρ, θ to be bounded away from 0. The solution (ρ, u, θ) hence is in the homogeneous Sobolev spaces $\dot{H}^s$.

Notation. Henceforth we will use Ψ_m (with various sup-indices to specify the equations in which they appear) to denote any pseudo-differential operator of order $m \in \mathbb{N}$. The reason to unify the lower order terms in this way is because the specific forms of those terms change when we apply various operators to either system (2.1) or its linearized form (2.5). However, it will be clear from the calculation that their specific forms only contribute to counting the number of derivatives of $(\hat{\rho},\, \hat{u},\, \hat{\theta})$ needed in the estimates.

Here we recall some definitions and notations related to pseudo-differential operators. We say $p(x, \xi)$ is the symbol of an operator T if for any f in the Schwartz space,

$$Tf = (2\pi)^{-d} \int_{\mathbb{R}^d} p(x, \xi)\, \hat{f}(\xi)\, e^{i\xi\cdot(x-y)}\, \mathrm{d}\xi\,.$$

We denote it as

$$T \sim p(x, \xi) \quad or \quad T = \Psi_p\,.$$

In this paper we consider classical pseudo-differential operators with symbols in the class $S^m_{1,0}$ (abbreviated as S^m). These symbols satisfy that

$$\sup_x |\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \le C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|}\,,$$

where α, β are muti-indices in $\mathbb{R}^d$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. We recall the following version of the Calderón-Vaillancourt theorem and some commutator estimates of pseudo-differential operators:

Theorem 2.1 ([7])**.** *Let m be a real number. Let $p(\xi, x) \in S^m$ be the symbol of a pseudo-differential operator Ψ_p. Then Ψ_p is a bounded linear operator from $H^m(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Moreover, there exists a constant $c = c(d)$ such that*

$$\|\Psi_p f\|_{L^2} \le c\, \|\Psi_p\|\, \|f\|_{H^m} \quad \textit{for every } f \in H^m(\mathbb{R}^d)\,, \tag{2.7}$$

where

$$\|\Psi_p\| = \sup_{|\alpha|,|\beta| \le [d/2]+1} \|\langle \xi \rangle^{-m+|\beta|} \partial_x^\alpha \partial_\xi^\beta p(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}\,. \tag{2.8}$$

Remark 2.1. In this paper we will use $m = -3, -2, -1, 0, 1, 2, 3$ with $H^0 = L^2$.

Theorem 2.2 (cf [9])**.** *Let $a(x, \xi) \in S^{m_1}(\mathbb{R}^d), b(x, \xi) \in S^{m_2}(\mathbb{R}^d)$. Let $a\sharp b(x, \xi)$ be the symbol of the composite operator $\Psi_a \circ \Psi_b$. Then*

(a) $a\sharp b \in S^{m_1+m_2}(\mathbb{R}^{2d})$.

(b) The (α, β)-seminorm $C^\sharp_{\alpha\beta}$ of $a\sharp b$ depends on the seminorms of a, b up to order $|\alpha| + |\beta| + |m_1| + d + 1$.

(c) We have

$$a\sharp b = ab - \tfrac{1}{i} \nabla_\xi a \cdot \nabla_x b + \mathcal{R}_2\,,$$

where $\mathcal{R}_2 \in S^{m_1+m_2-2}$ and the (α, β)-seminorm of $\mathcal{R}_2$ depends on the seminorms of a, b up to order $|\alpha| + |\beta| + |m_1| + d + 3$.

(d) The symbol of the commutator $[\Psi_a, \Psi_b] = \Psi_a \Psi_b - \Psi_b \Psi_a$ *is*

$$\tfrac{1}{i}(\nabla_\xi b \cdot \nabla_x a - \nabla_\xi a \cdot \nabla_x b) + \mathcal{R}_{a,b}\,,$$

where $\mathcal{R}_{a,b} \in S^{m_1+m_2-2}$ *and the* (α,β)*-seminorm of* $\mathcal{R}_{a,b}$ *depends on the seminorms of* a, b *up to order* $|\alpha| + |\beta| + |m_1| + |m_2| + d + 3$.

(e) The composition $\Psi_{a\sharp b}$ *is a bounded linear operator from* $H^{m_1+m_2}(\mathbb{R}^d)$ *to* $L^2(\mathbb{R}^d)$*. Its operator bound depends on the seminorms of* a, b *up to order* $2([d/2]+1) + |m_1| + d + 3$.

(f) The commutator $[\Psi_a, \Psi_b]$ *is a bounded linear operator from* $H^{m_1+m_2-2}(\mathbb{R}^d)$ *to* $L^2(\mathbb{R}^d)$*. Its operator bound depends on the seminorms of* a, b *up to order* $2([d/2]+1) + |m_1| + |m_2| + d + 3$.

Remark 2.2. Part (e) and (f) are immediate results of Theorem 2.1 and Part (b)(d) of Theorem 2.2.

Remark 2.3. The above operator bounds are not optimal. They suffice for the estimates in this paper since we are not pursuing optimal regularity. The sharp estimates for operators with limited-regularity symbols are given in [15].

Remark 2.4. In the calculation in the rest of this section where the above bounds for operators and their commutators are essentially used, all the terms (except those related to $q_1(x,\xi), q_2(x,\xi)$ in Step 4 in the proof of Theorem 2.3) are of the following form:

$$[\Psi_{b(\xi)},\ a(x)] \quad or \quad \Psi_{b(\xi)} \circ \Psi_{a(x)} \quad or \quad \Psi_{b(\xi)} \circ \Psi_{a(x)} - \Psi_{b(\xi)\,a(x) - \frac{1}{i}\nabla_\xi b \cdot \nabla_x a}\,,$$

where order of $b(\xi)$ do not exceed 4 and the x-dependence of $a(\cdot)$ is through the given state $(\hat\rho, \hat u, \hat\theta)$ up to its first derivative. Note that in the nonlinear setting $(\hat\rho, \hat u, \hat\theta)$ is the solution (ρ, u, θ). Therefore by Theorem 2.1 and 2.2, a sufficient condition for the regularity of the solution is $(\rho, u, \theta)(\cdot, t) \in C_b^{2d+9}(\mathbb{R}^d)$, that is, the space of functions whose derivatives up to order $2d+9$ are continuous and bounded.

For those related to $q_1(x,\xi), q_2(x,\xi)$ in Step 4, Remark 2.9 shows that q_1, q_2 depend only on the initial data and they are in S^0. Therefore $(\rho, u, \theta)(\cdot, t) \in C_b^{2d+9}(\mathbb{R}^d)$ for $t \geq 0$ also suffices.

With these notations the linear system (2.5) has the form

$$\begin{aligned}
&\partial_t \tilde\rho = -\epsilon\, \Delta_x^2 \tilde\rho + \Psi_1^\rho(\tilde\rho, \tilde u) + f_1\,,\\
&\partial_t \tilde u = -\epsilon\, \Delta_x^2 \tilde u + \frac{1}{\hat\rho}\nabla_x \cdot \Big[\mu(\hat\theta) D_x \tilde u\Big] + \hat\tau_1 \Delta_x \nabla_x \tilde\theta + \Psi_2^u(\tilde\rho, \tilde\theta) + \Psi_1^u(\tilde\rho, \tilde\theta) - \hat u \cdot \nabla_x \tilde u + f_2\,,\\
&\partial_t \tilde\theta = -\epsilon\, \Delta_x^2 \tilde\theta + \tfrac{2}{d}\frac{1}{\hat\rho}\nabla_x \cdot \Big[\kappa(\hat\theta)\nabla_x \tilde\theta\Big] + \hat\tau_4 \Delta_x \nabla_x \cdot \tilde u + \Psi_2^\theta(\tilde\rho, \tilde u, \tilde\theta) + \Psi_1^{\theta,1}(\tilde u, \tilde\theta) + f_3\,,
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
\Psi_1^\rho(\tilde\rho, \tilde u) &= -\hat\rho \nabla_x \cdot \tilde u - \hat u \cdot \nabla_x \tilde\rho\,,\\
\Psi_2^u(\tilde\rho, \tilde\theta) &= \hat A^\rho : \nabla_x^2 \tilde\rho + \hat A^\theta : \nabla_x^2 \tilde\theta\,,\\
\Psi_1^u(\tilde\rho, \tilde\theta) &= \hat B^\rho \cdot \nabla_x \tilde\rho + \hat B^\theta \cdot \nabla_x \tilde\theta - \nabla_x \tilde\theta - \frac{\hat\theta}{\hat\rho}\nabla_x \tilde\rho\,,\\
\Psi_2^\theta(\tilde\rho, \tilde u, \tilde\theta) &= \hat A^u \,\vdots\, \nabla_x^2 \tilde u + \hat\tau_5 D_x \hat u : \nabla_x^2 \tilde\theta + \hat\tau_6 D_x \hat u : \nabla_x^2 \tilde\rho\,,\\
\Psi_1^{\theta,1}(\tilde u, \tilde\theta) &= \hat B^u : \nabla_x \tilde u + \hat\tau_2 \nabla_x \hat\theta \cdot D_x \tilde u \cdot \nabla_x \hat\theta + \hat\tau_3 \nabla_x \hat\rho \cdot D_x \tilde u \cdot \nabla_x \hat\theta\\
&\quad + \tfrac{1}{d}\frac{\mu(\hat\theta)}{\hat\rho} D_x \hat u : D_x \tilde u - \tfrac{2}{d}\hat\theta \nabla_x \cdot \tilde u - \hat u \cdot \nabla_x \tilde\theta\,.
\end{aligned} \tag{2.10}$$

One can see that the symbols of these first and second order operators are first and second order polynomials in ξ respectively with their coefficients depending algebraically upon $(\hat{\rho}, \hat{\theta}, \nabla_x \hat{\rho}, D_x \hat{u}, \nabla_x \hat{\theta})$. We drop the tildes on $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ and write the regularized system (2.9) as

$$\partial_t(\rho, u, \theta) = -\epsilon\, \Delta_x^2(\rho, u, \theta) + \mathcal{L}(\hat{\rho}, \hat{u}, \hat{\theta})(\rho, u, \theta) + (f_1, f_2, f_3)\,, \tag{2.11}$$

where the linear operator $\mathcal{L}$ is defined through (2.9) and has the form

$$\mathcal{L}(\hat{\rho}, \hat{u}, \hat{\theta})(\rho, u, \theta) = \begin{pmatrix} \mathcal{L}_1(\hat{\rho}, \hat{u}, \hat{\theta})(\rho, u, \theta) \\ \mathcal{L}_2(\hat{\rho}, \hat{u}, \hat{\theta})(\rho, u, \theta) \\ \mathcal{L}_3(\hat{\rho}, \hat{u}, \hat{\theta})(\rho, u, \theta) \end{pmatrix} = \begin{pmatrix} \Psi_1^{\rho,0}(\rho, u) \\ \Psi_D^u u + \Psi_3^u \theta + \Psi_2^{u,0}(\rho, \theta) + \Psi_1^{u,0}(\rho, u, \theta) \\ \Psi_D^\theta \theta + \Psi_3^\theta u + \Psi_2^{\theta,0}(\rho, u, \theta) + \Psi_1^{\theta,0}(u, \theta) \end{pmatrix}, \tag{2.12}$$

where

$$\Psi_1^{\rho,0}(\rho, u) = \Psi_1^\rho(\rho, u)\,, \quad \Psi_2^{u,0}(\rho, \theta) = \Psi_2^u(\rho, \theta)\,, \quad \Psi_2^{\theta,0}(\rho, u, \theta) = \Psi_2^\theta(\rho, u, \theta)\,,$$

$$\Psi_1^{u,0}(\rho, u, \theta) = \Psi_1^u(\rho, u, \theta) - \hat{u} \cdot \nabla_x \tilde{u}\,, \quad \Psi_1^{\theta,0}(u, \theta) = \Psi_1^{\theta,1}(u, \theta)\,,$$

$$\Psi_D^u u = \frac{1}{\hat{\rho}} \nabla_x \cdot \left[\mu(\hat{\theta}) D_x u\right]\,, \quad \Psi_3^u \theta = \hat{\tau}_1 \Delta_x \nabla_x \theta\,,$$

$$\Psi_D^\theta \theta = \tfrac{2}{d} \frac{1}{\hat{\rho}} \nabla_x \cdot \left[\kappa(\hat{\theta}) \nabla_x \theta\right]\,, \quad \Psi_3^\theta u = \hat{\tau}_4 \Delta_x \nabla_x \cdot u\,.$$

Here Ψ_1^ρ, Ψ_2^u, Ψ_2^θ are defined in (2.10).

The discussion of degeneracies in the introduction suggests that we decompose the velocity field u into its divergence free part Pu and its gradient part Qu. Note that Qu is completely determined by the scalar function ϕ where $Qu = \nabla_x \phi$. Thus, instead of using Qu, we will derive the equation for $(-\Delta_x)^{1/2}\phi$. Let $\mathcal{R} = (\mathcal{R}_1, \cdots, \mathcal{R}_d)$ be the Riesz transform. Then we have

$$(-\Delta_x)^{1/2}\phi = -\mathcal{R} \cdot u\,.$$

To avoid the singularities of the symbols of P and Q at the origin, we consider the truncated solutions $P(1 - \psi_R(D))\, u$ and $-(1 - \psi_R(D))\mathcal{R} \cdot u$. Here $R > 0$ is a fixed number and the symbol ψ_R is a smooth function with a compact support which satisfies

$$\psi_R(\xi) = \begin{cases} 1, & |\xi| \le R\,, \\ 0, & |\xi| \ge 2R. \end{cases} \tag{2.13}$$

The symbols of P and $\mathcal{R}$ are

$$P \sim I - \frac{\xi \otimes \xi}{|\xi|^2}\,, \qquad \mathcal{R} \sim \frac{i\xi}{|\xi|}\,.$$

We will also use their index form $P = (P_{ij})_{d \times d}$ and $\mathcal{R} = (\mathcal{R}_1, \cdots, \mathcal{R}_d)$.

Let

$$u_P = P(1 - \psi_R(D))\, u\,, \qquad \Psi_{p_0} u = -(1 - \psi_R(D))\mathcal{R} \cdot u\,, \qquad u_R = \psi_R(D) u\,. \tag{2.14}$$

Apply the operator $P(1 - \psi_R(D))$ to the u-equation in (2.11). Then u_P satisfies

$$\begin{aligned} \partial_t u_P = &-\epsilon\, \Delta_x^2 u_P + P(1 - \psi_R(D)) \left(\tfrac{1}{\hat{\rho}} \nabla_x \cdot \left[\mu(\hat{\theta}) D_x u\right] + \hat{\tau}_1 \Delta_x \nabla_x \theta + \Psi_2^u(\rho, \theta)\right) \\ &+ P(1 - \psi_R(D)) \left(\Psi_1^u(\rho, \theta) - \hat{u} \cdot \nabla_x u\right) + P(1 - \psi_R(D)) f_2. \end{aligned} \tag{2.15}$$

Now compute the second and third order terms in the above equation. First,

$$
\begin{aligned}
P(1-\psi_R(D))\left(\tfrac{1}{\hat\rho}\nabla_x\cdot\Big[\mu(\hat\theta)D_x u\Big]\right) &= P(1-\psi_R(D))\left(\tfrac{\mu(\hat\theta)}{\hat\rho}\nabla_x\cdot D_x u\right) + \Psi_1^{u_P,1}(u_P,\Psi_{p_0}u,u_R)\\
&= \tfrac{\mu(\hat\theta)}{\hat\rho}\Delta_x u_P + \Psi_1^{u_P,2}(u_P,\Psi_{p_0}u,u_R)\,,
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{aligned}
&\Psi_1^{u_P,2}(u_P,\Psi_{p_0}u,u_R)\\
&= \left(\left[P(1-\psi_R(D)),\,\tfrac{\mu(\hat\theta)}{\hat\rho}\right]\nabla_x\cdot D_x + P(1-\psi_R(D))\left(\tfrac{1}{\hat\rho}\nabla_x\mu(\hat\theta)\cdot D_x\right)\right)(u_P+\mathcal{R}\Psi_{p_0}u+u_R)\,,
\end{aligned}
\tag{2.17}
$$

which shows $\Psi_1^{u_P,2}\in OPS^1$.

Next by the definition of P we have $P\nabla_x=0$. Thus,

$$
P(1-\psi_R(D))\left(\hat\tau_1\Delta_x\nabla_x\theta\right) = \left[P(1-\psi_R(D)),\,\hat\tau_1\Delta_x\right]\nabla_x\theta = \Psi_2^{u_P,1}\theta + \Psi_1^{u_P,3}\theta\,,
\tag{2.18}
$$

where $\Psi_2^{u_P,1}\theta$ is homogeneous of order 2. Its symbol is given by the leading term of $P(1-\psi_R(D))\left(\hat\tau_1\Delta_x\nabla_x\theta\right)$ which has the form

$$
\begin{aligned}
&-\sum_{j=1}^{d}\frac{\partial}{\partial\xi_j}\left((1-\psi_R(\xi))\left(I-\frac{\xi\otimes\xi}{|\xi|^2}\right)\right)\cdot\left(\frac{\partial\hat\tau_1}{\partial x_j}|\xi|^2\xi\right)\\
&= (\nabla_\xi\psi_R(\xi)\cdot\nabla_x\hat\tau_1)(|\xi|^2 I-\xi\otimes\xi)\cdot\xi + (1-\psi_R(\xi))\nabla_x\hat\tau_1\cdot(|\xi|^2 I-\xi\otimes\xi)\\
&= (1-\psi_R(\xi))\nabla_x\hat\tau_1\cdot(|\xi|^2 I-\xi\otimes\xi)\\
&= \nabla_x\hat\tau_1\cdot(|\xi|^2 I-\xi\otimes\xi) - \psi_R(\xi)\nabla_x\hat\tau_1\cdot(|\xi|^2 I-\xi\otimes\xi)\,.
\end{aligned}
$$

Therefore,

$$
\Psi_2^{u_P,1}\sim\nabla_x\hat\tau_1\cdot(|\xi|^2 I-\xi\otimes\xi)\,.
$$

The remainder operator $\Psi_1^{u_P,3}\in OPS^1$ is

$$
\Psi_1^{u_P,3} = \left[P(1-\psi_R(D)),\,\hat\tau_1\Delta_x\right]\nabla_x - \Psi_2^{u_P,1}\,.
$$

The second-order term $P(1-\psi_R(D))\Psi_2^u(\rho,\theta)$ in (2.15) has the form

$$
\begin{aligned}
P(1-\psi_R(D))\Psi_2^u(\rho,\theta) &= P(1-\psi_R(D))\left(\hat A^\rho:\nabla_x^2\rho\right) + P(1-\psi_R(D))\left(\hat A^\theta:\nabla_x^2\theta\right)\\
&= \Psi_2^{u_P,2}\rho + \Psi_2^{u_P,3}\theta + \Psi_1^{u_P,4}\rho + \Psi_1^{u_P,5}\theta\,,
\end{aligned}
\tag{2.19}
$$

where $\hat A^\rho,\hat A^\theta$ are defined in (2.6) and $\Psi_2^{u_P,2},\Psi_2^{u_P,3}$ are the leading order operators which are homogeneous of order 2. By (2.6),

$$
P(1-\psi_R(D))\hat A^\rho:\nabla_x^2\rho = P(1-\psi_R(D))\left(\hat\tau_3\nabla_x\hat\theta\Delta_x\rho\right) + P(1-\psi_R(D))\left(\tfrac{d-2}{d}\hat\tau_3\nabla_x\hat\theta\cdot\nabla_x^2\rho\right)\,.
$$

Again by $P\nabla_x=0$, the second term on the right-hand side of the above equation satisfies

$$
P(1-\psi_R(D))\left(\tfrac{d-2}{d}\hat\tau_3\nabla_x\hat\theta\cdot\nabla_x^2\rho\right) = \left[P(1-\psi_R(D)),\,\tfrac{d-2}{d}\hat\tau_3\nabla_x\hat\theta\cdot\nabla_x\right]\nabla_x\rho\,.
$$

Therefore it is a first order operator.

The symbol of $\Psi_2^{u_P,2}$ is then given by

$$
\Psi_2^{u_P,2}\sim -\hat\tau_3\nabla_x\hat\theta\cdot(|\xi|^2 I-\xi\otimes\xi)\,.
$$

The first order remainder $\Psi_1^{u_P,4}$ is given by $\sum_{j=1}^{d}[P_{ij}(1-\psi_R(D)),\, \hat\tau_3\partial_{x_j}\hat\theta\Delta_x]$. Here $P=(P_{ij})_{d\times d}$ is the index form.

Similarly,

$$P(1-\psi_R(D))\left(\hat A^\theta : \nabla_x^2\theta\right) = \Psi_2^{u_P,3}\theta + \Psi_1^{u_P,5}\theta\,,$$

where

$$\Psi_2^{u_P,3}\sim -\tfrac{1}{\hat\rho}\left[\left(\tfrac{1}{d}\partial_\rho\tau_1+\tau_3\right)\nabla_x\hat\rho + \left(\tfrac{1}{d}\partial_\theta\tau_1+\tau_2\right)\nabla_x\hat\theta\right]\cdot(|\xi|^2 I-\xi\otimes\xi)\,.$$

Inserting (2.16), (2.18), and (2.19) into (2.15), we have

$$\partial_t u_P = -\epsilon\,\Delta_x^2 u_P + \tfrac{\mu(\hat\theta)}{\hat\rho}\Delta_x u_P + \Psi_2^{u_P}(\rho,\theta) + \Psi_1^{u_P}(\rho,\theta,u_P,\Psi_{p_0}u,u_R) + P(1-\psi_R)f_2, \tag{2.20}$$

where

$$\begin{aligned}
\Psi_2^{u_P}(\rho,\theta) &= \Psi_2^{u_P,1}\theta + \Psi_2^{u_P,2}\rho + \Psi_2^{u_P,3}\theta\,,\\
\Psi_2^{u_P,1}\sim -\nabla_x\hat\tau_1\cdot(|\xi|^2I-\xi\otimes\xi)\,,&\quad \Psi_2^{u_P,2}\sim -\hat\tau_3\nabla_x\hat\theta\cdot(|\xi|^2I-\xi\otimes\xi)\,.\\
\Psi_2^{u_P,3}&\sim -\tfrac{1}{\hat\rho}\left[\left(\tfrac{1}{d}\partial_\rho\tau_1+\tau_3\right)\nabla_x\hat\rho + \left(\tfrac{1}{d}\partial_\theta\tau_1+\tau_2\right)\nabla_x\hat\theta\right]\cdot(|\xi|^2I-\xi\otimes\xi)\,,\\
\Psi_1^{u_P}(\rho,\theta,u_P,\Psi_{p_0}u,u_R) &= \Psi_1^{u_P,1}(u_P,\Psi_{p_0}u,u_R) + \Psi_1^{u_P,3}\theta + \Psi_1^{u_P,4}\rho + \Psi_1^{u_P,5}\theta\\
&\quad + P(1-\psi_P)(\Psi_1^u(\rho,\theta) - \hat u\cdot\nabla_x(u_P+\mathcal{R}\Psi_{p_0}u+u_R))\,.
\end{aligned}\tag{2.21}$$

The equation for $\Psi_{p_0}u$ is derived in a similar way as for u_P. We apply $-(1-\psi_R(D))\mathcal{R}\cdot$ to the u-equation in (2.11). Then $\Psi_{p_0}u$, defined in (2.14), satisfies

$$\begin{aligned}
\partial_t\Psi_{p_0}u &= -\epsilon\,\Delta_x^2\Psi_{p_0}u - (1-\psi_R(D))\mathcal{R}\cdot\left(\tfrac{1}{\hat\rho}\nabla_x\cdot\left[\mu(\hat\theta)D_xu\right] + \hat\tau_1\Delta_x\nabla_x\theta + \Psi_2^u(\rho,\theta)\right)\\
&\quad - (1-\psi_R(D))\mathcal{R}\cdot(\Psi_1^u(\rho,\theta)-\hat u\cdot\nabla_x u) + \Psi_{p_0}f_2\,.
\end{aligned}\tag{2.22}$$

Now compute the second and third order terms in the above equation. First,

$$\begin{aligned}
(1-\psi_R(D))\mathcal{R}\cdot\left(\tfrac{1}{\hat\rho}\nabla_x\cdot\left[\mu(\hat\theta)D_xu\right]\right) &= (1-\psi_R(D))\mathcal{R}\cdot\left(\tfrac{\mu(\hat\theta)}{\hat\rho}\nabla_x\cdot D_xu\right) + \Psi_1^{p_0,1}(u_P,\Psi_{p_0}u,u_R)\\
&= \tfrac{2(d-1)}{d}\tfrac{\mu(\hat\theta)}{\hat\rho}\Delta_x\Psi_{p_0}u + \Psi_1^{p_0,2}(u_P,\Psi_{p_0}u,u_R)\,,
\end{aligned}\tag{2.23}$$

where

$$\begin{aligned}
&\Psi_1^{p_0,2}(u_P,\Psi_{p_0}u,u_R)\\
&= \left(\left[(1-\psi_R(D))\mathcal{R}_j,\, \tfrac{\mu(\hat\theta)}{\hat\rho}\right]\nabla_x\cdot D_x + (1-\psi_R(D))\mathcal{R}_j\left(\tfrac{1}{\hat\rho}\nabla_x\mu(\hat\theta)\cdot D_x\right)\right)u_j\,.
\end{aligned}\tag{2.24}$$

Here u_j is the j-the component of $u=u_P+\mathcal{R}\Psi_{p_0}u+u_R$.

Next,

$$(1-\psi_R(D))\mathcal{R}\cdot(\hat\tau_1\Delta_x\nabla_x\theta) = \hat\tau_1(-\Delta_x)^{3/2}(1-\psi_R(D))\,\theta + [(1-\psi_R(D))\mathcal{R}_j,\,\hat\tau_1\Delta_x]\,\partial_{x_j}\theta\,. \tag{2.25}$$

The leading symbol of $[(1-\psi_R(D))\mathcal{R}_j,\, \hat\tau_1\Delta_x]\,\partial_{x_j}$ has the form

$$\begin{aligned}
&i\sum_{j,k=1}^{d}\partial_{\xi_k}\left((1-\psi_D(\xi))\frac{i\xi_j}{|\xi|}\right)\cdot(\partial_{x_k}\hat\tau_1)(-i|\xi|^2\xi_j)\\
&= i\sum_{j,k=1}^{d}\partial_{\xi_k}(1-\psi_D(\xi))(\partial_{x_k}\hat\tau_1)|\xi|^3 + i\sum_{j,k=1}^{d}(\partial_{x_k}\hat\tau_1)(1-\psi_D(\xi))|\xi|^2\xi\cdot\partial_{\xi_k}\left(\frac{\xi}{|\xi|}\right)\\
&= i\sum_{j,k=1}^{d}\partial_{\xi_k}(1-\psi_D(\xi))(\partial_{x_k}\hat\tau_1)|\xi|^3\,,
\end{aligned}$$

because $\xi\cdot\partial_{\xi_k}\left(\frac{\xi}{|\xi|}\right)=0$ for $j=1,\cdots,d$ and ξ being away from zero. Note that $\partial_{\xi_k}(1-\psi_D(\xi))$ is compactly supported. Thus

$$[(1-\psi_R(D))\mathcal{R}_j,\, \hat\tau_1\Delta_x]\,\partial_{x_j} \stackrel{\triangle}{=} \Psi_1^{p_0,3}\in OPS^1\,. \tag{2.26}$$

The second-order term $(1-\psi_R)\mathcal{R}\cdot\Psi_2^u(\rho,\theta)$ in (2.22) has the form

$$(1-\psi_R(D))\mathcal{R}\cdot\Psi_2^u(\rho,\theta) = (1-\psi_R(D))\mathcal{R}\cdot\left(\hat A^\rho : \nabla_x^2\rho\right) + (1-\psi_R(D))\mathcal{R}\cdot\left(\hat A^\theta : \nabla_x^2\theta\right)\,, \tag{2.27}$$

where $\hat A^\rho, \hat A^\theta$ are defined in (2.6). We have

$$\begin{aligned}
-(1-\psi_R(D))\mathcal{R}\cdot\left(\hat A^\rho : \nabla_x^2\rho\right) &= \Psi_2^{p_0,2}\rho + \Psi_1^{p_0,4}\rho\,,\\
-(1-\psi_R(D))\mathcal{R}\cdot\left(\hat A^\theta : \nabla_x^2\theta\right) &= \Psi_2^{p_0,3}\theta + \Psi_1^{p_0,5}\theta\,,
\end{aligned} \tag{2.28}$$

where $\Psi_2^{p_0,2}, \Psi_2^{p_0,3}$ are the leading order terms. Their symbols are

$$\begin{aligned}
\Psi_2^{p_0,2} &\sim \tfrac{2(d-1)}{d}\hat\tau_3\nabla_x\hat\theta\cdot i\xi|\xi|(1-\psi_R)\,,\\
\Psi_2^{p_0,3} &\sim A_1(\hat\rho,\hat\theta,\nabla_x\hat\rho,\nabla_x\hat\theta)\cdot i\xi|\xi|(1-\psi_R)\,,
\end{aligned} \tag{2.29}$$

where

$$A_1(\hat\rho,\hat\theta,\nabla_x\hat\rho,\nabla_x\hat\theta) = \tfrac{1}{\hat\rho}\left[\left(\tfrac{d+1}{d}\partial_\rho\tau_1 + \tfrac{2(d-1)}{d}\tau_3\right)\nabla_x\hat\rho + \left(\tfrac{d+1}{d}\partial_\theta\tau_1 + \tfrac{2(d-1)}{d}\tau_2\right)\nabla_x\hat\theta\right]\,. \tag{2.30}$$

Overall, the equation for $\Psi_{p_0}u$ is

$$\begin{aligned}
\partial_t(\Psi_{p_0}u) = {}& -\epsilon\,\Delta_x^2(\Psi_{p_0}u) + \tfrac{2(d-1)}{d}\hat\mu\Delta_x(\Psi_{p_0}u) - \hat\tau_1(-\Delta_x)^{\frac32}(1-\psi_R(D))\theta\\
&+ \Psi_2^{p_0}(\rho,\theta) + \Psi_1^{p_0}(\rho,\, Pu,\, \Psi_{p_0}u,\, u_R,\, \theta) + \Psi_{p_0}f_2\,,
\end{aligned} \tag{2.31}$$

where $\Psi_2^{p_0}(\rho,\theta) = \Psi_2^{p_0,2}\rho + \Psi_2^{p_0,3}\theta$ is defined in (2.29) and $\Psi_1^{p_0}$ is

$$\begin{aligned}
\Psi_1^{p_0}(\rho,\, Pu,\, \Psi_{p_0}u,\, u_R,\, \theta) = {}& \Psi_1^{p_0,2}(u_P,\Psi_{p_0}u,u_R) + [(1-\psi_R(D))\mathcal{R}_j,\, \hat\tau_1\Delta_x]\,\partial_{x_j}\theta\\
&+ \Psi_1^{p_0,4}\rho + \Psi_1^{p_0,5}\theta - (1-\psi_R(D))\mathcal{R}\cdot\left(\Psi_1^u(\rho,\theta) - \hat u\cdot\nabla_x u\right)\,,
\end{aligned} \tag{2.32}$$

with various first-order operators defined in (2.24), (2.26), (2.28), and $u = u_P + \mathcal{R}\Psi_{p_0}u + u_P$.

The third component of the velocity field u_R satisfies the equation

$$\begin{aligned}
\partial_t u_R = {}& -\epsilon\,\Delta_x^2 u_R + \psi_R(D)\left(\tfrac{1}{\hat\rho}\nabla_x\cdot\left[\mu(\hat\theta)D_x u\right] + \hat\tau_1\Delta_x\nabla_x\theta + \Psi_2^u(\rho,\theta)\right)\\
&+ \psi_R(D)\left(\Psi_1^u(\rho,\theta) - \hat u\cdot\nabla_x u\right) + \psi_R(D)f_2\,.
\end{aligned} \tag{2.33}$$

Since $\psi_R(\xi)$ is compactly supported, we only need an L^2 estimate for u_R.

Next, we rewrite the equation for θ as

$$\partial_t\theta = -\epsilon\,\Delta_x^2\theta + \hat\kappa\Delta_x\theta + \hat\tau_4(-\Delta_x)^{\frac{3}{2}}(1-\psi_R)(\Psi_{p_0}u) + \Psi_2^{\theta,3}(\Psi_{p_0}u) + \Psi_2^{\theta,4}\theta + \Psi_2^{\theta,5}u_P + \Psi_2^{\theta,6}\rho + \Psi_1^{\theta}(\rho,\, u_P,\, \Psi_{p_0}u,\, u_R,\, \theta) + f_3\,,$$

where $\hat\kappa = \frac{2}{d}\frac{\kappa(\hat\theta)}{\hat\rho}$. Here symbols of the second order operators are

$$\Psi_2^{\theta,3} \sim -A_2(\hat\rho,\hat\theta,\nabla_x\hat\rho,\nabla_x\hat\theta)\cdot i\xi|\xi|(1-\psi_R(D))\,, \qquad \Psi_2^{\theta,4} \sim -\hat\tau_5 D_x\hat u : (\xi\otimes\xi)\,, \tag{2.34}$$
$$\Psi_2^{\theta,5} \sim -A_3|\xi|^2\,, \qquad \Psi_2^{\theta,6} \sim -\hat\tau_6 D_x\hat u : \xi\otimes\xi\,,$$

with

$$A_3 = \tfrac{2}{d}\tfrac{1}{\hat\rho}\left[(\partial_\rho\tau_4+\tau_6)\,\nabla_x\hat\rho + (\partial_\rho\tau_4+\tau_5+\tau_7)\,\nabla_x\hat\theta\right]\,, \tag{2.35}$$
$$A_2 = A_3 + \tfrac{2}{d}\tfrac{1}{\hat\rho}\left[\left(\tfrac{d-2}{d}\partial_\rho\tau_4 + \tfrac{d-2}{d}\tau_6\right)\nabla_x\hat\rho + \left(\tfrac{d-2}{d}\partial_\theta\tau_4 + \tfrac{d-2}{d}\tau_5 - \tau_7\right)\nabla_x\hat\theta\right]\,.$$

The lower order term $\Psi_1^{\theta}(\rho,\, u_P,\, \Psi_{p_0}u,\, u_R,\, \theta)$ is given by

$$\tfrac{2}{d}\tfrac{1}{\hat\rho}\nabla_x\kappa(\hat\theta)\cdot\nabla_x\theta + \hat\tau_4(-\Delta_x)^{-3/2}\psi_R(D)\Psi_{p_0}u + \Psi_1^{\theta,2}(\rho, u_P, \mathcal{R}\Psi_{p_0}u,\, u_R,\, \theta)\,, \tag{2.36}$$

where $u = u_P + \mathcal{R}\Psi_{p_0}u + u_R$, and

$$\Psi_1^{\theta,2}(u_P,\mathcal{R}\Psi_{p_0}u,u_R,\theta) = \hat B^u : \nabla_x u + \hat\tau_2\nabla_x\hat\theta\cdot D_x u\cdot\nabla_x\hat\theta + \hat\tau_3\nabla_x\hat\rho\cdot D_x u\cdot\nabla_x\hat\theta \tag{2.37}$$
$$+ \tfrac{1}{d}\tfrac{\mu(\hat\theta)}{\hat\rho}D_x\hat u : D_x u - \tfrac{2}{d}\hat\theta\,\nabla_x\cdot u - \hat u\cdot\nabla_x\theta + \hat A^u \vdots \nabla_x^2 u_R\,.$$

Overall we have as follows the subsystem for (ρ, u_P) where there is no dispersive effect, the one for $(\Psi_{p_0}u,\theta)$ where there is strict dispersion, and the one for u_R:

$$\partial_t\rho = -\epsilon\,\Delta_x^2\rho + \Psi_1^{\rho}(\rho,u_P,\Psi_{p_0}u,u_R) + f_1\,, \tag{2.38}$$
$$\partial_t u_P = -\epsilon\,\Delta_x^2 u_P + \hat\mu\Delta_x u_P + \Psi_2^{u_P}(\rho,\theta) + \Psi_1^{u_P}(\rho,\theta,u_P,\Psi_{p_0}u,u_R) + P(1-\psi_R)f_2\,,$$

$$\partial_t(\Psi_{p_0}u) = -\epsilon\,\Delta_x^2(\Psi_{p_0}u) + \tfrac{2(d-1)}{d}\hat\mu\Delta_x(\Psi_{p_0}u) - \hat\tau_1(-\Delta_x)^{\frac{3}{2}}(1-\psi_R(D))\theta \tag{2.39}$$
$$+ \Psi_2^{p_0}(\rho,\theta) + \Psi_1^{p_0}(\rho,\, Pu,\, \Psi_{p_0}u,\, u_R,\, \theta) + \Psi_{p_0}f_2\,,$$
$$\partial_t\theta = -\epsilon\,\Delta_x^2\theta + \hat\kappa\Delta_x\theta + \hat\tau_4(-\Delta_x)^{\frac{3}{2}}(1-\psi_R)(\Psi_{p_0}u) + \Psi_2^{\theta,3}(\Psi_{p_0}u)$$
$$+ \Psi_2^{\theta,4}\theta + \Psi_2^{\theta,5}u_P + \Psi_2^{\theta,6}\rho + \Psi_1^{\theta}(\rho,\, u_P,\, \Psi_{p_0}u,\, u_R,\, \theta) + f_3\,,$$

$$\partial_t u_R = -\epsilon\,\Delta_x^2 u_R + \psi_R\left(\tfrac{1}{\hat\rho}\nabla_x\cdot\left[\mu(\hat\theta)D_x u\right] + \hat\tau_1\Delta_x\nabla_x\theta + \Psi_2^u(\rho,\theta)\right) \tag{2.40}$$
$$+ \psi_R\left(\Psi_1^u(\rho,\theta) - \hat u\cdot\nabla_x u\right) + \psi_R f_2\,,$$

where $\hat\mu = \frac{\mu(\hat\theta)}{\hat\rho}$, $u = u_P + \mathcal{R}\Psi_{p_0}u + u_R$. Various Ψ_k's are defined in (2.21), (2.29), (2.32), (2.34), and (2.36).

Thus we have divided the full linearized system (2.9) into three subsystems (2.38), (2.39), and (2.40), according to the degeneracy of dispersion and possible singularity at $\xi = 0$. As mentioned in the introduction, we will apply the framework in [13] to obtain an energy estimate for the strictly dispersive system (2.39). The energy estimate of (2.39) is not closed by itself due to the coupling of the two subsystems through the lower order terms. Therefore we need to explore the structure of system (2.38) and (2.40) to close the energy estimate for the full system. This will be done in Section 2.4.

2.3. **Assumptions for the Estimate.** In order to obtain bounds on the solutions of linear system (2.9) we make the following assumptions on $(\hat\rho, \hat u, \hat\theta)$ and the initial data. These assumptions will indicate the proper space for the well-posedness result.

$\mathcal{A}_1$. *Asymptotic flatness.* There exist constants $c_A, T_0 > 0$ such that $\forall (x,t) \in \mathbb{R}^d \times [0, T_0]$,

$$\Big|\partial_t(\hat\rho, \hat u, \hat\theta)(x,t)\Big| + \Big|\nabla_x(\hat\rho, \hat u, \hat\theta)(x,t)\Big| + \Big|\partial_t\nabla_x(\hat\rho, \hat u, \hat\theta)(x,t)\Big| \leq \frac{c_A}{\langle x\rangle^2} \tag{2.41}$$

with $\langle x\rangle^2 \triangleq 1 + |x|^2$.

$\mathcal{A}_2$. *Regularity.* For the same T_0, suppose

$$(\hat\rho - \bar\rho,\ \hat u,\ \hat\theta - \bar\theta) \in W^{1,\infty}([0,T_0]; H^N(\mathbb{R}^d))\,,$$

where $N = N(d)$ (see Theorem 1.1 and Remark 2.4). Again use c_A to denote the norm $\|(\hat\rho - \bar\rho,\ \hat u,\ \hat\theta - \bar\theta)\|_{W^{1,\infty}([0,T_0];H^N(\mathbb{R}^d))}$.

$\mathcal{A}_3$. *Lower bounds.* There exists a constant $c_0 > 0$ such that $\hat\rho, \hat\theta \geq c_0 > 0$. This together with the uniform bounds on $\hat\rho, \hat\theta$ guarantees the existence of a constant $\tau_0 > 0$ such that $\frac{1}{\tau_0} \geq \hat\tau_1/\hat\tau_4 \geq \tau_0 > 0$.

$\mathcal{A}_4$. *Nontrapping condition.* Let $h^{in}(\xi, x) = \sqrt{\hat\tau_1(x,0)\hat\tau_4(x,0)}\,|\xi|^3$ as defined in (1.8) and $H_{h^{in}}$ be the corresponding Hamiltonian flow. Then $H_{h^{in}}$ is nontrapping, that is, if $(\Xi, X)(t; \xi, x)$ is a solution to

$$\begin{aligned} \frac{\mathrm{d}\Xi}{\mathrm{d}t} &= -\nabla_x h^{in}(\Xi, X)\,, && \Xi(0) = \xi\,, \\ \frac{\mathrm{d}X}{\mathrm{d}t} &= \nabla_\xi h^{in}(\Xi, X)\,, && X(0) = x\,, \end{aligned} \tag{2.42}$$

then for any $\xi \neq 0$,

$$|X(t)| \to \infty \quad \text{as} \quad t \to \pm\infty\,.$$

Remark 2.5. We need $\hat\rho, \hat\theta$ to be away from zero so that system (2.39) will have strict dispersion and dissipation which are essential to the calculation in this paper.

Remark 2.6. It is sufficient to verify the nontrapping condition for $|\xi| = 1$ because the Hamiltonian flow satisfies the dilation scaling $\Xi(t; \lambda\xi, x) = \lambda\Xi(\lambda^2 t; \xi, x)$, $X(t; \lambda\xi, x) = X(\lambda^2 t; \xi, x)$.

Remark 2.7. Here we give some examples of the initial data that satisfy the non-trapping condition. The trivial case is when $\hat\tau_1(x,0)$ and $\hat\tau_4(x,0)$ are both positive constants. Then (2.42) becomes

$$\Xi(t) = \xi\,, \qquad \frac{\mathrm{d}X}{\mathrm{d}t} = 3\xi|\xi|\,\sqrt{\hat\tau_1(x,0)\hat\tau_4(x,0)}\,, \qquad X(0) = x\,,$$

which obviously satisfies the nontrapping condition. However, the derivation of $\hat\tau_1, \hat\tau_4$ indicates that in general they depend on θ and ρ.

A less trivial condition that is physically possible and guarantees the nontrapping condition is

$$|\nabla_x \sqrt{\hat\tau_1(x,0)\hat\tau_4(x,0)}|\,|x| \leq \tfrac{3}{2}\sqrt{\hat\tau_1(x,0)\hat\tau_4(x,0)} \qquad \text{for all } x\,. \tag{2.43}$$

For the derivation of the transport coefficients from kinetic equations, $\hat\tau_1$ and $\hat\tau_4$ are roughly proportional to $\frac{\theta^r}{\rho}$ for some r. Therefore, a sufficient condition for (2.43) to hold is

$$|\nabla_x \theta^{in}|\,|x| \leq \frac{3\theta^{in}}{4\,r}\,, \qquad |\nabla_x \rho^{in}|\,|x| \leq \frac{3\rho^{in}}{4}\,, \tag{2.44}$$

because then

$$|\nabla_x \sqrt{\hat{\tau}_1(x,0)\hat{\tau}_4(x,0)}|\,|x| \sim \left|\nabla_x \frac{(\theta^{in})^r}{\rho^{in}}\right|\,|x| \quad \le r\frac{(\theta^{in})^{r-1}}{\rho^{in}}|\nabla_x \theta^{in}|\,|x| + \frac{2(\theta^{in})^r}{(\rho^{in})^2}|\nabla_x \rho^{in}|\,|x|$$
$$\le \frac{3}{2}\frac{(\theta^{in})^r}{\rho^{in}} \sim |\sqrt{\hat{\tau}_1(x,0)\hat{\tau}_4(x,0)}|\,.$$

Note that (2.44) is automatically satisfied for large $|x|$ and small $|x|$ given the asymptotic flatness of the data and the bounds of ρ^{in}, θ^{in}. So we just need to assume that (2.44) holds over a compact annulus in $\mathbb{R}^d$.

Now suppose that (2.43) holds and we show that it implies the nontrapping condition. First notice that (2.42) gives

$$\sqrt{\hat{\tau}_1(X,0)\hat{\tau}_4(X,0)}\;|\Xi|^3 = const\,.$$

Therefore $|\Xi|^3$ has a positive lower bound for all time because $\hat{\tau}_1, \hat{\tau}_4$ are bounded from above for all x. By (2.42),

$$\frac{\mathrm{d}}{\mathrm{d}t}(X\cdot\Xi) = (3\sqrt{\hat{\tau}_1(X,0)\hat{\tau}_4(X,0)} - (X\cdot\nabla_X\sqrt{\hat{\tau}_1(X,0)\hat{\tau}_4(X,0)}))\,|\Xi|^3\,,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}|X|^2 = 6\sqrt{\hat{\tau}_1(X,0)\hat{\tau}_4(X,0)}\,|\Xi|\,(X\cdot\Xi)\,.$$

Therefore $X\cdot\Xi$ increases in time with a positive rate provided (2.43) holds. Then there will be a time t_0 when $X\cdot\Xi(t_0) > 1$. Thus for all $t \ge t_0$, $|X|^2$ increases with time with a positive rate and $|X|$ tends to ∞ as $t\to\infty$. The same argument holds for $t\to-\infty$.

Notation. Henceforth constants that depend only on the initial data and c_0 in $\mathcal{A}_3$ will have a 0 subscript. Constants that depend on the constants that appear in $\mathcal{A}_1$ and $\mathcal{A}_2$ will have an A subscript.

The asymptotic flatness assumption $\mathcal{A}_1$ and the nontrapping condition $\mathcal{A}_4$ are needed for the following lemma, which is due to Chihara [4]. We refer to [4, 6, 13] and references therein for discussions about the necessity of the nontrapping condition for the L^2-well-posedness of dispersive equations.

Lemma 2.1 ([4])**.** *Let C^{in} and α^{in} be the constants in (1.6) and (1.7) such that*

$$\sqrt{\hat{\tau}_1(x,0)\hat{\tau}_4(x,0)} \ge \sqrt{\alpha^{in}}\,, \qquad |\nabla_x(\hat{\rho},\hat{u},\hat{\theta})(x,0)| \le \frac{2C^{in}}{\langle x\rangle^2}\,,$$

and $H_{h^{in}}$ nontrapping. Then there exists a symbol $p(x,\xi)\in S^0$ such that

- *$p(x,\xi)$ is real;*
- *there exist a fixed $0<t_0<\infty$ and a function $P(\cdot)\in C^\infty(\mathbb{R})$ such that*

$$p(x,\xi) = P(\tilde{p}_1(x,\,\hat{\rho}(x,0),\,\hat{\theta}(x,0),\,\xi) + \tilde{p}_2(x,\xi)) \tag{2.45}$$

  where $\tilde{p}_1$ depends locally in $\hat{\rho},\hat{\theta}$ and is supported away from $x=0$ while $\tilde{p}_2(x,\xi)$ depends on the path of the bicharacteristic flow (X,Ξ) on $[0,t_0]$ and is supported around $x=0$;
- *there exist constants c_1, c_2 which depend on C^{in} and α^{in}*

$$H_{(1-\psi_R)h^{in}}p \ge c_1\frac{|\xi|^2}{\langle x\rangle^2} - c_2\,, \qquad \forall(\xi,x)\in\mathbb{R}^d\times\mathbb{R}^d\,, \tag{2.46}$$

where $H_{(1-\psi_R)h^{in}}p$ is the Poisson bracket $\{(1-\psi_R)h^{in},\, p\}$ such that

$$H_{(1-\psi_R)h^{in}}p = \sum_{j=1}^{d}(\partial_{\xi_j}((1-\psi_R)h^{in})\partial_{x_j}p - \partial_{\xi_j}((1-\psi_R)h^{in})\partial_{x_j}p)\,,$$

which is also the leading symbol of the commutator $[1-\psi_R)h^{in},\, p]$ up to $-i$.

- *for any $\alpha, \beta \in \mathbb{N}^d$,*
$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{-|\beta|}\,,$$
where $C_{\alpha\beta}$ depends on $\|\nabla_x^{|\alpha|+1}(\hat\rho,\hat\theta)(x,0)\|_{L^\infty(\mathbb{R}^d)}$.

Remark 2.8. We will show in Lemma 2.2 that given the assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$, the bound (2.46) propagates for a short time.

2.4. **Linear Estimate.** The main result of this section is the following bound on solutions of the linear system (2.11). Because this estimate is a priori, we assume that (ρ, u, θ) is a smooth solution to (2.11). Note that the regularity assumed in $\mathcal{A}_2$ is enough for the proof.

Theorem 2.3. *Let $(\hat\rho,\hat u,\hat\theta)(t,x) \in C([0,T_0];\dot H^\infty)$ be functions that satisfy assumptions $\mathcal{A}_1-\mathcal{A}_4$ where $\dot H^\infty$ is the homogeneous Sobolev space. Then for every solution $(\rho,u,\theta) \in C([0,T_0];H^\infty)$ of the linear system (2.11) there exists $0 < T \le T_0$ depending on the constants c_0, c_A in the assumptions and $c > 0$ depending on C^{in} and c_0 such that*

$$\begin{aligned}&\sup_{[0,T]}\left(\|\rho\|_{H^1}^2 + \|(u,\theta)\|_{L^2}^2\right)(t) + \int_0^T \|\nabla_x(u,\theta)\|_{L^2}^2(s)\,\mathrm{d}s \\ &\le c\left(\|\rho^{in}\|_{H^1}^2 + \|(u^{in},\theta^{in})\|_{L^2}^2 + \int_0^T (\|f_1\|_{H^1}^2 + \|(f_2,f_3)\|_{L^2}^2)(s)\mathrm{d}s\right).\end{aligned} \tag{2.47}$$

Both c and T are independent of ϵ.

Proof. The proof has five steps. We begin with estimates for $(\Psi_{p_0}u,\theta)$ in (2.39). This subsystem has nondegenerate dispersive terms. Let

$$\vec\omega = (\Psi_{p_0}u,\, \theta)^T = (\omega_1,\omega_2)^T\,. \tag{2.48}$$

Then the system for $\vec\omega$ has the form

$$\partial_t\vec\omega = -\epsilon\,\Delta_x^2\vec\omega + \Psi_D\vec\omega + \Psi_{L_0}\vec\omega + \Psi_{B_0}\vec\omega + \Psi_2^\omega(\rho,u_P) + \Psi_1^\omega(\rho,\, u_P,\, u_R,\, \vec\omega) + \Psi_0^\omega(f_2,f_3)\,, \tag{2.49}$$

with

$$\Psi_D\vec\omega = \begin{pmatrix} 2\frac{d-1}{d}\hat\mu\Delta_x\omega_1 \\ \hat\kappa\Delta_x\omega_2\end{pmatrix}, \quad \Psi_{L_0}\vec\omega = \begin{pmatrix} -\hat\tau_1(-\Delta_x)^{\frac32}(1-\psi_R)\omega_2 \\ \hat\tau_4\,(-\Delta_x)^{\frac32}(1-\psi_R)\omega_1\end{pmatrix},$$

$$\Psi_{B_0}\vec\omega = \begin{pmatrix} \Psi_2^{p_0,3}\omega_2 \\ \Psi_2^{\theta,3}\omega_1 + \Psi_2^{\theta,4}\omega_2\end{pmatrix}, \quad \Psi_2^\omega(\rho,u_P) = \begin{pmatrix} \Psi_2^{p_0,2}\rho \\ \Psi_2^{\theta,5}u_P + \Psi_2^{\theta,6}\rho\end{pmatrix},$$

$$\Psi_1^\omega(\rho,\vec\omega,u_P) = \begin{pmatrix} \Psi_1^{p_0}(\rho,\, u_P,\, u_R,\, \vec\omega) \\ \Psi_1^\theta\,(\rho,\, u_P,\, u_R,\, \vec\omega)\end{pmatrix}, \qquad \Psi_0^\omega(f_2,f_3) = \begin{pmatrix}\Psi_{p_0}f_2 \\ f_3\end{pmatrix},$$

where definitions of various operators are given in (2.21), (2.29), (2.32), (2.34), and (2.36).

The symbol of Ψ_{L_0} is given by

$$L_0 = \begin{pmatrix} 0 & -\hat\tau_1\,|\xi|^3(1-\psi_R) \\ \hat\tau_4\,|\xi|^3\,(1-\psi_R) & 0\end{pmatrix}.$$

Notation. Henceforth we will work with system (2.49) for $\vec{\omega}$ and we will use Ψ_m and $\Psi_{m,j}$ without any sup-index to denote m-th order operators. We will use Ψ_{R_j} for $j \in \mathbb{N}$ to denote operators related to commutators of $\epsilon\Delta_x^2$ with various operators.

<u>*Step 1. Diagonalization of* Ψ_{L_0}.</u> The matrix L_0 has eigenvalues $\lambda_\pm = \pm i\sqrt{\hat{\tau}_1\hat{\tau}_4}\,|\xi|^3\,(1-\psi_R)$. Let

$$A = \begin{pmatrix} -i & \sqrt{\hat{\tau}_1/\hat{\tau}_4} \\ i & \sqrt{\hat{\tau}_1/\hat{\tau}_4} \end{pmatrix}, \qquad L = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} = \lambda_+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.50}$$

Then

$$A^{-1} = \tfrac{1}{2} \begin{pmatrix} i & -i \\ \sqrt{\hat{\tau}_4/\hat{\tau}_1} & \sqrt{\hat{\tau}_4/\hat{\tau}_1} \end{pmatrix}, \qquad AL_0 = LA\,.$$

Notation. For any $w(x)$ nonnegative, let H^s_w denote the weighted Sobolev space defined as

$$H^s_w \triangleq \left\{u(x)\,:\, wu \in H^s\right\},$$

with the norm

$$\|u\|_{H^s_w} = \|wu\|_{H^s}\,.$$

Note that for the weight function $w = \langle x\rangle^2 = 1+|x|^2$, the norm $\|u\|_{H^s_{\langle x\rangle^2}}$ is equivalent to the norm $\sum_{|\alpha|\le s}\|\langle x\rangle^2 D_x^\alpha u\|_{L^2}$ for every $u \in H^s_{\langle x\rangle^2}$. Both A and A^{-1} are zeroth order symbols. The operator Ψ_A is a multiplication — i.e. $\Psi_A\vec{\omega} = A\vec{\omega}$. Then as an operator, A is invertible on H^s, $H^s_{\langle x\rangle^{-2}}$, and $H^s_{\langle x\rangle^2}$ for every $s \le N$ by assumptions $\mathcal{A}_2$ and $\mathcal{A}_3$.

Define

$$\vec{\beta} = A\vec{\omega}\,. \tag{2.51}$$

The equation for $\vec{\beta}$ is obtained by multiplying (2.49) by A.

$$\begin{aligned}\partial_t\vec{\beta} = \partial_t(A\vec{\omega}) = {}&-\epsilon\, A\Delta_x^2\vec{\omega} + A\Psi_D\vec{\omega} + A\Psi_{L_0}\vec{\omega} + A\Psi_{B_0}\vec{\omega} + (\partial_t A)\vec{\omega} \\ &+ A\Psi_2\rho + A\Psi_2 u_P + \Psi_1(\rho,\, u_P,\, \vec{\omega}) + \Psi_0 u_R + \Psi_0(f_2, f_3)\,.\end{aligned} \tag{2.52}$$

The term $\Psi_0 u_R$ is given by Ψ_1^θ in (2.37) and $\Psi_1^{u_Q}$ in (2.32).

Now we rewrite each term on the right-hand side of (2.52) to obtain a system for $\vec{\beta}$. First,

$$\epsilon\, A\Delta_x^2\vec{\omega} = \epsilon\,\Delta_x^2 A\vec{\omega} + \epsilon\,\left(\Psi_{R_1}A^{-1}\right)A\vec{\omega} = \epsilon\,\Delta_x^2\vec{\beta} + \epsilon\,\Psi_{R_2}\vec{\beta}\,,$$

where

$$\Psi_{R_1} = [A,\ \Delta_x^2] - \Delta_x^2 A\,, \qquad \Psi_{R_2} = \Psi_{R_1}A^{-1}\,.$$

Second,

$$A\Psi_{B_0}\vec{\omega} = A\Psi_{B_0}A^{-1}\vec{\beta}\,.$$

The leading symbol of $A\Psi_{B_0}A^{-1}$ is

$$B_1 = \tfrac{1}{2}\begin{pmatrix} -i & \sqrt{\hat{\tau}_1/\hat{\tau}_4} \\ i & \sqrt{\hat{\tau}_1/\hat{\tau}_4} \end{pmatrix}\begin{pmatrix} 0 & iA_1\cdot\xi|\xi|(1-\psi_R) \\ -iA_2\cdot\xi|\xi|(1-\psi_R) & -\hat{\tau}_5 D_x\hat{u} : (\xi\otimes\xi) \end{pmatrix}\begin{pmatrix} i & -i \\ \sqrt{\hat{\tau}_4/\hat{\tau}_1} & \sqrt{\hat{\tau}_4/\hat{\tau}_1} \end{pmatrix}, \tag{2.53}$$

where A_1, A_2 are defined in (2.30) and (2.35) respectively. The explicit form of B_1 is then calculated as

$$B_1(x,\xi,t) = \tfrac{1}{2}\begin{pmatrix} A_{11}\cdot\xi|\xi|(1-\psi_R) + A_{12} : (\xi\otimes\xi) & A_{13}\cdot\xi|\xi|(1-\psi_R) + A_{12} : (\xi\otimes\xi) \\ -A_{13}\cdot\xi|\xi|(1-\psi_R) + A_{12} : (\xi\otimes\xi) & -A_{11}\cdot\xi|\xi|(1-\psi_R) + A_{12} : (\xi\otimes\xi) \end{pmatrix}, \tag{2.54}$$

where

$$A_{11}=\sqrt{\hat\tau_1/\hat\tau_4}A_2+\sqrt{\hat\tau_4/\hat\tau_1}A_1\,,\quad A_{12}=-\hat\tau_5\,D_x\hat u\,,$$
$$A_{13}=-\sqrt{\hat\tau_1/\hat\tau_4}A_2+\sqrt{\hat\tau_4/\hat\tau_1}A_1\,.$$

Notice that $A_{ij}, 1\le i\le 2, 1\le j\le 3$ are all linear combinations of $\nabla_x\hat\rho, \nabla_x\hat u, \nabla_x\hat\theta$. Therefore by the assumptions $\mathcal{A}_1$ and $\mathcal{A}_3$, there exists $0<T_1\le T_0$ such that the symbol B_1 satisfies

$$|B_1(x,\xi,t)|\le\frac{c_{0,1}|\xi|^2}{\langle x\rangle^2}\,,\qquad\forall(x,\xi,t)\in\mathbb{R}^d\times\mathbb{R}^d\times[0,T_1]\,,\tag{2.55}$$

Here the constant $c_{0,1}$ depends only on the initial data and c_0 in $\mathcal{A}_3$ while T_1 depends on c_A in $\mathcal{A}_1$ and $\mathcal{A}_2$.

Each entry of the matrix of the remainder operator $\Psi_{r_0}=A\Psi_{B_0}A^{-1}-\Psi_{B_1}$ is a linear combination of two commutators

$$[A_1\cdot\nabla_x(-\Delta_x)^{1/2}(1-\psi_R(D)),\ \sqrt{\hat\tau_4/\hat\tau_1}]\quad\text{and}\quad[\hat\tau_5D_x\hat u:\nabla_x^2,\ \sqrt{\hat\tau_4/\hat\tau_1}]\,.$$

Thus it is a first-order operator.

Next we have

$$\begin{aligned}&A\Psi_DA^{-1}\\&=\tfrac12\begin{pmatrix}-i&\sqrt{\hat\tau_1/\hat\tau_4}\\ i&\sqrt{\hat\tau_1/\hat\tau_4}\end{pmatrix}\begin{pmatrix}2\frac{d-1}{d}\hat\mu\Delta_x&0\\0&\hat\kappa\Delta_x\end{pmatrix}\begin{pmatrix}i&-i\\\sqrt{\hat\tau_4/\hat\tau_1}&\sqrt{\hat\tau_4/\hat\tau_1}\end{pmatrix}\\&=\begin{pmatrix}\frac{d-1}{d}\hat\mu\Delta_x+\frac12\hat\kappa\sqrt{\hat\tau_1/\hat\tau_4}\,\Delta_x(\sqrt{\hat\tau_4/\hat\tau_1}\,\cdot)&-\frac{d-1}{d}\hat\mu\Delta_x+\frac12\hat\kappa\sqrt{\hat\tau_1/\hat\tau_4}\,\Delta_x(\sqrt{\hat\tau_4/\hat\tau_1}\,\cdot)\\-\frac{d-1}{d}\hat\mu\Delta_x+\frac12\hat\kappa\sqrt{\hat\tau_1/\hat\tau_4}\,\Delta_x(\sqrt{\hat\tau_4/\hat\tau_1}\,\cdot)&\frac{d-1}{d}\hat\mu\Delta_x+\frac12\hat\kappa\sqrt{\hat\tau_1/\hat\tau_4}\,\Delta_x(\sqrt{\hat\tau_4/\hat\tau_1}\,\cdot)\end{pmatrix}\\&=\begin{pmatrix}\left(\frac{d-1}{d}\hat\mu+\frac12\hat\kappa\right)\Delta_x&0\\0&\left(\frac{d-1}{d}\hat\mu+\frac12\hat\kappa\right)\Delta_x\end{pmatrix}+\begin{pmatrix}0&\left(-\frac{d-1}{d}\hat\mu+\frac12\hat\kappa\right)\Delta_x\\\left(-\frac{d-1}{d}\hat\mu+\frac12\hat\kappa\right)\Delta_x&0\end{pmatrix}+\Psi_{r_1}\,,\end{aligned}$$

where

$$\Psi_{r_1}=\tfrac12\hat\kappa\sqrt{\hat\tau_1/\hat\tau_4}\ \left[\Delta_x,\ \sqrt{\hat\tau_4/\hat\tau_1}\right]\begin{pmatrix}1&1\\1&1\end{pmatrix}\,.$$

Therefore,

$$A\Psi_D\vec\omega=\left(\tfrac{d-1}{d}\hat\mu+\tfrac12\hat\kappa\right)\Delta_x\vec\beta+\begin{pmatrix}0&\left(-\frac{d-1}{d}\hat\mu+\frac12\hat\kappa\right)\Delta_x\\\left(-\frac{d-1}{d}\hat\mu+\frac12\hat\kappa\right)\Delta_x&0\end{pmatrix}\vec\beta+\Psi_{r_1}A^{-1}\vec\beta\,.$$

Notice that although the second term on the right side of the above equation has second order entries, those entries are all off-diagonal. Combine this term with Ψ_{B_1} and use Ψ_{B_2} to denote this new second order operator. Then

$$B_2=B_1+\begin{pmatrix}0&\left(-\frac{d-1}{d}\hat\mu+\frac12\hat\kappa\right)\Delta_x\\\left(-\frac{d-1}{d}\hat\mu+\frac12\hat\kappa\right)\Delta_x&0\end{pmatrix}\,,\tag{2.56}$$

where B_1 is defined in (2.53). Note that the diagonal elements of B_2 are the same as those of B_1. Therefore they satisfy the condition (2.55).

Next we study the structure of $A\Psi_{L_0}$. Using the fact that $AL_0=LA$, we have

$$A\Psi_{L_0}=\Psi_LA+(\Psi_{AL_0}-\Psi_{LA})+(\Psi_{LA}-\Psi_LA)=\Psi_LA+(\Psi_{LA}-\Psi_LA)\,.$$

By (2.50),

$$\begin{aligned}\Psi_{LA} - \Psi_L A &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -i & \sqrt{\hat\tau_1/\hat\tau_4} \\ i & \sqrt{\hat\tau_1/\hat\tau_4} \end{pmatrix} \Psi_{\lambda_+} - \Psi_{\lambda_+} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -i & \sqrt{\hat\tau_1/\hat\tau_4} \\ i & \sqrt{\hat\tau_1/\hat\tau_4} \end{pmatrix} \cdot \right) \\ &= \begin{pmatrix} -i & \sqrt{\hat\tau_1/\hat\tau_4} \\ -i & -\sqrt{\hat\tau_1/\hat\tau_4} \end{pmatrix} \Psi_{\lambda_+} - \Psi_{\lambda_+} \left(\begin{pmatrix} -i & \sqrt{\hat\tau_1/\hat\tau_4} \\ -i & -\sqrt{\hat\tau_1/\hat\tau_4} \end{pmatrix} \cdot \right) \\ &= [\sqrt{\hat\tau_1/\hat\tau_4},\ \Psi_{\lambda_+}] \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

Let B_3 be the leading symbol of $\Psi_{LA} - \Psi_L A$. Then,

$$B_3 = -(3\sqrt{\hat\tau_1 \hat\tau_4}) \nabla_x (\sqrt{\hat\tau_1/\hat\tau_4}) \cdot (\xi|\xi|)(1-\psi_R) \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} . \tag{2.57}$$

Therefore there exist two constants $0 < T_2 \le T_1$ and $c_{0,2} > 0$ such that

$$|B_3(x,\xi,t)| \le \frac{c_{0,2}\,|\xi|^2}{\langle x \rangle^2}, \qquad \forall (x,\xi,t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0,T_2],$$

where T_2 depends on c_A in $\mathcal{A}_1$ and $\mathcal{A}_2$ while $c_{0,2}$ depends only on the initial data and c_0 in $\mathcal{A}_3$.

The remainder term Ψ_{r_2} is

$$\begin{aligned}\Psi_{r_2} &= \Psi_{LA} - \Psi_L A - \Psi_{B_3} \\ &= \left([\sqrt{\hat\tau_1/\hat\tau_4},\ \Psi_{\lambda_+}] + (3\sqrt{\hat\tau_1 \hat\tau_4}) \nabla_x (\sqrt{\hat\tau_1/\hat\tau_4}) \cdot (\xi|\xi|)(1-\psi_R) \right) \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \sqrt{\hat\tau_1 \hat\tau_4} \left([\sqrt{\hat\tau_1/\hat\tau_4},\ \Psi_{i|\xi|^3(1-\psi_R(\xi))}] + 3\nabla_x (\sqrt{\hat\tau_1/\hat\tau_4}) \cdot (\xi|\xi|)(1-\psi_R) \right) \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} .\end{aligned}$$

Combine Ψ_{B_3} with Ψ_{B_2} and define

$$B = B_2 + B_3 \triangleq \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{2.58}$$

where B_2, B_3 are defined in (2.56) and (2.57) respectively. Then there exists $c_{0,3}$ depending only on the initial data and c_0 in $\mathcal{A}_3$ such that the diagonal of B, denoted as B_{diag}, satisfies that

$$|B_{diag}(x,\xi,t)| \le \frac{c_{0,3}|\xi|^2}{\langle x \rangle^2}, \qquad \forall (x,\xi,t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0,T_2]. \tag{2.59}$$

Thus the second-order remainder terms B_{diag} on the diagonal are important only locally. Although they cannot be controlled by the dissipation terms because they are of the same order, we will show below that B_{diag} can be bounded due to the local regularization of the dispersive terms.

For the rest of the terms on the right-hand side of (2.52), we have

$$A\Psi_2(\rho, u_P) = \Psi_2(\rho, u_P), \qquad A\Psi_1(\rho, u_P) = \Psi_1(\rho, u_P),$$
$$A\Psi_0 u_R = \Psi_0 u_R, \qquad A\Psi_0(f_2, f_3) = \Psi_0(f_2, f_3).$$

The lower order term of $\vec\beta$ has several parts:

$$\Psi_1 \vec\beta = \Psi_{r_0}\vec\beta + \Psi_{r_1} A^{-1}\vec\beta + \Psi_{r_2}\vec\beta + A\Psi_1 A^{-1}\vec\beta.$$

Overall the system for $\vec{\beta}$ has the form

$$\begin{aligned}\partial_t\vec{\beta} = -\epsilon\,\Delta_x^2\vec{\beta} + \epsilon\,\Psi_{R_2}\vec{\beta} + \left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x\vec{\beta} + \Psi_L\,\vec{\beta} + \Psi_B\,\vec{\beta} \\ + \Psi_2(\rho, Pu) + \Psi_1(\rho, Pu, \vec{\beta}) + \Psi_0 u_R + \Psi_0(f_2, f_3)\,.\end{aligned} \tag{2.60}$$

<u>*Step 2. Diagonalization of* Ψ_B.</u> Write

$$\Psi_B = \Psi_{B_{diag}} + \Psi_{B_{anti}} = \begin{pmatrix}\Psi_{B_{11}} & 0 \\ 0 & \Psi_{B_{22}}\end{pmatrix} + \begin{pmatrix}0 & \Psi_{B_{12}} \\ \Psi_{B_{21}} & 0\end{pmatrix},$$

where B is defined in (2.58). We show in the following that $\Psi_{B_{anti}}$ can be eliminated using a normal form reduction. To this end, let

$$h(\xi, x, t) = \sqrt{\hat{\tau}_1\hat{\tau}_4}|\xi|^3\,, \qquad h_R = h(1-\psi_R)\,, \qquad \tilde{h}(\xi, x, t) = h^{-1}(\xi, x, t)\,(1-\psi_R(\xi))\,, \tag{2.61}$$

where ψ_R is defined in (2.13). Then $\Psi_{\tilde{h}} : H^{-3}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and $\Psi_{\tilde{h}}\Psi_{h_R} = I + \Psi_{r_3}$, where

$$\begin{aligned}\Psi_{r_3} &= \Psi_{\tilde{h}}\Psi_{h_R} - I \\ &= \frac{1}{\hat{\tau}_1\hat{\tau}_4}(-\Delta_x)^{-3/2}(1-\psi_R(D))\left(\hat{\tau}_1\hat{\tau}_4(-\Delta_x)^{3/2}(1-\psi_R(D))\right) - I \\ &= ((1-\psi(D))^2 - I) + \frac{1}{\hat{\tau}_1\hat{\tau}_4}[(-\Delta_x)^{-3/2}(1-\psi_R(D)),\ \hat{\tau}_1\hat{\tau}_4(-\Delta_x)^{3/2}(1-\psi_R(D))] \\ &= (\psi_R^2(D) - 2\psi_R(D)) + \frac{1}{\hat{\tau}_1\hat{\tau}_4}[(-\Delta_x)^{-3/2}(1-\psi_R(D)),\ \hat{\tau}_1\hat{\tau}_4(-\Delta_x)^{3/2}(1-\psi_R(D))] \\ &\in OPS^{-1}\,.\end{aligned} \tag{2.62}$$

Here we do not distinguish the identity operator I and the multiplicative identity 1.

Define the operators

$$T_{12} = \tfrac{i}{2}\,\Psi_{B_{12}}\Psi_{\tilde{h}}\,, \qquad T_{21} = -\tfrac{i}{2}\,\Psi_{B_{21}}\Psi_{\tilde{h}}\,, \qquad T = \begin{pmatrix}0 & T_{12} \\ T_{21} & 0\end{pmatrix}, \tag{2.63}$$

and the diagonalizing transformation Λ

$$\Lambda = I - T\,. \tag{2.64}$$

It can be shown by classical estimates for pseudo-differential operators that the norms $\|T\|_{H^s\to H^s}$, $\|T\|_{H^s_{\langle x\rangle^2}\to H^s_{\langle x\rangle^2}}$, and $\|T\|_{H^s_{\langle x\rangle^{-2}}\to H^s_{\langle x\rangle^{-2}}}$ are of size R^{-1} for any $s \in \mathbb{R}$. We provide the details of the proof in the appendix. By taking $R > 0$ large we can then assume that Λ is invertible on H^s, $H^s_{\langle x\rangle^2}$, and $H^s_{\langle x\rangle^{-2}}$ with its operator norms bounded between 1/2 and 2. Thus the inverse of Λ also has its operator norms on these spaces bounded between 1/2 and 2. In the following calculation we only need the case where $s = 0$.

In order to diagonalize Ψ_B, we apply the transformation Λ to system (2.60) and study the resulting terms. First

$$\epsilon\Lambda\Delta_x^2 + \epsilon\Lambda\Psi_{R_2} = \epsilon\Delta_x^2\Lambda + \epsilon(\Lambda\Delta_x^2 - \Delta_x^2\Lambda)\Lambda^{-1}\Lambda + \epsilon(\Lambda\Psi_{R_2}\Lambda^{-1})\Lambda = \epsilon\Delta_x^2\Lambda + \epsilon\Psi_{R_3}\Lambda\,,$$

where

$$\Psi_{R_3} = [\Lambda,\, \Delta_x^2]\Lambda^{-1} + \Lambda\Psi_{R_2}\Lambda^{-1} = [\Delta_x^2,\, T]\Lambda^{-1} + \Lambda\Psi_{R_2}\Lambda^{-1} \in OPS^3\,.$$

Second,

$$\Lambda\partial_t\vec{\beta} = \partial_t(\Lambda\vec{\beta}) + [\Lambda, \partial_t]\Lambda^{-1}\Lambda\vec{\beta}\,,$$

where the norm $\|[\Lambda, \partial_t]\Lambda^{-1}\|_{L^2\to L^2} = \|[\partial_t,\, T]\Lambda^{-1}\|_{L^2\to L^2}$ depends on $\|(\hat{\rho}-\bar{\rho},\, \hat{\theta}-\bar{\theta},\, \hat{u})\|_{W^{2,\infty}}$ and $\|\partial_t(\hat{\rho}-\bar{\rho},\, \hat{\theta}-\bar{\theta},\, \hat{u})\|_{W^{2,\infty}}$.

Next we have

$$\Lambda\Psi_{B_{diag}} - \Psi_{B_{diag}}\Lambda = -T\Psi_{B_{diag}} + \Psi_{B_{diag}}T \in OPS^1 .$$

This gives

$$\Lambda\Psi_{B_{diag}} = \Psi_{B_{diag}}\Lambda + \Psi_{1,1}\Lambda ,$$

with $\Psi_{1,1} = (\Lambda\Psi_{B_{diag}} - \Psi_{B_{diag}}\Lambda)\Lambda^{-1} \in OPS^1$.

Similarly,

$$\Lambda\Psi_{B_{anti}} = \Psi_{B_{anti}}\Lambda + \Psi_{1,2}\Lambda , \qquad \Lambda\left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x = \left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x\Lambda + \Psi_{1,3}\Lambda ,$$
$$\Lambda\Psi_1\vec{\beta} = \Psi_{1,4}\Lambda\vec{\beta} , \qquad \Lambda\Psi_1(\rho, u_P) = \Psi_{1,5}(\rho, u_P) , \qquad \Lambda\Psi_2 = \Psi_2 + \Psi_{1,6} ,$$
$$\Lambda\Psi_0 u_R = \Psi_{0,1}u_R , \qquad \Lambda\Psi_0(f_2, f_3) = \Psi_{0,2}(f_2, f_3) .$$

For the term $\Lambda\Psi_L - \Psi_L\Lambda$, we have

$$\begin{aligned}\Lambda\Psi_L - \Psi_L\Lambda &= \Psi_L T - T\Psi_L \\ &= i\begin{pmatrix}\Psi_{h_R} & 0 \\ 0 & -\Psi_{h_R}\end{pmatrix}\begin{pmatrix}0 & T_{12} \\ T_{21} & 0\end{pmatrix} - i\begin{pmatrix}0 & T_{12} \\ T_{21} & 0\end{pmatrix}\begin{pmatrix}\Psi_{h_R} & 0 \\ 0 & -\Psi_{h_R}\end{pmatrix} \\ &= i\begin{pmatrix}0 & \Psi_{h_R}T_{12} + T_{12}\Psi_{h_R} \\ -(\Psi_{h_R}T_{21} + T_{21}\Psi_{h_R}) & 0\end{pmatrix} ,\end{aligned}$$

where

$$\begin{aligned}\Psi_{h_R}T_{12} &= T_{12}\Psi_{h_R} + \Psi_{1,7} \quad \text{with} \quad \Psi_{1,7} = [\Psi_{h_R}, T_{12}] , \\ \Psi_{h_R}T_{21} &= T_{21}\Psi_{h_R} + \Psi_{1,8} \quad \text{with} \quad \Psi_{1,8} = [\Psi_{h_R}, T_{21}] .\end{aligned}$$

We have

$$\begin{aligned}i(\Psi_{h_R}T_{12} + T_{12}\Psi_{h_R}) &= 2iT_{12}\Psi_{h_R} + \Psi_{1,7} = -\Psi_{B_{12}} + \Psi_{1,9} , \\ -i(\Psi_{h_R}T_{21} + T_{21}\Psi_{h_R}) &= -2iT_{21}\Psi_h + \Psi_{1,8} = -\Psi_{B_{21}} + \Psi_{1,10} ,\end{aligned}$$

where

$$\begin{aligned}\Psi_{1,9} &= \Psi_{1,7} - \Psi_{B_{12}}\Psi_{r_3} , \\ \Psi_{1,10} &= \Psi_{1,8} - \Psi_{B_{21}}\Psi_{r_3} ,\end{aligned}$$

with Ψ_{r_3} being defined in (2.62).

Therefore,

$$\Lambda\Psi_L + \Lambda\Psi_{B_{anti}} = \Psi_L\Lambda + \Psi_{1,11} .$$

Let

$$\vec{z} = \Lambda\vec{\beta} . \tag{2.65}$$

Then the system for $\vec{z}$ has the form

$$\begin{aligned}\partial_t\vec{z} = &-\epsilon\,\Delta_x^2\vec{z} + \epsilon\Psi_{R_3}\vec{z} + \left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x\vec{z} + \Psi_L\vec{z} \\ &+ \Psi_{B_{diag}}\vec{z} + \Psi_2\rho + \Psi_2 Pu + \Psi_1(\rho, Pu, \vec{z}) + \Psi_0 u_R + \Psi_0(f_2, f_3) .\end{aligned} \tag{2.66}$$

Step 3. Regularized u_P and ρ. Note that we cannot close the energy estimate by directly using the ρ-equation. The reason is due to the presence of the term $\rho\nabla_x\cdot u$ in the ρ-equation. Roughly speaking, this suggests the regularity of ρ depends on the regularity of $\nabla_x\cdot u$ thus the regularity of $\nabla_x\vec{z}$. Therefore, the term $\Psi_2\rho$ in (2.66) behaves like $\Psi_3(\vec{z})$ which cannot be bounded by the energy method. Hence we cannot use the ρ-equation directly. Instead, we notice that $\rho\nabla_x\cdot u = \rho\nabla_x\cdot Qu$ while Qu is among the strictly dispersive part. Moreover, if we consider the whole system for $(\rho, u_P, u_Q, u_R, \theta)$, this term is off-diagonal just as $\Psi_{B_{anti}}$ in Step 3. Therefore we can once again apply the reduction to eliminate $\rho\nabla_x\cdot u$ (up to a zeroth-order term in $\vec{z}$).

The same situation happened for u_P. In the u_P-equation we have $\Psi_2\theta$ which can now be written as $\Psi_2\vec{z}$ while in the $\vec{z}$-equation there is $\Psi_2 u_P$. Even though they both have dissipative regularization, direct integration by parts will not able produce a closed estimate. We also apply the reduction to eliminate $\Psi_2\theta$ in the u_P-equation (up to a first-order term in θ or $\vec{z}$). Again this term is off-diagonal so that the reduction process can be carried out.

To this end, write θ in terms of $\vec{z} = (z_1, z_2)^T$. Recall that $\vec{z} = \Lambda A\vec{\omega}$ where Λ and A are both invertible. Solve $\vec{\omega}$ in terms of $\vec{z}$ to obtain

$$\vec{\omega} = A^{-1}\Lambda^{-1}\vec{z}\,.$$

In order to obtain explicit bounds, we write Λ^{-1} as

$$\Lambda^{-1} = (I-T)^{-1} = I + T + (I-T)^{-1}T^2 \stackrel{\triangle}{=} I + \tilde{\Lambda}\,,$$

where $\tilde{\Lambda} = \begin{pmatrix} \tilde{\Lambda}_{11} & \tilde{\Lambda}_{12} \\ \tilde{\Lambda}_{21} & \tilde{\Lambda}_{22} \end{pmatrix}$ is bounded from H^{-1} to L^2. Thus

$$\vec{\omega} = A^{-1}\vec{z} + A^{-1}\tilde{\Lambda}\vec{z} = \begin{pmatrix} \frac{1}{2}iz_1 - \frac{1}{2}iz_2 \\ \frac{1}{2}\sqrt{\hat{\tau}_1\hat{\tau}_4}z_1 - \frac{1}{2}\sqrt{\hat{\tau}_1\hat{\tau}_4}z_2 \end{pmatrix} + \Psi_1\vec{z}\,.$$

Hence,

$$\Psi_2\theta = \Psi_2\left(\frac{\sqrt{\tau_4/\tau_1}}{2}z_1\right) + \Psi_2\left(\frac{\sqrt{\tau_4/\tau_1}}{d}z_2\right) + \Psi_1\vec{z} \ \stackrel{\triangle}{=} \Psi_{\Gamma_1}z_1 + \Psi_{\Gamma_2}z_2 + \Psi_1\vec{z}\,.$$

Now define

$$T_1 = i\Psi_{\Gamma_1}\Psi_{\tilde{h}}\,, \qquad T_2 = -i\Psi_{\Gamma_2}\Psi_{\tilde{h}}\,.$$

Furthermore by (2.61) and (2.62),

$$\begin{aligned} T_1\Psi_{ih_R} + \Psi_{\Gamma_1} &= -\Psi_{\Gamma_1}\Psi_{\tilde{h}}\Psi_{h_R} + \Psi_{\Gamma_1} = -\Psi_{\Gamma_1}\Psi_{r_3}\,, \\ -T_2\Psi_{ih_R} + \Psi_{\Gamma_2} &= -\Psi_{\Gamma_2}\Psi_{\tilde{h}}\Psi_{h_R} + \Psi_{\Gamma_2} = -\Psi_{\Gamma_2}\Psi_{r_3}\,, \end{aligned}$$

Upon applying T_1 and T_2 to the equations of system (2.66) respectively, we find that T_1z_1 and T_2z_2 obey

$$\begin{aligned} \partial_t(T_1z_1) &= -\epsilon\,\Delta_x^2(T_1z_1) + \epsilon\,\Psi_{R_4}z_1 + T_1\Psi_{ih_R}z_1 + \Psi_1(\rho, u_P, \vec{z}) + \Psi_0u_R + \Psi_0(f_2, f_3)\,, \\ \partial_t(T_2z_2) &= -\epsilon\,\Delta_x^2(T_2z_2) + \epsilon\,\Psi_{R_5}z_2 - T_2\Psi_{ih}z_2 + \Psi_1(\rho, u_P, \vec{z}) + \Psi_0u_R + \Psi_0(f_2, f_3)\,, \end{aligned} \tag{2.67}$$

where

$$R_4 = [\Delta_x^2,\, T_1]\,, \qquad R_5 = [\Delta_x^2,\, T_2]\,.$$

Add the equations in (2.67) to the equation for u_P in system (2.38) and define

$$\vec{y} = u_P + T_1z_1 + T_2z_2\,. \tag{2.68}$$

Then the equation for $\vec{y}$ has the form

$$\partial_t\vec{y} = -\epsilon\,\Delta_x^2\vec{y} + \epsilon\,\Psi_{R_4}z_1 + \epsilon\,\Psi_{R_5}z_2 + \hat{\mu}\Delta_x\vec{y} + \Psi_2\rho + \Psi_1(\rho, \vec{y}, \vec{z}) + \Psi_0u_R + \Psi_0(f_2, f_3)\,. \tag{2.69}$$

Similarly, to eliminate the term $\hat{\rho}\nabla_x\cdot u$ in the mass equation, write

$$\hat{\rho}\nabla_x\cdot u = \hat{\rho}\nabla_x\cdot u_Q + \Psi_0u_R \stackrel{\triangle}{=} \Psi_{\Gamma_3}z_1 + \Psi_{\Gamma_4}z_2 + \Psi_0u_R\,.$$

Define the operators

$$T_3 = i\Psi_{\Gamma_3}\Psi_{\tilde{h}}\,, \qquad T_4 = -i\Psi_{\Gamma_4}\Psi_{\tilde{h}}\,.$$

Let

$$\varrho = \rho + T_3z_1 + T_4z_2\,. \tag{2.70}$$

Then ϱ satisfies

$$\partial_t \varrho = -\epsilon\, \Delta_x^2 \varrho + \epsilon\, \Psi_{R_6} z_1 + \epsilon\, \Psi_{R_7} z_2 - \hat{u} \cdot \nabla_x \varrho + \Psi_0(\varrho,\, \vec{y},\, \vec{z},\, u_R) + \Psi_0 f_1 + \Psi_{-1}(f_2, f_3)\,, \tag{2.71}$$

where

$$R_6 = [\Delta_x^2,\, T_3]\,, \qquad R_7 = [\Delta_x^2,\, T_4]\,.$$

Step 4. A further transformation. Following [13] we need a further transformation. Before defining this transformation, we prove the following lemma which extends Lemma 2.1 to the time-dependent case.

Lemma 2.2. *There exists $T^* > 0$, depending only on the constants in $\mathcal{A}_1$, $\mathcal{A}_3$, and Lemma 2.1 such that for every $t \in [0, T^*)$ one has*

$$H_{h_R} p = \{h_R, p\}\,(\xi, x, t) \geq \frac{c_1}{2} \frac{|\xi|^2}{\langle x \rangle^2} - \tilde{c}_2\,, \qquad \forall (\xi, x) \in \mathbb{R}^d \times \mathbb{R}^d\,, \quad |\xi| \geq R\,,$$

where h_R is defined in (2.61), $p(x, \xi)$ is defined in Lemma 2.1, and $\tilde{c}_2$ depends only on c_A and c_0.

Proof. By definition,

$$H_{h_R} p = \sum_{j=1}^d \left(\partial_{\xi_j} h\, \partial_{x_j} p - \partial_{x_j} h\, \partial_{\xi_j} p \right)\, (1 - \psi_R) + r\,,$$

$$H_{h_R^{in}} p = \sum_{j=1}^d \left(\partial_{\xi_j} h^{in}\, \partial_{x_j} p - \partial_{x_j} h^{in}\, \partial_{\xi_j} p \right)\, (1 - \psi_R) + r^{in}\,,$$

where

$$r = \sum_{j=1}^d h\, \partial_{\xi_j} (1 - \psi_R)\, \partial_{x_j} p\,, \qquad r^{in} = \sum_{j=1}^d h^{in}\, \partial_{\xi_j} (1 - \psi_R)\, \partial_{x_j} p\,.$$

Thus r, r^{in} are compactly supported. For each $|\xi| > R$ we have

$$\partial_{\xi_j} h = 3 \sqrt{\hat{\tau}_1 \hat{\tau}_4} |\xi| \xi_j\,, \qquad \partial_{x_j} h = \partial_{x_j} \left(\sqrt{\hat{\tau}_1 \hat{\tau}_4} \right) |\xi|^3\,.$$

Thus, it follows from assumption $\mathcal{A}_1$ and $\mathcal{A}_2$ that there exist $c_{0,4}, c_{0,5} > 0$ such that

$$|\partial_{\xi_j} h(\xi, x, t) - \partial_{\xi_j} h(\xi, x, 0)| \leq \frac{c_{0,4} T^*}{\langle x \rangle^2} |\xi|^2\,,$$

$$|\partial_{x_j} h(\xi, x, t) - \partial_{x_j} h(\xi, x, 0)| \leq \frac{c_{0,5} T^*}{\langle x \rangle^2} |\xi|^3\,.$$

where $c_{0,4}$ and $c_{0,5}$ depend only on the initial data. Therefore, we can find $c_{0,6}, c_{0,7} > 0$ such that

$$\begin{aligned} \left| H_{h_R} p - H_{h_R^{in}} p \right| &\leq \sum_{j=1}^d \left| \partial_{\xi_j} h - \partial_{\xi_j} h^{in} \right| \, \left| \partial_{x_j} p \right| + \sum_{j=1}^d \left| \partial_{x_j} h - \partial_{x_j} h^{in} \right| \, \left| \partial_{\xi_j} p \right| + |r| + |r^{in}| \\ &\leq \frac{c_{0,6} T^*}{\langle x \rangle^2} |\xi|^2 + c_{0,7}\,. \end{aligned}$$

Again $c_{0,6}$ and $c_{0,7}$ depend only on the initial data. Choosing T^* small enough and applying Lemma 2.1, we have

$$H_{h_R}p \geq H_{h_R^{in}}p - \left| H_{h_R}p - H_{h_R^{in}}p \right| \geq (c_1 - c_{0,6}T^*)\,\frac{|\xi|^2}{\langle x\rangle^2} - c_2 - c_{0,7} \geq \frac{c_1}{2}\frac{|\xi|^2}{\langle x\rangle^2} - \tilde{c}_2$$

where $\tilde{c}_2 = c_2 + c_{0,7}$ depends only on the initial data and c_0. □

To construct a further transformation, let

$$q_1(x,\xi) = \exp\big(Mp(x,\xi)(1-\psi_R)(\xi)\big)\,, \qquad q_2(x,\xi) = \exp\big(-Mp(x,\xi)(1-\psi_R)(\xi)\big)\,,$$

where $M > 0$ is to be chosen.

Remark 2.9. Note that since $p(x,\xi) \in S^0$ by Lemma 2.1, we also have $q_1, q_2 \in S^0$ and their (α,β)-seminorms are bounded by seminorms of $p(x,\xi)$ up to order (α,β). By Lemma 2.1 this requires that the initial data $(\hat{\rho}(x,0), \hat{\theta}(x,0)) \in C_b^{|\alpha|+1}$.

Let

$$\Psi_{q_1}\Psi_{q_2} = I + \Psi_{r_4}\,, \qquad \Psi_{q_2}\Psi_{q_1} = I + \Psi_{r_5}\,,$$

where

$$\Psi_{r_4} = \Psi_{q_1}\Psi_{q_2} - \Psi_{q_1 q_2}$$
$$\Psi_{r_5} = \Psi_{q_2}\Psi_{q_1} - \Psi_{q_2 q_1}\,.$$

Since $q_1, q_2 \in S^0$, we have $r_4, r_5 \in S^{-1}$. Meanwhile, since q_1, q_2 are supported away from the origin, we have $\Psi_{r_4} = \Psi_{r_4}\Psi_{1-\psi_R(2\xi)}$. Therefore $\|\Psi_{r_4}\|_{L^2\to L^2}$ is of size R^{-1}. Thus for R large Ψ_{q_1} and Ψ_{q_2} are invertible on L^2.

Now we have

$$\Psi_{h_R}\Psi_{q_1} - \Psi_{q_1}\Psi_{h_R} = \Psi_{-i\{h_R,q_1\}} + \Psi_{1,1}\,,$$

where h_R is defined in (2.61). The Poisson bracket has the form

$$\begin{aligned}\{h_R, q_1\} &= \sum_{j=1}^{d}\big(\partial_{\xi_j}h_R\,\partial_{x_j}q_1 - \partial_{x_j}h_R\,\partial_{\xi_j}q_1\big)\\ &= \sum_{j=1}^{d}\Big(\partial_{\xi_j}h_R\,\partial_{x_j}p - \partial_{x_j}h_R\,\partial_{\xi_j}p\Big)\,q_1\,M - \sum_{j=1}^{d}\big(\partial_{x_j}h_R\,\partial_{\xi_j}(1-\psi_R)\big)\,(q_1 M p)\,.\end{aligned}$$

Hence,

$$\Psi_{\{h_R,q_1\}} = \Psi_{MH_{h_R}p}\,\Psi_{q_1} + \Psi_{0,1}\,.$$

Therefore

$$\Psi_{ih_R}\Psi_{q_1} - \Psi_{q_1}\Psi_{ih_R} = \Psi_{MH_{h_R}p}\Psi_{q_1} + \Psi_{1,2}\,,$$

where $\Psi_{1,2} = \Psi_{1,1} + i\Psi_{0,1}$.

A similar computation shows that

$$\Psi_{ih_R}\Psi_{q_2} - \Psi_{q_2}\Psi_{ih_R} = -\Psi_{MH_{h_R}p}\Psi_{q_2} + \Psi_{1,3}\,.$$

Consider a final change of variable $\vec{z} \to \vec{\alpha}$ where

$$\vec{\alpha} = \begin{pmatrix}\Psi_{q_1} & 0\\ 0 & \Psi_{q_2}\end{pmatrix}\vec{z} \triangleq \Psi\vec{z}\,, \tag{2.72}$$

where $\vec{z}$ is defined in (2.65). Recall that Ψ is invertible and $\Psi^{-1} = \begin{pmatrix}\Psi_{q_1}^{-1} & 0\\ 0 & \Psi_{q_2}^{-1}\end{pmatrix}$ which is bounded on L^2.

To compute the system for $\vec{\alpha}$, apply Ψ to system (2.66). Then

$$\partial_t \vec{\alpha} = \Psi \partial_t \vec{z} = -\epsilon\, \Psi \Delta_x^2 \vec{z} + \epsilon\, \Psi \Psi_{R_3} \vec{z} + \Psi \left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right) \Delta_x \vec{z} + \Psi \Psi_L \vec{z} + \Psi \Psi_{B_{diag}} \vec{z} + \Psi \Psi_2(\varrho, \vec{y}) + \Psi \Psi_1(\varrho, \vec{y}, \vec{z}) + \Psi \Psi_0(\varrho, \vec{y}, \vec{z}, u_R) + \Psi \Psi_0(f_2, f_3)\,.$$

Evaluate each term on the right as follows. First,

$$-\epsilon \Psi \Delta_x^2 \vec{z} + \epsilon \Psi \Psi_{R_3} \vec{z} = -\epsilon \Delta_x^2 \vec{\alpha} - \epsilon([\Psi,\, \Delta_x^2]\Psi^{-1} - \Psi \Psi_{R_3} \Psi^{-1})\, \vec{\alpha} \stackrel{\triangle}{=} -\epsilon \Delta_x^2 \vec{\alpha} + \Psi_{R_8} \vec{\alpha}\,. \tag{2.73}$$

Second,

$$\Psi \left(\tfrac{d-1}{d}\hat{\mu} + \hat{\kappa}\right) \Delta_x \vec{z} = \left(\tfrac{d-1}{d}\hat{\mu} + \hat{\kappa}\right) \Delta_x \vec{\alpha} + [\Psi,\, \left(\tfrac{d-1}{d}\hat{\mu} + \hat{\kappa}\right) \Delta_x] \Psi^{-1} \vec{\alpha}\,,$$
$$\stackrel{\triangle}{=} \left(\tfrac{d-1}{d}\hat{\mu} + \hat{\kappa}\right) \Delta_x \vec{\alpha} + \Psi_{1,5}\, \vec{\alpha}\,.$$

Similarly,

$$\Psi \Psi_{B_{diag}} \vec{z} = \Psi_{B_{diag}} \vec{\alpha} + [\Psi,\, \Psi_{B_{diag}}] \Psi^{-1} \vec{\alpha} \stackrel{\triangle}{=} \Psi_{B_{diag}} \vec{\alpha} + \Psi_{1,6} \vec{\alpha}\,.$$

Next,

$$\Psi \Psi_2(\varrho, \vec{y}) = \Psi_2(\varrho, \vec{y})\,, \qquad \Psi \Psi_1(\varrho, \vec{y}) = \Psi_{1,7}(\varrho, \vec{y})\,, \qquad \Psi \Psi_1 \vec{z} = \Psi \Psi_1 \Psi^{-1} \vec{\alpha} = \Psi_{1,8} \vec{\alpha}\,,$$
$$\Psi \Psi_0 \vec{z} = \Psi \Psi_0 \Psi^{-1} \vec{\alpha} = \Psi_{0,1} \vec{\alpha}\,, \qquad \Psi \Psi_0 u_R = \Psi_{0,2} u_R\,, \qquad \Psi \Psi_0(f_2, f_3) = \Psi_{0,2}(f_2, f_3)\,.$$

For the dispersive part,

$$\Psi \Psi_L \vec{z} = \Psi_L \vec{\alpha} + \begin{pmatrix} \Psi_{q_1} \Psi_{ih_R} - \Psi_{ih_R} \Psi_{q_1} & 0 \\ 0 & -(\Psi_{q_2} \Psi_{ih_R} - \Psi_{ih_R} \Psi_{q_2}) \end{pmatrix} \vec{z}$$
$$= \Psi_L \vec{\alpha} + \begin{pmatrix} -\Psi_{MH_{h_R} p} & 0 \\ 0 & -\Psi_{MH_{h_R} p} \end{pmatrix} \vec{\alpha} + \Psi_{1,9} \vec{\alpha}\,,$$

where $\Psi_{1,9}$ is given by $\Psi_{1,2}$, $\Psi_{1,3}$, and Ψ^{-1}.

Overall the system for $\vec{\alpha}$ has the form

$$\begin{aligned} \partial_t \vec{\alpha} = -\epsilon\, \Delta_x^2 \vec{\alpha} + \epsilon\, \Psi_{R_8} \vec{\alpha} + \left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right) \Delta_x \vec{\alpha} + \Psi_L \vec{\alpha} + \begin{pmatrix} -\Psi_{MH_{h_R} p} & 0 \\ 0 & -\Psi_{MH_{h_R} p} \end{pmatrix} \vec{\alpha} \\ + \Psi_{B_{diag}} \vec{\alpha} + \Psi_2(\varrho, \vec{y}) + \Psi_1(\varrho, \vec{\alpha}, \vec{y}) + \Psi_0 u_R + \Psi_0(f_2, f_3)\,. \end{aligned} \tag{2.74}$$

Step 5. Energy estimate. Now we can derive the energy estimate for $(\varrho, \vec{\alpha}, \vec{y}, u_R)$. To this end, we multiply (2.74) by $\vec{\alpha}$ and integrate over $\mathbb{R}^d$ to obtain

$$\begin{aligned} \frac{d}{dt} \langle \vec{\alpha}, \vec{\alpha} \rangle &= \langle \partial_t \vec{\alpha}\,,\, \vec{\alpha} \rangle + \langle \vec{\alpha}\,,\, \partial_t \vec{\alpha} \rangle \\ &= -\epsilon\, \langle \Delta_x^2 \vec{\alpha}\,,\, \vec{\alpha} \rangle - \epsilon\, \langle \vec{\alpha}\,,\, \Delta_x^2 \vec{\alpha} \rangle + \epsilon\, \langle \Psi_{R_8} \vec{\alpha}\,,\, \vec{\alpha} \rangle + \epsilon\, \langle \vec{\alpha}\,,\, \Psi_{R_8} \vec{\alpha} \rangle \\ &\quad + \left\langle \left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right) \Delta_x \vec{\alpha}\,,\, \vec{\alpha} \right\rangle + \left\langle \vec{\alpha}\,,\, \left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right) \Delta_x \vec{\alpha} \right\rangle \\ &\quad + \langle \Psi_L \vec{\alpha}\,,\, \vec{\alpha} \rangle + \langle \vec{\alpha}\,,\, \Psi_L \vec{\alpha} \rangle \\ &\quad + \left\langle \begin{pmatrix} -\Psi_{MH_{h_R} p} & 0 \\ 0 & -\Psi_{MH_{h_R} p} \end{pmatrix} \vec{\alpha}\,,\, \vec{\alpha} \right\rangle + \left\langle \vec{\alpha}\,,\, \begin{pmatrix} -\Psi_{MH_{h_R} p} & 0 \\ 0 & -\Psi_{MH_{h_R} p} \end{pmatrix} \vec{\alpha} \right\rangle \\ &\quad + \langle \Psi_{B_{diag}} \vec{\alpha}\,,\, \vec{\alpha} \rangle + \langle \vec{\alpha}\,,\, \Psi_{B_{diag}} \vec{\alpha} \rangle + \langle \Psi_2 \vec{y}\,,\, \vec{\alpha} \rangle + \langle \vec{\alpha}\,,\, \Psi_2 \vec{y} \rangle \\ &\quad + \langle \Psi_2 \varrho\,,\, \vec{\alpha} \rangle + \langle \vec{\alpha}\,,\, \Psi_2 \varrho \rangle + \langle \Psi_1 \vec{\alpha}\,,\, \vec{\alpha} \rangle + \langle \vec{\alpha}\,,\, \Psi_1 \vec{\alpha} \rangle \\ &\quad + \langle \Psi_1 \vec{y}\,,\, \vec{\alpha} \rangle + \langle \vec{\alpha}\,,\, \Psi_1 \vec{y} \rangle + \langle \Psi_1 \varrho\,,\, \vec{\alpha} \rangle + \langle \vec{\alpha}\,,\, \Psi_1 \varrho \rangle \\ &\quad + \langle \Psi_0 u_R\,,\, \vec{\alpha} \rangle + \langle \vec{\alpha}\,,\, \Psi_0 u_R \rangle + \langle \vec{\alpha}\,,\, \Psi_0(f_2, f_3) \rangle\,. \end{aligned}$$

We estimate each term above. For the terms containing ϵ,

$$-\epsilon \left\langle \Delta_x^2 \vec{\alpha}\,,\, \vec{\alpha} \right\rangle - \epsilon \left\langle \vec{\alpha}\,,\, \Delta_x^2 \vec{\alpha} \right\rangle = -2\epsilon \left\| \Delta_x \vec{\alpha} \right\|_{L^2}^2 .$$

By (2.73), there exist $c_{A,1}, c_{A,2} > 0$ which depend on c_A such that

$$\epsilon \left| \left\langle \Psi_{R_8} \vec{\alpha}\,,\, \vec{\alpha} \right\rangle \right| + \epsilon \left| \left\langle \vec{\alpha}\,,\, \Psi_{R_8} \vec{\alpha} \right\rangle \right| \le \epsilon\, c_{A,1} \|\vec{\alpha}\|_{H^{\frac{3}{2}}}^2 \le 2\epsilon \left\| \Delta_x \vec{\alpha} \right\|_{L^2}^2 + \epsilon\, c_{A,2} \|\vec{\alpha}\|_{L^2}^2 .$$

For the dissipative term,

$$\left\langle \left(\tfrac{d-1}{d} \hat{\mu} + \tfrac{1}{2} \hat{\kappa} \right) \Delta_x \vec{\alpha}\,,\, \vec{\alpha} \right\rangle + \left\langle \vec{\alpha}\,,\, \left(\tfrac{d-1}{d} \hat{\mu} + \tfrac{1}{2} \hat{\kappa} \right) \Delta_x \vec{\alpha} \right\rangle \le -c_{0,8} \|\nabla_x \vec{\alpha}\|_{L^2(\mathbb{R}^2)}^2 + c_{A,3} \|\vec{\alpha}\|_{L^2}^2 ,$$

where $c_{0,8} > 0$ depends only on c_0.

Next we have

$$\left\langle \Psi_L \vec{\alpha}\,,\, \vec{\alpha} \right\rangle + \left\langle \vec{\alpha}\,,\, \Psi_L \vec{\alpha} \right\rangle = \left\langle (\Psi_L + \Psi_L^*) \vec{\alpha}\,,\, \vec{\alpha} \right\rangle ,$$

where Ψ_L^* is the adjoint operator of Ψ_L. Recall that

$$\Psi_L = \begin{pmatrix} \Psi_{ih_R} & 0 \\ 0 & -\Psi_{ih_R} \end{pmatrix} , \qquad h_R = \sqrt{\hat{\tau}_1 \hat{\tau}_4} |\xi|^3 (1 - \psi_R) .$$

Then the adjoint operator Ψ_L^* is

$$\Psi_L^* = \begin{pmatrix} \Psi_{ih_R}^* & 0 \\ 0 & -\Psi_{ih_R}^* \end{pmatrix} , \qquad \Psi_{ih_R}^* = -\Psi_{i|\xi|^3(1-\psi_R(\xi))} \big(\sqrt{\hat{\tau}_1 \hat{\tau}_4} \cdot \big) .$$

Thus

$$\Psi_L + \Psi_L^* = \begin{pmatrix} \Psi_{ih_R} + \Psi_{ih_R}^* & 0 \\ 0 & -(\Psi_{ih_R} + \Psi_{ih_R}^*) \end{pmatrix} ,$$

where

$$\Psi_{ih_R} + \Psi_{ih_R}^* = \big[\sqrt{\hat{\tau}_1 \hat{\tau}_4},\ \Psi_{i|\xi|^3(1-\psi_R(\xi))} \big] = -\sum_{|\alpha|=1} \partial_\xi^\alpha \partial_x^\alpha\, h_R + l_r , \tag{2.75}$$

where $\Psi_{l_r} \in OPS^1$. A typical term in the summation of (2.75) where $|\alpha| = 1$ is

$$-\partial_{\xi_j} \partial_{x_j} i h_R = -\partial_{\xi_j} \partial_{x_j} \big(\sqrt{\hat{\tau}_1 \hat{\tau}_4} |\xi|^3 (1 - \psi_R) \big) = -3\, \partial_{x_j} \big(\sqrt{\hat{\tau}_1 \hat{\tau}_4} \big)\, \xi_j |\xi| (1 - \psi_R) + \Psi_0 .$$

Therefore,

$$\Psi_L + \Psi_L^* = \hat{B}_{diag} + \Psi_1 ,$$

where there exists $0 < T_4 \le T_3$ and $c_{0,9} > 0$ such that

$$\hat{B}_{diag} = \begin{pmatrix} \hat{B}_{11} & 0 \\ 0 & \hat{B}_{22} \end{pmatrix} , \quad |\hat{B}_{kk}(x, \xi, t)| \le \frac{c_{0,9}}{\langle x \rangle^2} |\xi|^2 , \quad \forall (x, \xi, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T_4] , \quad k = 1, 2 . \tag{2.76}$$

Since $\hat{B}_{diag}$ is real, we can combine $\frac{1}{2} \hat{B}_{diag}$ with B_{diag} and still denote it $B_{diag} = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}$.
By (2.59) and (2.76) we have

$$|B_{kk}| \le \frac{c_{0,10} |\xi|^2}{\langle x \rangle^2} \qquad \text{for } k = 1, 2.$$

Therefore we can choose M large enough such that

$$-M H_{h_R} p + |B_{kk}| \le c_{0,11} - \tfrac{1}{2} c_{0,12} \frac{|\xi|^2}{\langle x \rangle^2} , \qquad |\xi| \ge 2R ,$$

where $c_{0,11}, c_{0,12}$ depend only on the initial data and c_0. The choice of M depends only on the initial data. Let $c' = c_{0,12}$. By the *sharp Gårding* inequality,

$$\left\langle \begin{pmatrix} -\Psi_{MH_{h_R}p} + B_{11} & 0 \\ 0 & -\Psi_{MH_{h_R}p} + B_{22} \end{pmatrix} \vec{\alpha}\,,\, \vec{\alpha} \right\rangle + \left\langle \vec{\alpha}\,,\, \begin{pmatrix} -\Psi_{MH_{h_R}p} + B_{11} & 0 \\ 0 & -\Psi_{MH_{h_R}p} + B_{22} \end{pmatrix} \vec{\alpha} \right\rangle$$
$$\leq c_{A,4} \|\vec{\alpha}\|^2_{H^{\frac{1}{2}}} - \mathrm{Re} \left\langle \begin{pmatrix} \Psi_{c'|\xi|^2/\langle x\rangle^2} & 0 \\ 0 & \Psi_{c'|\xi|^2/\langle x\rangle^2} \end{pmatrix} \vec{\alpha}\,,\, \vec{\alpha} \right\rangle$$
$$\leq \eta \|\vec{\alpha}\|^2_{H^1} + c_{A,\eta} \|\vec{\alpha}\|^2_{L^2} - \mathrm{Re} \left\langle \begin{pmatrix} \Psi_{c'|\xi|^2/\langle x\rangle^2} & 0 \\ 0 & \Psi_{c'|\xi|^2/\langle x\rangle^2} \end{pmatrix} \vec{\alpha}\,,\, \vec{\alpha} \right\rangle ,$$

For the operator $\Psi_{c'|\xi|^2/\langle x\rangle^2}$, because

$$\Psi_{c'|\xi|^2/\langle x\rangle^2} = \frac{1}{\langle x\rangle^2} \Psi_{c'|\xi|^2} + \Psi_1\,, \qquad \Psi_{c'|\xi|^2} = -c' \Delta_x\,,$$

and

$$-\frac{1}{\langle x\rangle^2} \Delta_x = -\nabla_x \cdot \left(\frac{1}{\langle x\rangle^2} \nabla_x \right) + \Psi_1\,,$$

we have

$$\mathrm{Re} \left\langle \begin{pmatrix} \Psi_{c'|\xi|^2/\langle x\rangle^2} & 0 \\ 0 & \Psi_{c'|\xi|^2/\langle x\rangle^2} \end{pmatrix} \vec{\alpha}\,,\, \vec{\alpha} \right\rangle \geq c' \int_{\mathbb{R}^d} \frac{1}{\langle x\rangle^2} |\nabla_x \vec{\alpha}|^2 \,\mathrm{d}x \,-\, \left(\eta \,\|\nabla_x \vec{\alpha}\|^2_{L^2} + c_\eta \,\|\vec{\alpha}\|^2_{L^2} \right) .$$

For the first order terms $\Psi_1 (\vec{\alpha}, \vec{y}, \varrho)$, let $\eta > 0$ be small enough. Then

$$\begin{aligned} |\langle \Psi_1 \vec{\alpha}\,,\, \vec{\alpha} \rangle| + |\langle \vec{\alpha}\,,\, \Psi_1 \vec{\alpha} \rangle| &\leq \eta \,\|\nabla_x \vec{\alpha}\|^2_{L^2} + c_{A,\eta} \,\|\vec{\alpha}\|^2_{L^2}\,, \\ |\langle \Psi_1 \vec{y}\,,\, \vec{\alpha} \rangle| + |\langle \vec{\alpha}\,,\, \Psi_1 \vec{y} \rangle| &\leq \eta \,\|\nabla_x \vec{y}\|^2_{L^2} + c_{A,\eta} \,\|\vec{\alpha}\|^2_{L^2}\,, \\ |\langle \Psi_1 \varrho\,,\, \vec{\alpha} \rangle| + |\langle \vec{\alpha}\,,\, \Psi_1 \varrho \rangle| &\leq \eta \,\|\nabla_x \vec{\alpha}\|^2_{L^2} + c_{A,\eta} \,\|\varrho\|^2_{L^2}\,, \\ |\langle \Psi_0(f_2, f_3)\,,\, \vec{\alpha} \rangle| + |\langle \vec{\alpha}\,,\, \Psi_0(f_2, f_3) \rangle| &\leq c_{A,15} \|\vec{\alpha}\|^2_{L^2} + \|(f_2, f_3)\|^2_{L^2}\,, \end{aligned}$$

with $c_{A,\eta}$ depending on η and the bounds c_A.

For the term $\Psi_2 \vec{y}$, there exists $0 < T_5 \leq T_4$ such that the leading symbol of the operator Ψ_2 in $\Psi_2 \vec{y}$, denoted as F_1, satisfies

$$|F_1(x, \xi, t)| \leq \frac{c_{0,13}}{\langle x\rangle^2} |\xi|^2\,, \qquad \text{for every } (x, \xi, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T_5]\,,$$

because F_1 is a linear combination of $\nabla_x(\hat{\rho}, \hat{u}, \hat{\theta})$. Therefore we obtain the bound

$$|\langle \Psi_{F_1} \vec{y}\,,\, \vec{\alpha} \rangle| + |\langle \vec{\alpha}\,,\, \Psi_{F_1} \vec{y} \rangle| \leq \eta \,\|\nabla_x \vec{y}\|^2_{L^2} + c_{0,\eta} \int_{\mathbb{R}^d} \frac{1}{\langle x\rangle^2} |\nabla_x \vec{\alpha}|^2 \,\mathrm{d}x\,,$$

where $c_{0,\eta}$ depends on η and the data.

For the $\Psi_2 \varrho$ term, we need to estimate the H^1 norm of ϱ. First, the L^2 estimate of ϱ shows that

$$\tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\varrho\|^2_{L^2} \leq c_{A,5} \|\varrho\|^2_{L^2} + \|\vec{\alpha}\|^2_{L^2} + \|\vec{y}\|^2_{L^2} + \|u_R\|^2_{L^2} + \|(f_1, f_2, f_3)\|^2_{L^2}\,. \tag{2.77}$$

Differentiating the equation (2.71) for ϱ with respect to x_l with $1 \le l \le d$, multiplying by $\partial_l \varrho$, and integrating over $\mathbb{R}^d$ we obtain the following equality

$$\begin{aligned}&\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_l \varrho\|_{L^2}^2 - \tfrac{1}{2}\int_{\mathbb{R}^d} |\partial_l \varrho|^2 \nabla_x \cdot u \,\mathrm{d}x + \int_{\mathbb{R}^d} \partial_l \varrho \nabla_x \varrho \cdot \partial_l u \,\mathrm{d}x \\ &= -\epsilon \left\langle \Delta_x^2(\partial_l \varrho)\,,\, \partial_l \varrho \right\rangle + \left\langle \epsilon\, \partial_l \Psi_{R_6} z_1\,,\, \partial_l \varrho \right\rangle + \left\langle \epsilon\, \partial_l \Psi_{R_7} z_2\,,\, \partial_l \varrho \right\rangle + \left\langle \Psi_1 \varrho\,,\, \partial_l \varrho \right\rangle + \left\langle \Psi_1 \vec{y}\,,\, \partial_l \varrho \right\rangle \\ &\quad + \left\langle \Psi_1 \vec{z}\,,\, \partial_l \varrho \right\rangle + \left\langle \Psi_1 u_R\,,\, \partial_l \varrho \right\rangle + \left\langle \Psi_1 f_1\,,\, \partial_l \varrho \right\rangle + \left\langle \Psi_0(f_2, f_3)\,,\, \partial_l \varrho \right\rangle .\end{aligned}$$

Therefore,

$$\begin{aligned}\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_l \varrho\|_{L^2}^2 \le c_{A,\eta}\, \|\partial_l \varrho\|_{L^2}^2 + \epsilon\, \|\vec{\alpha}\|_{L^2}^2 + \eta\, \|\nabla_x \vec{y}\|_{L^2}^2 + \eta\, \|\nabla_x \vec{\alpha}\|_{L^2}^2 + \|u_R\|_{L^2}^2 \\ + \|f_1\|_{H^1}^2 + \|(f_2, f_3)\|_{L^2}^2 .\end{aligned} \tag{2.78}$$

By combining (2.77) and (2.78) we obtain the energy estimate for $\|\varrho\|_{H^1}^2$ as

$$\begin{aligned}\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|_{H^1}^2 \le c_{A,\eta}\, \|\varrho\|_{H^1}^2 + \eta\, \|\nabla_x \vec{y}\|_{L^2}^2 + \eta\, \|\nabla_x \vec{\alpha}\|_{L^2}^2 + \|\vec{\alpha}\|_{L^2}^2 + \|\vec{y}\|_{L^2}^2 + \|u_R\|_{L^2}^2 . \\ + \|f_1\|_{H^1}^2 + \|(f_2, f_3)\|_{L^2}^2 .\end{aligned} \tag{2.79}$$

Upon adding the above estimates together we conclude that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\vec{\alpha}\|_{L^2}^2 + \|\varrho\|_{H^1}^2\right) + \hat{c}\int_{\mathbb{R}^d} \|\nabla_x \vec{\alpha}\|^2 \,\mathrm{d}x \\ &\le -\tfrac{1}{2}\epsilon\, \|\Delta_x \vec{\alpha}\|_{L^2}^2 + \tilde{c}\left(\|\vec{\alpha}\|_{L^2}^2 + \|\varrho\|_{H^1}^2 + \|u_R\|_{L^2}^2\right) + \eta\, \|\nabla_x \vec{y}\|_{L^2}^2 + \eta\, \|\nabla_x \vec{\alpha}\|_{L^2}^2 , \\ &\quad + \|f_1\|_{H^1}^2 + \|(f_2, f_3)\|_{L^2}^2 .\end{aligned} \tag{2.80}$$

where $\tilde{c}$ depends on c_A, c_0, and $\hat{c}$ depends on μ, κ, α_0, and d.

Next we check the energy estimates of $\vec{y}$. From equation (2.69), it is easy to see that the energy estimate for $\vec{y}$ is as follows.

$$\begin{aligned}\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{y}\|_{L^2}^2 + c_{0,14}\|\nabla_x \vec{y}\|_{L^2}^2 \le \eta\epsilon\, \|\Delta_x \vec{\alpha}\|_{L^2}^2 + c_{A,\eta}\, \|\vec{\alpha}\|_{L^2}^2 + c_{A,\eta}\, \|\varrho\|_{L^2}^2 \\ + c_{A,6}\left(\|\vec{y}\|_{L^2}^2 + \|u_R\|_{L^2}^2\right) + \|(f_2, f_3)\|_{L^2}^2 .\end{aligned} \tag{2.81}$$

Take η sufficiently small and add up (2.80) and (2.81).

Multiply the equation for u_R in (2.40) and integrate over $\mathbb{R}^d$. The L^2 estimate for u_R is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_R\|_{L^2}^2 \le c_{A,7}\,\left(\|\vec{\alpha}\|_{L^2}^2 + \|\vec{y}\|_{L^2}^2 + \|\varrho\|_{L^2}^2 + \|u_R\|_{L^2}^2\right) + \|f_2\|_{L^2}^2 . \tag{2.82}$$

Then the energy estimate for the entire system is written as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varrho\|_{H^1}^2 + \|(\vec{\alpha},\, \vec{y},\, u_R)\|_{L^2}^2\right) + c_{0,15}\|\nabla_x(\vec{\alpha},\, \vec{y},\, u_R)\|_{L^2}^2 \\ &\le c_{A,8}\left(\|\varrho\|_{H^1}^2 + \|(\vec{\alpha},\, \vec{y}, u_R)\|_{L^2}^2\right) + \|f_1\|_{H^1}^2 + \|(f_2, f_3)\|_{L^2}^2 .\end{aligned} \tag{2.83}$$

By taking $0 < T \le T_5$ small enough, we can make $c_{A,8} \le 2C_0$ where $C_0 = \|(\hat{\rho} - \bar{\rho},\, \hat{\theta} - \bar{\theta},\, \hat{u})(x,0)\|_{H^N}$. By Gronwall's inequality we then have

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\varrho\|_{H^1}^2 + \|(\vec{\alpha},\, \vec{y},\, u_R)\|_{L^2}^2\right) + \int_0^T \|\nabla_x \vec{\alpha},\, \vec{y})\|_{L^2}^2(s)\,\mathrm{d}s \\ &\le e^{2C_0 T}\left(\|\varrho(0)\|_{H^1}^2 + \|(\vec{\alpha}(0),\, \vec{y}(0),\, u_R(0))\|_{L^2}^2 + \int_0^T (\|f_1\|_{H^1}^2 + \|(f_2, f_3)\|_{L^2}^2)(s)\,\mathrm{d}s\right)\end{aligned}$$

To finish the proof, notice that $\|\varrho\|_{H^1}^2 + \|\vec{\alpha}\|_{L^2}^2 + \|\vec{y}\|_{L^2}^2 + \|u_R\|_{L^2}^2$ is equivalent to $\|\rho\|_{H^1}^2 + \|u\|_{L^2}^2 + \|\theta\|_{L^2}^2$ with the coefficients depending only on the data. We thereby conclude that there exist $T, c > 0$ where T depends on c_A, c_0 and c depends only on the initial data and c_0 such that

$$\begin{aligned}\sup_{[0,T]} \left(\|\rho\|_{H^1}^2 + \|(u,\theta)\|_{L^2}^2\right)(t) + \int_0^T \|\nabla_x(u,\theta)\|_{L^2}^2(s)\,\mathrm{d}s \\ \le c\left(\|\rho^{in}\|_{H^1}^2 + \|(u^{in},\theta^{in})\|_{L^2}^2) + \int_0^T (\|f_1\|_{H^1}^2 + \|(f_2,f_3)\|_{L^2}^2)(s)\,\mathrm{d}s\right).\end{aligned}$$

This completes the proof of Theorem 2.3. □

3. A Priori Estimate

Based on the linear estimate (2.47), we can now establish the a priori estimate for the nonlinear regularized system (2.3) with the abstract form

$$\begin{aligned}\partial_t U &= -\epsilon\,\Delta_x^2 U + \mathcal{L}(U)U\,, \\ U(x,0) &= (\rho^{in}, u^{in}, \theta^{in})\,,\end{aligned} \tag{3.1}$$

where $U = (\rho, u, \theta)$ and $\mathcal{L}$ is given by (2.12).

We begin by defining the following norms. Let $s > s_1$ be two integers such that

$$s_1 > d/2 + 6, \qquad s = \max\{s_1 + 6, N + d/2 + 5\}\,, \tag{3.2}$$

where $N = N(d)$ is large enough (see Remark 1.2 and 2.4). Let $\bar\rho$, $\bar\theta > 0$ be the constants such that the initial data $(\rho^{in}, u^{in}, \theta^{in})$ satisfy (1.6). For $(\rho, u, \theta) : \mathbb{R}^d \times [0,T] \to \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$ satisfying

$$\begin{aligned}\rho - \bar\rho \in C([0,T]; H^{s+1})\,, \qquad (u,\,\theta-\bar\theta) \in C([0,T]; H^s)\,, \\ \langle x\rangle^2 \partial_x^\alpha(\rho,u,\theta) \in C([0,T]; H^1\times L^2\times L^2)\,, \qquad \forall \alpha\in\mathbb{N}^d\,,\ 1\le|\alpha|\le s_1\,,\end{aligned} \tag{3.3}$$

define

$$\begin{aligned}&|\!|\!|(\rho-\bar\rho,\ u,\ \theta-\bar\theta)|\!|\!|_{s,T} \\ &= \sup_{[0,T]}\left(\|\rho(t)-\bar\rho\|_{H^{s+1}} + \|(u,\theta-\bar\theta)(t)\|_{H^s} + \sum_{1\le|\alpha|\le s_1}\left(\|\langle x\rangle^2\partial_x^\alpha\rho(t)\|_{H^1} + \|\langle x\rangle^2\partial_x^\alpha(u,\theta)(t)\|_{L^2}\right)\right).\end{aligned} \tag{3.4}$$

Define

$$\lambda = \|\rho^{in}-\bar\rho\|_{H^{s+1}} + \|(u^{in},\theta^{in}-\bar\theta)\|_{H^s} + \sum_{1\le|\alpha|\le s_1}\left(\|\langle x\rangle^2\partial_x^\alpha\rho^{in}\|_{H^1} + \|\langle x\rangle^2\partial_x^\alpha(u^{in},\theta^{in})\|_{L^2}\right) < \infty\,.$$

Suppose there exists a constant $\alpha_0 > 0$ such that $\rho^{in}, \theta^{in}, \mu(\theta^{in}), \kappa(\theta^{in}) \ge 2\alpha_0 > 0$. Given $T, M, s > 0$, define the set $X_{s,T,M}$ by

$$\begin{aligned}&X_{s,T,M} \\ &= \left\{(\rho,u,\theta)(t,x)\,:\, |\!|\!|(\rho-\bar\rho,u,\theta-\bar\theta)|\!|\!|_{s,T}\le M\,,\quad \rho,\theta\ge\alpha_0>0,\quad (\rho,u,\theta)(0) = (\rho^{in},u^{in},\theta^{in})\right\}.\end{aligned}$$

Let

$$M_0 = 4c\lambda\,, \tag{3.5}$$

with c being the constant in (2.47). Suppose $2c > 1$. Then the a priori estimate for system (2.3) states

Lemma 3.1. *Let $\bar\rho, \bar\theta > 0$ be two constants and $\bar U = (\bar\rho, 0, \bar\theta)$. Let $U = (\rho, u, \theta)$ be a solution to system (2.3) satisfying (3.3) with $\rho, \theta \geq \alpha_0 > 0$. Then there exists $T_{\alpha_0} > 0$ independent of ϵ such that $\|U - \bar U\|_{s, T_{\alpha_0}} \leq M_0$, where M_0 is defined in (3.5).*

Proof. First, by the linear estimate for (3.1), there exists $T > 0$ independent of ϵ such that

$$\sup_{[0,T]} \Big(\|\rho(t) - \bar\rho\|_{H^1} + \|(u, \theta - \bar\theta)(t)\|_{L^2} \Big) \leq M_0 \,.$$

Next we check the bounds of $(\rho - \bar\rho,\, u,\, \theta - \bar\theta)$ in higher order norms and norms with the weight $\langle x\rangle^2$. We use an induction proof. Suppose that there exists $0 < T_6 \leq T$ such that Lemma 3.1 holds for $s-1$ on $[0, T_6]$. For any multi-index α with $1 \leq |\alpha| \leq s$, apply ∂_x^α to the nonlinear system (2.3). The resulting system for $\partial_x^\alpha U = (\partial_x^\alpha \rho, \partial_x^\alpha u, \partial_x^\alpha \theta)$ is

$$\begin{aligned}
\partial_t(\partial_x^\alpha \rho) &= -\epsilon\, \Delta_x^2(\partial_x^\alpha \rho) + \mathcal{L}_1(U)(\partial_x^\alpha U) + \Psi_0\left(\partial_x^\gamma \rho,\, \partial_x^\gamma u\right) + \Psi_0\left(\partial_x^{\tilde\gamma} u\right) + f_{\alpha,0}\,,\\
\partial_t(\partial_x^\alpha u) &= -\epsilon\, \Delta_x^2(\partial_x^\alpha u) + \mathcal{L}_2(U)(\partial_x^\alpha U) + \Psi_2(\partial_x^\gamma \rho) + \Psi_1(\partial_x^\gamma U) + \Psi_0(\partial_x^\gamma U) + f_{\alpha,1}\,,\\
\partial_t(\partial_x^\alpha \theta) &= -\epsilon\, \Delta_x^2(\partial_x^\alpha \theta) + \mathcal{L}_3(U)(\partial_x^\alpha U) + \Psi_2(\partial_x^\gamma u) + \Psi_1(\partial_x^\gamma U) + \Psi_0(\partial_x^\gamma U) + f_{\alpha,2}\,,
\end{aligned} \tag{3.6}$$

where γ denotes any multi-index satisfying $|\gamma| = |\alpha|$, $\tilde\gamma$ is a multi-index such that $|\tilde\gamma| = |\alpha| - 1$. The forcing term $f_{\alpha,0}$ has the form

$$f_{\alpha,0} = \sum_{\substack{\alpha_1+\alpha_2=\alpha,\\ |\alpha_1|\leq|\alpha|-1,\\ |\alpha_2|\leq|\alpha|-2}} \left((\partial_x^{\alpha_1}\rho)\, \nabla_x \cdot (\partial_x^{\alpha_2} u) + (\nabla_x \partial_x^{\alpha_2} \rho) \cdot (\partial_x^{\alpha_2} u) \right) \,.$$

A typical term in the above sum is $(\partial_x^{\alpha_1}\rho)\,(\partial_x^{\alpha_2} u)$ with $|\alpha_1| + |\alpha_2| \leq s+1$, $|\alpha_1| \leq s-1$, and $|\alpha_2| \leq s-1$. Therefore, there exists $i \in \{1, 2\}$ such that $|\alpha_i| \leq \frac{s}{2} + 1$. For such α_i we have

$$(s-2) - |\alpha_i| \geq (s-2) - (\frac{s}{2} + 1) = \frac{s}{2} - 3 > \frac{d}{2}\,,$$

by the choice of s in (3.2). Therefore the term associated with α_i derivatives is in $W^{1,\infty}$. Thus we have

$$f_{\alpha,0} \in L^\infty(0, T_6; H^1)\,.$$

Similarly, we can show that

$$f_{\alpha,1}, f_{\alpha,2} \in L^\infty(0, T_6; L^2)\,.$$

The additional second order terms in the u-equation in (3.6) have the form

$$\Psi_2(\partial_x^\gamma \theta) = \sum_{\substack{\gamma_1+\gamma_2=\alpha,\\ |\gamma_1|=1, |\gamma_2|=|\alpha|-1}} \tfrac{d-1}{d}\, \partial_x^{\gamma_1} \left(\frac{\tau_1}{\rho}\right) \Delta_x \nabla_x \partial_x^{\gamma_2} \theta\,.$$

The additional second order terms in the θ-equation in (3.6) are

$$\Psi_2(\partial_x^\gamma u) = \sum_{\substack{\gamma_1+\gamma_2=\alpha,\\ |\gamma_1|=1, |\gamma_2|=|\alpha|-1}} \tfrac{4(d-1)}{d^2}\, \partial_x^{\gamma_1} \left(\frac{\tau_4}{\rho}\right) \Delta_x \nabla_x \cdot \partial_x^{\gamma_2} u\,.$$

It is then clear that the coefficients of these additional second order operators satisfy the assumptions $\mathcal{A}_1$, $\mathcal{A}_2$. The coefficient of Ψ_1, Ψ_0 in the above system depend on $\nabla_x^{\gamma_3} U$ for any γ_3 such that $|\gamma_3| \leq 3$. The assumptions $\mathcal{A}_1$, $\mathcal{A}_2$ are also satisfied for $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ when the solution

U satisfies (3.3) with s, s_1 given in (3.2). Therefore, we conclude that the same linear estimate (2.47) applies for $\partial_x^\alpha(\rho, u, \theta)$ for every $0 \le |\alpha| \le s$ — that is, there exists $0 < T_7 \le T_6$ such that

$$\sup_{[0,T_7]} \left(\|\rho - \bar{\rho}\|^2_{H^{|\alpha|+1}} + \|(u, \theta - \bar{\theta})\|^2_{H^{|\alpha|}} \right) + \int_0^{T_7} \|\nabla_x(u,\theta)\|^2_{H^{|\alpha|}}(s)\,\mathrm{d}s$$
$$\le c \left(\|\rho^{in} - \bar{\rho}\|^2_{H^{|\alpha|+1}} + \|(u^{in}, \theta^{in} - \bar{\theta})\|^2_{H^{|\alpha|}} + \int_0^{T_7} \|f_{\alpha,0}\|^2_{H^1}(s) + \|(f_{\alpha,1}, f_{\alpha,2})\|^2_{L^2}(s)\,\mathrm{d}s \right),$$

where c depends only on the data and α_0. Because $f_{\alpha,0} \in L^\infty(0, T_6; H^1(\mathbb{R}^d))$ and $(f_{\alpha,1}, f_{\alpha,2}) \in L^\infty(0, T_6; L^2(\mathbb{R}^d))$, the last term including the forcing can be made arbitrarily small by taking T_7 small. Therefore there exists $T_8 > 0$ independent of ϵ such that

$$\sup_{[0,T_8]} \left(\|\rho - \bar{\rho}\|_{H^{s+1}} + \|(u, \theta - \bar{\theta})\|_{H^s} \right) \le M_0/2\,. \tag{3.7}$$

Next, we estimate the bounds on $\langle x\rangle^2 \partial_x^\alpha(\rho, u, \theta)$ for $1 \le |\alpha| \le s_1$. We will show that the system for $\langle x\rangle^2 \partial_x^\alpha(\rho, u, \theta)$ has a similar structure as those for (ρ, u, θ) and $\partial_x^\alpha(\rho, u, \theta)$ so that the linear estimate (2.47) again applies.

For each $1 \le l \le d$ and multi-index β such that $1 \le |\beta| \le s_1 + 3$, the system satisfied by $x_l\, \partial_x^\beta U$ has the form

$$\begin{aligned}
\partial_t(x_l \partial_x^\beta \rho) &= -\epsilon\, \Delta_x^2(x_l \partial_x^\beta \rho) + \mathcal{L}_1(U)(x_l \partial_x^\beta U) + \Psi_0(x_l \partial_x^\gamma \rho,\, x_l \partial_x^\gamma u) + \Psi_0(x_l \partial_x^{\tilde\gamma} u) + f_{l,0}\,,\\
\partial_t(x_l \partial_x^\beta u) &= -\epsilon\, \Delta_x^2(x_l \partial_x^\beta u) + \mathcal{L}_2(U)(x_l \partial_x^\beta U) + \Psi_2(x_l \partial_x^\gamma \rho,\, x_l \partial_x^\gamma \theta) + \Psi_1(x_l \partial_x^\gamma U)\\
&\quad + \Psi_0(x_l \partial_x^\gamma U) + f_{l,1}\,,\\
\partial_t(x_l \partial_x^\beta \theta) &= -\epsilon\, \Delta_x^2(x_l \partial_x^\beta \theta) + \mathcal{L}_3(U)(x_l \partial_x^\beta U) + \Psi_2(x_l \partial_x^\gamma U) + \Psi_1(x_l \partial_x^\gamma U)\\
&\quad + \Psi_0(x_l \partial_x^\gamma U) + f_{l,2}\,,
\end{aligned}$$

with $\gamma, \tilde\gamma \in \mathbb{N}^d$ such that $|\gamma| = |\beta|$ and $|\tilde\gamma| = |\beta| - 1$. Similar as in (3.6), we have the coefficients of Ψ_2, Ψ_1, Ψ_0 satisfying $\mathcal{A}_1, \mathcal{A}_2$. The norm of the forcing terms $\|(f_{l,0}, f_{l,1}, f_{l,2})\|_{H^1 \times L^2 \times L^2(\mathrm{d}x)}$ depends only on the H^{s_1+6} norm of (ρ, u, θ) and $H^{|\beta|-1}$ norms of $(x_l\rho, x_l u, x_l\theta)$. It is thereby bounded when $s \ge s_1 + 6$. Consequently, the linear estimate holds for $(x_l \partial_x^\beta \rho, x_l \partial_x^\beta u, x_l \partial_x^\beta \theta)$ for each $l = 1, 2, \cdots, d$, that is, there exists $0 < T_9 \le T_8$ such that for every $1 \le l \le d$ and multi-index γ such that $0 \le |\gamma| \le s_1 + 2$,

$$\sup_{[0,T_9]} \left(\|x_l\, \nabla_x \rho\|^2_{H^{|\gamma|}} + \|x_l\, \nabla_x(u,\theta)\|^2_{H^{|\gamma|-1}} \right) + \int_0^{T_9} \|\nabla_x\,(x_l\, u, x_l\, \theta))\|^2_{H^{|\gamma|}}(s)\,\mathrm{d}s$$
$$\le c \left(\|x_l\, \nabla_x \rho^{in}\|^2_{H^{|\gamma|}} + \|x_l\, \nabla_x(u^{in}, \theta^{in})\|^2_{H^{|\gamma|-1}} + \int_0^{T_9} (\|f_{l,0}\|^2_{H^1} + \|(f_{l,1}, f_{l,2})\|^2_{L^2})(s)\,\mathrm{d}s \right).$$

Similarly, for each $1 \le l \le d$ and multi-index α satisfying $1 \le |\alpha| \le s_1$, the system for $x_l^2\, \partial_x^\alpha U$ has the form

$$\begin{aligned}
\partial_t(x_l^2 \partial_x^\alpha \rho) &= -\epsilon\, \Delta_x^2(x_l^2 \partial_x^\alpha \rho) + \mathcal{L}_1(U)(x_l^2 \partial_x^\alpha U) + \Psi_0(x_l^2 \partial_x^\gamma \rho,\, x_l^2 \partial_x^\gamma u) + \Psi_0(x_l^2 \partial_x^{\tilde\gamma} u) + g_{l,0}\,,\\
\partial_t(x_l^2 \partial_x^\alpha u) &= -\epsilon\, \Delta_x^2(x_l^2 \partial_x^\alpha u) + \mathcal{L}_2(U)(x_l^2 \partial_x^\alpha U) + \Psi_2(x_l^2 \partial_x^\gamma \rho,\, x_l^2 \partial_x^\gamma \theta) + \Psi_1(x_l^2 \partial_x^\gamma U)\\
&\quad + \Psi_0(x_l^2 \partial_x^\gamma U) + g_{l,1}\,,\\
\partial_t(x_l^2 \partial_x^\alpha \theta) &= -\epsilon\, \Delta_x^2(x_l^2 \partial_x^\alpha \theta) + \mathcal{L}_3(U)(x_l^2 \partial_x^\alpha U) + \Psi_2(x_l^2 \partial_x^\gamma U) + \Psi_1(x_l^2 \partial_x^\gamma U)\\
&\quad + \Psi_0(x_l^2 \partial_x^\gamma U) + g_{l,2}\,,
\end{aligned}$$

where $|\gamma| = |\alpha|$, $|\tilde{\gamma}| = |\alpha| - 1$. The forcings are

$$g_{l,0} = g_{l,0}\left(x_l^2 \partial_x^{\sigma_0} u\,, x_l^2 \partial_x^{\sigma_1}\rho\,, x_l \partial_x^{\sigma_2}(\rho, u, \theta),\, \partial_x^{\sigma_3}(\rho, u, \theta)\right)_{|\sigma_0| \le |\alpha|-2, |\sigma_1| \le |\alpha|-1, |\sigma_2| \le s_1+3,\, |\sigma_3| \le s_1+2}\,,$$

$$(g_{l,1}, g_{l,2}) = (g_{l,1}, g_{l,2})\left(x_l^2 \partial_x^{\sigma_1}(\rho, u, \theta),\, x_l \partial_x^{\sigma_2}(\rho, u, \theta),\, \partial_x^{\sigma_3}(\rho, u, \theta)\right)_{|\sigma_1| \le |\alpha|-1, |\sigma_2| \le s_1+3,\, |\sigma_3| \le s_1+2}\,.$$

Again the linear estimate applies. Therefore, there exists $0 < T_{10} \le T_9$ sufficiently small such that for every $1 \le |\alpha| \le s_1 - 1$, we have

$$\sup_{[0,T_{10}]} \left(\|x_l^2\, \nabla_x \rho\|_{H^{|\alpha|}}^2 + \|x_l^2\, \nabla_x(u,\theta)\|_{H^{|\alpha|-1}}^2\right) + \int_0^{T_{10}} \left\|\nabla_x\left(x_l^2\, u, x_l^2\, \theta\right)\right\|_{H^{|\alpha|}}^2 (s)\,\mathrm{d}s$$
$$\le c\left(\|x_l^2\, \nabla_x \rho^{in}\|_{H^{|\alpha|}}^2 + \|x_l^2\, \nabla_x(u^{in}, \theta^{in})\|_{H^{|\alpha|-1}}^2 + \int_0^{T_{10}} (\|g_{l,0}\|_{H^1}^2 + \|(g_{l,1}, g_{l,2})\|_{L^2}^2)(s)\,\mathrm{d}s\right).$$

Thus, by taking T_{10} sufficiently small, we have

$$\sup_{[0,T_{10}]} \left(\sum_{1 \le |\alpha_1| \le s_1+1} \|\langle x\rangle^2\, \partial_x^{\alpha_1}\rho\|_{L^2} + \sum_{1 \le |\alpha_2| \le s_1} \|\langle x\rangle^2\, \partial_x^{\alpha_2}(u,\theta)\|_{L^2}\right) \le M_0/2\,. \tag{3.8}$$

Upon adding (3.7) and (3.8), we conclude that there exists $T_{\alpha_0} > 0$ independent of ϵ such that

$$|\!|\!|(\rho - \bar{\rho}, u, \theta - \bar{\theta})|\!|\!|_{T_{\alpha_0}} \le M_0\,. \tag{3.9}$$

We thereby finish the proof of Lemma 3.1. □

4. Local Existence Proof

Based on Lemma 3.1, we can now prove the local existence of classical solutions to the nonlinear system (1.1). To show this, we first establish the existence of solutions to the regularized DNS system in Lemma 4.1. Then using Lemma 3.1, we show that the sequence of solutions to the regularized system exists on a time interval which is independent of ϵ. Finally, in the main theorem the convergence of this sequence of solutions is proved. Uniqueness is also proved in the main Theorem.

Let $U = (\rho, u, \theta)$, $\bar{U} = (\bar{\rho}, 0, , \bar{\theta})$, $U^{in} = (\rho^{in}, u^{in}, \theta^{in})$ where U^{in} satisfies the condition (1.6). Define the operator $\Gamma = \Gamma_\epsilon$ on $X_{s,T,M}$ by

$$\Gamma(\rho, u, \theta) = \Gamma(U) = e^{-\epsilon t \Delta_x^2} U^{in} + \int_0^t e^{-\epsilon(t-t')\Delta_x^2}\big(\mathcal{L}(U)U\big)(t')\,\mathrm{d}t'\,. \tag{4.1}$$

Then

$$\Gamma(U) - \bar{U} = e^{-\epsilon t \Delta_x^2}(U^{in} - \bar{U}) + \int_0^t e^{-\epsilon(t-t')\Delta_x^2}\big(\mathcal{L}(U)(U - \bar{U})\big)(t')\,\mathrm{d}t'\,. \tag{4.2}$$

We apply the contraction mapping theorem to show the local existence of the solution to the regularized DNS system for each $0 < \epsilon < 1$.

Lemma 4.1. *For each $\epsilon \in (0,1)$ there exists $T_\epsilon = O(\epsilon^3)$ such that the operator Γ defined in (4.1) defines a contraction mapping on X_{s,T_ϵ,M_0} where M_0 is defined in (3.5). Therefore, the regularized system (3.1) has a unique solution in X_{s,T_ϵ,M_0}.*

Proof. Suppose $U \in X_{s,T,M_0}$ for some $T > 0$. We study the semigroup generated by $-\epsilon\Delta_x^2$. Let β be a multi-index such that $|\beta| = 3$. Then for any $g \in L^2$,

$$\|\partial_x^\beta\, e^{-\epsilon t \Delta_x^2} g\|_{L^2} \le \frac{C}{\epsilon^{3/4} t^{3/4}} \|g\|_{L^2}\,,$$

where $C > 0$ is a generic constant. Because $\mathcal{L}$ is of order three, for any multi-indices α_1, α_2 such that $|\alpha_1| \le s+1$, $|\alpha_2| \le s$, we have

$$\begin{aligned}
&\sup_{[0,T_{\epsilon,1}]} \|\partial_x^{\alpha_1}\left(\Gamma(\rho) - \bar\rho\right)\|_{L^2} \\
&\le \|\partial_x^{\alpha_1}(\rho^{in} - \bar\rho)\|_{L^2} + \int_0^{T_{\epsilon,1}} \left\|\partial_x^{\alpha_1} e^{-\epsilon(T_{\epsilon,1}-t')\Delta_x^2}\big(\mathcal{L}_1(U)(U-\bar U)\big)\right\|_{L^2}(t')\,\mathrm{d}t' \\
&\le \|\partial_x^{\alpha_1}(\rho^{in} - \bar\rho)\|_{L^2} + \frac{c_{0,s} T_{\epsilon,1}^{1/4}}{\epsilon^{3/4}} M_0^2\,,
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{[0,T_{\epsilon,1}]} \|\partial_x^{\alpha_2}\left(\Gamma(u,\theta) - (0,\bar\theta)\right)\|_{L^2} \\
&\le \|\partial_x^{\alpha_2}(u^{in}, \theta^{in} - \bar\theta)\|_{L^2} + \int_0^{T_{\epsilon,1}} \left\|\partial_x^{\alpha_2} e^{-\epsilon(T_{\epsilon,1}-t')\Delta_x^2}\left(\mathcal{L}_2, \mathcal{L}_3\right)(U)(U-\bar U)\right\|_{L^2}(t')\,\mathrm{d}t' \\
&\le \|\partial_x^{\alpha_2}(u^{in}, \theta^{in} - \bar\theta)\|_{L^2} + \frac{c_{0,s} T_{\epsilon,1}^{1/4}}{\epsilon^{3/4}} M_0^2\left(1 + M_0^{k_s}\right)\,,
\end{aligned}$$

where $c_{0,s} > 0$ depends on α_0, s, and $k_s > 0$ depends only on s. Therefore,

$$\sup_{[0,T_{\epsilon,1}]} \|\Gamma(U) - \bar U\|_{H^s} \le \|U^{in} - \bar U\|_{H^s} + \frac{c_{0,s} T_{\epsilon,1}^{1/4}}{\epsilon^{3/4}} M_0^2\left(1 + M_0^{k_s}\right) \le \lambda + M_0/4 \le M_0/2\,, \tag{4.3}$$

by choosing $T_{\epsilon,1} = O(\epsilon^3)$ small enough.

Next, we show that the weighted norm $\sum_{1\le|\alpha|\le s_1} \|\langle x\rangle^2 \partial_x^\alpha \Gamma(U)\|_{L^2}$ is also bounded by $M_0/2$ for a sufficiently short time. The argument is similar to the one in Section 3. Notice that for any $1 \le l \le d$, $|\alpha| \le s_1 + 3$, a direct calculation shows that $x_l\, \partial_x^\alpha \Gamma(U)$ satisfies

$$\partial_t\left(x_l\, \partial_x^\alpha \Gamma(U)\right) = -\epsilon\, \Delta_x^2\left(x_l \partial_x^\alpha \Gamma(U)\right) + \mathcal{L}(U)\left(x_l\, \partial_x^\alpha U\right) + F\,,$$

where $F = F\left(\partial_x^\beta\, \Gamma(U),\, \partial_x^\gamma\, U\right)_{|\gamma|,|\beta|\le s_1+6}$ is a C^∞ function in its variables. Because $\Gamma(U)$ and U are bounded in $L^\infty([0,T_{\epsilon,1}]; H^s)$ for $s \ge s_1 + 6$, the function F is bounded in $L^\infty([0,T_{\epsilon,1}]; L^2)$. Moreover, there exists a polynomial $Q_1(x)$ such that

$$\sup_{[0,T_{\epsilon,1}]} \|F\|_{L^2} \le Q_1(M_0)\,.$$

Since $x_l\, \partial_x^\alpha U$ is bounded in L^2 for any $\alpha \le s_1$, similar calculation as above shows that by taking $T_{\epsilon,2} = O(\epsilon^3)$ small enough we have

$$\begin{aligned}
\sup_{[0,T_{\epsilon,2}]} \sum_{1\le|\alpha|\le s_1} \|x_l\, \partial_x^\alpha \Gamma(U)\|_{L^2} &\le \sum_{1\le|\alpha|\le s_1} \|x_l\, \partial_x^\alpha U^{in}\|_{L^2} + \frac{c_{0,s_1} T_{\epsilon,2}^{1/4}}{\epsilon^{3/4}} M_0^2\left(1 + M_0^{k_{s_1}}\right) + T_{\epsilon,2} Q_1(M_0) \\
&\le \lambda + M_0/4 \le M_0/2\,,
\end{aligned}$$

Here $c_{0,s_1} > 0$ depends on α_0, s_1, and $k_{s_1} > 0$ depends only on s_1.

Similarly, for each $1 \le l \le d$, $x_l^2\, \partial_x^\alpha \Gamma(U)$ satisfies

$$\partial_t \left(x_l^2\, \partial_x^\alpha \Gamma(U)\right) = -\epsilon\, \Delta_x^2 \left(x_l^2\, \partial_x^\alpha \Gamma(U)\right) + \mathcal{L}(U) \left(x_l^2\, \partial_x^\alpha U\right) + G\,,$$

where $G = G\left(x_l\, \partial_x^{\sigma_1}\Gamma(U),\, x_l\, \partial_x^{\sigma_2} U,\, \partial_x^\beta\, \Gamma(U),\, \partial_x^\gamma U\right)_{|\sigma_1|,|\sigma_2|,|\gamma|,|\beta| \le s_1+3}$ is a C^∞ function in its variables. Because $x_l\, \partial_x^\sigma \Gamma(U)$ is shown above to be bounded in $L^\infty([0, T_{\epsilon,2}]; L^2)$ for any $|\sigma| \le s_1 + 3$, and $\Gamma(U), U$ are bounded in $L^\infty([0, T_{\epsilon,2}]; H^s)$ for $s \ge s_1 + 6$, the function G is bounded in $L^\infty([0, T_{\epsilon,2}]; L^2)$. Moreover, there exists a polynomial $Q_2(x) > 0$ such that

$$\sup_{[0,T_{\epsilon,2}]} \|G\|_{L^2} \le Q_2(M_0)\,.$$

Using the fact that $x_l^2\, \partial_x^\alpha U$ is bounded in L^2 for any $\alpha \le s_1$, a calculation similar to the one above shows that for $0 < T_{\epsilon,3} \le T_{\epsilon,2}$,

(4.4)

$$\begin{aligned}\sup_{[0,T_{\epsilon,3}]} \sum_{1\le|\alpha|\le s_1} \|x_l\, \partial_x^\alpha \Gamma(U)\|_{L^2} &\le \sum_{1\le|\alpha|\le s_1} \|x_l\, \partial_x^\alpha U^{in}\|_{L^2} + \frac{c_{0,s_1} T_{\epsilon,3}^{1/4}}{\epsilon^{3/4}} M_0^2 \left(1 + M_0^{k_{s_1}}\right) + T_{\epsilon,3} Q_2(M_0) \\ &\le \lambda + M_0/4 \le M_0/2\,,\end{aligned}$$

by choosing $T_{\epsilon,3} = O(\epsilon^3)$ small enough. Here $c_{0,s_1} > 0$ depends on α_0, s_1, and $k_{s_1} > 0$ depends only on s_1.

For the positivity of $\Gamma(\rho)$ and $\Gamma(\theta)$, notice that $\Gamma(U)$ satisfies the linear equation

$$\partial_t \Gamma(U) = -\epsilon\, \Delta_x^2 \Gamma(U) + \mathcal{L}(U)U\,,$$

with $\mathcal{L}(U)U$ sufficiently smooth and $0 < \epsilon < 1$. Therefore, for the initial data $\rho^{in}, \theta^{in} \ge 2\alpha_0 > 0$, if we choose $T_{\epsilon,4}$ small, we have $\Gamma(\rho), \Gamma(\theta) \ge \alpha_0 > 0$.

Upon combining the positivity with (4.3) and (4.4), we conclude that Γ maps X_{T_ϵ,M_0} into itself for $T_\epsilon = \min\{T_{\epsilon,k}\}_{k=1}^4$ sufficiently small.

To show $\Gamma(U)$ is a contraction mapping on X_{T_ϵ,M_0}, for any $U_1, U_2 \in X_{T_\epsilon,M_0}$ consider the difference equation for $\Gamma(U_1) - \Gamma(U_2)$:

$$\partial_t \Big(\Gamma(U_1) - \Gamma(U_2)\Big) = -\epsilon\, \Delta_x^2 \Big(\Gamma(U_1) - \Gamma(U_2)\Big) + \Big(\mathcal{L}(U_1)U_1 - \mathcal{L}(U_2)U_2\Big)\,.$$

Similar calculation shows that there exists a polynomial $Q_3(M_0)$ such that

$$|||\Gamma(U_1) - \Gamma(U_2)|||_{X_{T_\epsilon,M_0}} \le c_{\epsilon,s,s_1} T_\epsilon^{1/4} Q_3(M_0) |||U_1 - U_2|||_{X_{T_\epsilon,M_0}}\,.$$

Therefore, by choosing T_ϵ sufficiently small, $\Gamma : X_{T_\epsilon,M_0} \to X_{T_\epsilon,M_0}$ is a contraction mapping. Therefore, there exists a solution $(\rho^\epsilon, u^\epsilon, \theta^\epsilon) \in X_{T_\epsilon,M_0}$ to the regularized nonlinear system (3.1). □

We show in the following lemma that the lifespan of U can be extended from T_ϵ to $T_0 > 0$ which is independent of ϵ.

Lemma 4.2. *There exists $T_0 > 0$ independent of ϵ such that the solution $U = (\rho, u, \theta)$ to the regularized system (2.3) exists on $[0, T_0]$ and satisfies that $|||U|||_{T_0} \le M_0$.*

Proof. The only assumption for the a priori estimate in Lemma 3.1 is that ρ and θ have a positive lower bound on a time interval which is independent of ϵ. Therefore if we can find $T_{0,1}$ independent of ϵ such that $\rho, \theta \ge \alpha_0 > 0$ on $[0, T_{0,1}]$, then the a priori estimate will hold on $T_0 = \min\{T_{\alpha_0}, T_{0,1}\}$ where T_{α_0} is given in Lemma 3.1. Since both T_{α_0} and $T_{0,1}$ are independent of ϵ, the system will not blow up before T_0 for any $0 < \epsilon < 1$. Thus we have the existence of the solution over $[0, T_0]$. In order to prove the existence of $T_{0,1}$, we first note that the right-hand side of the ρ-equation in system (2.3) is bounded by a function of M_0 uniformly in ϵ. This bound does not require either ρ or θ being bounded from below. Thus $\|\partial_t \rho\|_{L^\infty}$ is uniformly bounded by this function of M_0. By taking $T_{0,2}$ small enough we have $\rho \geq \alpha_0 > 0$ on $[0, T_{0,2}]$. Next we use this lower bound of ρ and the boundedness of U in the θ-equation. The right-hand side of the θ-equation thus $\|\partial_t \theta\|_{L^\infty}$ is bounded uniformly in ϵ by a function of M_0 and α_0. Thus there exists $0 < T_{0,1} \leq T_{0,2}$ such that $\theta \geq \alpha_0 > 0$ on $[0, T_{0,1}]$, which then concludes the proof. □

Finally, we state and prove the Main theorem.

Main Theorem. *There exists $N = N(d)$ such that if the Hamiltonian flow generated by the symbol*

$$h^{in}(\xi, x) = \sqrt{\hat{\tau}_1(x,0)\hat{\tau}_4(x,0)}|\xi|^3$$

is nontrapping and there are two constants $\bar{\rho}, \bar{\theta} > 0$ such that given the initial data $(\rho^{in}, u^{in}, \theta^{in})$ satisfying the condition

$$\|\rho^{in} - \bar{\rho}\|_{H^{s+1}} + \|(u^{in}, \theta^{in} - \bar{\theta})\|_{H^s} + \sum_{1 \leq |\alpha| \leq s_1} \left(\|\langle x\rangle^2 \partial_x^\alpha \rho^{in}\|_{H^1} + \|\langle x\rangle^2 \partial_x^\alpha (u^{in}, \theta^{in})\|_{L^2} \right) < \infty\,,$$

where $s_1 > d/2+6$, $s = \max\{s_1+6, N+d/2+5\}$, then there exists $T_0 > 0$ independent of ϵ such that system (2.2) has a unique solution $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ in X_{T_0, M_0}. Moreover, there exists (ρ, u, θ) such that $\rho - \bar{\rho} \in C([0, T_0]; H^{s+1}), (u, \theta - \bar{\theta}) \in C([0, T_0]; H^s) \cap L^2([0, T_0]; H^{s+1})$ satisfying

$$\left.\begin{aligned} \rho^\epsilon - \bar{\rho} &\longrightarrow \rho - \bar{\rho} && \text{in } C([0,T_0]; H^{s+1})\,, \\ (u^\epsilon,\, \theta^\epsilon - \bar{\theta}) &\longrightarrow (u,\, \theta - \bar{\theta}) && \text{in } C([0,T_0]; H^{s})\,, \\ \langle x\rangle^2 \partial_x^\alpha \rho^\epsilon &\longrightarrow \langle x\rangle^2 \partial_x^\alpha \rho && \text{in } C([0,T_0]; H^{1})\,, \\ \langle x\rangle^2 \partial_x^\alpha (u^\epsilon, \theta^\epsilon) &\longrightarrow \langle x\rangle^2 \partial_x^\alpha (u, \theta) && \text{in } C([0,T_0]; L^{2})\,, \end{aligned}\right\} \quad \text{as } \epsilon \to 0\,,$$

for any $1 \leq |\alpha| \leq s_1$ and (ρ, u, θ) is the unique solution to the original DNS system (1.1).

Proof. The existence of $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ has been shown in Lemma 4.1 and Lemma 3.1. The convergence of approximate solutions $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ is shown by the standard high-low technique [18]. First, we will show that $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ converges in $C(0, T_0; L^2)$. This gives the convergence of $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ in $C(0, T_0; H^{s'+1} \times H^{s'} \times H^{s'})$ for any $0 < s' < s$ by interpolation and uniform bounds of $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$. Then we apply classical methods for hyperbolic equations to prove the convergence in $C(0, T_0; H^{s+1} \times H^s \times H^s)$.

For $\epsilon, \epsilon' > 0$, let $\varrho = \rho^\epsilon - \rho^{\epsilon'}, v = u^\epsilon - u^{\epsilon'}, \eta = \theta^\epsilon - \theta^{\epsilon'}$ and study the system for (ϱ, v, η).

$$\begin{aligned} \partial_t \varrho &= -\epsilon\, \Delta_x^2 \varrho - (\epsilon - \epsilon')\, \Delta_x^2 \rho^{\epsilon'} + \mathcal{L}_1(\rho^\epsilon, u^\epsilon, \theta^\epsilon)(\varrho, v) + \Psi_{0,1}(\varrho, v)\,, \\ \partial_t v &= -\epsilon\, \Delta_x^2 v - (\epsilon - \epsilon')\, \Delta_x^2 u^{\epsilon'} + \mathcal{L}_2(\rho^\epsilon, u^\epsilon, \theta^\epsilon)(\varrho, v, \eta) + \Psi_{1,2}(\varrho, v, \eta)\,, \\ \partial_t \eta &= -\epsilon\, \Delta_x^2 \eta - (\epsilon - \epsilon')\, \Delta_x^2 \theta^{\epsilon'} + \mathcal{L}_3(\rho^\epsilon, u^\epsilon, \theta^\epsilon)(\varrho, v, \eta) + \Psi_{1,3}(\varrho, v, \eta)\,. \end{aligned} \tag{4.5}$$

where $\Psi_{k,j}$ are k^{th}-order operators with their coefficients depending on $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ and $(\rho^{\epsilon'}, u^{\epsilon'}, \theta^{\epsilon'})$ for $k = 0, 1$ and $j = 1, 2, 3$. The terms in $\Psi_{1,2}(\varrho, v, \eta)$ involving μ and τ_1 have the form

$$\frac{1}{\rho^\epsilon} \nabla_x \cdot \left[\eta \frac{\mu(\theta^\epsilon) - \mu(\theta^{\epsilon'})}{\theta^\epsilon - \theta^{\epsilon'}} D_x u^{\epsilon'} \right] - \varrho \frac{1}{\rho^\epsilon \rho^{\epsilon'}} \nabla_x \cdot [\mu(\theta^{\epsilon'}) D_x u^{\epsilon'}]\,.$$

The terms in $\Psi_{1,2}(\varrho, v, \eta)$ involving τ_1 have the form

$$\tfrac{d-1}{d}\frac{1}{\rho^\epsilon}\frac{\tau_1(\theta^{\epsilon'})-\tau_1(\theta^\epsilon)}{\theta^\epsilon-\theta^{\epsilon'}}\eta\Delta_x\nabla_x\theta^{\epsilon'} + \varrho\tfrac{d-1}{d}\frac{1}{\rho^\epsilon\rho^{\epsilon'}}\tau_1(\theta^{\epsilon'})\Delta_x\nabla_x\theta^{\epsilon'}\,.$$

The terms in $\Psi_{1,2}(\varrho, v, \eta)$ involving A^ρ defined in (2.2) have the form

$$\begin{aligned}
&\sum_{m=1}^d \frac{A^\rho(\rho^\epsilon,\theta^\epsilon,\nabla_x\rho^\epsilon,\nabla_x\theta^\epsilon)-A^\rho(\rho^\epsilon,\theta^\epsilon,\nabla_x\rho^\epsilon,\nabla_x\theta^{\epsilon'})}{\partial_{x_m}\theta^\epsilon-\partial_{x_m}\theta^{\epsilon'}}\partial_{x_m}\eta : \nabla_x^2\rho^{\epsilon'}\\
&+\sum_{m=1}^d \frac{A^\rho(\rho^\epsilon,\theta^\epsilon,\nabla_x\rho^\epsilon,\nabla_x\theta^{\epsilon'})-A^\rho(\rho^\epsilon,\theta^\epsilon,\nabla_x\rho^{\epsilon'},\nabla_x\theta^{\epsilon'})}{\partial_{x_m}\rho^\epsilon-\partial_{x_m}\rho^{\epsilon'}}\partial_{x_m}\varrho : \nabla_x^2\rho^{\epsilon'}\\
&+\frac{A^\rho(\rho^\epsilon,\theta^\epsilon,\nabla_x\rho^{\epsilon'},\nabla_x\theta^{\epsilon'})-A^\rho(\rho^\epsilon,\theta^{\epsilon'},\nabla_x\rho^{\epsilon'},\nabla_x\theta^{\epsilon'})}{\theta^\epsilon-\theta^{\epsilon'}}\eta : \nabla_x^2\rho^{\epsilon'}\\
&+\frac{A^\rho(\rho^\epsilon,\theta^{\epsilon'},\nabla_x\rho^{\epsilon'},\nabla_x\theta^{\epsilon'})-A^\rho(\rho^{\epsilon'},\theta^{\epsilon'},\nabla_x\rho^{\epsilon'},\nabla_x\theta^{\epsilon'})}{\rho^\epsilon-\rho^{\epsilon'}}\varrho : \nabla_x^2\rho^{\epsilon'}\,.
\end{aligned}$$

The rest of the terms in $\Psi_{1,2}(\varrho, v, \eta)$ involving A^θ, B^ρ, B^θ, as well as terms in $\Psi_{1,3}(\varrho, v, \eta)$ have similar forms as above. The zeroth-order operator in the ϱ-equation in (4.5) has the form

$$\Psi_{0,1}(\varrho, v) = -\varrho\nabla_x\cdot u^{\epsilon'} - v\cdot\nabla_x\rho^{\epsilon'}\,,$$

It is clear that given $(\rho^\epsilon-\bar\rho,\, u^\epsilon,\, \theta^\epsilon-\bar\theta)$, $(\rho^{\epsilon'}-\bar\rho,\, u^{\epsilon'},\, \theta^{\epsilon'}-\bar\theta) \in X_{T_0,M_0}$, the linear estimate applies to the above system. Therefore we have

(4.6)
$$\begin{aligned}
\sup_{[0,T_0]}\left(\|\varrho\|_{H^1}^2 + \|(v,\eta)\|_{L^2}^2\right) &\le c(\epsilon-\epsilon')\int_0^{T_0}\left(\|\Delta_x^2\rho^\epsilon(\cdot,s)\|_{H^1}^2 + \|\Delta_x^2 u^\epsilon(\cdot,s)\|_{L^2}^2 + \|\Delta_x^2\theta^\epsilon(\cdot,s)\|_{L^2}^2\right)\mathrm{d}s\\
&\le c(\epsilon-\epsilon')T_0M_0\,.
\end{aligned}$$

Thus $(\rho^\epsilon-\bar\rho,\, u^\epsilon,\, \theta^\epsilon-\bar\theta)$ is a Cauchy sequence in $C([0,T_0];L^2)$. Because it is also a bounded sequence in $L^\infty([0,T_0];H^{s+1})\times L^\infty([0,T_0];H^s\times H^s)$ we conclude that it is a Cauchy sequence in $C([0,T_0];H^{s'+1})\times C([0,T_0];H^{s'}\times H^{s'})$ for any $0<s'<s$. Therefore there exists (ρ,u,θ) such that

$$\begin{gathered}
\rho^\epsilon-\bar\rho \to \rho-\bar\rho \quad \textit{in}\ C([0,T_0];H^{s'+1})\,,\\
(u^\epsilon,\, \theta^\epsilon-\bar\theta) \to (\rho-\bar\rho,\, u,\, \theta-\bar\theta) \quad \textit{in}\ C([0,T_0];H^{s'}).
\end{gathered}$$

In addition, by the weak compactness of $(\rho^\epsilon-\bar\rho,\, u^\epsilon,\, \theta^\epsilon-\bar\theta)$ in $L^\infty([0,T_0];H^{s+1}\times H^s\times H^s)$ we also have that $(\rho,u,\theta)\in L^\infty([0,T_0];H^{s+1}\times H^s\times H^s)$.

By Fatou's Lemma, for all $\alpha\in\mathbb{N}^d$, $1\le|\alpha|\le s_1$

$$\begin{gathered}
\langle x\rangle^2\,\partial_x^\alpha(\rho-\bar\rho)\in L^\infty([0,T_0];H^1)\,,\\
\langle x\rangle^2\,\partial_x^\alpha(u,\theta-\bar\theta)\in L^\infty([0,T_0];L^2)\,,
\end{gathered}$$

By interpolation it is clear that for each $1\le l\le d$ and each $\alpha\in\mathbb{N}^d$ with $1\le|\alpha|\le s_1$ one has

$$\left.\begin{aligned}
x_l\,\partial_x^\alpha\rho^\epsilon &\longrightarrow x_l\,\partial_x^\alpha\rho && \text{in } C([0,T_0];H^1)\,,\\
x_l\,\partial_x^\alpha u^\epsilon &\longrightarrow x_l\,\partial_x^\alpha u && \text{in } C([0,T_0];L^2)\,,\\
x_l\,\partial_x^\alpha\theta^\epsilon &\longrightarrow x_l\,\partial_x^\alpha\theta && \text{in } C([0,T_0];L^2)\,,
\end{aligned}\right\}\quad \text{as } \epsilon\to 0\,.$$

Apply ∂_x^α to the system (4.5) for (ϱ, v, η) and multiply the result by $\langle x\rangle^2$. Using a similar argument as the L^2 convergence (4.6), we can show that for each $\alpha \in \mathbb{N}^d$ with $1 \le |\alpha| \le s_1$ one has

$$\left.\begin{aligned} \langle x\rangle^2 \partial_x^\alpha \rho^\epsilon &\longrightarrow \langle x\rangle^2 \partial_x^\alpha \rho \qquad \text{in } C([0,T_0]; H^1)\,, \\ \langle x\rangle^2 \partial_x^\alpha u^\epsilon &\longrightarrow \langle x\rangle^2 \partial_x^\alpha u \qquad \text{in } C([0,T_0]; L^2)\,, \\ \langle x\rangle^2 \partial_x^\alpha \theta^\epsilon &\longrightarrow \langle x\rangle^2 \partial_x^\alpha \theta \qquad \text{in } C([0,T_0]; L^2)\,, \end{aligned}\right\} \quad \text{as } \epsilon \to 0\,.$$

Based on the above results, we see that if we let $\epsilon \to 0$ then (ρ, u, θ) is a smooth solution to the nonlinear system with $(\rho - \bar\rho, u, \theta - \bar\theta) \in C([0,T_0]; H^{s'+1} \times H^{s'} \times H^{s'})$ for any $0 < s' < s$.

To show that $(\rho - \bar\rho) \in C([0,T_0]; H^{s+1})$ and $(u, \theta - \bar\theta) \in C([0,T_0]; H^s)$, we apply the standard technique for quasi-linear equations [18]. The proof consists of two steps. First we show that

$$(\rho - \bar\rho) \in C_w([0,T_0]; H^{s+1})\,, \qquad (u,\, \theta - \bar\theta) \in C_w([0,T_0]; H^s)\,, \tag{4.7}$$

where $C_w([0,T]; X)$ denotes the weak topology in the sense that if $f \in C_w([0,T]; X)$ then $\langle f, \phi\rangle \in C[0,T]$ for any $\phi \in X'$. The weak continuity (4.7) follows from the uniform bounds of the solution in $L^\infty(0,T_0; H^{s+1} \times H^s \times H^s)$, the strong continuity of the solution in $C([0,T_0]; H^{s'+1} \times H^{s'} \times H^{s'})$ for any $s' < s$, and the density argument.

To prove that the weak continuity is in fact a strong one, we need to show that $\|(\rho - \bar\rho)(t)\|_{H^{s+1}}$ and $\|(u,\, \theta - \bar\theta)(t)\|_{H^s}$ are continuous for $t \in [0,T_0]$. To this end, we work with their equations. To simplify the notation, denote $\tilde\rho = \rho - \bar\rho$, $\tilde\theta = \theta - \bar\theta$. Recall that by the energy estimates

$$\|\tilde\rho\|_{L^\infty(0,T_0; H^{s+1})} < \infty\,, \quad \|(u, \tilde\theta)\|_{L^\infty(0,T_0; H^s)} + \|(u, \tilde\theta)\|_{L^2(0,T_0; H^{s+1})} < \infty\,. \tag{4.8}$$

Now take the s-th derivatives of the equations for $(u, \tilde\theta)$. We have

$$\partial_t(\partial_x^\alpha u) = \tfrac{d-1}{d} \tfrac{\tau_1}{\rho} \Delta_x \nabla_x (\partial_x^\alpha \tilde\theta) + \Psi_2(\partial_x^{\alpha_1} u) + \Psi_2(\partial_x^{\alpha_2} \tilde\theta) + \Psi_2(\partial_x^{\alpha_3} \tilde\rho) + F_1(\partial_x^{\alpha_4} u,\, \partial_x^{\alpha_5} \tilde\theta,\, \partial_x^{\alpha_6} \tilde\rho)\,,$$

$$\partial_t(\partial_x^\alpha \theta) = \tfrac{4(d-1)}{d} \tfrac{\tau_4}{\rho} \Delta_x \nabla_x \cdot (\partial_x^\alpha u) + \Psi_2(\partial_x^{\alpha_1} u) + \Psi_2(\partial_x^{\alpha_2} \tilde\theta) + \Psi_2(\partial_x^{\alpha_3} \tilde\rho) + F_2(\partial_x^{\alpha_4} u,\, \partial_x^{\alpha_5} \tilde\theta,\, \partial_x^{\alpha_6} \tilde\rho)\,,$$

where $|\alpha| = |\alpha_1| = |\alpha_2| = |\alpha_3| = s$, $|\alpha_4| \le s-1$, $|\alpha_5| \le s-1$, $|\alpha_6| \le s+1$, and $F_1(\cdot)$ is a polynomial in terms of its variables. It is easy to see from the equation that $(F_1, F_2) \in L^\infty(0,T_0; L^2)$. Here we do not require Ψ_2 to be homogenous of order 2. To annihilate the leading-order dispersive terms in the above system, multiply the u-equation by $\tau_4 \partial_x^\alpha u$, the $\tilde\theta$-equation by $\frac14 \tau_1 \partial_x^\alpha \tilde\theta$, integrate them over $\mathbb{R}^d$, and add these two integrated equations together. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d} \tau_4 |\partial_x^\alpha u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} \tfrac14 \tau_4 |\partial_x^\alpha \tilde\theta|^2 \,\mathrm{d}x\right) = F_3\,,$$

where

$$F_3 = \langle \partial_x^\alpha u,\ \Psi_2(\partial_x^{\alpha_1}\rho,\, \partial_x^{\alpha_2} u,\, \partial_x^{\alpha_3}\tilde\theta) + F_1\rangle + \langle \partial_x^\alpha \tilde\theta,\ \Psi_2(\partial_x^{\alpha_1}\rho,\, \partial_x^{\alpha_2} u,\, \partial_x^{\alpha_3}\tilde\theta) + F_2\rangle\,,$$

with $|\alpha| = |\alpha_1| = |\alpha_2| = |\alpha_3| = s$. Here again we do not need Ψ_2 to be homogeneous. It simply denotes that the leading order of the operator is 2. Note that because τ_1, τ_4 depend on t, there are commutator terms containing $\frac{\mathrm{d}}{\mathrm{d}t}\tau_1$ and $\frac{\mathrm{d}}{\mathrm{d}t}\tau_4$ but they are of zeroth order in terms of $(\partial_x^\alpha u, \partial_x^\alpha \tilde\theta)$. By integration by parts and (4.8), we have $F_3 \in L^1(0,T_0)$. Therefore,

$$\left(\int_{\mathbb{R}^d} \tau_4 |\partial_x^\alpha u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} \tfrac14 \tau_4 |\partial_x^\alpha \tilde\theta|^2 \,\mathrm{d}x\right) \in W^{1,1}(0,T_0) \hookrightarrow C[0,T_0]\,,$$

by Sobolev imbedding in $\mathbb{R}^1$. To conclude that $\|(u,\, \theta - \bar\theta)\|_{H^s} \in C[0,T_0]$ we just need to note that τ_1, τ_4 are continously differentiable functions with positive lower bounds.

To prove that $\|\rho-\bar{\rho}\|_{H^{s+1}} \in C[0,T_0]$, we consider the regularized ρ as in Step 4 in Section 2. First, the equation for $\partial_x^\alpha \rho$ is

$$\partial_t(\partial_x^\alpha \rho) + \rho \nabla_x \cdot (\partial_x^\alpha u) + u \cdot \nabla_x(\partial_x^\alpha \rho) = \Psi_0(\partial_x^{\alpha_1}\rho, \partial_x^{\alpha_2} u) + F_0(\partial_x^{\gamma_1}\rho, \partial_x^{\gamma_2} u)\,,$$

where $|\alpha| = |\alpha_1| = |\alpha_2| = s$ and $|\gamma_1|, |\gamma_2| \leq s-1$. Using the equations for $\partial_x^\alpha u$ and $\partial_x^\alpha \theta$, we have the equation for the regularzied ϱ as

$$\partial_t(\partial_x^\alpha \varrho) + u \cdot \nabla_x(\partial_x^\alpha \varrho) = \Psi_0(\partial_x^{\alpha_1}\rho, \partial_x^{\alpha_2} u, \partial_x^{\alpha_3}\theta)\,.$$

Note that we have included all the lower-order terms in Ψ_0. Now we differential the above equation once to get

$$\partial_t(\partial_x^\sigma \varrho) + u \cdot \nabla_x(\partial_x^\sigma \varrho) = \Psi_0(\partial_x^{\sigma_1}\rho, \partial_x^{\sigma_2} u, \partial_x^{\sigma_3}\theta) \in L^2(0,T;L^2)\,,$$

where $|\sigma| = |\sigma_1| = |\sigma_2| = |\sigma_3| = s+1$. Multiply the above equation by $\partial_x^\sigma \varrho$, integrate over $\mathbb{R}^d$ and use integration by parts. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\sigma \varrho\|_{L^2} = F_4 \in L^1(0,T_0)\,.$$

Thus $\|\varrho\|_{H^{s+1}} \in C[0,T_0]$. Notice that ρ differs from ϱ by $T_3 z_1$ and $T_4 z_2$ as in Step 4 in Section 2 where T_3, T_4 are operators of order -2 and z_1, z_4 are given by $(\partial_x^\alpha u, \partial_x^\alpha \theta)$ with $|\alpha| = s$. Hence the continuity of (u,θ) in H^s ensures the continuity of $\|T_3 z_1 + T_4 z_2\|_{H^{s+1}}$. We thereby conclude the continuity of $\|(\rho-\bar{\rho})(t)\|_{H^{s+1}}$ for $t \in [0,T_0]$ thus the strong continuity of $(\rho-\bar{\rho}, u, \theta-\bar{\theta})$ in $C([0,T_0]; H^{s+1}\times H^s \times H^s)$.

To show the uniqueness of the classical solution, notice that if $U_1 = (\rho_1, u_1, \theta_1)$, $U_2 = (\rho_2, u_2, \theta_2)$ are two solutions with the same initial, then the difference $U_1 - U_2$ satisfies

$$\partial_t(U_1 - U_2) = \mathcal{L}(U_1)U_1 - \mathcal{L}(U_2)U_2\,, \qquad (U_1-U_2)(x,0) = 0\,, \tag{4.9}$$

Similar as we have done for (4.5), the difference equation (4.9) has the form

$$\partial_t(U_1 - U_2) = \mathcal{L}(U_1)(U_1-U_2) + \Psi_1(U_1-U_2)\,.$$

By the linear estimate in Section 2 with $\epsilon = 0$, we have

$$\sup_{[0,T_0]} \|U_1 - U_2\|_{L^2} \leq c\,\|(U_1-U_2)(x,0)\|_{H^1} = 0\,,$$

which proves the uniqueness.

The proof for the stability follows similarly by comparing the equations for $\partial_x^{\alpha_1} U_1$, $\partial_x^\alpha U_2$ and the equations for $\langle x\rangle^2 \partial_x^\alpha U_1$, $\langle x\rangle^2 \partial_x^{\alpha_2} U_2$ where $\alpha_1, \alpha_2 \in \mathbb{N}^d$ with $1 \leq |\alpha_1| \leq s$, $1 \leq |\alpha_2| \leq s_1$ and the linear estimates for the differences.

Hence, there exists a unique solution (ρ, u, θ) to the original DNS system (1.1) such that

$$\rho - \bar{\rho} \in C\left([0,T_0]; H^{s+1}\right) \cap C^1\left((0,T_0]; H^s\right) \cap C\left([0,T_0]; H^{s_1}_{\langle x\rangle^2}\right)\,,$$

$$(u, \theta-\bar{\theta}) \in C\left([0,T_0]; H^s\right) \cap C^1\left((0,T_0]; H^{s-3}\right) \cap C\left([0,T_0]; H^{s_1-1}_{\langle x\rangle^2}\right)\,,$$

with $s_1 > d/2 + 6$, $s = \max\{s_1 + 6; N + d/2 + 5\}$. □

Appendix A. Operator Bounds in Weighted Spaces

In this appendix we show the operator bounds of T in the weighted spaces $H^s_{\langle x\rangle^2}$, where T is defined in (2.63) and $s \in \mathbb{R}$. Recall that $T \in OPS^{-1}$. Denote the symbol of T as $a(x,\xi)$. Then $a(x,\xi)$ is supported outside of $B(0,R)$ in ξ. Recall the definition of the space $H^s_{\langle x\rangle^2}$

$$H^s_{\langle x\rangle^2} = \{f : \langle x\rangle^2 f \in H^s\}, \qquad s \in \mathbb{R}.$$

We can use the bound of T in the unweighted spaces H^s to obtain its bound in the weighted space $H^s_{\langle x\rangle^2}$. Indeed, for f in the Schwartz space,

$$\|Tf\|_{L^2_{\langle x\rangle^2}} = \|\langle x\rangle^2 Tf\|_{L^2} = \|\langle x\rangle^2 T(\tfrac{1}{\langle x\rangle^2}(\langle x\rangle^2 f))\|_{L^2}.$$

One way to see that $\langle x\rangle^2 T \circ \frac{1}{\langle x\rangle^2} \in OPS^{-1}$ is via the asymptotic expansion of $T \circ \frac{1}{\langle x\rangle^2}$. Symbol of the operator $\langle x\rangle^2 T \circ \frac{1}{\langle x\rangle^2}$ satisfies

$$\langle x\rangle^2 a\sharp\tfrac{1}{\langle x\rangle^2} \equiv \langle x\rangle^2 \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}(\partial_\xi^\alpha a)(\partial_x^\alpha \tfrac{1}{\langle x\rangle^2}) \qquad \mathrm{mod}\ S^{-\infty},$$

where a is the symbol of T and $a\sharp\frac{1}{\langle x\rangle^2}$ denotes the symbol of the composite operator $T(\frac{1}{\langle x\rangle^2}\cdot)$. Since $\langle x\rangle^2(\partial_x^\alpha \frac{1}{\langle x\rangle^2})$ is bounded for all α and x, the above symbol is in the same class S^{-1} as $a(x,\xi)$. However, the asymptotic expansion does not show directly that the S^0-seminorms are of order R^{-1} because $\langle x\rangle^2 a\sharp\frac{1}{\langle x\rangle^2}$ is not necessarily supported outside of $B(0,R)$. Thus, to show that the operator bound is of size R^{-1}, we need a precise form of $\langle x\rangle^2 a\sharp\frac{1}{\langle x\rangle^2}$. To this end, define

$$\mathcal{L}_\eta = (1+|y|^2)^{-1}(I-\Delta_\eta), \qquad \mathcal{L}_y = (1+|\eta|^2)^{-1}(I-\Delta_y).$$

Then for any $N \in \mathbb{N}$,

$$\mathcal{L}_\eta^N e^{iy\cdot\eta} = \mathcal{L}_y^N e^{iy\cdot\eta} = e^{iy\cdot\eta}.$$

Thus the integral form [9] of the symbol $\langle x\rangle^2 a\sharp\frac{1}{\langle x\rangle^2}$ is

$$\begin{aligned} \langle x\rangle^2\, a\sharp\tfrac{1}{\langle x\rangle^2} &= \langle x\rangle^2 \int_{\mathbb{R}^{2d}} a(x,\xi+\eta)\,\frac{1}{\langle x-y\rangle^2}\,\mathcal{L}_\eta^N \mathcal{L}_y^N\, e^{iy\cdot\eta}\,\mathrm{d}\eta\,\mathrm{d}y \\ &= \langle x\rangle^2 \int_{\mathbb{R}^{2d}} \left(\mathcal{L}_\eta^{*,N}\, a(x,\xi+\eta)\right)\left(\mathcal{L}_y^{*,N}\left(\tfrac{1}{\langle x-y\rangle^2}\right)\right) e^{iy\cdot\eta}\,\mathrm{d}\eta\,\mathrm{d}y. \end{aligned} \tag{A.1}$$

where

$$\mathcal{L}_\eta^{*,N} = (1+|\eta|)^{-N}(I-\Delta_\eta)^N, \qquad \mathcal{L}_y^{*,N} = (I-\Delta_y)^N\left((1+|y|^2)^{-N}\cdot\right). \tag{A.2}$$

Thus,

$$\left|\langle x\rangle^2\, a\sharp\tfrac{1}{\langle x\rangle^2}\right| \le \langle x\rangle^2 \int_{\mathbb{R}^d} |\mathcal{L}_y^{*,N}\left(\tfrac{1}{\langle x-y\rangle^2}\right)|\,\mathrm{d}y \int_{\mathbb{R}^d} |\mathcal{L}_\eta^{*,N}\, a(x,\xi+\eta))|\,\mathrm{d}\eta.$$

Note that since $a \in S^{-1}$ and is supported outside of $B(0,R)$, we have for every $N > d/2$,

$$\int_{\mathbb{R}^d} |\mathcal{L}_\eta^{*,N}\, a(x,\xi+\eta))|\,\mathrm{d}\eta \le C_N \sum_{|\beta|\le 2N} \int_{\mathbb{R}^d} (1+|\eta|^2)^{-N} |\partial_\eta^\beta a(x,\xi+\eta)|\,\mathrm{d}\eta \le C_N R^{-1}, \tag{A.3}$$

because $|\partial_\eta^\beta a(x,\eta)| \le C_\beta R^{-1}$ for all $|\beta| \ge 0$.

The estimate for the term $\int_{\mathbb{R}^d} |\mathcal{L}_y^{*,N}\left(\frac{1}{\langle x-y\rangle^2}\right)|\,\mathrm{d}y$ is as follows: we only need to check the worst term where all the derivatives go to $\frac{1}{\langle x-y\rangle^2}$, which is $\langle x\rangle^2\int_{\mathbb{R}^d}|(I-\Delta_y)^N(\frac{1}{\langle x-y\rangle^2})|\,(1+|y|^2)^{-N}\,\mathrm{d}y$. Then for $N\geq\frac{d+2}{2}$,

$$
\begin{aligned}
&\langle x\rangle^2\int_{\mathbb{R}^d}|(I-\Delta_y)^N(\tfrac{1}{\langle x-y\rangle^2})|\,(1+|y|^2)^{-N}\,\mathrm{d}y\\
&=\langle x\rangle^2\int_{\mathbb{R}^d}|(I-\Delta_y)^N(\tfrac{1}{\langle y\rangle^2})|\,(1+|x-y|^2)^{-N}\,\mathrm{d}y\\
&\leq c_N\langle x\rangle^2\int_{\mathbb{R}^d}\frac{1}{\langle y\rangle^2}(1+|x-y|^2)^{-N}\,\mathrm{d}y\\
&\leq c_N\int_{\mathbb{R}^d}\frac{\langle x-y\rangle^2\langle y\rangle^2}{\langle y\rangle^2}(1+|x-y|^2)^{-N}\,\mathrm{d}y\\
&\leq c_N\,,
\end{aligned}
\tag{A.4}
$$

Here c_N denotes a constant depending on N. It may change from line to line by multiples of some generic constants. Thus by (A.3) and (A.4), the $(0,0)$-seminorm of $\langle x\rangle^2 a\sharp\frac{1}{\langle x\rangle^2}$ is of size R^{-1}.

Estimates of general (α,β)-seminorms follow in the same way. By (A.1),

$$
\begin{aligned}
&\partial_x^\alpha\partial_\xi^\beta\langle x\rangle^2\,a\sharp\tfrac{1}{\langle x\rangle^2}\\
&=\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}c_{\alpha_i}\partial_x^{\alpha_1}\left(\langle x\rangle^2\right)\int_{\mathbb{R}^{2d}}\left(\mathcal{L}_\eta^{*,N}\,\partial_x^{\alpha_2}\partial_\xi^\beta a(x,\xi+\eta)\right)\left(\mathcal{L}_y^{*,N}\partial_x^{\alpha_3}\left(\tfrac{1}{\langle x-y\rangle^2}\right)\right)e^{iy\cdot\eta}\,\mathrm{d}\eta\,\mathrm{d}y\,.
\end{aligned}
$$

Each term in the above summation satisfies

$$
\begin{aligned}
&\langle\xi\rangle^\beta\left|\partial_x^{\alpha_1}\left(\langle x\rangle^2\right)\int_{\mathbb{R}^{2d}}\left(\mathcal{L}_\eta^{*,N}\,\partial_x^{\alpha_2}\partial_\xi^\beta a(x,\xi+\eta)\right)\left(\mathcal{L}_y^{*,N}\partial_x^{\alpha_3}\left(\tfrac{1}{\langle x-y\rangle^2}\right)\right)e^{iy\cdot\eta}\,\mathrm{d}\eta\,\mathrm{d}y\right|\\
&\leq\langle\xi\rangle^\beta\int_{\mathbb{R}^d}(1+|\eta|^2)^{-N}\left|(I-\Delta_\eta)^N\,\partial_x^{\alpha_2}\partial_\xi^\beta a(x,\xi+\eta)\right|\,\mathrm{d}\eta\\
&\quad\times\left|\partial_x^{\alpha_1}\langle x\rangle^2\right|\int_{\mathbb{R}^d}\left|(I-\Delta_y)^N\,\left((1+|y|^2)^{-N}\partial_x^{\alpha_3}\tfrac{1}{\langle x-y\rangle^2}\right)\right|\,\mathrm{d}\eta\,,
\end{aligned}
\tag{A.5}
$$

where the first factor on the right-hand side of (A.5) has the bound

$$
\begin{aligned}
&\langle\xi\rangle^{|\beta|}\int_{\mathbb{R}^d}(1+|\eta|^2)^{-N}\left|(I-\Delta_\eta)^N\,\partial_x^{\alpha_2}\partial_\xi^\beta a(x,\xi+\eta)\right|\,\mathrm{d}\eta\\
&\leq C_N\langle\xi\rangle^{|\beta|}\int_{|\xi+\eta|\geq R}(1+|\eta|^2)^{-N}(1+|\xi+\eta|^2)^{-\beta-1}\,\mathrm{d}\eta\\
&=C_N\langle\xi\rangle^{|\beta|}\int_{|\eta|\geq R}(1+|\xi-\eta|^2)^{-N}(1+|\eta|^2)^{-\beta-1}\,\mathrm{d}\eta\leq C_NR^{-1}\,,
\end{aligned}
$$

for $N>\frac{d+|\beta|}{2}$. The second factor on the right-hand side of (A.5) is bounded in the same way as in (A.4). Thus the (α,β)-seminorms of $\langle x\rangle^2 a\sharp\frac{1}{\langle x\rangle^2}$ is of size R^{-1} for all α,β. By the Calderón-Vaillancourt theorem, the operator norms $\|\langle x\rangle^2 T\circ\frac{1}{\langle x\rangle^2}\|_{H^s\to H^s}$ are of size R^{-1} for $s=0$. This is equivalent to that $\|T\|_{H^s_{\langle x\rangle^2}\to H^s_{\langle x\rangle^2}}$ is of size R^{-1} for $s=0$. Note that similar calculations as in (A.4) holds if we change $\langle x\rangle^2$ to $\langle x\rangle^{-2}$. Therefore the norm $\|T\|_{H^s_{\langle x\rangle^{-2}}\to H^s_{\langle x\rangle^{-2}}}$ is also of size R^{-1} for $s=0$.

General weighted H^s-norms can be obtained by considering the composition $\Lambda^s(\langle x\rangle^2 T\circ\frac{1}{\langle x\rangle^2})$. or $\Lambda^s(\langle x\rangle^{-2}T\circ\langle x\rangle^2)$. Let

$$w=\langle x\rangle^2 a\sharp\tfrac{1}{\langle x\rangle^2}\,,\qquad T_w=\langle x\rangle^2 T\circ\frac{1}{\langle x\rangle^2}\,,$$

or

$$w=\frac{1}{\langle x\rangle^2}\,a\sharp\langle x\rangle^2\,,\qquad T_w=\frac{1}{\langle x\rangle^2}\,T\circ\langle x\rangle^2\,,$$

We just proved that

$$\sup_x|\langle\xi\rangle^{|\beta|}\partial_x^\alpha\partial_\xi^\beta w|\le C_{\alpha,\beta}R^{-1}\,,\qquad\text{for any multi-indices }\alpha,\beta. \tag{A.6}$$

Now we want to show that the symbol of $\Lambda^s\circ T_w$, denoted as $\langle\xi\rangle^s\sharp w$, satisfies

$$\sup_x|\langle\xi\rangle^{-s+|\beta|}\partial_x^\alpha\partial_\xi^\beta(\langle\xi\rangle^s\sharp w)|\le C_{\alpha,\beta}R^{-1}\,,\qquad\text{for any multi-indices }\alpha,\beta. \tag{A.7}$$

We apply a similar estimate as in (A.5). First,

$$\begin{aligned}&\partial_x^\alpha\partial_\xi^\beta(\langle\xi\rangle^s\sharp w)\\&=\sum_{\beta_1+\beta_2=\beta}c_{\beta_i}\int_{\mathbb{R}^{2d}}\left(\mathcal{L}_\eta^{*,N}\,\partial_\xi^{\beta_1}\langle\xi+\eta\rangle^s\right)\left(\mathcal{L}_y^{*,N}\partial_x^\alpha\partial_\xi^{\beta_2}w(x-y,\xi)\right)e^{iy\cdot\eta}\,\mathrm{d}\eta\,\mathrm{d}y\,.\end{aligned} \tag{A.8}$$

Note that

$$\left|\mathcal{L}_\eta^{*,N}\,\partial_\xi^{\beta_1}\langle\xi+\eta\rangle^s\right|\le C_{N,\beta_1}(1+|\eta|^2)^{-N}\langle\xi+\eta\rangle^{s-|\beta_1|}\,,$$

and by (A.6),

$$\left|\mathcal{L}_y^{*,N}\partial_x^\alpha\partial_\xi^{\beta_2}w(x-y,\xi)\right|\le C_{N,\beta_2,\alpha}R^{-1}(1+|y|^2)^{-N}\langle\xi\rangle^{-|\beta_2|}\,.$$

Thus each term in the summation on the right-hand side of (A.8) satisfies

$$\begin{aligned}&\langle\xi\rangle^{-s+|\beta|}\left|\int_{\mathbb{R}^{2d}}\left(\mathcal{L}_\eta^{*,N}\,\partial_\xi^{\beta_1}\langle\xi+\eta\rangle^s\right)\left(\mathcal{L}_y^{*,N}\partial_x^\alpha\partial_\xi^{\beta_2}w(x-y,\xi)\right)e^{iy\cdot\eta}\,\mathrm{d}\eta\,\mathrm{d}y\right|\\&\le\langle\xi\rangle^{-s+|\beta|}\int_{\mathbb{R}^d}\left|\left(\mathcal{L}_\eta^{*,N}\,\partial_\xi^{\beta_1}\langle\xi+\eta\rangle^s\right)\right|\,\mathrm{d}\eta\int_{\mathbb{R}^d}\left|\left(\mathcal{L}_y^{*,N}\partial_x^\alpha\partial_\xi^{\beta_2}w(x-y,\xi)\right)\right|\,\mathrm{d}y\\&\le C_{N,\beta}R^{-1}\langle\xi\rangle^{-s+|\beta_1|}\int_{\mathbb{R}^d}(1+|\eta|^2)^{-N}\langle\xi+\eta\rangle^{s-|\beta_1|}\,\mathrm{d}\eta\\&\le C_{N,\beta}R^{-1}\,,\end{aligned}$$

by taking $N>\frac{|s|+d}{2}$. Thus (A.7) holds for any $s\in\mathbb{R}$. This implies $\|\Lambda^s T_w\|_{H^s\to L^2}$ is of size R^{-1} for any $s\in\mathbb{R}$. This is equivalent to that T_w is bounded in H^s and its operator norm is of size R^{-1}. This is further equivalent to that $\|T\|_{H^s_{\langle x\rangle^2}}$ and $\|T\|_{H^s_{\langle x\rangle^{-2}}}$ are of size R^{-1} for all $s\in\mathbb{R}$, which thereby completes the estimates of T in the weighted Sobolev spaces.

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