SINGULAR LIMIT OF A DISPERSIVE NAVIER-STOKES SYSTEM WITH AN ENTROPY STRUCTURE

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ABSTRACT. We prove a low Mach number limit for a dispersive fluid system [3] which contains third-order corrections to the compressible Navier-Stokes. We show that the classical solutions to this system in the whole space \( \mathbb{R}^n \) converge to classical solutions to ghost-effect systems [7]. Our analysis follows the framework in [4], which is built on the methodology developed by Métivier and Schochet [6] and Alazard [1] for systems up to the second order. The key new ingredient is the application of the entropy structure of the dispersive fluid system. This structure enables us to treat cases not covered in [4] and to simplify the analysis in [4].

1. INTRODUCTION

In this paper we establish a low Mach number limit of a dispersive Navier-Stokes system (DNS) derived from kinetic equations in [3]. The governing equations for this system are

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho U) &= 0, \\
\partial_t (\rho U) + \nabla \cdot \left( \rho U \otimes U \right) + \nabla p &= \nabla \cdot \Sigma + \nabla \cdot \tilde{\Sigma}, \\
\partial_t (\rho e) + \nabla \cdot \left( \rho e U + \rho \Theta U \right) &= \nabla \cdot \left( \Sigma U - q \right) + \nabla \cdot \left( \tilde{\Sigma} U + \tilde{q} \right),
\end{align*}
\]

(1.1)

where \((\rho(x, t), U(x, t), \Theta(x, t))\) denote the density, velocity, and temperature of the fluid at time \(t \in \mathbb{R}^+\) and position \(x \in \mathbb{R}^3\) respectively. The terms \(p, q, \Sigma\) represent the pressure, heat flux, and viscous stress tensor (for a Newtonian fluid) respectively with

\[
\begin{align*}
p &= p(\rho, \Theta) = \rho \Theta, \\
q(\Theta) &= -\tilde{k}(\Theta) \nabla \Theta, \\
\Sigma &= \tilde{\mu}(\Theta) \nabla U + (\nabla U)^T - \frac{2}{3} (\nabla \cdot U) I.
\end{align*}
\]

The total energy density is given by

\[
\rho e = \frac{1}{2} \rho |U|^2 + \frac{3}{2} \rho \Theta.
\]

The quantities \(\tilde{\Sigma}\) and \(\tilde{q}\) denote dispersive corrections to the stress tensor and heat flux respectively. They are given by

\[
\begin{align*}
\tilde{\Sigma} &= \hat{\tau}_1(\Theta) \left( \nabla^2 \Theta - \frac{1}{3} (\Delta \Theta) I \right) + \hat{\tau}_2(\Theta) \left( \nabla \Theta \otimes \nabla \Theta - \frac{1}{3} |\nabla \Theta|^2 I \right) \\
&\quad + \hat{\tau}_3(\rho, \Theta) \left( \nabla U \otimes (\nabla U)^T - (\nabla U)^T \nabla U \right), \\
\tilde{q} &= \hat{\tau}_4(\Theta) \left( \nabla U + \frac{1}{3} |\nabla \Theta| \cdot U \right) + \hat{\tau}_5(\Theta) \nabla \Theta \cdot (\nabla U + (\nabla U)^T - \frac{2}{3} (\nabla \cdot U) I) \\
&\quad + \hat{\tau}_6(\rho, \Theta) \left( \nabla U - (\nabla U)^T \right) \cdot \nabla \Theta.
\end{align*}
\]
Here we assume that the transport coefficients \( \hat{\tau}_1, \cdots, \hat{\tau}_6 \) are \( C^\infty \) functions of their variables. By linearizing the DNS system (1.1) around constant states one can see that the coupling of \( \nabla_x \cdot \hat{\Sigma} \) and \( \nabla_x \cdot \hat{\varphi} \) forms a dispersive relation.

One main feature of the DNS system is that it possesses an entropy structure: by the derivation in [3], the transport coefficients in \( \hat{\Sigma} \) and \( \hat{\varphi} \) satisfy
\[
\hat{\tau}_4 = \frac{\Theta}{2} \hat{\tau}_1, \quad \hat{\tau}_2 + 2\hat{\tau}_5 = \partial_\Theta \left( \frac{\hat{\tau}_4}{\Theta^2} \right),
\]
(1.2)
such that
\[
\hat{\Sigma} : \frac{\nabla_x U}{\Theta} + \hat{\varphi} \cdot \frac{\nabla_x \Theta}{\Theta^2} = \nabla_x \cdot \left( \frac{\hat{\tau}_1}{2\Theta} \nabla_x \Theta \cdot \left( \nabla_x U + (\nabla_x U)^T - \frac{2}{3}(\nabla_x \cdot U)I \right) \right).
\]
Note that (1.2) implies that \( \hat{\tau}_1 \) and \( \hat{\tau}_4 \) have the same sign. Under assumption (1.2) one can show that the DNS system dissipates the Euler entropy in the same way as the Navier-Stokes system.

This paper continues our previous work [4] on dispersive Navier-Stokes systems, where a low Mach number limit is established for \( \hat{\tau}_1, \hat{\tau}_4 > 0 \) without using the entropy structure. The positivity assumption of \( \hat{\tau}_1, \hat{\tau}_4 \) is essential for the analysis in [4] because it guarantees that operators in the form \( T = I - \hat{\tau}_1 \Delta_x \) are invertible. It is the invertibility of \( T \) that provides the annihilation of the most singular terms in the energy estimates. Thus this type of analysis does not generalize to the case where \( \hat{\tau}_1, \hat{\tau}_4 < 0 \). However, the original DNS system derived in [3] indeed has negative \( \hat{\tau}_1, \hat{\tau}_4 \) and the corresponding transport coefficient in the limiting ghost-effect system is also negative [7]. Our goal in this paper is to show that the low Mach limit for the negative \( \hat{\tau}_1, \hat{\tau}_4 \) can be established by using the extra entropy structure (1.2). Moreover, our analysis here applies to the case with positive \( \hat{\tau}_1, \hat{\tau}_4 \) as well and it is more straightforward than that in [4]. The main reason is that the relation between \( \hat{\tau}_1 \) and \( \hat{\tau}_4 \) in (1.2) shows that the DNS system has an antisymmetric structure which allows us to perform direct energy estimates. Our result in this paper shows the DNS system indeed recovers genuine ghost-effect systems instead of merely ghost-effect-type systems in [4].

We also want to point out that compared with the original system derived from kinetic equations in [3], system (1.1) is simplified in the sense that terms involving second order derivatives of \( \rho \) are not included within \( \nabla_x \cdot \hat{\Sigma} \) and \( \nabla_x \cdot \hat{\varphi} \). This renders a second assumption that \( \hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_4, \hat{\tau}_5 \) are independent of \( \rho \) since the entropy structure needs to be satisfied.

The limiting \textit{ghost effect system} can be formally derived from kinetic equations using a Hilbert expansion method (cf. Sone [7]). This is a system \textit{beyond} classical fluid equations that describes the phenomenon in which the temperature field of the fluid has finite variations, and the flow is driven by the gradient of the temperature field [7]. Let \((\rho, \vartheta, P^*)\) be the density, temperature, and pressure fields of the fluid. Then the ghost effect system in its general form is given by
\[
\begin{aligned}
\nabla_x (\rho \vartheta) &= 0, \\
\partial_t \vartheta + \nabla_x \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x P^* &= \nabla_x \cdot \hat{\Sigma} + \nabla_x \cdot \hat{\varphi}, \\
&+ \frac{2}{3} \partial_t (\rho \vartheta) + \frac{5}{2} \rho \vartheta \nabla_x \cdot u = -\nabla_x \cdot q,
\end{aligned}
\]
(1.3)
where \( P^* \) is the pressure, which is an independent variable for the ghost effect system. \( P^* \) can be viewed as a Lagrangian multiplier as in the case of incompressible dynamics. The viscous and heat conducting terms are written as
\[
\Sigma = \tilde{\mu}(\vartheta) \left( \nabla_x u + (\nabla_x u)^T - \frac{2}{3}(\nabla_x \cdot u)I \right), \quad q = -\tilde{\kappa}(\vartheta) \nabla_x \vartheta,
\]
where \( \tilde{\mu}(\theta), \tilde{\kappa}(\theta) > 0 \) are the viscosity and heat conductivity respectively. The thermal stress tensor \( \tilde{\Sigma} \) has the form

\[
\tilde{\Sigma} = \tilde{\tau}_1(\theta) \left( \nabla_x^2 \theta - \frac{1}{3} \Delta_x \theta I \right) + \tilde{\tau}_2(\theta) \left( \nabla_x \theta \otimes \nabla_x \theta - \frac{1}{3} |\nabla_x \theta|^2 I \right),
\]

where \( \tilde{\tau}_1(\cdot), \tilde{\tau}_2(\cdot) \in C^\infty(\mathbb{R}) \) are transport coefficients with \( \tilde{\tau}_1 \leq 0 \). The term \( \nabla_x \cdot \tilde{\Sigma} \) contains third-order derivatives and is not derivable from the Navier-Stokes system.

In order to derive (1.3) from (1.1), we introduce the same scaling as in [4]. Let \( \epsilon \) be the Knudsen number. We have

\[
\begin{align*}
\tilde{\mu} &= \epsilon \tilde{\mu}, & \tilde{\kappa} &= \epsilon \tilde{\kappa}, & \tilde{\tau}_i &= \epsilon^2 \tilde{\tau}_i, & i = 1, \ldots, 6, \\
p &= p_0 + O(\epsilon), & U \sim O(\epsilon), & \nabla_x \Theta \sim O(1), & p = \rho \Theta,
\end{align*}
\]

where \( p_0 \) is a constant which is fixed as 1. Thus, we consider the case when \( p \) is around 1 and \( \Theta \) has a finite variation. To emphasize the dependence of \( (\rho, U, \Theta) \) on \( \epsilon \), we denote the solution as \( (\rho_\epsilon, U_\epsilon, \Theta_\epsilon) \) and the pressure as \( p_\epsilon \). In order to ensure the positivity of both \( p_\epsilon \) and \( \Theta_\epsilon \), as usual we write \( (p_\epsilon, U_\epsilon, \Theta_\epsilon) \) as

\[
\begin{align*}
p_\epsilon &= e^{\rho_\epsilon}, & U_\epsilon &= \epsilon u^\epsilon, & \Theta_\epsilon &= e^{\Theta_\epsilon}, & \rho_\epsilon &= e^{\rho_\epsilon},
\end{align*}
\]

where the superscripts denote the fluctuation functions. We will consider a long-time scale such that

\[
t = \frac{1}{\epsilon}.
\]

For simplicity we subsequently drop the hat from \( \hat{t} \) and use \( t \) in the system for the fluctuations. The dispersive Navier-Stokes system (1.1) thereby transforms into its scaled analogue

\[
\begin{align*}
\frac{3}{5}(\partial_t + u^\epsilon \cdot \nabla_x)p^\epsilon + \frac{1}{\epsilon} \nabla_x \cdot (u^\epsilon - \frac{2}{5} e^{-ep^\epsilon} \kappa(\theta^\epsilon) \nabla_x \theta^\epsilon) &= \frac{2}{5} \epsilon e^{-ep^\epsilon} \left( (\tilde{\Sigma} + \tilde{\Sigma} \Theta) : \nabla_x u^\epsilon + \nabla_x \cdot \tilde{q} \right) \\
+ \frac{2}{5} \epsilon e^{-ep^\epsilon} \kappa(\theta^\epsilon) \nabla_x p^\epsilon \cdot \nabla_x \theta^\epsilon, \\
e^{-\rho_\epsilon}(\partial_t + u^\epsilon \cdot \nabla_x)u^\epsilon + \frac{1}{\epsilon} \nabla_x p^\epsilon &= e^{-\rho_\epsilon} \left( \nabla_x \cdot (\tilde{\Sigma} + \tilde{\Sigma} \Theta) \nabla_x u^\epsilon + \nabla_x \cdot \tilde{q} \right), \\
\frac{3}{5}(\partial_t + u^\epsilon \cdot \nabla_x)\theta^\epsilon + \epsilon \nabla_x u^\epsilon = e^2 e^{-ep^\epsilon} \left( (\tilde{\Sigma} + \tilde{\Sigma} \Theta) : \nabla_x u^\epsilon + \nabla_x \cdot \tilde{q} \right) \\
- e^{-ep^\epsilon} \nabla_x \cdot q,
\end{align*}
\]

\[
(p^\epsilon, u^\epsilon, \theta^\epsilon)(x, 0) = (p_\epsilon^{in}, u_\epsilon^{in}, \theta_\epsilon^{in})(x),
\]

where the constitutive relations inherited from those for (1.1) are

\[
\Theta_\epsilon = e^{\Theta_\epsilon}, \quad q = -\kappa(\theta^\epsilon) \nabla_x \theta^\epsilon, \quad \Sigma = \mu(\theta^\epsilon) \left( \nabla_x u^\epsilon + (\nabla_x u^\epsilon)^T - \frac{2}{3} \nabla_x \cdot u^\epsilon \right),
\]

\[
\tilde{\Sigma} = \tilde{\tau}_1(\theta^\epsilon) \left( \nabla_x^2 \theta^\epsilon - \frac{1}{3} (\Delta_x \theta^\epsilon) I + \tau_2(\theta^\epsilon) \left( \nabla_x \theta^\epsilon \otimes \nabla_x \theta^\epsilon - \frac{1}{3} \nabla_x \theta^\epsilon \right)^2 I \right) \\
+ \epsilon^2 \tau_3(\theta^\epsilon) \left( \nabla_x u^\epsilon \left( \nabla_x u^\epsilon \right)^T - \left( \nabla_x u^\epsilon \right)^T \nabla_x u^\epsilon \right),
\]

\[
\tilde{q} = \tau_4(\theta^\epsilon) \left( \Delta_x u^\epsilon + \frac{1}{2} \nabla_x \nabla_x \cdot u^\epsilon \right) + \tau_6(\epsilon p^\epsilon, \theta^\epsilon) \left( \nabla_x u^\epsilon \left( \nabla_x u^\epsilon \right)^T - \left( \nabla_x u^\epsilon \right)^T \nabla_x u^\epsilon \right) \cdot \nabla_x \theta^\epsilon \\
+ \tau_5(\theta^\epsilon) \nabla_x \theta^\epsilon \cdot (\nabla_x u^\epsilon + (\nabla_x u^\epsilon)^T - \frac{2}{3} \nabla_x \cdot u^\epsilon),
\]

while

\[
\begin{align*}
\kappa(\theta^\epsilon) &= \tilde{\kappa}(\Theta_\epsilon) \Theta_\epsilon, & \mu(\theta^\epsilon) &= \tilde{\mu}(\Theta_\epsilon), & \rho_\epsilon &= P_\epsilon / \Theta_\epsilon = e^{ep^\epsilon - \rho_\epsilon}, \\
\tau_1(\theta^\epsilon) &= \tilde{\tau}_1(\Theta_\epsilon) \Theta_\epsilon, & \tau_4(\theta^\epsilon) &= \tilde{\tau}_4(\Theta_\epsilon), \\
\tau_2(\theta^\epsilon) &= \tilde{\tau}_1(\Theta_\epsilon) \Theta_\epsilon + \tilde{\tau}_2(\Theta_\epsilon) \Theta_\epsilon^2, & \tau_3(\epsilon p^\epsilon, \theta^\epsilon) &= \tilde{\tau}_3(\rho_\epsilon, \Theta_\epsilon), \\
\tau_5(\theta^\epsilon) &= \tilde{\tau}_5(\Theta_\epsilon) \Theta_\epsilon, & \tau_6(\epsilon p^\epsilon, \theta^\epsilon) &= \tilde{\tau}_6(\rho_\epsilon, \Theta_\epsilon). \tag{1.8}
\end{align*}
\]
If we assume that for all indices \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) such that \(|\alpha| \leq 3\),
\[
(\nabla_x p^\epsilon, \nabla_x^a u^\epsilon, \nabla_x^a \theta^\epsilon) (t, x) \to (0, \nabla_x^a u, \nabla_x^a \theta)(t, x) \quad \text{as } \epsilon \to 0,
\]
then (1.6) converges formally to the system
\[
\begin{align*}
\nabla_x \cdot \left( \frac{5}{2} u - \kappa(\theta) \nabla_x \theta \right) &= 0, \\
\epsilon^{-\theta} (\partial_t u + u \cdot \nabla_x u) + \nabla_x P^s = \nabla_x \cdot \Sigma + \nabla_x \cdot \tilde{\Sigma}, \\
\frac{5}{2}(\partial_t + u \cdot \nabla_x) \theta + \nabla_x \cdot u &= -\nabla_x \cdot q,
\end{align*}
\]
where \( P^s \) is the pressure which is an independent variable. By (1.9) we have
\[
\Sigma = \mu(\theta \cdot 0) \left( \nabla_x u + (\nabla_x u)^T - \frac{2}{3}(\nabla_x \cdot u) I \right), \\
\tilde{\Sigma} = \tau_1(\theta) \left( \nabla_x^2 \theta - \frac{1}{3} \Delta_x \Theta I \right) + \tau_2(\theta) \left( \nabla_x \theta \otimes \nabla_x \theta - \frac{1}{3} |\nabla_x \theta|^2 I \right).
\]
Here \( \mu(\theta) \) is the viscosity, \( \kappa(\theta) \) is the heat conductivity, and \( \tau_1(\theta), \tau_2(\theta) \in C^\infty(\mathbb{R} \times \mathbb{R}) \) are transport coefficients with \( \tau_1 \leq 0 \) as defined in (1.8).

We can write the limiting system (1.10) in the form of a ghost system as (1.3) by taking \((\varrho, u, \theta) = (\varrho, u, e^\theta)\) as the variables. System (1.10) then has the form
\[
\begin{align*}
\varrho \theta &= 1, \\
\partial_t \varrho + \nabla_x \cdot (\varrho u) &= 0, \\
\partial_t (\varrho u) + \nabla_x \cdot (\varrho u \otimes u) + \nabla_x P^s &= \nabla_x \cdot \Sigma + \nabla_x \cdot \tilde{\Sigma}, \\
\nabla_x \cdot \left( \frac{5}{2} u - \kappa(\theta) \nabla_x \theta \right) &= 0,
\end{align*}
\]
where \( \Sigma \) and \( \tilde{\Sigma} \) are defined in (1.11).

Our goal in this paper is to rigorously justify the above formal calculation of the low Mach number limit from the dispersive system (1.6) to the ghost effect system (1.10) in the whole space.

2. Main Theorem and Outline of the Proof

In this section we will state the main theorem and give an outline of its proof. We adopt the same notation as in [4].

**Definition 2.1.** Let \( \alpha_0, \theta \) be two constants and \( \alpha_0 > 0 \). For each \( \epsilon, t, s > 0 \), define the norms
\[
\| (p, u, \theta - \theta) \|_{\epsilon, s, t} := \sup_{[0, t]} \left( \| (p, u)(t) \|_{H^s} + \| \Lambda_{\epsilon}^{2s+1} (\epsilon p, \epsilon u, \theta - \theta)(t) \|_{H^{s+1}} \right)
\]
\[
+ \alpha_0 \left( \int_0^t \| \nabla_x (u, p) \|_{H^s}^2 + \| \nabla_x \Lambda_{\epsilon}^{2s+1} (\epsilon u, \theta) \|_{H^{s+1}}^2 \| (p, u, \theta - \theta(t)) \|_{H^{s+1}}^2 d\tau \right)^{\frac{1}{2}},
\]
and
\[
\| (p, u, \theta - \theta) \|_{\epsilon, s, 0} := \| (p^m, u^m) \|_{H^s} + \| \Lambda_{\epsilon}^{2s+1} (\epsilon p^m, \epsilon u^m, \theta^m - \theta) \|_{H^{s+1}},
\]
where \( H^s = H^s(\mathbb{R}^3) \) is the usual Sobolev space such that derivatives up to order \( s \) of a function are square-integrable. The invertible operator \( \Lambda_{\epsilon} \) is defined by
\[
\Lambda_{\epsilon} := (I - \epsilon^2 \Delta_x)^{1/2}
\]
for any \( \epsilon > 0 \).
Definition 2.2. Let \((p^\epsilon, u^\epsilon, \theta^\epsilon)\) be the solution to (1.6). We call \(\psi = (\epsilon p^\epsilon, \epsilon u^\epsilon, \theta^\epsilon)\) the slow motion because it varies on the time scale of order one. We call \((p^\epsilon, u^\epsilon)\) the fast motion because it varies on the time scale of order \(\epsilon\).

The main theorem states

Main Theorem. Let \(s \geq 6\). Suppose that there exist constants \(M_0, A, c_0, \sigma\) such that the initial data of the fluctuations \((p^\infty, u^\infty, \theta^\infty)\) satisfy

\[
\| (p^\infty, u^\infty, \theta^\infty - \overline{\theta}) \|_{L^s} \leq M_0, \\
(\theta^\infty - \overline{\theta}, \Pi(\epsilon^{-\sigma} u^\infty)) \rightarrow (\theta^\infty - \overline{\theta}, u^\infty) \quad \text{in} \quad H^s(\mathbb{R}^3), \\
|\nabla \theta^\infty| \leq c_0 |x|^{-2-\sigma},
\]

where \(\Pi\) is the projection onto the divergence free part of \(\epsilon^{-\sigma} u^\infty\). Then there exists \(T > 0\) such that for any \(\epsilon \in (0,1]\), the Cauchy problem for system (1.6) has a unique solution

\[
(p^\epsilon, u^\epsilon, \theta^\epsilon - \overline{\theta}) \in C([0,T]; H^s(\mathbb{R}^3)) \cap L^\infty(0,T; H^{3s+2}(\mathbb{R}^3)) \cap L^2(0,T; H^{3s+3}(\mathbb{R}^3)),
\]

for all \(s_1 < 3s + 1\). Moreover, there exists a positive constant \(M\) depending only on \(T, M_0\) such that

\[
\| (p^\epsilon, u^\epsilon, \theta^\epsilon - \overline{\theta}) \|_{L^s,T} \leq M.
\]

Furthermore, the sequence of solutions \((p^\epsilon, u^\epsilon, \theta^\epsilon)\) to system (1.6) converges weakly-* in \(L^\infty(0,T; H^s(\mathbb{R}^3))\) and strongly in \(L^2(0,T; H^{s'}_{loc}(\mathbb{R}^3))\) for all \(s < s\) to the limit \((0, u, \theta)\), where \((u, \theta)\) satisfies the ghost effect system (1.10).

The low-Mach number limit of fluid equations has a long history. The interested reader is referred to the references given in (for example) [1, 6]. Our proofs of the main theorem follows the same framework as in [4], which is built upon the methodology developed by Métivier and Schochet [6] and Alazard [1, 2]. In [6] Métivier and Schochet proved the incompressible limit for the non-isentropic Euler equations for classical solutions with general initial data. In [1] Alazard proved the low Mach number limit for the compressible Navier-Stokes for classical solutions with general initial data. In [2] he proved the low Mach number limit for the compressible Navier-Stokes with general equations of state and a large source term in the temperature equation. The analysis in [1] considers different types of estimates in different ranges of wave numbers. Here we will close the energy estimate directly by studying the energy bounds of the slow motion and the fast motion in order without separating the wave numbers.

The main difficulty in closing the energy estimate in this paper remains the same as in [4]: we need to simultaneously remove the most singular terms in (1.6), which include the terms associated with \(\epsilon\) and the third-order terms in the dispersive part. Again we need to take into account of the anti-symmetric structure of those terms. Specifically, let \(U^\epsilon = (p^\epsilon, u^\epsilon, \theta^\epsilon)\) be the solution to system (1.6) and \(\psi^\epsilon = (\epsilon p^\epsilon, \epsilon u^\epsilon, \theta^\epsilon)\), then (1.6) can be reformulated as

\[
A_1(\psi^\epsilon)(\partial_t + u \cdot \nabla_x)U^\epsilon + \frac{1}{\epsilon} A_2(\psi^\epsilon)U^\epsilon = A_3(\psi^\epsilon)U^\epsilon + \mathcal{R},
\]

where \(A_1(\psi^\epsilon)\) is a diagonal matrix, \(A_2(\psi^\epsilon)U^\epsilon\) are formed by certain combinations of the singular terms with the leading orders from the dispersive terms, \(A_3(\psi^\epsilon)U^\epsilon\) are the dissipative terms, and \(\mathcal{R}\) includes the rest of the dispersive terms. In [4], when the coefficients \(\tau_1, \tau_2\) are unrelated, the term \(A_2(\psi^\epsilon)U^\epsilon\) is not readily anti-symmetric. However, it is anti-symmetrizable by a certain symmetrizer matrix composed of symmetric positive operators. As pointed out in the introduction, the positivity of the operators in the symmetrizer matrix essentially depends on the fact that \(\tau_1, \tau_2\) are both positive.
Thus this symmetrizing method fails when \( \hat{\tau}_1, \hat{\tau}_4 < 0 \). The key observation in resolving this difficulty is that if one makes use of the intrinsic entropy structure of the DNS system, then \( \hat{\tau}_1 \) and \( \hat{\tau}_4 \) are related by (1.2). It turns out this particular relation shows \( A_2(\psi^\epsilon) \psi^\epsilon \) is actually anti-symmetric. As a consequence, energy estimates can then be carried out in a rather straightforward way because the most singular term \( A_2(\psi^\epsilon) \psi^\epsilon \) naturally disappears in the \( L^2 \)-type estimates. We want to mention that this argument applies to the case where \( \hat{\tau}_1, \hat{\tau}_4 \) are both positive as well.

Proof of the main theorem consists of two main steps: uniform bounds of the solution \( (p^\epsilon, u^\epsilon, \theta^\epsilon) \) in an appropriate norm and local energy decay of the fast motion \( (p^\epsilon, u^\epsilon) \). We will only show the details for deriving the uniform bounds because the local energy decay does not depend on the signs of \( \hat{\tau}_1, \hat{\tau}_4 \) and is proved in exactly the same way as in [4]. The proposition for the uniform bounds states

**Proposition 2.3.** For each fixed \( \epsilon > 0 \), let \( (p^\epsilon, u^\epsilon, \theta^\epsilon) \in C([0,T]; H^s(\mathbb{R}^3)) \) be the solution to the scaled DNS system (1.6). Let

\[
\Omega = \| (p^\epsilon, u^\epsilon, \theta^\epsilon - \theta) \|_{\epsilon, s, T}, \quad \Omega_0 = \| (p^\epsilon_{in}, u^\epsilon_{in}, \theta^\epsilon_{in} - \theta) \|_{\epsilon, s, 0}.
\]

Then there exists an increasing function \( C(\cdot) \) such that

\[
\|(p^\epsilon, u^\epsilon, \theta^\epsilon - \theta)\|_{\epsilon, s, T} \leq C(\Omega_0) e^{(\sqrt{T} + \epsilon)} C(\Omega),
\]

which further indicates [6] that there exists \( T_0 > 0 \) independent of \( \epsilon \) such that \( \|(p^\epsilon, u^\epsilon, \theta^\epsilon - \theta)\|_{\epsilon, s, T} \) are uniformly bounded in \( \epsilon \) over \([0, T_0]\).

The rest of the paper is devoted to the proof of Proposition 2.3. The outline of the proof is as follows. In Section 3 we derive the a priori estimate (Proposition 3.1) for the slow motion \( \psi^\epsilon = (\epsilon p^\epsilon, \epsilon u^\epsilon, \theta^\epsilon) \). This estimate implies addition bounds on various time derivatives of the fast motion, the slow motion \( \psi^\epsilon \), and another slowly varying quantity \( \text{curl}(e^{\theta^\epsilon} u^\epsilon) \). Estimates of these additional bounds only depend on the order of the system. In particular, they are not affected by the signs of \( \hat{\tau}_1, \hat{\tau}_4 \). Thus these estimates are identical to those in [4] and are omitted in this paper. In Section 4 we derive the a priori estimate for the fast motion \( (p^\epsilon, u^\epsilon) \). This is done in two steps: first we show the bounds for certain time derivatives of the fast motion (Proposition 4.1). Then we prove the desired \( H^s \)-bounds for the fast motion using Proposition 4.1. Again since the second step does not depend on the signs of \( \hat{\tau}_1, \hat{\tau}_4 \), we will omit its details in this paper. Overall, in this paper we will focus on details for the a priori estimate of the slow motion (Proposition 3.1) and the bounds for time derivatives of the fast motion (Proposition 4.1).

3. A Priori Estimates – Slow Motion

In this section we derive a priori estimates for slow motions which vary on the time scale of order one. Estimates in the next section for fast motions depend heavily on those for slow motions.

Assume there exist two constants \( \alpha_1, \alpha_2 > 0 \) such that

\[
0 < \alpha_1 < h(\epsilon p_0, g(\theta_0), \mu(\theta_0), \kappa(\theta_0)) < \alpha_2 < \infty.
\]  

(3.1)

For \( t > 0 \) and \( s > 5 \), recall the norm \( \| \cdot \|_{\epsilon, s, t} \) defined by

\[
\| (p, u, \theta) \|_{\epsilon, s, t} := \sup_{[0,t]} \left( \| (p, u) \|_{H^s} + \| \Lambda_{\epsilon}^{2s+1}(\epsilon p, \epsilon u, \theta) \|_{H^{s+1}} \right) 
\]

\[
+ \alpha_0 \left( \int_0^t \left( \| \nabla_x(u, p) \|_{H^s} + \| \nabla_x \Lambda_{\epsilon}^{2s+1}(\epsilon u, \theta) \|_{H^{s+1}} \right)^2 (\tau) \right)^{\frac{1}{2}},
\]

(3.2)
We also define
\[
R = R(t) = \|(p, u)\|_{H^s} + \|\Lambda_\epsilon^{2s+1} (ep, eu, \theta)\|_{H^{s+1}},
\]
\[
R' = R'(t) = \left( \|(\nabla_x u, \nabla_x p)\|_{H^s} + \|\nabla_x \Lambda_\epsilon^{2s+1} (eu, \theta)\|_{H^{s+1}} \right)^{1/2}.
\]
Thus \(\|(p, u, \theta)\|_{c,s,T} = \sup_{[0,t]} R(s) + \left( \int_0^t (R'(s))^2 \, ds \right)^{1/2}\). We will repeatedly use the following basic calculus inequalities (see for example [3]): for \(|k_1| + |k_2| + \cdots + |k_n| = s\) with \(s > d + 2 = 5\),
\[
\|\nabla_x^{k_1} u_1 \nabla_x^{k_2} u_2 \cdots \nabla_x^{k_n} u_n\|_{L^2} \leq \|u_1\|_{H^s} \|u_2\|_{H^s} \cdots \|u_n\|_{H^s}, \quad \forall n \in \mathbb{N},
\]
\[
\|f, \nabla_x^m g\|_{L^2(\mathbb{R}^3)} \leq c_m (\|\nabla_x f\|_{L^\infty} \|\nabla_x^{m-1} g\|_{L^2} + \|\nabla_x^m f\|_{L^2} \|g\|_{L^\infty})
\]
\[
\leq c_m (\|\nabla_x f\|_{L^\infty} \|g\|_{H^{s-1}} + \|f\|_{H^s} \|g\|_{L^\infty}),
\]
\[
\|\nabla_x^m (fg)\|_{L^2(\mathbb{R}^3)} \leq c_m (\|f\|_{L^\infty} \|\nabla_x^m g\|_{L^2} + \|\nabla_x^m f\|_{L^2} \|g\|_{L^\infty})
\]
\[
\leq c_m (\|\nabla_x f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}),
\]
for any \(1 \leq |m| \leq s\).

### 3.1. A Priori Estimate of the Slow Motion

The a priori estimate for the slow motion \((ep, eu, \theta)\) is stated in the following proposition.

**Proposition 3.1.** Let \((p, u, \theta)\) be a solution to the system (1.6). Suppose \(s > 5\). There exists a constant \(\alpha_0 > 0\) such that if we define the norm \(\|\cdot\|_{c,s,T}\) as
\[
\|(p, u, \theta)\|_{c,s,T}^2 = \sup_{[0,t]} \|\Lambda_\epsilon^{2s+1} (ep, eu, \theta)\|_{H^{s+1}} + \alpha_0 \left( \int_0^t \|\nabla_x \Lambda_\epsilon^{2s+1} (eu, \theta)\|_{H^{s+1}}^2 (\tau) \, d\tau \right)^{1/2},
\]
then for all \(T \in [0, 1]\), we have
\[
\|(p, u, \theta)\|_{c,s,T} \leq C(\Omega_0) e^{(\sqrt{T+\epsilon})C(\Omega)},
\]
where
\[
\Omega = \|(p, u, \theta)\|_{c,s,T}, \quad \Omega_0 = \|(p, u, \theta)\|_{c,s,0}.
\]

**Proof.** Define
\[
(\hat{p}, \hat{u}, \hat{\theta}) = (ep - \theta, eu, \theta).
\]
Algebraic calculation using (1.6) shows that the system for \((\hat{p}, \hat{u}, \hat{\theta})\) has the form
\[
(\partial_t + u \cdot \nabla_x) \hat{p} + \frac{1}{\epsilon} \nabla_x \cdot \hat{u} = 0,
\]
\[
g(\theta)(\partial_t + u \cdot \nabla_x) \hat{u} + \frac{1}{\epsilon} \nabla_x \hat{p} + \frac{1}{\epsilon} \nabla_x \hat{\theta} = h(\hat{p}) \nabla_x \cdot \Sigma + \epsilon h(\hat{p}) \nabla_x \cdot \hat{\Sigma},
\]
\[
\frac{3}{2} (\partial_t + u \cdot \nabla_x) \hat{\theta} + \frac{1}{\epsilon} \nabla_x \cdot \hat{\theta} = \epsilon h(\hat{p}) \nabla_x \cdot \hat{\Sigma} + \epsilon h(\hat{p}) \hat{\Sigma} \cdot \nabla_x \hat{u}
\]
\[
+ h(\hat{p}) \Sigma : \nabla_x \hat{u} - h(\hat{p}) \nabla_x \cdot q,
\]
\[
(p, u, \theta)(x, 0) = (p^{in}, u^{in}, \theta^{in})(x).
\]
Thus the leading-orders in the dispersive terms form an antisymmetric structure, which will vanish in 

Note that by (1.2) and (1.8), we have

where

\[ q = -\kappa(\theta) \nabla \dot{\theta}, \quad \Sigma = \mu(\theta) (\nabla \dot{u} + (\nabla \dot{u})^T - \frac{2}{3}(\nabla \cdot \dot{u}) \mathbf{I}), \]

\[ \tilde{\Sigma} = \tau_1(\theta) \left( \nabla \dot{\theta}^2 - \frac{1}{3}(\Delta \dot{\theta}) \mathbf{I} \right) + \tau_2(\theta) \left( \nabla \theta \otimes \nabla \dot{\theta} - \frac{1}{3}(\nabla \theta \cdot \nabla \dot{\theta}) \mathbf{I} \right) \]

\[ + \tau_3(\epsilon p, \theta) (\nabla_x \dot{u}^T - \frac{2}{3}(\nabla \cdot \dot{u}) \mathbf{I}), \]

\[ \bar{q} = \tau_4(\theta) (\Delta \dot{u} + \frac{1}{3} \nabla_x \nabla_x \cdot \dot{u}) + \tau_6(\epsilon p, \theta) (\nabla_x \dot{u} - (\nabla_x \dot{u})^T) \cdot \nabla \theta, \]

\[ + \tau_5(\theta) \nabla_x \theta \cdot (\nabla_x \dot{u} + (\nabla_x \dot{u})^T - \frac{2}{3}(\nabla \cdot \dot{u}) \mathbf{I}). \]

Note that the equation for \( \dot{p} = \epsilon p - \theta \) is simply the linearized density equation. Let \( \Psi_k \) denote any \( k \)-th order differential operator. The general form of \( \Psi_k w \) is

\[ \Psi_k w = a(x) \nabla_x^\alpha w, \]

where \( |\alpha| = k \). Use this notation and apply \( \nabla_x^\alpha \) with \( |\alpha| \leq s + 1 \) to (3.11). We have

\[ g(\theta)(\partial_t + u \cdot \nabla_x) \dot{u}_\alpha + \frac{1}{\epsilon} \nabla_x \cdot \dot{u}_\alpha = R_{1,\alpha}, \]

\[ \frac{3}{2}(\partial_t + u \cdot \nabla_x) \dot{\theta}_\alpha + \frac{1}{\epsilon} \nabla_x \cdot \dot{\theta}_\alpha = \frac{3}{2} \epsilon h(\dot{\theta}) \tau_1(\theta) \nabla_x \dot{u}_\alpha + R_{2,\alpha}, \]

\[ \frac{3}{2}(\partial_t + u \cdot \nabla_x) \tau_1(\theta) \nabla_x \dot{u}_\alpha + \frac{1}{\epsilon} \nabla_x \cdot \tau_1(\theta) \dot{u}_\alpha = \frac{3}{2} \epsilon h(\dot{\theta}) \tau_1(\theta) \nabla_x \dot{u}_\alpha + \frac{3}{2} \epsilon h(\dot{\theta}) \tau_1(\theta) \nabla_x \dot{u}_\alpha + R_{3,\alpha}, \]

\[ (\dot{\theta}_\alpha, \dot{u}_\alpha, \theta_\alpha)(x, 0) = (\dot{\theta}_\alpha^0, \dot{u}_\alpha^0, \theta_\alpha^0)(x), \]

where \( |\beta| = |\alpha| \) and \( |\nu| \leq \max\{ |\alpha| - 1, 0 \} \) and \( (\dot{\theta}_\alpha, \dot{u}_\alpha, \theta_\alpha) = (\nabla_x^\alpha \dot{p}, \nabla_x^\alpha \dot{u}, \nabla_x^\alpha \dot{\theta}) \). The coefficients of \( \Psi_k \) depend only on \( (\dot{p}, \dot{u}, \dot{\theta}) \) and \( \nabla_x^\gamma (\dot{p}, \dot{u}, \dot{\theta}) \) with \( |\gamma| \leq 3 \). The commutator terms are

\[ R_{1,\alpha} = -[\nabla_x^\alpha, u] \cdot \nabla_x \dot{p}, \quad R_{3,\alpha} = -[\nabla_x^\alpha, \frac{3}{2} u] \cdot \nabla_x \dot{\theta}. \]

\[ R_{2,\alpha} = -[\nabla_x^\alpha, g(\theta)] \partial_t \dot{u} - [\nabla_x^\alpha, g(\theta) u] \cdot \nabla_x \dot{u}, \]

Note that by (1.2) and (1.8), we have

\[ \frac{2}{3} \epsilon h(\dot{\theta}) \tau_1(\theta) = \frac{4}{3} \epsilon h(\dot{\theta}) \tau_4(\theta). \]

Thus the leading-orders in the dispersive terms form an antisymmetric structure, which will vanish in \( L^2 \)-type energy estimates. Now we perform the \( L^2 \) energy estimate by multiplying (3.12) by \( (\dot{p}_\alpha, \dot{u}_\alpha, \theta_\alpha) \) and integrating over \( \mathbb{R}^3 \). The estimates for each equation are as follows. By (3.4), we only need to check the leading order terms. For the \( \dot{p} \)-equation, we have

\[ \frac{1}{2} \frac{d}{dt} \| \dot{\theta}_\alpha \|^2_{L^2} + \frac{1}{\epsilon} \langle \dot{p}_\alpha, \nabla_x \cdot \dot{u}_\alpha \rangle \leq \frac{1}{2} \| \nabla_x u \|_{L^\infty} \| \dot{p}_\alpha \|^2_{L^2} + \| \dot{p}_\alpha \|_{L^2} \| R_{1,\alpha} \|_{L^2}. \]
Note that $R_{1,\alpha} = 0$ if $\alpha = 0$. For $1 \leq |\alpha| \leq s + 1$ we have
\[
\|R_{1,\alpha}\|_{L^2} \leq C_\alpha \left( \|\nabla_x u\|_{L^\infty} \|\nabla_x^{|\alpha|} \hat{p}\|_{L^2} + \|\nabla_x^{|\alpha|} u\|_{L^2} \|\nabla_x \hat{p}\|_{L^2} \right)
\leq C(||u||_{H^{s+1}} \|\nabla_x \hat{p}\|_{W^{1,\infty}} + ||u||_{W^{1,\infty}} \|\hat{p}\|_{H^{s+1}}),
\]
by (3.5). Thus,
\[
\frac{1}{2} \frac{d}{dt} \|\hat{p}_\alpha\|_{L^2}^2 + \frac{1}{\epsilon} \langle \hat{p}_\alpha, \nabla_x \cdot \hat{u}_\alpha \rangle \leq C(R)(1 + R').
\tag{3.14}
\]
Similarly, for any $m \in \mathbb{N}$,
\[
\frac{1}{2} \frac{d}{dt} \|\epsilon \nabla_x^m \rho_\alpha\|_{L^2}^2 + \frac{1}{\epsilon} \langle \epsilon \nabla_x^m \hat{p}_\alpha, \nabla_x \cdot (\epsilon \nabla_x^m \hat{u}_\alpha) \rangle
\leq \frac{1}{2} \left( \|\epsilon \nabla_x u\|_{L^\infty} + 1 \right) \|\epsilon \nabla_x^m \hat{p}_\alpha\|_{L^2}^2 + \frac{1}{2} \|\epsilon \nabla_x u\|_{L^\infty} \|\nabla_x \hat{p}\|_{L^2} + \frac{1}{2} \|\epsilon \nabla_x \hat{p}\|_{L^2},
\tag{3.15}
\]
where
\[
\|\epsilon \nabla_x^m R_{1,m+|\alpha|}\|_{L^2} \leq C(||\epsilon \nabla_x^m \nabla_x^\alpha u\|_{L^2} \|\nabla_x \hat{p}\|_{W^{1,\infty}} + ||u||_{W^{1,\infty}} \|\epsilon \nabla_x^m \nabla_x^\alpha \hat{p}\|_{L^2})
\leq C(||\epsilon \nabla_x^m \nabla_x^\alpha \hat{u}\|_{L^2} \|\nabla_x \hat{p}\|_{W^{1,\infty}} + ||u||_{W^{1,\infty}} \|\epsilon \nabla_x^m \nabla_x^\alpha \hat{p}\|_{H^{s+1}}). \tag{3.16}
\]
If we choose $m \leq 2s + 1$, then
\[
\|\epsilon \nabla_x^m R_{1,m+|\alpha|}\|_{L^2} \leq C(||\Lambda_x^{2s} \nabla_x \hat{u}\|_{H^{s+1}} \|\nabla_x \hat{p}\|_{W^{1,\infty}} + ||u||_{W^{1,\infty}} \|\Lambda_x^{2s+1} \nabla_x \hat{p}\|_{H^{s+1}})
\leq C(R)(1 + R'),
\tag{3.17}
\]
for any $m \leq 2s + 1$.

Next, we check the $\hat{u}$-equation. The $L^2$ estimate gives
\[
\frac{1}{2} \frac{d}{dt} \|\hat{u}_\alpha\|_{L^2}^2 + \frac{1}{\epsilon} \langle \epsilon \nabla_x \hat{p}_\alpha, \nabla_x \hat{u}_\alpha \rangle \leq C(R)(1 + R')
\tag{3.18}
\]
where $\alpha_0$ is determined by the (positive) lower bound of $\mu(\theta)$ and $\kappa(\theta)$. By (3.13),
\[
R_{2,\alpha} = [\nabla_x^\alpha, g(\theta)](u \cdot \nabla_x \hat{u}) + [\nabla_x^\alpha, g(\theta)](g(\theta)^{-1} \nabla_x p) + [\nabla_x^\alpha, g(\theta)](\Psi_2 + \epsilon \Psi_3)(\hat{u}, \hat{\theta})
\]
\[
+ [\nabla_x^\alpha, g(\theta)](\Psi_1 + \epsilon \Psi_0)(\hat{p}, \hat{u}, \hat{\theta}) - [\nabla_x^\alpha, g(\theta) u] \cdot \nabla_x \hat{u}.
\]
Note that $R_{2,\alpha} = 0$ for $\alpha = 0$. Thus, we need only to consider $|\alpha| \geq 2$. We show the details for estimates of the leading order terms, which are $[\nabla_x^\alpha, g(\theta)](g(\theta)^{-1} \nabla_x p)$, $[\nabla_x^\alpha, g(\theta)](\Psi_2 + \epsilon \Psi_3)(\hat{u}, \hat{\theta})$, and $[\nabla_x^\alpha, g(\theta) u] \cdot \nabla_x \hat{u}$. First,
\[
\|\nabla_x^\alpha, g(\theta)](g(\theta)^{-1} \nabla_x p)\|_{L^2} + \|\nabla_x^\alpha, g(\theta)](\Psi_2 + \epsilon \Psi_3)(\hat{u}, \hat{\theta})\|_{L^2} + \|\nabla_x^\alpha, g(\theta) u] \cdot \nabla_x \hat{u}\|_{L^2}
\leq C \left( ||g(\theta) - g(0)||_{W^{1,\infty}} ||g(\theta)^{-1} \nabla_x p||_{H^s} + ||g(\theta) - g(0)||_{H^{s+1}} ||g(\theta)^{-1} \nabla_x p||_{W^{1,\infty}} \right)
+ C \left( ||g(\theta) - g(0)||_{H^{s+1}} ||\Psi_2(\hat{u}, \hat{\theta})||_{W^{1,\infty}} + ||g(\theta) - g(0)||_{W^{1,\infty}} ||\Psi_2(\hat{u}, \hat{\theta})||_{H^s} \right)
+ C \left( ||g(\theta) - g(0)||_{H^{s+1}} ||\Psi_2(\hat{u}, \hat{\theta})||_{W^{1,\infty}} + ||g(\theta) - g(0)||_{W^{1,\infty}} ||\Psi_2(\hat{u}, \hat{\theta})||_{H^s} \right)
\leq C(R)(1 + R')$. 

We apply integration by part for the estimate related to $[\nabla^\alpha_x, g(\theta)](\epsilon\Psi_3(\hat{u}, \hat{\theta}))$. Specifically, suppose $\nabla^\alpha_x = \partial_{x_i} \nabla^{\alpha_i} = \partial_{x_i} \nabla^{\alpha_i}$ for some $i, j \in \{1, 2, 3\}$ and $\alpha', \alpha'' \in \mathbb{N}^3$ such that $|\alpha'| = |\alpha| - 1$ and $|\alpha''| = |\alpha| - 2$. Then

$$
\left(\nabla^\alpha_x, g(\theta)\right)(\epsilon\Psi_3(\hat{u}, \hat{\theta})) = \left(\partial_{x_i} \nabla^{\alpha_i}, g(\theta)\right)(\epsilon\Psi_3(\hat{u}, \hat{\theta})) + \left(\partial_{x_i} \left((\partial_{x_i} g(\theta)) \hat{u}\right), \nabla^{\alpha''}(\epsilon\Psi_3(\hat{u}, \hat{\theta}))\right)
$$

$$
\leq C \|\nabla_x \hat{u}\|_{H^{\alpha-1}}(\|g(\theta) - g(0)\|_{H^{\alpha}} + \|\epsilon\Psi_3(\hat{u}, \hat{\theta})\|_{L^\infty} \|\nabla_x \hat{u}\|_{H^{\alpha-1}} + \|\epsilon\Psi_3(\hat{u}, \hat{\theta})\|_{H^{\alpha-1}} + C \|\nabla_x \nabla_x g(\theta)\|_{L^\infty} \|\hat{u}\|_{H^{\alpha+1}} \|\epsilon\Psi_3(\hat{u}, \hat{\theta})\|_{H^{\alpha-1}}
$$

$$
\leq C(R)(1 + R') + \epsilon C(R)(R')^2.
$$

Thus,

$$
\frac{1}{2} \frac{d}{dt} \left(\nabla^\alpha_x, g(\theta)\right) + \frac{1}{\epsilon} \left(\nabla^\alpha_x, \nabla_x \hat{u}\right) + \frac{1}{\epsilon} \left(\nabla^\alpha_x, \nabla_x \hat{\theta}\right) + \alpha_0 \|\nabla_x \hat{u}\|_{L^2}^2
$$

$$
\leq \left(\nabla^\alpha_x, \frac{2}{\epsilon} \epsilon h(\hat{p}) \tau_1(\theta) \nabla_x \Delta_x (\epsilon \nabla_x)^m \hat{\theta}\right) + \epsilon C(R)(R')^2 + C(R)(1 + R').
$$

Applying $(\epsilon \nabla_x)^m$ will only affect the terms containing the fast motion in $R_{2, \alpha}$. We show the calculation for the leading-order terms involving the fast motion in $R_{2, \alpha}$:

$$
\|e^{m \nabla_x^{m+|\alpha|}} g(\theta)(u \cdot \nabla_x \hat{u})\|_{L^2} \leq \|e^{m \nabla_x^{m+|\alpha|}} g(\theta)\|_{L^\infty} \|u \cdot \nabla_x \hat{u}\|_{L^\infty} + \|g(\theta)\|_{L^\infty} \|e^{m \nabla_x^{m+|\alpha|}} u \cdot \nabla_x \hat{u}\|_{L^2}
$$

$$
\leq C\|\nabla_x \hat{u}\|_{H^{\alpha+1}} \|\nabla_x \hat{u}\|_{L^\infty} \|\Delta_x \hat{\theta}\|_{H^{\alpha}} + C\|\nabla_x \hat{u}\|_{L^\infty} \|\Delta_x \hat{\theta}\|_{H^{\alpha}} + \|\nabla_x \hat{u}\|_{L^\infty} \|\nabla_x \hat{u}\|_{L^\infty} \|\Delta_x \hat{\theta}\|_{H^{\alpha}}.
$$

The leading term (in the fast motion) in $e^{m \nabla_x^{m+|\alpha|}} g(\theta)(g(\theta)^{-1} \nabla_x \hat{u})$ is $(g(\theta)^{-1} \nabla_x g(\theta)) e^{m \nabla_x^{m+|\alpha|}} \hat{u}$. In order to control this term, we apply integration by parts:

$$
\left|e^{m \nabla_x^{m+|\alpha|}} \hat{u}, g(\theta) \right| \epsilon \left|e^{m \nabla_x^{m+|\alpha|}} \hat{u} \right| \epsilon \left|e^{m \nabla_x^{m+|\alpha|}} \hat{u} \right| \leq \left(\|\nabla_x \hat{u}\|_{H^{\alpha+1}} \|\Delta_x \hat{\theta}\|_{H^{\alpha}} + \|g(\theta)^{-1} \nabla_x g(\theta)\|_{L^\infty} \|\Delta_x \hat{\theta}\|_{H^{\alpha}} \right) \|\Delta_x \hat{\theta}\|_{H^{\alpha}}.
$$

Thus, if we restrict to $0 \leq m \leq 2s + 1$, then

$$
\frac{1}{2} \frac{d}{dt} \left(\nabla^\alpha_x, g(\theta)\right) + \frac{1}{\epsilon} \left(\nabla^\alpha_x, \nabla_x \hat{u}\right) + \frac{1}{\epsilon} \left(\nabla^\alpha_x, \nabla_x \hat{\theta}\right) + \alpha_0 \|\nabla_x \hat{u}\|_{L^2}^2
$$

$$
\leq \left(\nabla^\alpha_x, \frac{2}{\epsilon} \epsilon h(\hat{p}) \tau_1(\theta) \nabla_x \Delta_x (\epsilon \nabla_x)^m \hat{\theta}\right) + \epsilon C(R)(R')^2 + C(R)(1 + R').
$$

The structure of the $\hat{\theta}$ equation is similar to that of the $\hat{u}$ equation. Thus, the $L^2$ estimate for $\hat{\theta}$ gives

$$
\frac{1}{2} \frac{d}{dt} \left(\hat{\theta}, \frac{2}{\epsilon} \hat{\theta}\right) + \frac{1}{\epsilon} \left(\hat{\theta}, \nabla \cdot \hat{u}\right) + \alpha_0 \|\nabla \hat{\theta}\|_{L^2}^2
$$

$$
\leq \left(\hat{\theta}, \frac{2}{\epsilon} \epsilon h(\hat{p}) \tau_4(\theta) \nabla_x \Delta_x \cdot \hat{u}\right) + \epsilon C(R)(R')^2 + C(R)(1 + R').
$$
and
\[
\frac{1}{2} \frac{d}{dt} \left( (\epsilon \nabla_x)^m \tilde{\theta}_\alpha \right) + \frac{3}{2} (\epsilon \nabla_x)^m \tilde{\theta}_\alpha + \frac{1}{\epsilon} \left( (\epsilon \nabla_x)^m \tilde{\theta}_\alpha , \nabla_x \cdot (\epsilon \nabla_x)^m \tilde{u}_\alpha \right) + \alpha_0 \left\| \nabla_x (\epsilon \nabla_x)^m \tilde{\theta}_\alpha \right\| _{L^2}^2 \\
\leq \left( (\epsilon \nabla_x)^m \tilde{\theta}_\alpha , \frac{3}{2} \epsilon \delta (\hat{p}) \tau_3 (\hat{\theta}) \nabla_x \Delta_s \cdot (\epsilon \nabla_x)^m \tilde{u}_\alpha \right) + \epsilon C(R)(R')^2 + C(R)(1 + R').
\] (3.23)
for any \( m \in \mathbb{N} \).

Adding (3.14), (3.17), (3.18), (3.17), (3.22), and (3.23), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\| \Lambda^m (\hat{p}, \hat{u}) \right\| _{H^{s+1}}^2 + \frac{1}{2} \frac{d}{dt} \left\| \Lambda^m \hat{u} \right\| _{H^{s+1}}^2 + \alpha_0 \left\| \nabla_x (\Lambda^m \hat{u}, \Lambda^m \hat{\theta}) \right\| _{H^{s+1}}^2 \\
\leq \epsilon C(R)(R')^2 + C(R)(1 + R').
\]
By the lower bound (3.1) for \( g(\theta) \), we have that
\[
\left\| \Lambda^m (\hat{p}, \hat{u}, \hat{\theta}) \right\| _{H^{s+1}}^2 + \alpha_0 \int_0^T \left\| \nabla_x (\Lambda^m \hat{u}, \Lambda^m \hat{\theta}) \right\| _{H^{s+1}}^2 \leq C(\Omega_0) e^{(\sqrt{T} + \epsilon)C(\Omega)}.
\] (3.24)

Remark 3.1. Notice that (3.16), (3.19), and (3.20) indicate that the dependence of \( (\epsilon \nabla_x)^m (\hat{p}, \hat{u}, \hat{\theta}) \) on the fast motion does not change when \( m \) increases. In particular, the estimate for \( (\epsilon \nabla_x)^m (\hat{p}, \hat{u}, \hat{\theta}) \) depends only on \( \| u \|_{W^{1,\infty}} \) for any \( m \in \mathbb{N} \). This shows we can choose any \( m \in \mathbb{N} \) in order to close the estimate in Proposition 3.1 (with \( R, R', \Omega_0, \Omega \) redefined with the corresponding \( m \)).

3.2. Additional Bounds. In this part we list out additional bounds based on Proposition 3.1. The details for deriving these bounds are omitted because they are very similar to those in [4]. This is because these bounds do not depend on the signs of \( \tau_1, \tau_4 \). Instead they only depend on the order of the equations. First, the slow motion \( \text{curl}(e^{-\theta}u) \) satisfies the equation:
\[
(\partial_t + u \cdot \nabla_x) \left( \text{curl} \left( e^{-\theta}u \right) \right) = \text{curl} \left( e^{-\epsilon p} \left( \nabla_x \cdot \Sigma + \nabla_x \cdot \hat{\Sigma} \right) \right) + \text{curl}(e^{-\theta}u \partial_t \theta) \\
+ [\text{curl}, u] \cdot \nabla_x (e^{-\theta}u) + \text{curl}(e^{-\theta}u(u \cdot \nabla_x \theta)).
\] (3.25)

It has the following bound:

Lemma 3.1. Let \( s > 5 \). Let \((p, u, \theta)\) be a solution to the fluctuation equations (1.6). Then there exists an increasing function such that
\[
\sup_{[0,T]} \| \text{curl}(e^{-\theta}u) \| _{H^{s-1}}^2 + \alpha_0 \int_0^T \left\| \nabla_x \text{curl}(e^{-\theta}u) \right\| _{H^{s-1}}^2 d\tau \leq C(\Omega_0) e^{\sqrt{T}C(\Omega)}.
\] (3.26)

Next we show some bounds of terms \( \| (\epsilon \partial_t)^k (u, p) \| _{L^2} \) and \( \| \Lambda^{2(s-k)} (\epsilon \partial_t)^k (\epsilon p, \epsilon u, \theta) \| _{H^{s+1-k}} \) for \( 1 \leq k \leq s \). These bounds will be used in the next section. They are immediate consequences of Proposition 3.1.

Lemma 3.2. Let \((p, u, \theta)\) be the solution to (1.6). Let \( 1 \leq k \leq s \). Then
\[
\| (\epsilon \partial_t)^k (p, u) \| _{H^{s-k}} \leq C(R),
\]
\[
\| (\epsilon \partial_t)^k (p, u) \| _{H^{s+1-k}} + \| (\epsilon \partial_t)^{k-1} \partial_t \theta \| _{H^{s+1-k}} \leq C(R)(1 + R').
\] (3.27)
Lemma 3.3. Let \((p, u, \theta)\) be the solution to (1.6) and \(\psi = (ep, eu, \theta)\). Then for all \(1 \leq k \leq s\),
\[
\|\Lambda^2(s-k+1)(\epsilon \partial_t)^k \psi\|_{H^{s+1-k}} \leq C(R), \\
\|\nabla_x \Lambda^2(s-k+1)(\epsilon \partial_t)^k \psi\|_{H^{s+1-k}} \leq C(R)(1 + R'), \\
\sup_{[0,T]} \left(\|\Lambda^2(s-k+1)(\epsilon \partial_t)^k \psi\|_{H^{s+1-k}} + \|(\epsilon \partial_t)^k \text{curl}(e^{-\theta}u)\|_{H^{s-k-1}}\right) \leq C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)},
\]
where \(R, R', \Omega_0, \Omega\) are defined in (3.3) and (3.9).

We also recall the following bound in [1] (Remark 6.21):

Lemma 3.4 ([1]). Let \(F \in C^\infty(\mathbb{R}^1 \times \mathbb{R}^3 \times \mathbb{R}^1)\) such that \(F(0) = 0\). Let \(\psi = (ep, eu, \theta)\) be the solution to (1.6). Then
\[
\|F(\psi)\|_{L^\infty(0,T;H^s)} \leq C(\Omega_0) + \sqrt{T}C(\Omega) \leq C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}.
\]

4. A priori estimate – fast motion

In this section we establish estimates for \(p\) and the acoustic part of \(u\). Using the various bounds obtained for the slow motion in Section 3, we can show Sobolev bounds of \((\nabla_x \cdot u, \nabla_x p)\) by studying estimates for \((\epsilon \partial_t)^k(p, u)\) with \(1 \leq k \leq s\) which is summarized in the following proposition:

Proposition 4.1. Let \((p, u, \theta)\) be a solution to (1.6). Let \(s > 5\). Define
\[
(p_\gamma, u_\gamma, \theta_\gamma) := (\epsilon \partial_t)^\gamma(p, u, \theta),
\]
where \(1 \leq \gamma \leq s\). Then there exists an increasing function \(C(\cdot)\) such that
\[
\sup_{[0,T]} \left(\|p_\gamma\|_{L^2}^2 + \|u_\gamma\|_{H^1}^2\right) + a_0 \int_0^T \|\nabla_x(p_\gamma, u_\gamma)\|_{L^2}^2 \, d\tau \leq C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}.
\]

Proof. Applying the operator \((\epsilon \partial_t)^\gamma\) to the system (1.6), we have
\[
\frac{3}{2}(\partial_t + u \cdot \nabla_x)p_\gamma + \frac{1}{4} \nabla_x \cdot (u_\gamma \nabla_x \theta_\gamma) = (\epsilon \partial_t)^\gamma \left(\frac{5}{6}e(h(\epsilon)p)\left((\Sigma + \tilde{\Sigma}) \cdot \nabla_x u + \nabla_x \cdot \tilde{q}\right)\right) \\
- (\epsilon \partial_t)^\gamma \left(\frac{5}{6}h(\epsilon)\kappa(\theta)\nabla_x p \cdot \nabla_x \theta\right) + g_1, \\
g(\theta)(\partial_t + u \cdot \nabla_x)\theta_\gamma + \frac{1}{3} \nabla_x \theta_\gamma = h(\epsilon)\nabla_x \cdot \Sigma_\gamma + (\epsilon \partial_t)^\gamma \left(h(\epsilon)\nabla_x \cdot \tilde{\Sigma}\right) + g_2,
\]
where
\[
\Sigma_\gamma = \mu(\theta) (\nabla_x u_\gamma + (\nabla_x u_\gamma)^T - \frac{2}{3}(\nabla_x \cdot u_\gamma) I), \\
\tilde{\Sigma}_\gamma = \tau_1(\theta) \left(\nabla^2_{\theta} \theta_\gamma - \frac{1}{3}(\Delta_x \theta_\gamma) I\right) + \tau_2(\theta) \left(\nabla_x \theta \otimes \nabla_x \theta_\gamma - \frac{1}{3} \nabla_x \theta \cdot \nabla_x \theta_\gamma I\right), \\
+ \tau_3(ep, \theta) \left((\nabla_x (ep))(\nabla_x (ep))^T - (\nabla_x (ep))^T \nabla_x (ep)\right).
\]
The commutator terms are
\[
g_1 = \left[ \frac{3}{5} u, (\varepsilon \partial_t)^\gamma \right] \cdot \nabla_x p - \frac{1}{\varepsilon} \nabla_x \cdot \left[ \frac{2}{5} h(\varepsilon p) \kappa(\theta), (\varepsilon \partial_t)^\gamma \right] \nabla_x \theta
\]
\[
g_2 = [g(\theta), (\varepsilon \partial_t)^\gamma] \partial_t u + [g(\theta) u, (\varepsilon \partial_t)^\gamma] \cdot \nabla_x u + [\Psi_2 + \Psi_1, (\varepsilon \partial_t)^\gamma] u,
\]
\[
g_3 = [u, (\frac{2}{5} \varepsilon \partial_t)^\gamma] \nabla_x \theta,
\]
where \( \Psi_2 \) and \( \Psi_1 \) denote second and first-order homogeneous differential operators respectively. The \( p_\gamma \)-equation indicates that we need to consider the system for
\[
(p_\gamma, \tilde{u}_\gamma) = (p_\gamma, u_\gamma - \frac{2}{5} h(\varepsilon p) \kappa(\theta) \nabla_x \theta_\gamma).
\]
By the \( u_\gamma \) and \( \theta_\gamma \) equations, we derive that the system for \( (p_\gamma, \tilde{u}_\gamma) \) has the form
\[
\begin{align*}
\frac{2}{5} (\partial_t + u \cdot \nabla_x) p_\gamma + \frac{1}{\varepsilon} \nabla_x \cdot p_\gamma &= (\varepsilon \partial_t)^\gamma \left( \frac{2}{5} h(\varepsilon p) \kappa(\theta) \left( \left( \Sigma + \tilde{\Sigma} \right) : \nabla_u u + \nabla_x \cdot \eta \right) \right) \\
&- (\varepsilon \partial_t)^\gamma \left( \frac{2}{5} h(\varepsilon p) \kappa(\theta) \nabla_x p \cdot \nabla_x \theta \right) + g_1,
\end{align*}
\]
\[
g(\theta)(\partial_t u + u \cdot \nabla_x) \tilde{u}_\gamma + \frac{1}{\varepsilon} \nabla_x p_\gamma = \frac{4}{15} g(\theta) h(\varepsilon p) \kappa(\theta) \nabla_x \cdot \tilde{u}_\gamma + h(\varepsilon p) \nabla_x \cdot \Sigma \tilde{u}_\gamma + g_4,
\]
where
\[
\begin{align*}
\Sigma \tilde{u}_\gamma &= \mu(\theta) \left( \nabla_x \tilde{u}_\gamma + \left( \nabla_x \tilde{u}_\gamma \right)^T \right) - \frac{2}{3} \left( \nabla_x \cdot \tilde{u}_\gamma \right) I,
\end{align*}
\]
\[
g_4 = h(\varepsilon p) \nabla_x \Sigma_\gamma + g_2 + \frac{4}{15} g(\theta) h(\varepsilon p) \kappa(\theta) \nabla_x \Sigma_\gamma \cdot \left( \frac{2}{5} h(\varepsilon p) \kappa(\theta) \nabla_x \theta_\gamma \right) \\
&- (g(\theta) \nabla_x \theta_\gamma) \partial_t \left( \frac{2}{5} h(\varepsilon p) \kappa(\theta) \right) - \frac{4}{15} g(\theta) h(\varepsilon p) \kappa(\theta) \nabla_x (RHS \theta_\gamma) \\
&+ g(\theta) \left( \frac{2}{5} h(\varepsilon p) \kappa(\theta) \nabla_x, u \cdot \nabla_x \right) \theta_\gamma,
\]
where \( RHS \theta_\gamma \) is the right-hand side of the \( \theta_\gamma \)-equation in (4.3). In the \( L^2 \)-estimate we will only show the details for the leading-order terms for the slow motion and the fast motion. Denoting \( \psi = (\varepsilon u, \varepsilon p, \theta) \) as the slow motion, the leading-orders on the right-hand side of the \( p_\gamma \)-equation expressed in the abstract form in terms of the homogeneous differential operators are
\[
\Gamma_1 = \Psi_2 \theta_\gamma + \Psi_3 \tilde{u}_\gamma + \Psi_1 (u_\gamma, p_\gamma) + \Psi_1 ((\varepsilon \partial_t)^\gamma \partial_t \theta),
\]
where the coefficients of \( \Psi_k \)'s depend up to the first-order spatial derivatives of \( \psi, u, p \). In particular, the coefficient of \( \Psi_3 \) depends only on the slow motion. The leading-order terms on the right-hand side of the \( \tilde{u}_\gamma \)-equation are
\[
\Gamma_2 = \Psi_3 \theta_\gamma + \Psi_0 ((\varepsilon \partial_t)^\gamma \partial_t \theta) + \Psi_1 (u_\gamma) + \varepsilon \Psi_4 \tilde{u}_\gamma,
\]
where the coefficients of the operators depend up to the fourth order spatial derivatives of \( \psi \) and up to the second-order derivatives of \( u \). In particular, the coefficients of \( \Psi_3 \) and \( \varepsilon \Psi_4 \) depend only on the slow motion.

Now we compute the contribution of \( \Gamma_1, \Gamma_2 \) in the \( L^2 \)-estimate. To this end, we multiply (4.15) by \( p_\gamma \) and \( \tilde{u}_\gamma \) and integrate over \( \mathbb{R}^n \). The estimates for \( \Gamma_1, p_\gamma \) and \( \Gamma_2, \tilde{u}_\gamma \) are as follows.
\[
|\langle \Gamma_1, p_\gamma \rangle| \leq ||p_\gamma||_{L^2} ||\Psi_2 \theta_\gamma||_{L^2} + ||p_\gamma||_{L^2} ||\Psi_3 \tilde{u}_\gamma||_{L^2} + ||p_\gamma||_{L^2} ||\Psi_1 ||_{L^2} ||u_\gamma||_{L^2} + C(R) ||p_\gamma||_{L^2}^2
\]
\[
+ ||p_\gamma||_{L^2} ||\Psi_1 ((\varepsilon \partial_t)^\gamma \partial_t \theta)||_{L^2} \\
\leq C(R) \left( 1 + R' \right) + ||p_\gamma||_{L^2} ||\Psi_3 \tilde{u}_\gamma||_{L^2},
\]
where by integration by parts
\[
|\langle p_\gamma, \Psi_3 \tilde{u}_\gamma \rangle| \leq ||\delta_1(\psi)||_{L^\infty} ||\nabla_x p_\gamma||_{L^2} ||\nabla^2_x \tilde{u}_\gamma||_{L^2} + C(R) \left( 1 + R' \right).
\]
Here we have used $\delta_1(\psi)$ to denote the coefficient of the operator $\Psi_3$. Note that by Proposition 3.1, we have
\[ \|\delta_1(\psi)\|_{L^\infty} \leq C(\Omega_0)e^{(\sqrt{T+\epsilon})C(\Omega)}. \]
Overall, we have
\[ |(\Gamma_1, p_\gamma)| \leq C(R)(1 + R') + C(\Omega_0)e^{(\sqrt{T+\epsilon})C(\Omega)}\|\nabla_x p_\gamma\|_{L^2}\|\nabla_x^2 \tilde{u}_\gamma\|_{L^2}. \]
Similarly, the contribution from $\Gamma_2$ satisfies
\[ |(\Gamma_2, \tilde{u}_\gamma)| \leq C(R)(1 + R') + C(\Omega_0)e^{(\sqrt{T+\epsilon})C(\Omega)}\|\Lambda_\epsilon \nabla_x^2 \tilde{u}_\gamma\|_{L^2}^2. \]
The $L^2$-estimate of $(p_\gamma, \tilde{u}_\gamma)$ thus has the form
\[
\frac{1}{2} \frac{d}{d\tau} \left( \|p_\gamma\|_{L^2}^2 + \|\tilde{u}_\gamma\|_{L^2}^2 \right) + \alpha_0 \|\nabla_x \tilde{u}_\gamma\|_{L^2}^2(\tau) \\
\leq C(R)(1 + R') + C(\Omega_0)e^{(\sqrt{T+\epsilon})C(\Omega)} \left( \|\nabla_x p_\gamma\|_{L^2}\|\nabla_x^2 \tilde{u}_\gamma\|_{L^2} + \|\Lambda_\epsilon \nabla_x^2 \tilde{u}_\gamma\|_{L^2}^2 \right). \tag{4.8}
\]
This implies
\[
\sup_{[0,T]} \left( \|p_\gamma\|_{L^2}^2 + \|\tilde{u}_\gamma\|_{L^2}^2 \right) + \alpha_0 \int_0^T \|\nabla_x \tilde{u}_\gamma\|_{L^2}^2 \, d\tau \\
\leq C(\Omega_0)e^{(\sqrt{T+\epsilon})C(\Omega)} \left( 1 + \int_0^T \|\Lambda_\epsilon \nabla_x^2 \tilde{u}_\gamma\|_{L^2}^2 \, d\tau \right) + \epsilon_0 \int_0^T \|\nabla_x p_\gamma\|_{L^2}^2 \, d\tau,
\]
where $0 < \epsilon_0 < \frac{1}{\tilde{b}}$ is small enough and is to be determined. Note that the last inequality follows from the last inequality in (3.28).

To conclude, we need the bound for $\int_0^T \|\nabla_x p_\gamma\|_{L^2}^2 \, d\tau$. The proof is similar to the corresponding part in [4] since again the estimates involved here do not depend on the sign of $\hat{t}_1, \hat{t}_4$. However for the completeness of the proof of the theorem, we include the details here. To this end, we multiply the $u_\gamma$-equation in (4.3) by $\epsilon\nabla_x p_\gamma$ and integrate over $\mathbb{R}^3 \times [0, T]$. The resulting equation is
\[
\int_0^T \|\nabla_x p_\gamma\|_{L^2}^2(\tau) \, d\tau = - \int_0^T \int_{\mathbb{R}^3} g(\theta) \partial_t((\epsilon\partial_t)^\gamma(\epsilon u)) \cdot \nabla_x p_\gamma(x, \tau) \, dx \, d\tau \\
+ \int_0^T \int_{\mathbb{R}^3} \Gamma \cdot (\nabla_x p_\gamma)(x, \tau) \, dx \, d\tau, \tag{4.9}
\]
where
\[
\Gamma = -g(\theta)(\epsilon u) \cdot \nabla_x u_\gamma + \epsilon h(\epsilon p)(\nabla_x \cdot \Sigma_\gamma) + ch(\epsilon p)(\epsilon\partial_t)^\gamma(\nabla_x \cdot \hat{\Sigma}) + \epsilon g_2. \tag{4.10}
\]
Note that by Lemma 3.2, the first term in $\Gamma$ satisfies
\[
\|g(\theta)(\epsilon u) \cdot \nabla_x u_\gamma\|_{L^2(\mathbb{R}^3 \times [0, T])} \leq \epsilon \left( \int_0^T (C(R)(1 + R'))^2(\tau) \, d\tau \right)^{1/2} \leq \epsilon C(\Omega).
\]
By Lemma 3.3, the second term in $\Gamma$ satisfies
\[
\|\epsilon h(\epsilon p)(\nabla_x \cdot \Sigma_\gamma) + ch(\epsilon p)(\epsilon\partial_t)^\gamma(\nabla_x \cdot \hat{\Sigma})\|_{L^2(\mathbb{R}^3 \times [0, T])} \leq C(\Omega_0)e^{(\sqrt{T+\epsilon})C(\Omega)}. \]
By checking the leading-orders in \( g_2 \), we deduce that \( g_2 \) satisfies
\[
\|g_2\|_{L^2(\mathbb{R}^3 \times [0,T])} \leq \left( \int_0^T (C(R)(1 + R'))^2(\tau) \, d\tau \right)^{1/2} \leq C(\Omega).
\]
Here by Lemma 3.2 and Lemma 3.3 the lower-order terms in \( g_2 \) satisfy the same bound. Thus the last term in \( \Gamma \) satisfies
\[
\|\epsilon g_2\|_{L^2(\mathbb{R}^3 \times [0,T])} \leq \epsilon \left( \int_0^T (C(R)(1 + R'))^2(\tau) \, d\tau \right)^{1/2} \leq \epsilon C(\Omega),
\]
where \( R, R', \Omega \) are defined in (3.3) and (3.9). Combining these bounds we have
\[
\int_0^T \int_{\mathbb{R}^3} \Gamma \cdot (\nabla_\gamma p_\gamma)(x, \tau) \, dx \, d\tau \leq \|\nabla x p_\gamma\|_{L^2(\mathbb{R}^3 \times [0,T])} \leq C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)} + \frac{1}{8} \|\nabla x p_\gamma\|^2_{L^2(\mathbb{R}^3 \times [0,T])}. \tag{4.11}
\]

To estimate the first term on the right-hand side of (4.9), we integrate by parts in both \( t \) and \( x \). Then
\[
\int_0^T \int_{\mathbb{R}^3} g(\theta) \partial_t((\epsilon \partial_t)^{\gamma} (\epsilon u)) \cdot \nabla_x p_\gamma(x, \tau) \, dx \, d\tau
= \int_{\mathbb{R}^3} \left( (g(\theta) u_\gamma) \cdot ((\epsilon \partial_t)^{\gamma} \nabla_x(\epsilon p))(x, 0) \right) \, dx - \int_0^T \int_{\mathbb{R}^3} (g(\theta) u_\gamma) \cdot ((\epsilon \partial_t)^{\gamma} \nabla_x(\epsilon p))(x, 0) \, dx \, d\tau
+ \int_0^T \int_{\mathbb{R}^3} ((\epsilon \partial_t \nabla g(\theta)) \cdot u_\gamma) p_\gamma \, dx \, d\tau + \int_0^T \int_{\mathbb{R}^3} ((\epsilon \partial_t \nabla g(\theta)) \cdot u_\gamma) p_\gamma \, dx \, d\tau.
\tag{4.12}
\]

Estimates of each term are as follows. First, for each \( t \in [0, T] \), by Lemma 3.3, Lemma 3.4, and (4.8),
\[
\|(\epsilon \partial_t)^{\gamma} \nabla_x(\epsilon p))\|_{L^2(\mathbb{R}^3)} \leq C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)},
\]
\[
\|g(\theta)(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \leq C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}.
\]
\[
\|u_\gamma(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \|\nabla \theta\|_{L^2(\mathbb{R}^3)} + \|\frac{2}{5} h(\epsilon p) \kappa(\theta) \nabla x \theta\|_{L^2(\mathbb{R}^3)} \leq C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)} + \frac{1}{8} \|\nabla x p_\gamma\|^2_{L^2(\mathbb{R}^3 \times [0,T])}.
\]

Therefore for \( t = 0 \) and \( t = T \),
\[
\int_{\mathbb{R}^3} (g(\theta) u_\gamma) \cdot ((\epsilon \partial_t)^{\gamma} \nabla_x(\epsilon p))(x, t) \, dx \leq C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)} + \frac{1}{8} \|\nabla x p_\gamma\|^2_{L^2(\mathbb{R}^3 \times [0,T])}. \tag{4.13}
\]

By Lemma 3.3,
\[
\|p_\gamma\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \|\nabla x \cdot u_\gamma\|_{L^2(\mathbb{R}^3 \times [0,T])} \leq C(R)(1 + R') \leq C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}.
\]

Therefore,
\[
\int_0^T \int_{\mathbb{R}^3} ((\epsilon \partial_t \nabla g(\theta)) \cdot u_\gamma) p_\gamma \, dx \, d\tau \leq \sqrt{T} C(\Omega) \leq C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)},
\]
\[
\int_0^T \int_{\mathbb{R}^3} ((\epsilon \partial_t \nabla g(\theta)) \cdot u_\gamma) p_\gamma \, dx \, d\tau \leq \sqrt{T} C(\Omega) \leq C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}. \tag{4.14}
\]
To estimate the last two terms in (4.12) we need to use the $p_\gamma$-equation in (4.3), which is

$$
ep \partial_t p_\gamma = -\frac{2}{3}(\ep u) \cdot \nabla_x u - \frac{2}{3} \nabla_x u - \frac{\ep}{3} (\ep \partial_t)^{\gamma} (h(\ep) \nabla_x \cdot (\kappa(\theta) \nabla_x \theta)) \nabla_x \cdot u + \nabla_x \theta + \eta + \eta_0.$$

(4.15)

Using (4.15) we show bounds for the second last term in (4.12). First, by Lemma 3.3 and (?), the contribution from the first term in $\ep \partial_t p_\gamma$ is

$$\left| \int_0^T \int_{\mathbb{R}^3} (g(\theta)(\nabla_x \cdot u_\gamma)) \left( \frac{\ep}{3}(\ep u) \cdot \nabla_x p_\gamma \right) \, dx \, d\tau \right|$$

$$\leq \frac{\ep}{3} \sqrt{T} \| g(\theta) \|_{L^\infty([0, T])} \| \nabla_x \cdot u_\gamma \|_{L^\infty([0, T]; L^2(\mathbb{R}^3))} \| \nabla_x p_\gamma \|_{L^2([0, T])}$$

$$\leq C(\Omega_0) \ep \| \nabla_x p_\gamma \|_{L^2([0, T])}.$$

The contribution from the term $\frac{\ep}{3} \nabla_x \cdot u_\gamma$ in $\ep \partial_t p_\gamma$ is

$$\left| \int_0^T \int_{\mathbb{R}^3} (g(\theta)(\nabla_x \cdot u_\gamma)) \left( \frac{\ep}{3} \nabla_x \cdot u_\gamma \right) \, dx \, d\tau \right|$$

$$\leq \frac{\ep}{3} \| g(\theta) \|_{L^\infty([0, T])} \| \nabla_x \cdot u_\gamma \|_{L^2([0, T])} \| \nabla_x \cdot u_\gamma \|_{L^2([0, T])}$$

$$\leq C(\Omega_0) \ep \| \nabla_x p_\gamma \|_{L^2([0, T])}.$$

By Lemma 3.2 and Lemma 3.3, the contribution from the term $\ep g_1$ in $\ep \partial_t p_\gamma$ satisfies

$$\left| \int_0^T \int_{\mathbb{R}^3} (g(\theta)(\nabla_x \cdot u_\gamma)) \left( \ep g_1 \right) \, dx \, d\tau \right|$$

$$\leq \| g(\theta) \|_{L^\infty([0, T])} \| \nabla_x \cdot u_\gamma \|_{L^2([0, T])} \| g_1 \|_{L^2([0, T])}$$

$$\leq C(\Omega_0) \ep \| \nabla_x p_\gamma \|_{L^2([0, T])}.$$

The rest of the terms in $\ep \partial_t p_\gamma$ are all for the slow motion $(\ep \partial_t)^{\gamma} \psi = (\ep \partial_t)^{\gamma} (\ep p, \ep u, \theta)$. The highest order spatial derivatives for these terms are $(\ep \nabla_x) \nabla_x^2$. Therefore by Lemma 3.3, their $L^2([0, T])$-norms are bounded by $C(\Omega_0) \ep \| \nabla_x p_\gamma \|_{L^2([0, T])}$. Hence, their contribution to the second last term of (4.12) is also bounded by $C(\Omega_0) \ep \| \nabla_x p_\gamma \|_{L^2([0, T])}$. Altogether we have

$$\left| \int_0^T \int_{\mathbb{R}^3} (g(\theta)(\nabla_x \cdot u_\gamma)) \left( \ep \partial_t p_\gamma \right) \, dx \, d\tau \right| \leq C(\Omega_0) \ep \| \nabla_x p_\gamma \|_{L^2([0, T])}^2.$$

Hence we choose $\ep_0$ small enough such that

$$\ep_0 C(\Omega_0) \ep \| \nabla_x p_\gamma \|_{L^2([0, T])} \leq \frac{1}{8}.$$

Note that this can be done because $C(\cdot)$ is a fixed function depending only on the particular form of $g(\cdot)$ and we can consider the time period such that $\Omega < 1$. Thus $C(\Omega_0) \ep \| \nabla_x p_\gamma \|_{L^2([0, T])}$ is a fixed quantity. The last term in (4.12) is bounded in a similar way. Then we have

$$\int_0^T \| \nabla_x p_\gamma \|_{L^2(\tau)} \, d\tau \leq C(\Omega_0) \ep \| \nabla_x p_\gamma \|_{L^2([0, T])}^2 + \frac{\ep}{3} \| \nabla_x p_\gamma \|_{L^2([0, T])}.$$
which gives
\[ \int_0^T \| \nabla_x p_\tau \|_{L^2}^2(\tau) \, d\tau \leq C(\Omega_0) e^{(\sqrt{T+\epsilon})C(\Omega)}. \] (4.16)

Combining (4.16) with (4.8), we thereby finish the proof of Proposition 4.1.

The derivation of the a priori estimate for \( \| (p,u) \|_{H^s} \) from Proposition 4.1 follows the same way as in [4]. Again one uses the relation
\[ \nabla_x \cdot u = -\frac{3}{5}(\epsilon \partial_t)p - \frac{3}{5}(\epsilon u) \cdot \nabla_x p + \frac{2}{5}e^{-\epsilon p} \nabla_x \cdot (\kappa(\theta) \nabla_x \theta) + \frac{2}{5} \epsilon e^{-\epsilon p} \Sigma : \nabla_x (\epsilon u) \]
\[ + \frac{2}{5} \epsilon e^{-\epsilon p} \bar{\Sigma} : \nabla_x (\epsilon u) + \frac{2}{5} \epsilon^2 e^{-\epsilon p} \nabla_x \cdot \bar{q}, \] (4.17)
and apply it iteratively to transfer the time derivatives into spatial derivatives. Since the signs of \( \hat{\tau}_1, \hat{\tau}_4 \) do not contribute and the proof is rather identical to [4], we simply state the result without providing a detailed proof:

**Proposition 4.2.** Let \( (p,u,\theta) \) be a solution to the fluctuation equations (1.6). Then
\[ \sup_{[0,T]} \left( \| p \|_{H^s}^2 + \| \nabla_x \cdot u \|_{H^{s-1}}^2 \right) + \int_0^T \left( \| \nabla_x \nabla_x \cdot u \|_{H^{s-1}}^2 + \| \nabla_x p \|_{H^s}^2 \right)(\tau) \, d\tau \]
\[ \leq C(\Omega_0) e^{(\sqrt{T+\epsilon})C(\Omega)}. \]

**Conclusion.** Combining Proposition 3.1, Lemma 3.1, and Proposition 4.2, we complete the proof of Theorem 2.3.

**REFERENCES**