A LOW MACH NUMBER LIMIT OF A DISPERSIVE NAVIER-STOKES SYSTEM

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Abstract. We establish a low Mach number limit for classical solutions over the whole space of a compressible fluid dynamic system that includes dispersive corrections to the Navier-Stokes equations. The limiting system is a ghost effect system [26]. Our analysis builds upon the framework developed by Métivier and Schochet [20] and Alazard [2] for non-dispersive systems. The strategy involves establishing a priori estimates for the slow motion as well as a priori estimates for the fast motion. The desired convergence is obtained by establishing the local decay of the energy of the fast motion.

Key words. Low Mach number, compressible and viscous fluid, dispersive Navier-Stokes equations, kinetic theory, ghost effect systems.

AMS subject classifications. 35L65, 76N10, 35B45

1. Introduction. We establish a low Mach number limit for classical solutions over the whole space of a compressible fluid dynamic system that includes dispersive corrections to the Navier-Stokes equations. The limiting system is a so-called *ghost effect system* [26] which is not derivable from the Navier-Stokes system of gas dynamics but is derivable from kinetic equations. This work is part of a program [16, 17] that aims to identify fluid dynamic regimes and to construct a unified model that captures them all. Such a model can also be useful in transition regimes when classical fluid equations are inadequate to describe the dynamics of fluids while computations using kinetic models are expensive.

The governing equations for the dispersive system (DNS) under consideration are as follows

$$
\begin{aligned}
&\partial_t\rho + \nabla_x\cdot(\rho U) = 0\,,\\
&\partial_t(\rho U) + \nabla_x\cdot(\rho U\otimes U) + \nabla_x p(\rho,\theta) = \nabla_x\cdot\Sigma + \nabla_x\cdot\tilde{\Sigma}\,,\\
&\partial_t(\rho e) + \nabla_x\cdot(\rho e U + \rho\Theta U) = \nabla_x\cdot(\Sigma U - q) + \nabla_x\cdot(\tilde{\Sigma}U + \tilde{q})\,,\\
&(\rho, U, \Theta)(x,0) = (\rho^{\mathrm{in}}, U^{\mathrm{in}}, \Theta^{\mathrm{in}})(x)\,,
\end{aligned}
\tag{1.1}
$$

where $(\rho(x,t), U(x,t), \Theta(x,t))$ denote the density, velocity, and temperature of the fluid at time $t\in\mathbb{R}^+$ and position $x\in\mathbb{R}^3$ respectively, whereas $p = p(\rho,\Theta)$, $q = q(\Theta)$, $\Sigma = \Sigma(U,\Theta)$ denote the pressure, heat flux, and viscous stress tensor respectively with

$$p(\rho,\Theta) = \rho\Theta, \qquad q(\Theta) = -\hat{\kappa}(\Theta)\nabla_x\Theta\,,$$

and Σ is the stress for a Newtonian fluid

$$\Sigma = \hat{\mu}(\Theta)(\nabla_x U + (\nabla_x U)^T - \frac{2}{3}(\nabla_x\cdot U)I).$$

The total energy density is given by

$$\rho e = \frac{1}{2}\rho|U|^2 + \frac{3}{2}\rho\Theta.$$

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The quantities $\tilde{\Sigma}$ and $\tilde{q}$ denote dispersive corrections to the stress tensor and heat flux respectively and are given by

$$\begin{aligned}\tilde{\Sigma} = \hat{\tau}_1(\rho,\Theta)(\nabla_x^2\Theta - \tfrac{1}{3}(\Delta_x\Theta)I) + \hat{\tau}_2(\rho,\Theta)(\nabla_x\Theta\otimes\nabla_x\Theta - \tfrac{1}{3}|\nabla_x\Theta|^2 I)\\ +\hat{\tau}_3(\rho,\Theta)(\nabla_x U(\nabla_x U)^T - (\nabla_x U)^T\nabla_x U)\,,\\ \tilde{q} = \hat{\tau}_4(\rho,\Theta)(\Delta_x U + \tfrac{1}{3}\nabla_x\nabla_x\cdot U) + \hat{\tau}_5(\rho,\Theta)\nabla_x\Theta\cdot(\nabla_x U + (\nabla_x U)^T - \tfrac{2}{3}(\nabla_x\cdot U)I)\\ +\hat{\tau}_6(\rho,\Theta)\left(\nabla_x U - (\nabla_x U)^T\right)\cdot\nabla_x\Theta\,.\end{aligned}$$

Here we assume that the transport coefficients $\hat{\tau}_1, \cdots, \hat{\tau}_6$ are C^∞ functions of their variables and $\hat{\tau}_1, \hat{\tau}_4 > 0$. By linearizing the DNS system (1.1) around constant states one can see that the coupling of $\nabla_x\cdot\tilde{\Sigma}$ and $\nabla_x\cdot\tilde{q}$ forms a dispersive relation.

Compared with the original system derived from kinetic equations in [16], system (1.1) is simplified in that terms involving second order derivatives of ρ are not included within $\nabla_x\cdot\tilde{\Sigma}$ and $\nabla_x\cdot\tilde{q}$. In fact, unlike second order terms in Θ and U within these dispersive terms, which can be controlled by the diffusive regularization, the state of the art at the moment does not allow control of the second order terms in ρ. This is due to the fact that ρ satisfies a *purely* hyperbolic equation.

One feature of the DNS system is that it possesses an entropy structure provided the transport coefficients in $\tilde{\Sigma}$ and $\tilde{q}$ satisfy

$$\hat{\tau}_4 = \frac{\Theta}{2}\hat{\tau}_1, \qquad \frac{\hat{\tau}_2}{\Theta} + \frac{2\hat{\tau}_5}{\Theta^2} = \partial_\Theta\left(\frac{\hat{\tau}_4}{\Theta^2}\right), \tag{1.2}$$

such that

$$\tilde{\Sigma} : \frac{\nabla_x U}{\Theta} + \tilde{q}\cdot\frac{\nabla_x\Theta}{\Theta^2} = \nabla_x\cdot\left(\frac{\hat{\tau}_1}{2\Theta}\nabla_x\Theta\cdot\left(\nabla_x U + (\nabla_x U)^T - \frac{2}{3}(\nabla_x\cdot U)I\right)\right).$$

Under assumption (1.2) one can show that the DNS system dissipates the Euler entropy in the same way as the Navier-Stokes system. However we do not need assumption (1.2) in this paper because we only consider local-in-time behavior of the system. Indeed the low Mach number limit is established only by assuming that $\hat{\tau}_1, \cdots, \hat{\tau}_6$ are C^∞ functions of their variables and $\hat{\tau}_1, \hat{\tau}_4 > 0$.

The *ghost effect system* can be formally derived from kinetic equations using a Hilbert expansion method (cf. Sone [26]). This is a system *beyond* classical fluid equations that describes the phenomenon in which the temperature field of the fluid has finite variations, and the flow is driven by the gradient of the temperature field [26]. Let $(\varrho, \vartheta, P^*)$ be the density, temperature, and pressure fields of the fluid. Then the ghost effect system in its general form is given by

$$\begin{aligned}\nabla_x(\varrho\vartheta) = 0,\\ \partial_t\varrho + \nabla_x\cdot(\varrho u) = 0,\\ \partial_t(\varrho u) + \nabla_x\cdot(\varrho u\otimes u) + \nabla_x P^* = \nabla_x\cdot\Sigma + \nabla_x\cdot\tilde{\Sigma},\\ \tfrac{3}{2}\partial_t(\rho\vartheta) + \tfrac{5}{2}\rho\vartheta\nabla_x\cdot u = -\nabla_x\cdot q,\end{aligned} \tag{1.3}$$

where P^* is the pressure, which is an independent variable for the ghost effect system. P^* can be viewed as a Lagrangian multiplier as in the case of incompressible dynamics. The viscous and heat conducting terms are written as

$$\Sigma = \bar{\mu}(\vartheta)\left(\nabla_x u + (\nabla_x u)^T - \frac{2}{3}(\nabla_x\cdot u)I\right), \qquad q = -\bar{\kappa}(\vartheta)\nabla_x\vartheta,$$

where $\bar{\mu}(\vartheta), \bar{\kappa}(\vartheta) > 0$ are the viscosity and heat conductivity respectively. The thermal stress tensor $\tilde{\Sigma}$ has the form

$$\tilde{\Sigma} = \bar{\tau}_1(\vartheta)\left(\nabla_x^2\vartheta - \frac{1}{3}\Delta_x\vartheta\, I\right) + \bar{\tau}_2(\vartheta)\,\left(\nabla_x\vartheta \otimes \nabla_x\vartheta - \tfrac{1}{3}|\nabla_x\vartheta|^2 I\right),$$

where $\bar{\tau}_1(\cdot), \bar{\tau}_2(\cdot) \in C^\infty(\mathbb{R})$ are transport coefficients with $\bar{\tau}_1 > 0$. The term $\nabla_x \cdot \tilde{\Sigma}$ contains third-order derivatives and is not derivable from the Navier-Stokes system.

In order to derive (1.3) from (1.1) one introduces a small parameter ϵ to (1.1). In kinetic theory, ϵ is the Knudsen number which measures the relative size of the mean free path of gas molecules with respect to a typical macroscopic length. It satisfies the von Kármen relation,

$$\epsilon \propto \frac{Ma}{Re}\,.$$

Here Ma denotes the Mach number which is typically used to compare a typical flow velocity U_{ref} with a characteristic speed of sound c and is given by

$$Ma := \frac{U_{\mathrm{ref}}}{c}\,, \tag{1.4}$$

and Re is the Reynolds number. If one considers a fluid regime where the viscous and inertial effects are comparable to each other, then the Reynolds number is of order one. In such a regime, ϵ and Ma are of the same scale. We thereby use ϵ to denote the Mach number in this paper. In accordance with the kinetic theory of fluids

$$\hat{\mu} = \epsilon\tilde{\mu}, \qquad \hat{\kappa} = \epsilon\tilde{\kappa}, \qquad \hat{\tau}_i = \epsilon^2\tilde{\tau}_i,\ i = 1, \cdots, 6\,.$$

For the derivation of the ghost effect system we need to focus on the physical regime in which

$$p \sim p_0 + O(\epsilon)\,, \qquad U \sim O(\epsilon)\,, \qquad \nabla_x\Theta \sim O(1)\,, \qquad p = \rho\Theta\,,$$

where p_0 is a constant. The existence of solutions to dispersive system (DNS) is established for the case where ρ and θ are close to constants at infinity. Therefore in our case the limiting pressure $\varrho\vartheta$ will be a constant that is independent of time. Without loss of generality, we fix this constant p_0 as 1. Thus, we consider the case when p is around 1 while Θ has a finite variation. To emphasize the dependence of (ρ, U, Θ) on ϵ, we denote the solution as $(\rho_\epsilon, U_\epsilon, \Theta_\epsilon)$ and the pressure as p_ϵ.

In order to ensure the positivity of both p_ϵ and Θ_ϵ, as usual we write $(p, U_\epsilon, \Theta_\epsilon)$ as

$$p_\epsilon = e^{\epsilon p^\epsilon}\,, \qquad U_\epsilon = \epsilon u^\epsilon\,, \qquad \Theta_\epsilon = e^{\theta^\epsilon}\,, \qquad \rho_\epsilon = e^{\rho^\epsilon}\,, \tag{1.5}$$

where the superscripts denote the fluctuation functions. In this regime, a longer time scale needs to be considered in order to capture the evolution of the fluctuations, namely

$$t = \frac{1}{\epsilon}\hat{t}\,. \tag{1.6}$$

For simplicity we subsequently drop the hat from $\hat{t}$ and use t in the system for the fluctuations. The dispersive Navier-Stokes system (1.1) thereby transforms into its scaled analogue

$$
\begin{aligned}
&\tfrac{3}{5}(\partial_t+u^\epsilon\cdot\nabla_x)p^\epsilon+\tfrac{1}{\epsilon}\nabla_x\cdot\big(u^\epsilon-\tfrac{2}{5}e^{-\epsilon p^\epsilon}\kappa(\theta^\epsilon)\nabla_x\theta^\epsilon\big)=\tfrac{2}{5}\epsilon e^{-\epsilon p^\epsilon}\Big(\big(\Sigma+\tilde\Sigma\big):\nabla_x u^\epsilon+\nabla_x\cdot\tilde q\Big)\\
&\qquad\qquad+\tfrac{2}{5}e^{-\epsilon p^\epsilon}\kappa(\theta^\epsilon)\nabla_x p^\epsilon\cdot\nabla_x\theta^\epsilon\,,\\
&e^{-\theta^\epsilon}(\partial_t+u^\epsilon\cdot\nabla_x)u^\epsilon+\tfrac{1}{\epsilon}\nabla_x p^\epsilon=e^{-\epsilon p^\epsilon}\left(\nabla_x\cdot\Sigma+\nabla_x\cdot\tilde\Sigma\right)\,,\\
&\tfrac{3}{2}(\partial_t+u^\epsilon\cdot\nabla_x)\theta^\epsilon+\nabla_x\cdot u^\epsilon=\epsilon^2 e^{-\epsilon p^\epsilon}\Big(\big(\Sigma+\tilde\Sigma\big):\nabla_x u^\epsilon+\nabla_x\cdot\tilde q\Big)\\
&\qquad\qquad-e^{-\epsilon p^\epsilon}\nabla_x\cdot q\,,\\
&(p^\epsilon,u^\epsilon,\theta^\epsilon)(x,0)=(p_\epsilon^{\rm in},u_\epsilon^{\rm in},\theta_\epsilon^{\rm in})(x)\,,
\end{aligned}
\tag{1.7}
$$

where the constitutive relations inherited from those for (1.1) are

$$
\begin{aligned}
&\Theta_\epsilon=e^{\theta^\epsilon}\,,\quad q=-\kappa(\theta^\epsilon)\nabla_x\theta^\epsilon\,,\quad \Sigma=\mu(\theta^\epsilon)\left(\nabla_x u^\epsilon+(\nabla_x u^\epsilon)^T-\tfrac{2}{3}(\nabla_x\cdot u^\epsilon)I\right)\,,\\
&\tilde\Sigma=\tau_1(\epsilon p^\epsilon,\theta^\epsilon)\left(\nabla_x^2\theta^\epsilon-\tfrac{1}{3}(\Delta_x\theta^\epsilon)I\right)+\tau_2(\epsilon p^\epsilon,\theta^\epsilon)\left(\nabla_x\theta^\epsilon\otimes\nabla_x\theta^\epsilon-\tfrac{1}{3}|\nabla_x\theta^\epsilon|^2 I\right)\\
&\qquad+\epsilon^2\tau_3(\epsilon p^\epsilon,\theta^\epsilon)\left(\nabla_x u^\epsilon(\nabla_x u^\epsilon)^T-(\nabla_x u^\epsilon)^T\nabla_x u^\epsilon\right)\,,\\
&\tilde q=\tau_4(\epsilon p^\epsilon,\theta^\epsilon)\left(\Delta_x u^\epsilon+\tfrac{1}{3}\nabla_x\nabla_x\cdot u^\epsilon\right)+\tau_6(\epsilon p^\epsilon,\theta^\epsilon)\left(\nabla_x u^\epsilon-(\nabla_x u^\epsilon)^T\right)\cdot\nabla_x\theta^\epsilon\\
&\qquad+\tau_5(\epsilon p^\epsilon,\theta^\epsilon)\nabla_x\theta^\epsilon\cdot\left(\nabla_x u^\epsilon+(\nabla_x u^\epsilon)^T-\tfrac{2}{3}(\nabla_x\cdot u^\epsilon)I\right)\,,
\end{aligned}
\tag{1.8}
$$

while

$$
\begin{aligned}
&\kappa(\theta^\epsilon)=\tilde\kappa(\Theta_\epsilon)\,\Theta_\epsilon\,,\qquad \mu(\theta^\epsilon)=\tilde\mu(\Theta_\epsilon)\,,\qquad \rho_\epsilon=P_\epsilon/\Theta_\epsilon=e^{\epsilon p^\epsilon-\theta^\epsilon}\,,\\
&\tau_1(\epsilon p^\epsilon,\theta^\epsilon)=\tilde\tau_1(\rho_\epsilon,\Theta_\epsilon)\,\Theta_\epsilon\,,\qquad \tau_4(\epsilon p^\epsilon,\theta^\epsilon)=\tilde\tau_4(\rho_\epsilon,\Theta_\epsilon)\,,\\
&\tau_2(\epsilon p^\epsilon,\theta^\epsilon)=\tilde\tau_1(\rho_\epsilon,\Theta_\epsilon)\,\Theta_\epsilon+\tilde\tau_2(\rho_\epsilon,\Theta_\epsilon)\,\Theta_\epsilon^2\,,\quad \tau_3(\epsilon p^\epsilon,\theta^\epsilon)=\tilde\tau_3(\rho_\epsilon,\Theta_\epsilon)\,,\\
&\tau_5(\epsilon p^\epsilon,\theta^\epsilon)=\tilde\tau_5(\rho_\epsilon,\Theta_\epsilon)\Theta_\epsilon\,,\quad \tau_6(\epsilon p^\epsilon,\theta^\epsilon)=\tilde\tau_6(\rho_\epsilon,\Theta_\epsilon)\Theta_\epsilon\,.
\end{aligned}
\tag{1.9}
$$

If we assume that for all indices $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ such that $|\alpha|\le 3$,

$$
(\nabla_x p^\epsilon,\ \nabla_x^\alpha u^\epsilon,\ \nabla_x^\alpha\theta^\epsilon)(t,x)\to(0,\ \nabla_x^\alpha u,\ \nabla_x^\alpha\theta)(t,x)\qquad\text{as }\epsilon\to 0,
\tag{1.10}
$$

then (1.7) converges formally to the system

$$
\begin{aligned}
&\nabla_x\cdot\left(\tfrac{5}{2}u-\kappa(\theta)\nabla_x\theta\right)=0\,,\\
&e^{-\theta}\left(\partial_t u+u\cdot\nabla_x u\right)+\nabla_x P^*=\nabla_x\cdot\Sigma+\nabla_x\cdot\tilde\Sigma\,,\\
&\tfrac{3}{2}(\partial_t+u\cdot\nabla_x)\theta+\nabla_x\cdot u=-\nabla_x\cdot q\,,
\end{aligned}
\tag{1.11}
$$

where P^* is the pressure which is an independent variable. By (1.10) we have

$$
\begin{aligned}
&\Sigma=\mu(\theta)\left(\nabla_x u+(\nabla_x u)^T-\tfrac{2}{3}(\nabla_x\cdot u)I\right)\,,\qquad q=-\kappa(\theta)\nabla_x\theta\,,\\
&\tilde\Sigma=\tau_1(\varrho,\theta)\left(\nabla_x^2\theta-\tfrac{1}{3}\Delta_x\Theta I\right)+\tau_2(\varrho,\theta)\left(\nabla_x\theta\otimes\nabla_x\theta-\tfrac{1}{3}|\nabla_x\theta|^2 I\right)\,.
\end{aligned}
\tag{1.12}
$$

Here $\mu(\theta)$ is the viscosity, $\kappa(\theta)$ is the heat conductivity, and $\tau_1(\varrho,\theta),\tau_2(\varrho,\theta)\in C^\infty(\mathbb{R}\times\mathbb{R})$ are transport coefficients with $\tau_1>0$ as defined in (1.9).

We can write the limiting system (1.11) in the form of a ghost system as (1.3) by taking $(\varrho,u,\vartheta)=(\varrho,u,e^\theta)$ as the variables. System (1.11) then has the form

$$
\begin{aligned}
&\varrho\vartheta=1\,,\\
&\partial_t\varrho+\nabla_x\cdot(\varrho u)=0\,,\\
&\partial_t(\varrho u)+\nabla_x\cdot(\varrho u\otimes u)+\nabla_x P^*=\nabla_x\cdot\Sigma+\nabla_x\cdot\tilde\Sigma\,,\\
&\nabla_x\cdot\left(\tfrac{5}{2}u-\kappa(\theta)\nabla_x\vartheta\right)=0\,,
\end{aligned}
\tag{1.13}
$$

where Σ and $\tilde{\Sigma}$ are defined in (1.12).

Our goal in this paper is to rigorously justify the above formal calculation of the low Mach number limit from the dispersive system (1.7) to the ghost effect system (1.11) in the whole space.

We remark that the investigation of singular limits of compressible flows over bounded domains is physically relevant and presents new challenges in the analysis. Unlike the cases involving the whole domain or exterior domains where acoustic waves are damped locally due to dispersive effects of the wave equation [1, 2, 6, 20, 29], the main obstacle in the treatment of bounded domains is the persistency of the fast waves over these domains. Therefore in general one can only expect weak convergence of the solutions. It is worth noting [7] that there are situations where strong convergence can be achieved due to the interaction of acoustic waves with the boundary of the domain, where a thin boundary layer is created to damp the energy carried by these fast oscillations. This phenomenon has been observed for both asymptotics of fluid equations and hydrodynamic limits of kinetic equations [7, 21]. It is therefore natural to ask the question whether similar phenomenon happens for the DNS system. Before trying to answer this question, one needs to know what are the physical boundary conditions that should be imposed on the DNS system. These boundary conditions are typically derived from the underlying kinetic equations so that they are compatible with the given boundary conditions for the kinetic equations. Deriving admissible boundary conditions for the DNS system, establishing the well-posedness theory of this system and investigating its asymptotics over bounded domains are the long terms goals of this program.

The outline of this paper is as follows. In Section 2 we introduce the space setting and present main result of our article. In Section 3 we state the local well-posedness theorems with brief explanations of their proofs for the dispersive Navier-Stokes system (1.1) (or equivalently system (1.7)) and the ghost effect system (1.3). In Section 4 we prove a priori estimates for the slow motion $\psi^\epsilon = (\epsilon p^\epsilon, \epsilon u^\epsilon, \theta^\epsilon)$. In Section 5 we prove a priori estimates for the fast motion (p^ϵ, u^ϵ) based on the bounds derived in Section 3. In Section 6 we show the local decay of the energy of the fast motion, and in Section 7 we show the convergence of (1.7) to (1.11).

2. Main result. Before stating the main theorem, we introduce the following norms that appear naturally during the analysis.

Definition 2.1. *Let $\alpha_0, \underline{\theta}$ be two constants and $\alpha_0 > 0$. For each $\epsilon, t, s > 0$, define the norms*

$$\begin{aligned} \|\!|(p,u,\theta-\underline{\theta})\|\!|_{\epsilon,s,t} &:= \sup_{[0,t]} \left(\|(p,u)(t)\|_{H^s} + \|\Lambda_\epsilon^{2s+1}(\epsilon p, \epsilon u, \theta-\underline{\theta})(t)\|_{H^{s+1}} \right) \\ &\quad + \alpha_0 \left(\int_0^t \left(\|\nabla_x (u,p)\|_{H^s}^2 + \|\nabla_x \Lambda_\epsilon^{2s+1}(\epsilon u, \theta)\|_{H^{s+1}}^2 \right)(\tau) \mathrm{d}\tau \right)^{\frac{1}{2}}, \end{aligned} \tag{2.1}$$

and

$$\|\!|(p,u,\theta-\underline{\theta})\|\!|_{\epsilon,s,0} := \|(p^{in}, u^{in})\|_{H^s} + \|\Lambda_\epsilon^{2s+1}(\epsilon p^{in}, \epsilon u^{in}, \theta^{in}-\underline{\theta})\|_{H^{s+1}}, \tag{2.2}$$

where $H^s = H^s(\mathbb{R}^3)$ is the usual Sobolev norm such that derivatives up to order s of a function are square-integrable. The invertible operator Λ_ϵ is defined by

$$\Lambda_\epsilon := (I - \epsilon^2 \Delta_x)^{1/2}. \tag{2.3}$$

for any $\epsilon > 0$.

DEFINITION 2.2. *Let* $(p^\epsilon, u^\epsilon, \theta^\epsilon)$ *be the solution to (1.7). We call* $\psi = (\epsilon p^\epsilon, \epsilon u^\epsilon, \theta^\epsilon)$ *as the* slow motion *because it varies on the time scale of order one. We call* (p^ϵ, u^ϵ) *as the* fast motion *because it varies on the time scale of order* ϵ.

The main theorem states

THEOREM 2.3. *Let* $s \geq 6$. *Suppose that there exist positive constants* M_0, $\underline{\theta}$, c_0, σ *such that the initial data of the fluctuations* $(p_\epsilon^{\rm in}, u_\epsilon^{\rm in}, \theta_\epsilon^{\rm in})$ *satisfy*

$$\begin{aligned} &\|(p_\epsilon^{\rm in},\ u_\epsilon^{\rm in},\ \theta_\epsilon^{\rm in} - \underline{\theta})\|_{\epsilon,s,0} \leq M_0\,,\\ &(\theta_\epsilon^{\rm in} - \underline{\theta},\ \Pi(e^{-\theta_\epsilon^{\rm in}} u_\epsilon^{\rm in})) \to (\theta^{\rm in} - \underline{\theta},\ u^{\rm in}) \quad \text{in } H^s(\mathbb{R}^3)\,,\\ &|\theta_\epsilon^{\rm in} - \underline{\theta}| \leq c_0 |x|^{-1-\sigma}\,, \qquad |\nabla_x \theta_\epsilon^{\rm in}| \leq c_0 |x|^{-2-\sigma}\,, \end{aligned} \tag{2.4}$$

where Π *is the projection onto the divergence free part of* $e^{-\theta_\epsilon^{\rm in}} u_\epsilon^{\rm in}$. *Then there exists* $T > 0$ *such that for any* $\epsilon \in (0,1]$, *the Cauchy problem for system (1.7) has a unique solution*

$$(p^\epsilon, u^\epsilon, \theta^\epsilon - \underline{\theta}) \in C([0,T]; H^{s_1}(\mathbb{R}^3)) \cap L^\infty(0,T; H^{3s+2}(\mathbb{R}^3)) \cap L^2(0,T; H^{3s+3}(\mathbb{R}^3))\,,$$

for all $s_1 < 3s + 1$ *and there exists a positive constant* M *depending only on* T, M_0 *such that*

$$\|(p^\epsilon, u^\epsilon, \theta^\epsilon - \underline{\theta})\|_{\epsilon,s,T} \leq M\,. \tag{2.5}$$

Furthermore, the sequence of solutions $(p^\epsilon, u^\epsilon, \theta^\epsilon)$ *to system (1.7) converges weakly in* $L^\infty(0,T; H^s(\mathbb{R}^3))$ *and strongly in* $L^2(0,T; H^{s'}_{loc}(\mathbb{R}^3))$ *for all* $s' < s$ *to the limit* $(0, u, \theta)$, *where* (u, θ) *satisfies the ghost effect system (1.11).*

Our analysis builds on the framework of Métivier and Schochet [20] and Alazard [2]. In [20] Métivier and Schochet proved the incompressible limit for the non-isentropic Euler equations for classical solutions with general initial data. In [2] Alazard proved the low Mach number limit for the compressible Navier-Stokes for classical solutions with general initial data and in [3] he proved the low Mach number limit for the compressible Navier-Stokes with general equations of state and a large source term in the temperature equation. Due to the presence of the high-order dispersive terms, our result does not follow directly from [2, 3]. We need to take into account of the anti-symmetric structure of those terms. Specifically, let $U^\epsilon = (p^\epsilon, u^\epsilon, \theta^\epsilon)$ be the solution to system (1.1) and $\psi^\epsilon = (\epsilon p^\epsilon, \epsilon u^\epsilon, \theta^\epsilon)$, then (1.1) can be reformulated as

$$A_1(\psi^\epsilon)(\partial_t + u \cdot \nabla_x) U^\epsilon + \frac{1}{\epsilon} A_2(\psi^\epsilon) U^\epsilon = A_3(\psi^\epsilon) U^\epsilon + \mathcal{R}\,,$$

where $A_1(\psi^\epsilon)$ is a diagonal matrix, $A_2(\psi^\epsilon)U^\epsilon$ are formed by certain combinations of the singular terms with the leading orders from the dispersive terms, $A_3(\psi^\epsilon)U^\epsilon$ are the dissipative terms, and $\mathcal{R}$ includes the rest of the dispersive terms. In the case of compressible Navier-Stokes $A_2(\psi^\epsilon)$ is anti-symmetric. Here it is not readily anti-symmetric but it is anti-symmetrizable by a certain symmetrizer matrix composed of symmetric positive operators. Similar as in [2] we establish uniform bounds for the solution by considering the interactions of its fast and slow motion parts. The theorem for the uniform bounds states

THEOREM 2.4. *For each fixed* $\epsilon > 0$, *let* $(p^\epsilon, u^\epsilon, \theta^\epsilon) \in C([0,T]; H^s(\mathbb{R}^3))$ *be the solution to the scaled DNS system (1.7). Let*

$$\Omega = \|(p^\epsilon, u^\epsilon, \theta^\epsilon - \underline{\theta})\|_{\epsilon,s,T}\,, \qquad \Omega_0 = \|(p_\epsilon^{\rm in}, u_\epsilon^{\rm in}, \theta_\epsilon^{\rm in} - \underline{\theta})\|_{\epsilon,s,0}\,.$$

Then there exists an increasing function $C(\cdot)$ *such that*

$$\|\!|(p^\epsilon, u^\epsilon, \theta^\epsilon - \underline{\theta})\|\!|_{\epsilon,s,T} \leq C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}\,,$$

which further indicates [20] that there exists $T_0 > 0$ *independent of* ϵ *such that* $\|\!|(p^\epsilon, u^\epsilon, \theta^\epsilon - \underline{\theta})\|\!|_{\epsilon,s,T}$ *are uniformly bounded in* ϵ *over* $[0, T_0]$. The proof for Theorem 2.4 in this paper is presented slightly differently from [2]. We directly derive bounds for (p^ϵ, u^ϵ) without separating the frequency space into high and low frequencies. Once the uniform bounds are established, the convergence of the solutions is shown by directly applying the local energy decay method for fast waves in the whole space [20].

3. Local Well-Posedness. In this section we state the local well-posedness theorems with brief explanation of their proofs for the dispersive Navier-Stokes system (1.1) and its limiting system (1.11).

The proof of the local well-posedness of (1.1) follows directly by classical energy method for hyperbolic-parabolic systems. More specifically, although we have third-order dispersive terms, the leading orders of these terms form an anti-symmetric structure. Therefore they do not hamper the usual L^2-H^s estimates. The rest of the dispersive terms are of orders up to two. Although this is the same order as the dissipation, they do not introduce extra difficulties because they are of order $O(\epsilon^2)$ while the dissipative terms are of order $O(\epsilon)$. When ϵ is small we have enough control for these terms using dissipative regularization. Here we do need the viscosity coefficient $\hat{\mu}(\Theta)$ and $\hat{\kappa}(\Theta)$ to be bounded away from zero when Θ is bounded from below. This assumption is consistent with the forms of $\hat{\mu}$ and $\hat{\kappa}$ as derived from kinetic equations. For most collision kernels, $\hat{\mu}$ and $\hat{\kappa}$ are in the form of Θ^γ where $\gamma > 0$ depends on the specific type of the collision kernel. The local well-posedness theorem states

THEOREM 3.1. *Let* $s \geq 6$. *Suppose there exist two constants* $\rho_0, \Theta_0 > 0$ *such that the initial data* $(\rho_\epsilon^{\rm in}, U_\epsilon^{\rm in}, \Theta_\epsilon^{\rm in})$ *satisfies that*

$$\|(\rho_\epsilon^{\rm in} - \rho_0,\ U_\epsilon^{\rm in},\ \Theta_\epsilon^{\rm in} - \Theta_0)\|_{H^s(\mathbb{R}^3)} \leq C_0\,.$$

Then there exists $\epsilon_0 > 0$ *such that for any fixed* $0 < \epsilon < \epsilon_0$, *there exists* $\alpha_0 > 0$ *depending on* ρ_0, Θ_0 *and* $T_\epsilon > 0$ *depending on* ϵ *and* C_0 *such that (1.1) has a unique solution which satisfies*

$$\rho_\epsilon \in C([0, T_\epsilon]; H^s(\mathbb{R}^3))\,,\ (U_\epsilon, \Theta_\epsilon - \Theta_0) \in C([0, T_\epsilon]; H^s(\mathbb{R}^3)) \cap L^2([0, T_\epsilon]; H^{s+1}(\mathbb{R}^3))\,,$$

and

$$\sup_{[0,T_\epsilon]} \|(\rho_\epsilon, U_\epsilon, \Theta_\epsilon - \Theta_0)\|_{H^s} + \alpha_0 \int_0^{T_\epsilon} \|\nabla_x (U_\epsilon, \Theta_\epsilon)\|_{H^{s+1}}^2(\tau)\,\mathrm{d}\tau \leq 2C_0\,.$$

We also need the well-posedness of the ghost effect system (1.11). Again local well-posedness of classical solutions of this system can be established with the aid of classical energy estimates for hyperbolic-parabolic equations. Note that although there is a third-order term in θ in (1.11) and there is no anti-symmetric structure to balance this term, it still does not give rise to any major difficulty because the leading order of this term is in the form of a gradient, which can be incorporated into the pressure term. By doing so the rest of the terms can be treated as perturbations.

Similar proofs can be found for combustion models and Kazhikhov-Smagulov type models (see [10, 27] for example). The theorem states

THEOREM 3.2. *Let $s \geq 5$ and $\bar{\theta}$ be two constants. Let the initial data $(\theta^{\mathrm{in}}, u^{\mathrm{in}})$ satisfy that*

$$\theta^{\mathrm{in}} - \bar{\theta} \in H^{s+1}(\mathbb{R}^3)\,, \qquad u^{\mathrm{in}} \in H^s(\mathbb{R}^3)\,. \tag{3.1}$$

Then there exists $T > 0$ such that the ghost effect system (1.11) has a unique solution (P, θ, u) with

$$\begin{aligned} \theta - \bar{\theta} &\in C([0,T]; H^{s+1}(\mathbb{R}^3)) \cap L^2([0,T]; H^{s+2}(\mathbb{R}^3)) \cap C^\infty((0,T)\times\mathbb{R}^3)\,,\\ u &\in C([0,T]; H^{s}(\mathbb{R}^3)) \cap L^2([0,T]; H^{s+1}(\mathbb{R}^3)) \cap C^\infty((0,T)\times\mathbb{R}^3)\,,\\ \nabla_x P &\in C([0,T]; H^{s-2}(\mathbb{R}^3)) \cap C^\infty((0,T)\times\mathbb{R}^3)\,. \end{aligned} \tag{3.2}$$

4. A Priori Estimates – Slow Motion. In this section we derive a priori estimates for slow motions which varies on time scale of order one. Estimates in the next section about fast motions depend heavily on those for slow motions.

4.1. Linearization. From now on, we drop the ϵ-index for the variables. Consider the following linearized system of (1.7), where we add forcing terms to each equation and linearize the system by replacing certain terms by a given state (p_0, u_0, θ_0):

$$\begin{aligned} \tfrac{3}{5}(\partial_t + u_0\cdot\nabla_x)p + \frac{1}{\epsilon}\nabla_x\cdot\big(u - \tfrac{2}{5}h(\epsilon p_0)\kappa(\theta_0)\,\nabla_x\theta\big) &= \tfrac{2}{5}\epsilon h(\epsilon p_0)\left(\left(\Sigma + \tilde{\Sigma}\right):\nabla_x u_0 + \nabla_x\cdot\tilde{q}\right)\\ &\quad - \tfrac{2}{5}h(\epsilon p_0)\kappa(\theta_0)\nabla_x p_0\cdot\nabla_x\theta + f_1\,,\\ g(\theta_0)(\partial_t + u_0\cdot\nabla_x)\,u + \frac{1}{\epsilon}\nabla_x p &= h(\epsilon p_0)\left(\nabla_x\cdot\Sigma + \nabla_x\cdot\tilde{\Sigma}\right) + f_2\,,\\ \tfrac{3}{2}(\partial_t + u_0\cdot\nabla_x)\,\theta + \nabla_x\cdot u &= \epsilon^2 h(\epsilon p_0)\left(\left(\Sigma + \tilde{\Sigma}\right):\nabla_x u_0 + \nabla_x\cdot\tilde{q}\right)\\ &\quad - h(\epsilon p_0)\nabla_x\cdot q + f_3\,,\\ (p,u,\theta)(x,0) &= (p^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})(x)\,, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} &q = -\kappa(\theta_0)\nabla_x\theta\,,\quad h(\epsilon p_0) = e^{-\epsilon p_0}\,,\quad g(\theta_0) = e^{-\theta_0}\,,\\ &\Sigma = \mu(\theta_0)\left(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{3}(\nabla_x\cdot u)I\right)\,,\\ &\tilde{\Sigma} = \tau_1(\epsilon p_0,\theta_0)\left(\nabla_x^2\theta - \tfrac{1}{3}(\Delta_x\theta)I\right) + \tau_2(\epsilon p_0,\theta_0)\left(\nabla_x\theta_0\otimes\nabla_x\theta - \tfrac{1}{3}(\nabla_x\theta_0\cdot\nabla_x\theta)I\right)\\ &\qquad + \epsilon^2\tau_3(\epsilon p_0,\theta_0)\left((\nabla_x u_0)(\nabla_x u)^T - (\nabla_x u_0)^T\nabla_x u\right)\,,\\ &\tilde{q} = \tau_4(\epsilon p_0,\theta_0)\left(\Delta_x u + \tfrac{1}{3}\nabla_x\nabla_x\cdot u\right) + \tau_5(\epsilon p_0,\theta_0)\nabla_x\theta_0\cdot\left(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{3}(\nabla_x\cdot u)I\right)\\ &\qquad + \tau_6(\epsilon p_0,\theta_0)\left(\nabla_x u - (\nabla_x u)^T\right)\cdot\nabla_x\theta_0\,. \end{aligned}$$

Assume there exist two constants $\alpha_1, \alpha_2 > 0$ such that

$$0 < \alpha_1 < h(\epsilon p_0), g(\theta_0), \mu(\theta_0), \kappa(\theta_0), \tau_1(p_0,\theta_0), \tau_4(p_0,\theta_0) < \alpha_2 < \infty\,. \tag{4.2}$$

We call (p, u) the fast motion and $(\epsilon p, \epsilon u, \theta)$ the slow motion, because $(\epsilon p, \epsilon u, \theta)$ varies on the time scale of order $O(1)$, while (p, u) on the time scale of order $O(\epsilon)$.

DEFINITION 4.1. *For any $s \in \mathbb{R}^1$, define the weighted Sobolev norm $\|\cdot\|_{H^{s+1}_\epsilon}$ as*

$$\|w\|_{H^{s+1}_\epsilon} = \epsilon\|\nabla_x w\|_{H^s} + \|w\|_{H^s}\,,$$

for any $w \in H^{s+1}(\mathbb{R}^3)$. By the definition of Λ_ϵ in (2.3), it is clear that

$$\|w\|_{H^1_\epsilon} \sim \|\Lambda_\epsilon w\|_{L^2}\,,$$

for any $\epsilon > 0$ and $w \in H^{s+1}$.

First we prove a linear estimate for $(\epsilon p, \epsilon u, \theta)$ where (p, u, θ) is the solution of (4.1). To this end, we work out the system for

$$(\hat{p}, \hat{u}, \hat{\theta}) = (\epsilon p - \theta, \epsilon u, \theta)\,. \tag{4.3}$$

Algebraic calculation using (4.1) shows that the system for $(\hat{p}, \hat{u}, \hat{\theta})$ has the form

$$\begin{aligned}
(\partial_t + u_0\cdot\nabla_x)\hat{p} + \frac{1}{\epsilon}\nabla_x\cdot\hat{u} &= \tfrac{3}{5}\epsilon f_1 - \tfrac{2}{3} f_3\,,\\
g(\theta_0)(\partial_t + u_0\cdot\nabla_x)\,\hat{u} + \frac{1}{\epsilon}\nabla_x\hat{p} + \frac{1}{\epsilon}\nabla_x\hat{\theta} &= h(\epsilon p_0)\nabla_x\cdot\Sigma + \epsilon h(\epsilon p_0)\nabla_x\cdot\tilde{\Sigma} + \epsilon f_2\,,\\
\tfrac{3}{2}(\partial_t + u_0\cdot\nabla_x)\,\hat{\theta} + \frac{1}{\epsilon}\nabla_x\cdot\hat{u} &= \epsilon h(\epsilon p_0)\nabla_x\cdot\tilde{q} + \epsilon h(\epsilon p_0)\,\tilde{\Sigma} : \nabla_x(\epsilon u_0)\\
&\quad + h(\epsilon p_0)\Sigma : \nabla_x(\epsilon u_0) - h(\epsilon p_0)\nabla_x\cdot q\\
&\quad + f_3\,,\\
(p, u, \theta)(x, 0) &= (p^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})(x)\,.
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
&q = -\kappa(\theta_0)\nabla_x\hat{\theta}\,, \qquad \Sigma = \mu(\theta_0)\left(\nabla_x\hat{u} + (\nabla_x\hat{u})^T - \tfrac{2}{3}(\nabla_x\cdot\hat{u})I\right)\,,\\
&\tilde{\Sigma} = \tau_1(\epsilon p_0, \theta_0)\left(\nabla_x^2\hat{\theta} - \tfrac{1}{3}(\Delta_x\hat{\theta})I\right) + \tau_2(\epsilon p_0, \theta_0)\left(\nabla_x\theta_0\otimes\nabla_x\hat{\theta} - \tfrac{1}{3}(\nabla_x\theta_0\cdot\nabla_x\hat{\theta})I\right)\\
&\qquad + \tau_3(\epsilon p_0, \theta_0)\left(\nabla_x(\epsilon u_0)(\nabla_x\hat{u})^T - (\nabla_x(\epsilon u_0))^T\nabla_x\hat{u}\right)\,,\\
&\tilde{q} = \tau_4(\epsilon p_0, \theta_0)\left(\Delta_x\hat{u} + \tfrac{1}{3}\nabla_x\nabla_x\cdot\hat{u}\right) + \tau_6(\epsilon p_0, \theta_0)\left(\nabla_x\hat{u} - (\nabla_x\hat{u})^T\right)\cdot\nabla_x\theta_0\,.\\
&\qquad + \tau_5(\epsilon p_0, \theta_0)\nabla_x\theta_0\cdot\left(\nabla_x\hat{u} + (\nabla_x\hat{u})^T - \tfrac{2}{3}(\nabla_x\cdot\hat{u})I\right)
\end{aligned}$$

Note that the equation for $\hat{p} = \epsilon p - \theta$ is essentially the linearized density equation.

4.2. Linear Estimate for the Slow Motion. The L^2-estimate for system (4.4) states

THEOREM 4.2. *Let $(\hat{p}, \hat{u}, \hat{\theta})$ be a solution to the linear system (4.4). Then there exist an increasing function $C(\cdot)$ and a constant $\alpha_0 > 0$ such that*

$$\begin{aligned}
&\sup_{[0,T]}\|(\hat{p}, \hat{u}, \hat{\theta})\|^2_{H^1_\epsilon} + \alpha_0\int_0^T \|\nabla_x(\hat{u}, \hat{\theta})\|^2_{H^1_\epsilon}(\tau)\,\mathrm{d}\tau\\
&\le C(r_0)e^{TC(r)}\,\|(\epsilon p^{in}, \epsilon u^{in}, \theta^{in})\|^2_{H^1_\epsilon} + C(r)\int_0^T \left|\left\langle\hat{p},\ T_2(\tfrac{3}{5}\epsilon f_1 - \tfrac{2}{3}f_3)\right\rangle\right|(\tau)\,\mathrm{d}\tau\\
&\quad + C(r)\int_0^T |\langle\hat{u},\ T_2(\epsilon f_2)\rangle|\,\mathrm{d}\tau + C(r)\int_0^T |\langle\hat{\theta},\ T_1 f_3\rangle|(\tau)\,\mathrm{d}\tau\\
&\quad + C(r)\int_0^T \left|\left\langle\epsilon^3 f_2,\ \Psi_1\hat{u} + \Psi_0\hat{u}\right\rangle\right|\,\mathrm{d}\tau\,,
\end{aligned} \tag{4.5}$$

where the operators T_1, T_2 are defined in (4.9) *and the constants r, r_0 are given by*

$$\begin{aligned}
&\phi = (\epsilon p_0,\ \theta_0)\,, \quad r_0 = \|\phi(0)\|_{L^\infty}\,,\\
&r = \sup_{[0,T]}\left(\|(\phi, \partial_t\phi)\|_{W^{2,\infty}},\ \|(u_0,\ p_0)\|_{W^{1,\infty}}\right)\,.
\end{aligned} \tag{4.6}$$

Proof. The main structure of system (4.4) becomes more transparent by rewriting it in the following abstract way. We use Ψ_i to denote i^{th}-order homogeneous differential operator with their coefficients depending on $(\epsilon p_0, \epsilon u_0, \theta_0)$ and their derivatives up to the second order to obtain

$$
\begin{aligned}
(\partial_t + u_0 \cdot \nabla_x)\hat{p} + \frac{1}{\epsilon}\nabla_x \cdot \hat{u} &= \tfrac{3}{5}\epsilon f_1 - \tfrac{2}{3} f_3\,, \\
g(\theta_0)(\partial_t + u_0\cdot\nabla_x)\,\hat{u} + \frac{1}{\epsilon}\nabla_x\hat{p} + \frac{1}{\epsilon}\nabla_x\hat{\theta} &= h(\epsilon p_0)\mu(\theta_0)\left(\Delta_x + \tfrac{1}{3}\nabla_x\nabla_x\cdot\right)\hat{u} \\
&\quad + \tfrac{2}{3}\epsilon\nabla_x\nabla_x\cdot(h(\epsilon p_0)\tau_1(\epsilon p_0,\theta_0)\nabla_x\hat{\theta}) \\
&\quad + \epsilon\Psi_2^u(\hat{u},\hat{\theta}) + \Psi_1^u(\hat{u},\hat{\theta}) + \epsilon f_2\,, \\
\tfrac{3}{2}(\partial_t + u_0\cdot\nabla_x)\,\hat{\theta} + \frac{1}{\epsilon}\nabla_x\cdot\hat{u} &= h(\epsilon p_0)\kappa(\theta_0)\Delta_x\hat{\theta} + \epsilon\Psi_2^\theta(\hat{u},\hat{\theta}) + \Psi_1^\theta(\hat{u},\hat{\theta}) \\
&\quad + \tfrac{4}{3}\epsilon\,\nabla_x\cdot(\nabla_x\cdot(h(\epsilon p_0)\tau_4(\epsilon p_0,\theta_0)\nabla_x\hat{u})) \\
&\quad + f_3\,, \\
(\hat{p},\hat{u},\hat{\theta})(x,0) &= (\hat{p}^{\text{in}},\hat{u}^{\text{in}},\hat{\theta}^{\text{in}})(x),
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
\Psi_2^u(\hat{u},\hat{\theta}) &= h(\epsilon p_0)\,\nabla_x\tau_1(\epsilon p_0,\theta_0)\cdot(\nabla_x^2\hat{\theta} - \tfrac{1}{3}(\Delta_x\hat{\theta})I) \\
&\quad - \tfrac{2}{3}\nabla_x\left(h(\epsilon p_0)\,\tau_1(\epsilon p_0,\theta_0)\right)\cdot(\nabla_x^2\hat{\theta} + (\Delta_x\hat{\theta})I) \\
&\quad + \tfrac{2}{3}h(\epsilon p_0)\,\tau_2(\epsilon p_0,\theta_0)\nabla_x\theta_0\cdot\nabla_x^2\hat{\theta} \\
&\quad + h(\epsilon p_0)\,\tau_3(\epsilon p_0,\theta_0)\left(\nabla_x(\epsilon u_0) : \nabla_x^2\hat{u} - \nabla_x(\epsilon u_0) : \nabla_x(\nabla_x\hat{u})^T\right), \\
\Psi_1^u(\hat{u},\hat{\theta}) &= -\tfrac{2}{3}\epsilon\nabla_x^2(h(\epsilon p_0)\,\tau_1(\epsilon p_0,\theta_0))\cdot\nabla_x\hat{\theta} \\
&\quad + \epsilon h(\epsilon p_0)\,\nabla_x\tau_2(\epsilon p_0,\theta_0)\cdot\Big(\nabla_x\theta_0\otimes\nabla_x\hat{\theta} - \tfrac{1}{3}(\nabla_x\theta_0\cdot\nabla_x\hat{\theta})I\Big) \\
&\quad + \epsilon h(\epsilon p_0)\,\tau_2(\epsilon p_0,\theta_0)\cdot\Big((\Delta_x\theta_0)\nabla_x\hat{\theta} - \tfrac{1}{3}\nabla_x^2\theta_0\cdot\nabla_x\hat{\theta}\Big) \\
&\quad + \epsilon h(\epsilon p_0)\,\nabla_x\tau_3(\epsilon p_0,\theta_0)\cdot\left((\nabla_x(\epsilon u_0))\,(\nabla_x\hat{u})^T - (\nabla_x(\epsilon u_0))^T\,(\nabla_x\hat{u})\right) \\
&\quad + \tau_3(\epsilon p_0,\theta_0)\left(\Delta_x(\epsilon u_0)(\nabla_x\hat{u})^T - (\nabla_x\nabla_x\cdot(\epsilon u_0))^T\nabla_x\hat{u}\right)\,, \\
&\quad + h(\epsilon p_0)\nabla_x\mu(\theta_0)\cdot\left(\nabla_x\hat{u} + (\nabla_x\hat{u})^T - \tfrac{2}{3}(\nabla_x\cdot\hat{u})I\right)\,, \\
\Psi_2^\theta(\hat{u},\hat{\theta}) &= h(\epsilon p_0)\,\nabla_x\tau_4(\epsilon p_0,\theta_0)\cdot(\Delta_x\hat{u} + \tfrac{1}{3}\nabla_x\nabla_x\cdot\hat{u}) \\
&\quad - \tfrac{4}{3}\nabla_x(h(\epsilon p_0)\,\tau_4(\epsilon p_0,\theta_0))\cdot(\nabla_x\nabla_x\cdot\hat{u} + \Delta_x\hat{u}) \\
&\quad + h(\epsilon p_0)\tau_5(\epsilon p_0,\theta_0)\nabla_x\theta_0\cdot\left(\Delta_x\hat{u} + \tfrac{1}{3}\nabla_x\nabla_x\cdot\hat{u}\right) \\
&\quad + h(\epsilon p_0)\tau_6(\epsilon p_0,\theta_0)\nabla_x\theta_0\cdot(\Delta\hat{u} - \nabla_x\nabla_x\cdot\hat{u})\,, \\
\Psi_1^\theta(\hat{u},\hat{\theta}) &= \epsilon h(\epsilon p_0)(\nabla_x\tau_5(\epsilon p_0,\theta_0)\otimes\nabla_x\theta_0) : \left(\nabla_x\hat{u} + (\nabla_x\hat{u})^T - \tfrac{2}{3}(\nabla_x\cdot\hat{u})I\right) \\
&\quad + h(\epsilon p_0)\tau_5(\epsilon p_0,\theta_0)\nabla_x^2\theta_0 : \left(\nabla_x(\hat{u}) + (\nabla_x(\hat{u}))^T - \tfrac{2}{3}(\nabla_x\cdot(\hat{u}))I\right) \\
&\quad + \epsilon h(\epsilon p_0)(\nabla_x\tau_6(\epsilon p_0,\theta_0)\otimes\nabla_x\theta_0) : (\nabla_x\hat{u} - (\nabla_x\hat{u})^T) \\
&\quad - \tfrac{4}{3}\epsilon\nabla_x^2(h(\epsilon p_0)\,\tau_4(\epsilon p_0,\theta_0)) : \nabla_x\hat{u} + \nabla_x\kappa(\theta_0)\cdot\nabla_x\theta\,.
\end{aligned}
$$

Combining the leading order dispersive terms with the singular terms, we have
(4.8)
$$
\begin{aligned}
(\partial_t + u_0 \cdot \nabla_x)\hat{p} &= -\frac{1}{\epsilon}\nabla_x \cdot \hat{u} + \tfrac{3}{5}\epsilon f_1 - \tfrac{2}{3} f_3 \,, \\
g(\theta_0)(\partial_t + u_0 \cdot \nabla_x)\,\hat{u} + \frac{1}{\epsilon}\nabla_x \hat{p} &= -\frac{1}{\epsilon}\nabla_x \left(\left(I - \tfrac{2}{3}\epsilon^2 \nabla_x \cdot (h(\epsilon p_0)\tau_1(\epsilon p_0, \theta_0)\nabla_x)\right)\hat{\theta}\right) \\
&\quad + h(\epsilon p_0)\mu(\theta_0)\left(\Delta_x + \tfrac{1}{3}\nabla_x \nabla_x \cdot\right)\hat{u} + \epsilon \Psi_2^u(\hat{u}, \hat{\theta}) \\
&\quad + \Psi_1^u(\hat{u}, \hat{\theta}) + \epsilon f_2 \,, \\
\tfrac{3}{2}(\partial_t + u_0 \cdot \nabla_x)\,\hat{\theta} &= -\frac{1}{\epsilon}\nabla_x \cdot \left(\left(I - \tfrac{4}{3}\epsilon^2 \nabla_x \cdot (h(\epsilon p_0)\tau_4(\epsilon p_0, \theta_0)\nabla_x)\right)\hat{u}\right) \\
&\quad + h(\epsilon p_0)\kappa(\theta_0)\Delta_x \hat{\theta} + \epsilon \Psi_2^\theta(\hat{u}, \hat{\theta}) + \Psi_1^\theta(\hat{u}, \hat{\theta}) + f_3 \,, \\
(\hat{p}, \hat{u}, \hat{\theta})(x, 0) &= (\hat{p}^{\text{in}}, \hat{u}^{\text{in}}, \hat{\theta}^{\text{in}})(x) \,.
\end{aligned}
$$

Defining the operators T_1, T_2 as

$$
T_1 = I - \tfrac{2}{3}\epsilon^2 \nabla_x \cdot (h(\epsilon p_0)\tau_1(\theta_0)\nabla_x) \,, \qquad T_2 = I - \tfrac{4}{3}\epsilon^2 \nabla_x \cdot (h(\epsilon p_0)\tau_4(\theta_0)\nabla_x) \tag{4.9}
$$

it is apparent that T_1, T_2 are symmetric positive operators. Adopting this notation, system (4.8) now has the form

$$
\begin{aligned}
(\partial_t + u_0 \cdot \nabla_x)\hat{p} + \frac{1}{\epsilon}\nabla_x \cdot \hat{u} &= \tfrac{3}{5}\epsilon f_1 - \tfrac{2}{3} f_3 \,, \\
g(\theta_0)(\partial_t + u_0 \cdot \nabla_x)\,\hat{u} + \frac{1}{\epsilon}\nabla_x \hat{p} + \frac{1}{\epsilon}\nabla_x (T_1 \hat{\theta}) &= h(\epsilon p_0)\mu(\theta_0)\left(\Delta_x + \tfrac{1}{3}\nabla_x \nabla_x \cdot\right)\hat{u} \\
&\quad + \epsilon \Psi_2^u(\hat{u}, \hat{\theta}) + \Psi_1^u(\hat{u}, \hat{\theta}) + \epsilon f_2 \,, \\
\tfrac{3}{2}(\partial_t + u_0 \cdot \nabla_x)\,\hat{\theta} + \frac{1}{\epsilon}\nabla_x \cdot (T_2 \hat{u}) &= h(\epsilon p_0)\kappa(\theta_0)\Delta_x \hat{\theta} + \epsilon \Psi_2^\theta(\hat{u}, \hat{\theta}) \\
&\quad + \Psi_1^\theta(\hat{u}, \hat{\theta}) + f_3 \,, \\
(\hat{p}, \hat{u}, \hat{\theta})(x, 0) &= (\hat{p}^{\text{in}}, \hat{u}^{\text{in}}, \hat{\theta}^{\text{in}})(x) \,.
\end{aligned}
\tag{4.10}
$$

In order to perform the L^2 energy estimate, we multiply $T_2\hat{p}$, $T_2\hat{u}$, $T_1\hat{\theta}$ to the equations for $\hat{p}, \hat{u}, \hat{\theta}$ respectively, integrate over R^3, and sum up the three integrated equations. Note that although T_1, T_2 are positive symmetric operators, they have variable coefficients. Thus they do not commute either with time derivatives or spatial derivatives. To make notations shorter, let

$$
g_1(\phi_0) = \tfrac{2}{3} h(\epsilon p_0)\tau_1(\epsilon p_0, \theta_0) \,, \quad g_2(\phi_0) = \tfrac{4}{3} h(\epsilon p_0)\tau_4(\epsilon p_0, \theta_0) \,, \quad \langle \cdot \rangle = \int_{\mathbb{R}^3} \cdot \, \mathrm{d}x \,.
$$

First, the time derivative term in the integrated $\hat{p}$-equation is

$$
\frac{\mathrm{d}}{\mathrm{dt}}\langle \hat{p},\, T_2\hat{p}\rangle = \langle \partial_t \hat{p},\, T_2 \hat{p}\rangle + \langle \hat{p},\, \partial_t (T_2 \hat{p})\rangle = 2\langle \partial_t \hat{p},\, T_2\hat{p}\rangle + \langle \hat{p},\, [\partial_t, T_2]\hat{p}\rangle \,, \tag{4.11}
$$

where

$$
[\partial_t, T_2]\hat{p} = -\epsilon^2 \nabla_x \cdot ((\partial_t g_2(\phi_0))\nabla_x \hat{p}) \,.
$$

Therefore we have

$$
\left| \left\langle \hat{p},\; [\partial_t, T_2]\hat{p} \right\rangle \right| \leq C(r)\|\epsilon \nabla_x \hat{p}\|_{L^2}^2 \leq C(r)\|\hat{p}\|_{H^1_\epsilon}^2 \,,
$$

where $C(r) = C_1(\|\partial_t g_2(\phi_0)\|_{L^\infty})$ is a positive increasing function in its variable. The definition of r is given by (4.6). Thus, we have

$$\langle \partial_t \hat{p},\, T_2 \hat{p} \rangle = \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}} \langle \hat{p},\, T_2 \hat{p} \rangle - \tfrac{1}{2} \langle \hat{p},\, [\partial_t, T_2] \hat{p} \rangle \,, \tag{4.12}$$

with

$$|\langle \hat{p},\ [\partial_t, T_2] \hat{p} \rangle| \le C(r) \|\hat{p}\|^2_{H^1_\epsilon} \,. \tag{4.13}$$

The second term in the estimate for $\hat{p}$ is $\big\langle \hat{p},\, T_2(u_0 \cdot \nabla_x \hat{p}) \big\rangle$, which has the bound

$$\big|\big\langle \hat{p},\ T_2(u_0 \cdot \nabla_x \hat{p}) \big\rangle\big| \le \big|\big\langle \hat{p},\ (u_0 \cdot \nabla_x \hat{p}) \big\rangle\big| + \big|\big\langle \hat{p},\ \epsilon^2 \nabla_x \cdot (g_2(\phi_0) \nabla_x (u_0 \cdot \nabla_x \hat{p})) \big\rangle\big|$$

where

$$\big|\big\langle \hat{p},\ (u_0 \cdot \nabla_x \hat{p}) \big\rangle\big| = \tfrac{1}{2} \left| \int_{\mathbb{R}^3} (\nabla_x \cdot u_0)\, |\hat{p}|^2 \,\mathrm{d}x \right| \le C(r)\, \|\hat{p}\|^2_{L^2} \,,$$

and

$$\begin{aligned} &\big|\big\langle \hat{p},\ \epsilon^2 \nabla_x \cdot (g_2(\phi_0) \nabla_x (u_0 \cdot \nabla_x \hat{p})) \big\rangle\big| = \epsilon^2 \left|\big\langle \nabla_x \hat{p},\ g_2(\phi_0) \nabla_x (u_0 \cdot \nabla_x \hat{p}) \big\rangle\right| \\ &\le \epsilon^2 \int_{\mathbb{R}^3} |g_2(\phi_0) \nabla_x u_0|\, |\nabla_x \hat{p}|^2 \,\mathrm{d}x + \tfrac{1}{2} \epsilon^2 \int_{\mathbb{R}^3} |\nabla_x \cdot (g_2(\phi_0)\, u_0)| |\nabla_x \hat{p}|^2 \,\mathrm{d}x \,. \end{aligned}$$

Therefore,

$$\big|\big\langle \hat{p},\ T_2(u_0 \cdot \nabla_x \hat{p}) \big\rangle\big| \le C(r) \, \big(\|\hat{p}\|^2_{L^2} + \|\epsilon \nabla_x \hat{p}\|^2_{L^2} \big) \le C(r) \|\hat{p}\|^2_{H^1_\epsilon} \,. \tag{4.14}$$

Here $C(r) = C_2(\|u_0\|_{W^{1,\infty}}, \|g_2\|_{W^{1,\infty}})$.

The third term in the estimate for the $\hat{p}$-equation is

$$\left\langle \hat{p},\ \frac{1}{\epsilon} T_2 \nabla_x \cdot \hat{u} \right\rangle = \frac{1}{\epsilon} \left\langle T_2 \hat{p},\ \nabla_x \cdot \hat{u} \right\rangle \,. \tag{4.15}$$

Adding (4.12), (4.14), and (4.15) together, we obtain

$$\tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}} \langle \hat{p},\, T_2 \hat{p} \rangle + \frac{1}{\epsilon} \left\langle T_2 \hat{p},\ \nabla_x \cdot \hat{u} \right\rangle \le C(r)\, \|\hat{p}\|^2_{H^1_\epsilon} + \big|\big\langle \hat{p},\ T_2(\tfrac{3}{5} \epsilon f_1 - \tfrac{2}{3} f_3) \big\rangle\big| \,, \tag{4.16}$$

where $C(r) = C_3(\|\partial_t g_2\|_{L^\infty}, \|(u_0, g_2)\|_{W^{1,\infty}})$.

Next, we estimate each term in the integrated $\hat{u}$-equation. First, the time derivative term is treated in the similar way as in (4.11). In order to make the operator symmetric, we consider

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}} \Big\langle \hat{u},\ \sqrt{g(\theta_0)} T_2 (\sqrt{g(\theta_0)} \hat{u}) \Big\rangle \\ &= \frac{\mathrm{d}}{\mathrm{dt}} \Big\langle \hat{u},\, T_2 (g(\theta_0) \hat{u}) \Big\rangle + \frac{\mathrm{d}}{\mathrm{dt}} \Big\langle \hat{u},\, [\sqrt{g(\theta_0)}, T_2] (\sqrt{g(\theta_0)} \hat{u}) \Big\rangle \,, \end{aligned} \tag{4.17}$$

The first term on the right-hand side of (4.17) is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \Big\langle \hat{u},\, T_2 (g(\theta_0) \hat{u}) \Big\rangle &= \Big\langle \partial_t \hat{u},\, T_2 (g(\theta_0) \hat{u}) \Big\rangle + \Big\langle \hat{u},\, \partial_t \left(T_2 (g(\theta_0) \hat{u}) \right) \Big\rangle \\ &= 2 \Big\langle \hat{u},\, T_2 \big(g(\theta_0) \partial_t \hat{u} \big) \Big\rangle - \Big\langle \partial_t \hat{u},\, [g(\theta_0), T_2] \hat{u} \Big\rangle \\ &\quad + \Big\langle \hat{u},\, [\partial_t, T_2 (g(\theta_0) \,\cdot)] \hat{u} \Big\rangle \,, \end{aligned} \tag{4.18}$$

where

$$[\partial_t, T_2(g(\theta_0)\,\cdot)]\,\hat{u} = \partial_t(g(\theta_0))\,\hat{u} - \epsilon^2\nabla_x\cdot(\partial_t g_2(\phi_0)\,\nabla_x(g(\theta_0)\,\hat{u})) \\ - \epsilon^2\nabla_x\cdot(g_2(\phi_0)\,\nabla_x((\partial_t g(\theta_0))\,\hat{u}))\ .$$

Hence by integration by parts, the bound for $\left\langle \hat{u},\, [\partial_t, T_2(g(\theta_0)\,\cdot)]\hat{u}\right\rangle$ is

$$\left|\left\langle \hat{u},\, [\partial_t, T_2(g(\theta_0)\,\cdot)]\hat{u}\right\rangle\right| \le C(r)\,\left(\|\hat{u}\|_{L^2}^2 + \|\epsilon\nabla_x\hat{u}\|_{L^2}^2\right) \le C(r)\,\|\hat{u}\|_{H^1_\epsilon}^2\,, \tag{4.19}$$

with $C(r) = C_4(\|\partial_t g\|_{W^{1,\infty}}, \|g\|_{W^{1,\infty}}, \|\partial_t g_2\|_{L^\infty})$.

In order to estimate $\left\langle \partial_t\hat{u},\, [g(\theta_0), T_2]\hat{u}\right\rangle$, first we write

$$[g(\theta_0), T_2]\hat{u} = \epsilon^2\nabla_x\cdot(g_2(\phi_0)\,\nabla_x g(\theta_0)\otimes\hat{u}) + \epsilon^2 g_2(\phi_0)\nabla_x g(\theta_0)\cdot\nabla_x\hat{u} \\ \triangleq \epsilon^2(\Psi_1\hat{u} + \Psi_0\hat{u})\,.$$

Here again Ψ_1 and Ψ_0 are homogeneous operators of order 1 and 0 respectively. By the $\hat{u}$-equation,

$$\epsilon^2\partial_t\hat{u} = -\epsilon^2 u_0\cdot\nabla_x\hat{u} - \epsilon g(\theta_0)^{-1}\nabla_x\hat{p} - \epsilon g(\theta_0)^{-1}\nabla_x(T_1\hat{\theta}) \\ + \epsilon^2 g(\theta_0)^{-1}h(\epsilon p_0)\mu(\theta_0)\left(\Delta_x + \tfrac{1}{3}\nabla_x\nabla_x\cdot\right)\hat{u} + \epsilon^3\Psi_2^u(\hat{u},\hat{\theta}) + \epsilon^2\Psi_1^u(\hat{u},\hat{\theta}) + \epsilon^3 f_2 \\ = -\epsilon^2 u_0\cdot\nabla_x\hat{u} - \epsilon g(\theta_0)^{-1}\nabla_x\hat{p} - \epsilon g(\theta_0)^{-1}\nabla_x(T_1\hat{\theta}) + \epsilon^2\Psi_2^u\hat{u} + \epsilon^3\Psi_2^u\hat{\theta} \\ + \epsilon^2\Psi_1^u(\hat{u},\hat{\theta}) + \epsilon^3 f_2\,.$$

The bounds for each term in $\left\langle \epsilon^2\partial_t\hat{u},\, \Psi_1\hat{u}\right\rangle$ are

$$\left|\left\langle \epsilon^2 u_0\cdot\nabla_x\hat{u},\, \Psi_1\hat{u} + \Psi_0\hat{u}\right\rangle\right| \le C(r)\,\|\epsilon\hat{u}\|_{L^2}^2 + C(r)\,\|\epsilon\nabla_x\hat{u}\|_{L^2}^2\,,$$
$$\left|\left\langle \epsilon g(\theta_0)^{-1}\nabla_x\hat{p},\, \Psi_1\hat{u} + \Psi_0\hat{u}\right\rangle\right| \le C(r)\,\|\epsilon\nabla_x\hat{p}\|_{L^2}^2 + C(r)\,\|\hat{u}\|_{L^2}^2 + \tfrac{1}{8}\alpha_0\|\nabla_x\hat{u}\|_{L^2}^2\,,$$

$$\left|\left\langle \epsilon g(\theta_0)^{-1}\nabla_x(T_1\hat{\theta}),\ \Psi_1\hat{u} + \Psi_0\hat{u}\right\rangle\right| \\ \le \left|\left\langle \epsilon g(\theta_0)^{-1}\nabla_x\hat{\theta},\ \Psi_1\hat{u} + \Psi_0\hat{u}\right\rangle\right| + \left|\left\langle \epsilon^3 g(\theta_0)^{-1}\nabla_x\nabla_x\cdot(g_1(\phi_0)\nabla_x\hat{\theta}),\ \Psi_1\hat{u} + \Psi_0\hat{u}\right\rangle\right| \\ \le C(r)\,\left(\|\epsilon\nabla_x\hat{\theta}\|_{L^2}^2 + \|\hat{u}\|_{L^2}^2\right) + \tfrac{1}{8}\alpha_0\|\nabla_x\hat{u}\|_{L^2}^2 + \epsilon\,C(r)\,\left(\|\epsilon\Delta_x\hat{u}\|_{L^2}^2 + \|\epsilon\Delta_x\hat{\theta}\|_{L^2}^2\right)\,,$$

$$\left|\left\langle \epsilon^2\Psi_2\hat{u},\, \Psi_1\hat{u} + \Psi_0\hat{u}\right\rangle\right| \le C(r)\,\left(\|\epsilon\nabla_x\hat{u}\|_{L^2}^2 + \|\epsilon\hat{u}\|_{L^2}^2\right) + \tfrac{1}{8}\alpha_0\,\|\epsilon\Delta_x\hat{u}\|_{L^2}^2\,,$$
$$\left|\left\langle \epsilon^3\Psi_2\hat{\theta},\, \Psi_1\hat{u} + \Psi_0\hat{u}\right\rangle\right| \le C(r)\,\left(\|\epsilon\nabla_x\hat{u}\|_{L^2}^2 + \|\epsilon\hat{u}\|_{L^2}^2\right) + \epsilon\,\|\epsilon\Delta_x\hat{\theta}\|_{L^2}^2\,,$$
$$\left|\left\langle \epsilon^2\Psi_1(\hat{u},\hat{\theta}),\, \Psi_1\hat{u} + \Psi_0\hat{u}\right\rangle\right| \le C(r)\,\left(\|\epsilon\nabla_x\hat{u}\|_{L^2}^2 + \|\epsilon\hat{u}\|_{L^2}^2 + \|\epsilon\nabla_x\hat{\theta}\|_{L^2}^2\right)\,,$$

Combining these bounds we obtain

$$\left|\left\langle \partial_t\hat{u},\, [g(\theta_0), T_2]\hat{u}\right\rangle\right| \le C(r)\,\left(\|\epsilon\nabla_x\hat{p}\|_{L^2}^2 + \|\hat{u}\|_{H^1_\epsilon}^2 + \|\epsilon\nabla_x\hat{\theta}\|_{L^2}^2\right) \\ + \tfrac{1}{4}\alpha_0\|\nabla_x\hat{u}\|_{L^2}^2 + \tfrac{1}{4}\alpha_0\|\epsilon\Delta_x\hat{u}\|_{L^2}^2 + \tfrac{1}{4}\alpha_0\|\epsilon\Delta_x\hat{\theta}\|_{L^2}^2\,, \\ + \left|\left\langle \epsilon^3 f_2,\, \Psi_1\hat{u} + \Psi_0\hat{u}\right\rangle\right| \tag{4.20}$$

where α_0 is less than the lower bound of the viscosity coefficients such that the terms associated with α_0 can be absorbed into the dissipation terms and ϵ is chosen to be small enough such that $\epsilon C(r) \le \frac{1}{8}\alpha_0$. Here

$$C(r) = C_5(\|u_0\|_{L^\infty}, \|(g,\, g_1,\, g_2)\|_{W^{2,\infty}}, \|g^{-1}\|_{L^\infty}, \|(\tau_1, h)\|_{W^{1,\infty}}) .$$

Combining (4.19) and (4.20), we have

$$\begin{aligned}\Big\langle \hat{u},\, T_2\big(g(\theta_0)\partial_t\hat{u}\big)\Big\rangle &= \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big\langle \hat{u},\, T_2(g(\theta_0)\hat{u})\Big\rangle + \tfrac{1}{2}\Big\langle \hat{u},\, [g(\theta_0), T_2]\partial_t\hat{u}\Big\rangle \\ &\quad - \tfrac{1}{2}\Big\langle \hat{u},\, [\partial_t, T_2(g(\theta_0)\,\cdot)]\hat{u}\Big\rangle ,\end{aligned} \tag{4.21}$$

with

$$\begin{aligned}&\Big|\tfrac{1}{2}\Big\langle \hat{u},\, [g(\theta_0), T_2]\partial_t\hat{u}\Big\rangle\Big| + \Big|\tfrac{1}{2}\Big\langle \hat{u},\, [\partial_t, T_2(g(\theta_0)\,\cdot)]\hat{u}\Big\rangle\Big| \\ &\le C(r)\,\Big(\|\hat{u}\|^2_{H^1_\epsilon} + \|\epsilon\nabla_x\hat{p}\|^2_{L^2} + \|\epsilon\nabla_x\hat{\theta}\|^2_{L^2}\Big) + \tfrac{1}{4}\alpha_0\|\nabla_x\hat{u}\|^2_{L^2} \\ &\quad + \tfrac{1}{4}\alpha_0\|\epsilon\Delta_x\hat{u}\|^2_{L^2} + \tfrac{1}{4}\alpha_0\|\epsilon\Delta_x\hat{\theta}\|^2_{L^2} + \Big|\Big\langle \epsilon^3 f_2,\, \Psi_1\hat{u} + \Psi_0\hat{u}\Big\rangle\Big| ,\end{aligned}$$

where

$$\begin{aligned}&C(r) \\ &= C_6(\|\partial_t g\|_{W^{1,\infty}}, \|\partial_t g_2\|_{L^\infty}, \|u_0\|_{L^\infty}, \|(g, g_1, g_2)\|_{W^{2,\infty}}, \|g^{-1}\|_{L^\infty}, \|(\tau_1, h)\|_{W^{1,\infty}}) .\end{aligned}$$

The second term on the right-hand side of (4.17) is bounded in a similar way. By integration by parts,

$$\Big\langle \hat{u},\, [\sqrt{g(\theta_0)}, T_2](\sqrt{g(\theta_0)}\hat{u})\Big\rangle = \epsilon^2\,\langle \hat{u},\, \Psi_1\hat{u}\rangle + \epsilon^2\,\langle \hat{u},\, \Psi_0\hat{u}\rangle = \epsilon^2\,\langle \hat{u},\, \Psi_0\hat{u}\rangle ,$$

where Ψ_1, Ψ_0 are homogeneous operators of order 1 and 0 respectively. Their coefficients depend on $\nabla_x^\alpha(g(\theta_0), g_2(\epsilon p_0, \theta_0))$ with $|\alpha| \le 2$. Thus,

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big\langle \hat{u},\, [\sqrt{g(\theta_0)}, T_2](\sqrt{g(\theta_0)}\hat{u})\Big\rangle = \big\langle \epsilon^2\partial_t\hat{u},\, \Psi_0\hat{u}\big\rangle + \epsilon^2\,\langle \hat{u},\, \Psi_0\hat{u}\rangle . \tag{4.22}$$

The equation for $\hat{u}$ yields

$$\epsilon^2\partial_t\hat{u} = \epsilon\Psi_1\hat{p} + \epsilon^2\Psi_2\hat{u} + \epsilon^2\Psi_1\hat{u} + \epsilon^3\Psi_2\hat{\theta} + \epsilon^2\Psi_2\hat{\theta} + \epsilon^2\Psi_1\hat{\theta} + \epsilon^3 g^{-1}(\theta_0) f_2 .$$

Hence the bound for the first term on the right-hand side of (4.22) is

$$\big|\big\langle \epsilon^2\partial_t\hat{u},\, \Psi_0\hat{u}\big\rangle\big| \le C(r)\,\|(\hat{p}, \hat{u}, \hat{\theta})\|^2_{H^1_\epsilon} + \tfrac{1}{16}\alpha_0\|\Delta_x(\epsilon\hat{u})\|^2_{L^2} + \epsilon^3|\langle \hat{u},\, g^{-1}(\theta_0) f_2\rangle| .$$

Thus we have the bound for the second term of the right-hand side of (4.17) as

$$\begin{aligned}&\Big|\frac{\mathrm{d}}{\mathrm{dt}}\Big\langle \hat{u},\, [\sqrt{g(\theta_0)}, T_2](\sqrt{g(\theta_0)}\hat{u})\Big\rangle\Big| \\ &\le C(r)\,\|(\hat{p}, \hat{u}, \hat{\theta})\|^2_{H^1_\epsilon} + \tfrac{1}{16}\alpha_0\|\Delta_x(\epsilon\hat{u})\|^2_{L^2} + \epsilon^3|\langle \hat{u},\, g^{-1}(\theta_0) f_2\rangle| .\end{aligned} \tag{4.23}$$

Combining (4.21) and (4.23), we have

$$\Big\langle \hat{u},\, T_2\big(g(\theta_0)\partial_t\hat{u}\big)\Big\rangle = \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big\langle \hat{u},\, T_2(g(\theta_0)\hat{u})\Big\rangle + R_1 , \tag{4.24}$$

with

$$R_1 \le C(r)\,\|(\hat{p},\hat{u},\hat{\theta})\|^2_{H^1_\epsilon} + \tfrac14\alpha_0\|\nabla_x\hat{u}\|^2_{H^1_\epsilon} + \tfrac14\alpha_0\|\epsilon\Delta_x\hat{\theta}\|^2_{L^2} + \left|\left\langle \epsilon^3 f_2,\ \Psi_1\hat{u}+\Psi_0\hat{u}\right\rangle\right| .$$

The convection term is bounded as

$$\begin{aligned} &|\langle \hat{u},\ T_2\left(g(\theta_0)u_0\cdot\nabla_x\hat{u}\right)\rangle| \\ &\le |\langle \hat{u},\ g(\theta_0)u_0\cdot\nabla_x\hat{u}\rangle| + \epsilon^2\,|\langle \nabla_x\cdot(g_2(\phi_0)\nabla_x\hat{u}),\ g(\theta_0)u_0\cdot\nabla_x\hat{u}\rangle| \\ &\le C(r)\|\hat{u}\|^2_{H^1_\epsilon}\,, \end{aligned} \tag{4.25}$$

by integration by parts. Here $C(r) = C_8(\|(g,g_2,u_0)\|_{W^{1,\infty}})$.

The next two terms in the integrated $\hat{u}$-equation are

$$\left\langle \hat{u},\ \frac1\epsilon\nabla_x(T_2\hat{p})\right\rangle + \left\langle \hat{u},\ \frac1\epsilon T_2\nabla_x(T_1\hat{\theta})\right\rangle = -\left\langle \nabla_x\cdot\hat{u},\ \frac1\epsilon(T_2\hat{p})\right\rangle + \left\langle T_2\hat{u},\ \frac1\epsilon\nabla_x(T_1\hat{\theta})\right\rangle . \tag{4.26}$$

Next, the dissipative term satisfies

$$\begin{aligned} &-\left\langle \hat{u},\ T_2\left(h(\epsilon p_0)\mu(\theta_0)\left(\Delta_x + \tfrac13\nabla_x\nabla_x\cdot\right)\hat{u}\right)\right\rangle \\ &\ge 2\alpha_0\left(\|\nabla_x\hat{u}\|^2_{L^2} + \|\epsilon\Delta_x\hat{u}\|^2_{L^2}\right) - C(r)\,\|\hat{u}\|^2_{L^2}\,. \end{aligned} \tag{4.27}$$

The bounds for the remaining terms in the $\hat{u}$-equation are

$$\begin{aligned} &\left|\left\langle \hat{u},\ \epsilon T_2\Psi_2(\hat{u},\hat{\theta}) + T_2\Psi_1(\hat{u},\hat{\theta}) + \epsilon\, T_2 f_2\right\rangle\right| + \left|\left\langle \hat{u},\ \frac1\epsilon[T_2,\ \nabla_x]\hat{p}\right\rangle\right| \\ &\le \left|\left\langle \hat{u},\ \epsilon\Psi_2(\hat{u},\hat{\theta}) + \Psi_1(\hat{u},\hat{\theta})\right\rangle\right| + \epsilon^3\left|\left\langle \nabla_x\cdot(g_2(\phi_0)\nabla_x\hat{u}),\ \Psi_2(\hat{u},\hat{\theta})\right\rangle\right| \\ &\quad + \epsilon^3\left|\left\langle \nabla_x\cdot(g_2(\phi_0)\nabla_x\hat{u}),\ \Psi_1(\hat{u},\hat{\theta})\right\rangle\right| + \epsilon\,|\langle \hat{u},\ [\nabla_x\cdot(g_2(\phi_0)\nabla_x),\nabla_x]\hat{p}\rangle| \\ &\quad + |\langle \hat{u},\epsilon T_2 f_2\rangle| \\ &\le C(r)\|\hat{u}\|_{L^2}\|\Delta_x(\epsilon\hat{u},\epsilon\hat{\theta})\|_{L^2} + \epsilon C(r)\|\nabla_x(\epsilon\hat{u})\|_{L^2}\|\Delta_x(\epsilon\hat{u},\epsilon\hat{\theta})\|_{L^2} \\ &\quad + \epsilon C(r)\|\Delta_x(\epsilon\hat{u},\epsilon\hat{\theta})\|^2_{L^2} + \epsilon\,|\langle \hat{u},\ \nabla_x\cdot(\nabla_x\hat{p}\otimes\nabla_x g_2)\rangle| + |\langle \hat{u},\ T_2 f_2\rangle| \\ &\le C(r)\,\|(\hat{p},\hat{u},\hat{\theta})\|^2_{H^1_\epsilon} + \tfrac18\alpha_0\left(\|\nabla_x\hat{u}\|^2_{H^1_\epsilon} + \|\Delta_x(\epsilon\hat{\theta})\|^2_{L^2}\right) + |\langle \hat{u},\ \epsilon\, T_2 f_2\rangle|\,, \end{aligned} \tag{4.28}$$

by choosing ϵ small such that $\epsilon C(r) \le \frac1{16}\alpha_0$. Here $C(r) = C_9(\|(h,\tau_1,\tau_2,\tau_3,g_2)\|_{W^{1,\infty}})$.

Combining (4.24), (4.25), (4.26), (4.27), and (4.28), we have

$$\begin{aligned} &\tfrac12\frac{\mathrm{d}}{\mathrm{dt}}\left\langle \hat{u},\ T_2(g(\theta_0)\hat{u})\right\rangle - \left\langle \nabla_x\cdot\hat{u},\ \frac1\epsilon(T_2\hat{p})\right\rangle + \left\langle T_2\hat{u},\ \frac1\epsilon\nabla_x(T_1\hat{\theta})\right\rangle + \tfrac32\alpha_0\|\nabla_x\hat{u}\|^2_{H^1_\epsilon} \\ &\le C(r)\|(\hat{p},\hat{u},\hat{\theta})\|^2_{H^1_\epsilon} + \frac12\alpha_0\|\epsilon\Delta_x\hat{\theta}\|^2_{L^2} + |\langle \hat{u},\ \epsilon\, T_2 f_2\rangle| + \left|\left\langle \epsilon^3 f_2,\ \Psi_1\hat{u}+\Psi_0\hat{u}\right\rangle\right| . \end{aligned} \tag{4.29}$$

Similarly, the $\hat{\theta}$-equation yields

$$\begin{aligned} &\tfrac12\frac{\mathrm{d}}{\mathrm{dt}}\left\langle \tfrac32\hat{\theta},\ T_1\hat{\theta}\right\rangle - \left\langle T_2\hat{u},\ \frac1\epsilon\nabla_x(T_1\hat{\theta})\right\rangle + \tfrac32\alpha_0\|\nabla_x\hat{\theta}\|^2_{H^1_\epsilon} \\ &\le C(r)\|(\hat{p},\hat{u},\hat{\theta})\|^2_{H^1_\epsilon} + \frac12\alpha_0\|\epsilon\Delta_x\hat{u}\|^2_{L^2} + \left|\left\langle \hat{\theta},\ T_1 f_3\right\rangle\right| . \end{aligned} \tag{4.30}$$

Upon adding up (4.16), (4.29), and (4.30), the singular terms are canceled out, and the energy inequality is

(4.31)
$$\begin{aligned}&\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\left\langle \hat{p},\, T_2\hat{p}\right\rangle + \left\langle \hat{u},\, \sqrt{g(\theta_0)}T_2(\sqrt{g(\theta_0)}\hat{u})\right\rangle + \left\langle \tfrac{3}{2}\hat{\theta},\, T_1\hat{\theta}\right\rangle\right) + \alpha_0\|\nabla_x(\hat{u},\hat{\theta})\|^2_{H^1_\epsilon}\\
&\le C(r)\,\|(\hat{p},\hat{u},\hat{\theta})\|^2_{H^1_\epsilon} + \left|\langle \hat{p},\ T_2(\tfrac{3}{5}\epsilon f_1 - \tfrac{2}{3}f_3)\rangle\right| + |\langle \hat{u},\ T_2(\epsilon f_2)\rangle| + \left|\left\langle \hat{\theta},\ T_1 f_3\right\rangle\right|\\
&\quad + \left|\left\langle \epsilon^3 f_2,\ \Psi_1\hat{u} + \Psi_0\hat{u}\right\rangle\right| .\end{aligned}$$

By Gronwall's inequality and the equivalence of norms

$$\left\langle \hat{p},\, T_2\hat{p}\right\rangle + \left\langle \hat{u},\, \sqrt{g(\theta_0)}T_2(\sqrt{g(\theta_0)}\hat{u})\right\rangle + \left\langle \tfrac{3}{2}\hat{\theta},\, T_1\hat{\theta}\right\rangle \sim \|(\hat{p},\hat{u},\hat{\theta})\|^2_{H^1_\epsilon} \sim \|(\epsilon p,\epsilon u,\theta)\|^2_{H^1_\epsilon}\,,$$

we have

$$\begin{aligned}&\sup_{[0,t]}\|(\hat{p},\hat{u},\hat{\theta})\|_{H^1_\epsilon} + \alpha_0\left(\int_0^t \|\nabla_x(\hat{u},\hat{\theta})\|^2_{H^1_\epsilon}(\tau)\,\mathrm{d}\tau\right)^{1/2}\\
&\le C(r_0)e^{TC(r)}\,\|(\epsilon p^{in},\epsilon u^{in},\theta^{in})\|_{H^1_\epsilon} + C(r)\int_0^T \left|\langle \hat{p},\ T_2(\tfrac{3}{5}\epsilon f_1 - \tfrac{2}{3}f_3)\rangle\right|(\tau)\,\mathrm{d}\tau\\
&\quad + C(r)\int_0^T |\langle \hat{u},\ T_2(\epsilon f_2)\rangle|(\tau)\,\mathrm{d}\tau + C(r)\int_0^T |\langle \hat{\theta},\ T_1 f_3\rangle|(\tau)\,\mathrm{d}\tau\\
&\quad + C(r)\int_0^T\left|\left\langle \epsilon^3 f_2,\ \Psi_1\hat{u}+\Psi_0\hat{u}\right\rangle\right|\,\mathrm{d}\tau\,,\end{aligned}\tag{4.32}$$

which completes the proof. □

4.3. A Priori Estimate for Slow Motions. In this part we establish the a priori estimate for the nonlinear system (1.7). First, for $t > 0$ and $s > 5$, recall the norm $\|\!|\cdot|\!\|_{\epsilon,s,t}$ defined as

(4.33)
$$\begin{aligned}\|\!|(p,u,\theta)|\!\|_{\epsilon,s,t} &:= \sup_{[0,t]}\left(\|(p,u)\|_{H^s} + \|\Lambda_\epsilon^{2s+1}(\epsilon p,\epsilon u,\theta)\|_{H^{s+1}}\right)\\
&\quad + \alpha_0\left(\int_0^t \left(\|\nabla_x(u,p)\|^2_{H^s} + \|\nabla_x\Lambda_\epsilon^{2s+1}(\epsilon u,\theta)\|^2_{H^{s+1}}\right)(\tau)\,\mathrm{d}\tau\right)^{1/2},\end{aligned}$$

where $\alpha_0 > 0$ is given by Theorem 4.2. We also define

$$\begin{aligned}R &= \|(p,u)\|_{H^s} + \|\Lambda_\epsilon^{2s+1}(\epsilon p,\epsilon u,\theta)\|_{H^{s+1}}\,,\\
R' &= \|\nabla_x u\|_{H^s} + \|\nabla_x\Lambda_\epsilon^{2s+1}(\epsilon u,\theta)\|_{H^{s+1}}\,.\end{aligned}\tag{4.34}$$

The a priori estimate for the slow motion $(\epsilon p,\epsilon u,\theta)$ is stated in the following theorem.

THEOREM 4.3. *Let (p,u,θ) be a solution to the system (1.7). For $t > 0$, and $s > 5$, define the norm $\|\!|\cdot|\!\|^\flat_{\epsilon,s,t}$ as*

$$\begin{aligned}\|\!|(p,u,\theta)|\!\|^\flat_{\epsilon,s,t} :&= \sup_{[0,t]}\|\Lambda_\epsilon^{2s+1}(\epsilon p,\epsilon u,\theta)\|_{H^{s+1}}\\
&\quad + \alpha_0\left(\int_0^t \|\nabla_x\Lambda_\epsilon^{2s+1}(\epsilon u,\theta)\|^2_{H^{s+1}}(\tau)\,\mathrm{d}\tau\right)^{1/2},\end{aligned}\tag{4.35}$$

where $\alpha_0 > 0$ *is given by Theorem 4.2. Then for all* $T \in [0,1]$*, we have*

$$\|\!|(p,u,\theta)|\!\|^{\flat}_{\epsilon,s,T} \le C(\Omega_0)\, e^{(\sqrt{T}+\epsilon)C(\Omega)}\,, \tag{4.36}$$

where

$$\Omega = \|\!|(p,u,\theta)|\!\|_{\epsilon,s,T}\,, \qquad \Omega_0 = \|\!|(p,u,\theta)|\!\|_{\epsilon,s,0}\,. \tag{4.37}$$

The proof combines the linear estimate (4.5) with the estimates for commutator terms, which are based on the following lemmas:

LEMMA 4.4 (Calculus Inequality-Commutator). *Let* $\psi_m \in OPS^m$ *be a Fourier multiplier with* $m > 0$*. Then for all* $\sigma \ge 0$ *and* $f, g \in H^{m+\sigma}$ *there exists a generic constant* c_0 *such that*

$$\|[f,\ \psi_m]\, g\|_{H^\sigma(\mathbb{R}^3)} \le c_0 \left(\|f\|_{W^{1,\infty}}\, \|g\|_{H^{\sigma+m-1}} + \|f\|_{H^{\sigma+m}}\, \|g\|_{L^\infty}\right)\,. \tag{4.38}$$

As a consequence, for any $0 \le \epsilon \le 1$*,* $\alpha \ge 0$*, and* $f, g \in H^{\sigma+m+|\alpha|+1}$ *we have*

$$\|\Lambda_\epsilon^\alpha [f,\ \psi_m]\, g\|_{H^\sigma(\mathbb{R}^3)} \le c_0 \left(\|f\|_{W^{1,\infty}}\, \|\Lambda_\epsilon^\alpha g\|_{H^{\sigma+m-1}} + \|\Lambda_\epsilon^\alpha f\|_{H^{\sigma+m}}\, \|g\|_{L^\infty}\right)\,, \tag{4.39}$$

where the operator Λ_ϵ *is defined by (2.3).*

LEMMA 4.5 (Calculus Inequalities-Composition). *Let* $G \in C^\infty$ *such that* $G(0) = 0$*. Let* $v \in H^s(\mathbb{R}^3)$ *with* $s > 5$*. Then* $G(v) \in H^s(\mathbb{R}^3)$ *and there exists a constant* c_0 *which depends on* s *such that*

$$\|\nabla_x^k G(v)\|_{L^2} \le c_0 \|\nabla_v G\|_{C^{s-1}} \|v\|_{L^\infty}^{s-1} \|v\|_{H^s} \le c_0 \|\nabla_v G\|_{C^{s-1}} \|v\|_{H^s}^s\,,$$

for all $1 \le k \le s$*.*

We will frequently estimate terms of the form $|\langle \phi_1,\ T_\epsilon [f,\ \partial_x^\nu \Lambda_\epsilon^m]\, \phi_2\rangle|$ in the following calculations where $T_\epsilon = I - \epsilon^2 \nabla_x \cdot (a(x)\nabla_x)$. Here $a(x) \in W^{1,\infty}$, the index m is positive, and $\nu = (\nu_1, \nu_2, \nu_3)$ is a multi-index such that $|\nu| = k$. We collect its bounds in the following lemma.

LEMMA 4.6. *Let* $m > 0$ *and* $\nu = (\nu_1, \nu_2, \nu_3)$ *be a multi-index such that* $|\nu| = k$*. Then for all* $\sigma \ge 0$ *and* $f, g \in H^{m+k+\sigma}$*, there exists a generic constant* c_0 *such that*

$$\begin{aligned} \|[f,\ \partial_x^\nu \Lambda_\epsilon^m]\, g\|_{H^\sigma(\mathbb{R}^3)} &\le c_0 \epsilon^m \left(\|f\|_{W^{1,\infty}}\, \|g\|_{H^{\sigma+k+m-1}} + \|f\|_{H^{\sigma+k+m}}\, \|g\|_{L^\infty}\right) \\ &\quad + c_0 \left(\|f\|_{W^{1,\infty}}\, \|g\|_{H^{\sigma+k-1}} + \|f\|_{H^{\sigma+k}}\, \|g\|_{L^\infty}\right)\,, \end{aligned} \tag{4.40}$$

and for all $\phi_1, \phi_2 \in H^{k+m+2}$*,*

$$\begin{aligned} &|\langle \phi_1,\ T_\epsilon [f,\ \partial_x^\nu \Lambda_\epsilon^m]\, \phi_2\rangle| \\ &\le c_0 \epsilon^m \|\phi_1\|_{H^1_\epsilon} \|a\|_{L^\infty} \left(\|f\|_{W^{1,\infty}}\, \|\phi_2\|_{H_\epsilon^{k+m}} + \|f\|_{H_\epsilon^{k+m+1}}\, \|\phi_2\|_{L^\infty}\right) \\ &\quad + c_0 \|\phi_1\|_{H^1_\epsilon} \|a\|_{L^\infty} \left(\|f\|_{W^{1,\infty}}\, \|\phi_2\|_{H_\epsilon^{k}} + \|f\|_{H_\epsilon^{k+1}}\, \|\phi_2\|_{L^\infty}\right)\,, \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} &|\langle \phi_1,\ T_\epsilon [f,\ \partial_x^\nu \Lambda_\epsilon^m]\, \phi_2\rangle| \\ &\le c_0 \epsilon^m \|a\|_{W^{1,\infty}} \|\nabla_x \phi_1\|_{H^1_\epsilon} \left(\|f\|_{W^{1,\infty}}\, \|\phi_2\|_{H_\epsilon^{k+m-1}} + \|f\|_{H_\epsilon^{k+m}}\, \|\phi_2\|_{L^\infty}\right) \\ &\quad + c_0 \|a\|_{W^{1,\infty}} \|\nabla_x \phi_1\|_{H^1_\epsilon} \left(\|f\|_{W^{1,\infty}}\, \|\phi_2\|_{H_\epsilon^{k-1}} + \|f\|_{H_\epsilon^{k}}\, \|\phi_2\|_{L^\infty}\right)\,. \end{aligned} \tag{4.42}$$

Proof. The commutator term on the left-hand side of (4.40) can be rewritten as

$$[f,\ \partial_x^\nu \Lambda_\epsilon^m]\, g = [f,\ \partial_x^\nu (1+\epsilon^m(-\Delta_x)^{m/2})(1+\epsilon^m(-\Delta_x)^{m/2})^{-1}\Lambda_\epsilon^m]\, g\,.$$

Define the Fourier multiplier ψ_ϵ as

$$\psi_\epsilon = (1+\epsilon^m(-\Delta_x)^{m/2})^{-1}\Lambda_\epsilon^m\,.$$

Then $\psi_\epsilon \in S^0$ and its seminorms are uniformly bounded in ϵ. Therefore by (4.38)

$$\begin{aligned}\|[f,\ \partial_x^\nu \Lambda_\epsilon^m]\, g\|_{H^\sigma} &\le \|[f,\ \partial_x^\nu \psi_\epsilon]\, g\|_{H^\sigma} + \epsilon^m \|[f,\ \partial_x^\nu(-\Delta_x)^{m/2}\psi_\epsilon]\, g\|_{H^\sigma}\\ &\le c_0\epsilon^m \left(\|f\|_{W^{1,\infty}}\,\|g\|_{H^{\sigma+k+m-1}} + \|f\|_{H^{\sigma+k+m}}\,\|g\|_{L^\infty}\right)\\ &\quad + c_0\left(\|f\|_{W^{1,\infty}}\,\|g\|_{H^{\sigma+k-1}} + \|f\|_{H^{\sigma+k}}\,\|g\|_{L^\infty}\right)\,.\end{aligned}$$

Estimate (4.41) follows directly from (4.40) by Hölder inequality. Indeed,

$$\begin{aligned}&|\langle \phi_1,\ T_\epsilon[f,\ \partial_x^\nu\Lambda_\epsilon^m]\,\phi_2\rangle|\\ &\le |\langle \phi_1,\ [f,\ \partial_x^\nu\Lambda_\epsilon^m]\,\phi_2\rangle| + \epsilon^2|\langle a\nabla_x\phi_1,\ \nabla_x[f,\ \partial_x^\nu\Lambda_\epsilon^m]\,\phi_2\rangle|\\ &\le c_0\epsilon^m\|\phi_1\|_{H^1_\epsilon}\|a\|_{L^\infty}\left(\|f\|_{W^{1,\infty}}\,\|\phi_2\|_{H_\epsilon^{k+m}} + \|f\|_{H_\epsilon^{k+m+1}}\,\|\phi_2\|_{L^\infty}\right)\\ &\quad + c_0\|\phi_1\|_{H^1_\epsilon}\|a\|_{L^\infty}\left(\|f\|_{W^{1,\infty}}\,\|\phi_2\|_{H_\epsilon^{k}} + \|f\|_{H_\epsilon^{k+1}}\,\|\phi_2\|_{L^\infty}\right)\,.\end{aligned}$$

Estimate (4.42) follows by integration by part to transfer one derivative to ϕ_1.

$$\begin{aligned}&|\langle \phi_1,\, T_\epsilon[f,\, \partial_x^\nu\Lambda_\epsilon^m]\,\phi_2\rangle|\\ &\le |\langle \phi_1,\ [f,\ \partial_x^\nu\Lambda_\epsilon^m]\,\phi_2\rangle| + \epsilon^2|\langle a\nabla_x\phi_1,\ \nabla_x[f,\ \partial_x^\nu\Lambda_\epsilon^m]\,\phi_2\rangle|\,.\end{aligned}\tag{4.43}$$

We have

$$\begin{aligned}|\langle \phi_1,\ [f,\ \partial_x^\nu\Lambda_\epsilon^m]\,\phi_2\rangle| &\le |\langle \partial_{\nu_1}\phi_1,\ [f,\ \partial_x^{\nu_2}\Lambda_\epsilon^m]\,\phi_2\rangle| + |\langle \phi_1,\ (\partial_x^{\nu_1}f)\,\partial_x^{\nu_2}\Lambda_\epsilon^m\,\phi_2\rangle|\\ &\le |\langle \partial_{\nu_1}\phi_1,\ [f,\ \partial_x^{\nu_2}\Lambda_\epsilon^m]\,\phi_2\rangle| + |\langle \partial_{\nu_3}\phi_1,\ [\partial_x^{\nu_1}f,\ \partial_x^{\nu_4}\Lambda_\epsilon^m]\,\phi_2\rangle|\\ &\quad + |\langle \phi_1,\ (\partial_x^{\nu_5}f)\,\partial_x^{\nu_6}\Lambda_\epsilon^m\,\phi_2\rangle|\,,\end{aligned}$$

where $|\nu_1| = |\nu_3| = 1$, $|\nu_5| = 2$, $|\nu_4| = |\nu_6| = k-2$, and $|\nu_2| = k-1$. Therefore,

$$\begin{aligned}&|\langle \phi_1,\ [f,\ \partial_x^\nu\Lambda_\epsilon^m]\,\phi_2\rangle|\\ &\le c_0\epsilon^m\|\nabla_x\phi_1\|_{L^2}\left(\|f\|_{W^{2,\infty}}\,\|\phi_2\|_{H^{k+m-2}} + \|f\|_{H^{k+m-1}}\,\|\phi_2\|_{L^\infty}\right)\\ &\quad + c_0\|\nabla_x\phi_1\|_{L^2}\left(\|f\|_{W^{1,\infty}}\,\|\phi_2\|_{H^{k-2}} + \|f\|_{H^{k-1}}\,\|\phi_2\|_{L^\infty}\right)\\ &\quad + c_0\|\phi_1\|_{L^2}\|f\|_{W^{2,\infty}}\|\Lambda_\epsilon^m\phi_2\|_{H^{k-2}}\,.\end{aligned}\tag{4.44}$$

The last term in (4.43) is bounded as

$$\begin{aligned}&\epsilon^2|\langle a\nabla_x\phi_1,\ \nabla_x[f,\ \partial_x^\nu\Lambda_\epsilon^m]\,\phi_2\rangle|\\ &\le \epsilon^2|\langle a\Delta_x\phi_1,\ [f,\ \partial_x^\nu\Lambda_\epsilon^m]\,\phi_2\rangle| + \epsilon^2|\langle \nabla_x a\cdot\nabla_x\phi_1,\ [f,\ \partial_x^\nu\Lambda_\epsilon^m]\,\phi_2\rangle|\\ &\le c_0\epsilon^{m+1}\|a\|_{W^{1,\infty}}\|\epsilon\nabla_x\phi_1\|_{H^1}\left(\|f\|_{W^{1,\infty}}\,\|\phi_2\|_{H^{k+m-1}} + \|f\|_{H^{k+m}}\,\|\phi_2\|_{L^\infty}\right)\\ &\quad + c_0\epsilon\|a\|_{W^{1,\infty}}\|\epsilon\nabla_x\phi_1\|_{H^1}\left(\|f\|_{W^{1,\infty}}\,\|\phi_2\|_{H^{k-1}} + \|f\|_{H^{k}}\,\|\phi_2\|_{L^\infty}\right)\,.\end{aligned}\tag{4.45}$$

Combining (4.44) and (4.45) we have

$$
\begin{aligned}
&|\langle \phi_1,\, T_\epsilon[f,\, \partial_x^\nu \Lambda_\epsilon^m]\, \phi_2\rangle| \\
&\le c_0 \epsilon^m \|a\|_{W^{1,\infty}} \|\nabla_x \phi_1\|_{H^1_\epsilon} \left(\|f\|_{W^{1,\infty}} \|\phi_2\|_{H^{k+m-1}_\epsilon} + \|f\|_{H^{k+m}_\epsilon} \|\phi_2\|_{L^\infty} \right) \\
&\quad + c_0 \|a\|_{W^{1,\infty}} \|\nabla_x \phi_1\|_{H^1_\epsilon} \left(\|f\|_{W^{1,\infty}} \|\phi_2\|_{H^{k-1}_\epsilon} + \|f\|_{H^k_\epsilon} \|\phi_2\|_{L^\infty} \right) .
\end{aligned}
$$

□

The rest of this section is devoted to the proof of Theorem 4.3.

Proof. [Proof of Theorem 4.3] Let $\beta = (\beta_1, \beta_2, \beta_3)$ be a multi-index such that $|\beta| = s+1$. Let

$$
(p_\beta, u_\beta, \theta_\beta) \triangleq \left(\partial_x^\beta \Lambda_\epsilon^{2s} (\epsilon p - \theta),\ \partial_x^\beta \Lambda_\epsilon^{2s} (\epsilon u),\ \partial_x^\beta \Lambda_\epsilon^{2s} \theta \right) .
$$

Then $(p_\beta, u_\beta, \theta_\beta)$ satisfy the nonlinear system

(4.46)

$$
\begin{aligned}
(\partial_t + u\cdot\nabla_x) p_\beta + \frac{1}{\epsilon} \nabla_x \cdot u_\beta &= h_1 , \\
g(\theta)(\partial_t + u\cdot\nabla_x)\, u_\beta + \frac{1}{\epsilon}\nabla_x p_\beta + \frac{1}{\epsilon}\nabla_x \theta_\beta &= h(\epsilon p)\nabla_x\cdot\Sigma_\beta + \epsilon h(\epsilon p)\nabla_x\cdot\tilde\Sigma_\beta + h_2 , \\
\tfrac{3}{2}(\partial_t + u\cdot\nabla_x)\,\theta_\beta + \frac{1}{\epsilon}\nabla_x\cdot u_\beta &= \epsilon h(\epsilon p)\nabla_x\cdot\tilde q_\beta + h(\epsilon p)\nabla_x\cdot q_\beta + h_3 \\
&\quad + \epsilon h(\epsilon p)\Sigma_\beta : \nabla_x u + \epsilon^2 h(\epsilon p)\,\tilde\Sigma_\beta : \nabla_x u,
\end{aligned}
$$

where

$$
\begin{aligned}
q_\beta &= \kappa(\theta)\nabla_x\theta_\beta , \qquad \Sigma_\beta = \mu(\theta)\left(\nabla_x u_\beta + (\nabla_x u_\beta)^T - \tfrac{2}{3}(\nabla_x\cdot u_\beta) I \right) , \\
\tilde\Sigma_\beta &= \tau_1(\epsilon p,\theta)\left(\nabla_x^2\theta_\beta - \tfrac{1}{3}(\Delta_x\theta_\beta) I \right) + \tau_2(\epsilon p,\theta)\left(\nabla_x\theta\otimes\nabla_x\theta_\beta - \tfrac{1}{3}(\nabla_x\theta\cdot\nabla_x\theta_\beta) I \right) \\
&\quad + \tau_3(\epsilon p,\theta)\left(\nabla_x(\epsilon u)(\nabla_x u_\beta)^T - (\nabla_x \epsilon u)^T \nabla_x u_\beta \right) , \\
\tilde q_\beta &= \tau_4(\epsilon p,\theta)\left(\Delta_x u_\beta + \tfrac{1}{3}\nabla_x\nabla_x\cdot u_\beta \right) + \tau_6(\epsilon p,\theta)\left(\nabla_x(\epsilon u) - (\nabla_x(\epsilon u))^T \right)\cdot\nabla_x\theta_\beta , \\
&\quad + \tau_5(\epsilon p,\theta)\nabla_x\theta\cdot\left(\nabla_x u_\beta + (\nabla_x u_\beta)^T - \tfrac{2}{3}(\nabla_x\cdot u_\beta) I \right)
\end{aligned}
$$

and (h_1, h_2, h_3) are the commutator terms given by

$$
\begin{aligned}
h_1 &= -[\partial_x^\beta\Lambda_\epsilon^{2s},\, u]\cdot\nabla_x(\epsilon p - \theta) , \\
h_2 &= -\left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta) \right] \partial_t(\epsilon u) + \left(\partial_x^\beta\Lambda_\epsilon^{2s}\,(h(\epsilon p)\nabla_x\cdot\Sigma) - h(\epsilon p)\nabla_x\cdot\Sigma_\beta \right) \\
&\quad - \left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)u \right]\cdot\nabla_x(\epsilon u) + \left(\partial_x^\beta\Lambda_\epsilon^{2s}\left(\epsilon h(\epsilon p)\nabla_x\cdot\tilde\Sigma \right) - \epsilon h(\epsilon p)\nabla_x\cdot\tilde\Sigma_\beta \right) , \\
h_3 &= -\left[\partial_x^\beta\Lambda_\epsilon^{2s},\, u \right]\cdot\nabla_x\theta + \left(\partial_x^\beta\Lambda_\epsilon^{2s}\,(\epsilon h(\epsilon p)\nabla_x\cdot\tilde q) - \epsilon h(\epsilon p)\nabla_x\cdot\tilde q_\beta \right) \\
&\quad + \left(\partial_x^\beta\Lambda_\epsilon^{2s}\,(h(\epsilon p)\nabla_x\cdot q) - h(\epsilon p)\nabla_x\cdot q_\beta \right) \\
&\quad + \left(\partial_x^\beta\Lambda_\epsilon^{2s}\,(\epsilon h(\epsilon p)\Sigma:\nabla_x u) - \epsilon h(\epsilon p)\Sigma_\beta:\nabla_x u \right) \\
&\quad + \left(\partial_x^\beta\Lambda_\epsilon^{2s}\left(\epsilon^2 h(\epsilon p)\tilde\Sigma:\nabla_x u \right) - \epsilon^2 h(\epsilon p)\tilde\Sigma_\beta:\nabla_x u \right) ,
\end{aligned}
$$

where

$$
\begin{aligned}
q &= \kappa(\theta)\nabla_x\theta , \quad \Sigma = \mu(\theta)\left(\nabla_x(\epsilon u) + (\nabla_x\epsilon u)^T - \tfrac{2}{3}(\nabla_x\cdot(\epsilon u)) I \right) , \\
\tilde\Sigma &= \tau_1(\theta)\left(\nabla_x^2\theta - \tfrac{1}{3}(\Delta_x\theta) I \right) + \tau_2(\theta)\left(\nabla_x\theta\otimes\nabla_x\theta - \tfrac{1}{3}|\nabla_x\theta|^2 I \right) \\
&\quad + \tau_3(\epsilon p,\theta)\left(\nabla_x(\epsilon u)(\nabla_x(\epsilon u))^T - (\nabla_x(\epsilon u))^T\nabla_x(\epsilon u) \right) , \\
\tilde q &= \tau_4(\theta)\left(\Delta_x(\epsilon u) + \tfrac{1}{3}\nabla_x\nabla_x\cdot(\epsilon u) \right) + \tau_6(\epsilon p,\theta)\left(\nabla_x(\epsilon u) - (\nabla_x(\epsilon u))^T \right)\cdot\nabla_x\theta \\
&\quad + \tau_5(\theta)\nabla_x\theta\cdot\left(\nabla_x(\epsilon u) + (\nabla_x(\epsilon u))^T - \tfrac{2}{3}(\nabla_x\cdot(\epsilon u)) I \right) .
\end{aligned}
$$

By the linear estimate (4.5), we need to estimate the following terms: $|\langle p_\beta,\ T_2 h_1\rangle|$, $|\langle u_\beta,\ T_2 h_2\rangle|$, $|\langle \theta_\beta,\ T_1 h_3\rangle|$, and $\left|\left\langle \epsilon^2 h_2,\ \Psi_1 u_\beta + \Psi_0 u_\beta\right\rangle\right|$. First,

$$|\langle p_\beta,\ T_2 h_1\rangle| \le C(R)\|p_\beta\|_{H^1_\epsilon}\|h_1\|_{H^1_\epsilon}\,,$$

and by (4.40),

$$\begin{aligned}\|h_1\|_{H^1_\epsilon} &= \|[u,\, \partial_x^\beta \Lambda_\epsilon^{2s}]\cdot\nabla_x(\epsilon p-\theta)\|_{H^1_\epsilon}\\ &\le c_0\epsilon^{2s}\left(\|u\|_{W^{1,\infty}}\|\nabla_x(\epsilon p-\theta)\|_{H^{3s+1}_\epsilon} + \|u\|_{H^{3s+2}_\epsilon}\|\nabla_x(\epsilon p-\theta)\|_{L^\infty}\right)\\ &\quad + c_0\left(\|u\|_{W^{1,\infty}}\|\nabla_x(\epsilon p-\theta)\|_{H^{s+1}_\epsilon} + \|u\|_{H^{s+2}_\epsilon}\|\nabla_x(\epsilon p-\theta)\|_{L^\infty}\right)\,.\end{aligned}$$

Since

$$\begin{aligned}\epsilon^{2s}\|\nabla_x(\epsilon p,\theta)\|_{H^{3s+1}_\epsilon} \le \|\Lambda_\epsilon^{2s+1}\nabla_x(\epsilon p,\theta)\|_{H^s} \le \|\Lambda_\epsilon^{2s+1}(\epsilon p,\theta)\|_{H^{s+1}} \le R\,,\\ \|\nabla_x(\epsilon p,\theta)\|_{H^{s+1}_\epsilon} \le \|\Lambda_\epsilon\nabla_x(\epsilon p,\theta)\|_{H^s} \le \|\Lambda_\epsilon(\epsilon p,\theta)\|_{H^{s+1}} \le R\,,\\ \epsilon^{2s}\|u\|_{H^{3s+2}_\epsilon} \le \|\Lambda_\epsilon^{2s+1}u\|_{H^{s+1}} \le \|u\|_{H^{s+1}} + \epsilon\|\Lambda_\epsilon^{2s}\nabla_x u\|_{H^{s+1}} \le R+R'\,,\\ \|u\|_{H^{s+2}_\epsilon} \le \|u\|_{H^{s+1}} + \|\nabla_x(\epsilon u)\|_{H^{s+1}} \le R+R'\,,\end{aligned}$$

We have

$$\|h_1\|_{H^1_\epsilon} \le C(R)(1+R')\,,$$

and

$$|\langle p_\beta,\ T_2 h_1\rangle| \le C(R)(1+R')\,.$$

For the first term of h_2, we have

$$\begin{aligned}-\left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)\right]\partial_t(\epsilon u) &= \left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)\right](u\cdot\nabla_x(\epsilon u)) + \left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)\right]\left(g^{-1}(\theta)\nabla_x p\right)\\ &\quad - \left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)\right]\left(g^{-1}(\theta)h(\epsilon p)\nabla_x\cdot\Sigma\right)\\ &\quad - \left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)\right]\left(\epsilon g^{-1}(\theta)h(\epsilon p)\nabla_x\cdot\tilde\Sigma\right)\,.\end{aligned} \tag{4.47}$$

We have the following bounds related to these terms in (4.47). First by (4.42)

$$\begin{aligned}&\left|\left\langle u_\beta,\ T_2\left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)\right](u\cdot\nabla_x(\epsilon u))\right\rangle\right|\\ &\le C(R)\,\epsilon^{2s}\,\|\nabla_x u_\beta\|_{H^1_\epsilon}\left(\|u\cdot\nabla_x(\epsilon u)\|_{H^{3s}_\epsilon} + \|g(\theta)-g(0)\|_{H^{3s+1}_\epsilon}\right)\\ &\quad + C(R)\,\|\nabla_x u_\beta\|_{H^1_\epsilon}\left(\|u\cdot\nabla_x(\epsilon u)\|_{H^s_\epsilon} + \|g(\theta)-g(0)\|_{H^{s+1}_\epsilon}\right)\\ &\le C(R)\,\|\nabla_x u_\beta\|_{H^1_\epsilon}\left(\left\|\Lambda_\epsilon^{2s+1}(\epsilon u)\right\|_{H^s} + \left\|\Lambda_\epsilon^{2s+1}\theta\right\|_{H^s}\right)\\ &\le C(R)R'\,.\end{aligned}$$

The estimate related to the second term on the right-hand side of (4.47) is

$$\begin{aligned}&\left|\left\langle u_\beta,\ T_2\left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)\right]\left(g^{-1}(\theta)\nabla_x p\right)\right\rangle\right|\\ &\le C(R)\,\epsilon^{2s}\,\|\nabla_x u_\beta\|_{H^1_\epsilon}\left(\|g^{-1}(\theta)\nabla_x p\|_{H^{3s}_\epsilon} + \|g(\theta)-g(0)\|_{H^{3s+1}_\epsilon}\right)\\ &\quad + C(R)\,\|\nabla_x u_\beta\|_{H^1_\epsilon}\left(\|g^{-1}(\theta)\nabla_x p\|_{H^s_\epsilon} + \|g(\theta)-g(0)\|_{H^{s+1}_\epsilon}\right)\\ &\le C(R)\,\|\nabla_x u_\beta\|_{H^1_\epsilon}\left(\left\|\Lambda_\epsilon^{2s}(\epsilon p)\right\|_{H^{s+1}} + \|\Lambda_\epsilon^{2s+1}\theta\|_{H^s} + \|p\|_{H^s}\right)\\ &\le C(R)R'\,.\end{aligned}$$

We show the estimates related to the leading order terms in the rest of the terms in (4.47). The lower order terms are bounded similarly. The leading order term for the third term on the right-hand side of (4.47) satisfies

(4.48)
$$\begin{aligned}
&\left|\left\langle u_\beta,\ T_2\left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)\right] g^{-1}(\theta)h(\epsilon p)\mu(\theta)\,\Delta_x(\epsilon u)\right\rangle\right| \\
&\le C(R)\,\epsilon^{2s}\,\|\nabla_x u_\beta\|_{H^1_\epsilon}\,\left(\|g^{-1}(\theta)h(\epsilon p)\mu(\theta)\,\Delta_x(\epsilon u)\|_{H^{3s}_\epsilon}+\|g(\theta)-g(0)\|_{H^{3s+1}_\epsilon}\right) \\
&\quad + C(R)\,\|\nabla_x u_\beta\|_{H^1_\epsilon}\,\left(\|g^{-1}(\theta)h(\epsilon p)\mu(\theta)\,\Delta_x(\epsilon u)\|_{H^{s}_\epsilon}+\|g(\theta)-g(0)\|_{H^{s+1}_\epsilon}\right) \\
&\le C(R)\,\left(\left\|\Lambda_\epsilon^{2s+1}(\epsilon u)\right\|_{H^{s+1}}+\|\Lambda_\epsilon^{2s+1}\theta\|_{H^s}\right) \\
&\le C(R)R'\,.
\end{aligned}$$

The estimate for the leading order term in the last term of (4.47) is

(4.49)
$$\begin{aligned}
&\left|\left\langle u_\beta,\ T_2\left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)\right]\epsilon g^{-1}(\theta)h(\epsilon p)\tau_1(\theta)\,\Delta_x\nabla_x\theta\right\rangle\right| \\
&\le C(R)\,\epsilon^{2s}\,\|\nabla_x u_\beta\|_{H^1_\epsilon}\,\left(\|\epsilon g^{-1}(\theta)h(\epsilon p)\tau_1(\theta)\,\Delta_x\nabla_x\theta\|_{H^{3s}_\epsilon}+\epsilon\|g(\theta)-g(0)\|_{H^{3s+1}_\epsilon}\right) \\
&\quad + C(R)\,\|\nabla_x u_\beta\|_{H^1_\epsilon}\,\left(\|\epsilon g^{-1}(\theta)h(\epsilon p)\tau_1(\theta)\,\Delta_x\nabla_x\theta\|_{H^{s}_\epsilon}+\epsilon\|g(\theta)-g(0)\|_{H^{s+1}_\epsilon}\right) \\
&\le C(R)R'\left(\epsilon\|\nabla_x\Lambda_\epsilon^{2s+1}\theta\|_{H^{s+1}}+\epsilon\|\Lambda_\epsilon^{2s+1}\theta\|_{H^s}\right) \\
&\le C(R)R'(1+\epsilon\,R')\,.
\end{aligned}$$

Thus, the leading order terms are all bounded by $C(R)R'(1+\epsilon R')$. The lower order terms are estimated in the same way. Therefore the first term in h_2 is bounded by $C(R)R'(1+\epsilon R')$

$$\left|\left\langle u_\beta,\ T_2\left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)\right]\partial_t(\epsilon u)\right\rangle\right|\le C(R)R'(1+\epsilon R')\,. \tag{4.50}$$

The third term of h_2 has a similar structure as h_1. Therefore its estimate is similar to that for $|\langle p_\beta,\ T_2 h_1\rangle|$, which gives

$$\left|\left\langle u_\beta,\ T_2\left[\partial_x^\beta\Lambda_\epsilon^{2s},\, g(\theta)u\right]\cdot\nabla_x(\epsilon u)\right\rangle\right|\le C(R)(1+R')\,. \tag{4.51}$$

We are left with the estimates for the second and fourth terms on h_2. Again we show only the bounds of their leading orders. The leading order term of the second term $\left(\partial_x^\beta\Lambda_\epsilon^{2s}\left(h(\epsilon p)\nabla_x\cdot\Sigma\right)-h(\epsilon p)\nabla_x\cdot\Sigma_\beta\right)$ is

$$\left([h(\epsilon p),\ \partial_x^\beta\Lambda_\epsilon^{2s}]\mu(\theta)+h(\epsilon p)\,[\mu(\theta),\ \partial_x^\beta\Lambda_\epsilon^{2s}]\right)\left(\Delta_x(\epsilon u)+\tfrac{1}{3}\nabla_x\nabla_x\cdot(\epsilon u)\right).$$

Its estimate is similar to (4.48). Therefore,

$$\left|\left\langle u_\beta,\ T_2\left(\partial_x^\beta\Lambda_\epsilon^{2s}\left(h(\epsilon p)\nabla_x\cdot\Sigma\right)-h(\epsilon p)\nabla_x\cdot\Sigma_\beta\right)\right\rangle\right|\le C(R)R'\,. \tag{4.52}$$

The leading order term in $\left(\partial_x^\beta\Lambda_\epsilon^{2s}\left(\epsilon h(\epsilon p)\nabla_x\cdot\tilde\Sigma\right)-\epsilon h(\epsilon p)\nabla_x\cdot\tilde\Sigma_\beta\right)$ is

$$\left([\epsilon h(\epsilon p),\ \partial_x^\beta\Lambda_\epsilon^{2s}]\tau_1(\epsilon p,\theta)+\epsilon h(\epsilon p)\,[\tau_1(\epsilon p,\theta),\ \partial_x^\beta\Lambda_\epsilon^{2s}]\right)\left(\tfrac{2}{3}\nabla_x\Delta_x\theta\right).$$

Its estimate is similar to (4.49). Therefore,

$$\left|\left\langle u_\beta,\ T_2\left(\partial_x^\beta\Lambda_\epsilon^{2s}\left(\epsilon h(\epsilon p)\nabla_x\cdot\tilde\Sigma\right)-\epsilon h(\epsilon p)\nabla_x\cdot\tilde\Sigma_\beta\right)\right\rangle\right|\le C(R)R'(1+\epsilon R')\,. \tag{4.53}$$

Combining (4.50), (4.51), (4.52), and (4.53) we have

$$|\langle u_\beta,\ T_2 h_2\rangle| \le C(R)\, R'(1+\epsilon R')\,. \tag{4.54}$$

The θ-equation has similar structures to the ϵu-equation. Hence the estimate for h_3 is similar to the estimates for h_2

$$|\langle \theta_\beta,\ T_1 h_3\rangle| \le C(R)\, R'(1+\epsilon R')\,. \tag{4.55}$$

By the linear estimate for the slow motion (4.31), we have

$$\begin{aligned}&\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\Big\langle p_\beta,\, T_2 p_\beta\Big\rangle + \Big\langle u_\beta,\, \sqrt{g(\theta)}T_2(\sqrt{g(\theta)}u_\beta)\Big\rangle + \Big\langle \tfrac{3}{2}\theta_\beta,\, T_1\theta_\beta\Big\rangle\right)\\ &\quad + \alpha_0\|\nabla_x(u_\beta,\theta_\beta)\|^2_{H^1_\epsilon}\\ &\le C(R) + C(R)R' + \epsilon C(R)(R')^2\,.\end{aligned} \tag{4.56}$$

By the equivalence of the norms

$$\|(p_\beta,u_\beta,\theta_\beta)\|_{H^1_\epsilon} \sim \Big\langle p_\beta,\, T_2 p_\beta\Big\rangle + \Big\langle u_\beta,\, \sqrt{g(\theta)}T_2(\sqrt{g(\theta)}u_\beta)\Big\rangle + \Big\langle \tfrac{3}{2}\theta_\beta,\, T_1\theta_\beta\Big\rangle\,,$$

we have

$$\begin{aligned}|\!|\!|(p,u,\theta)|\!|\!|^{\flat}_{\epsilon,s,T} &\le C(\Omega_0) + C(\Omega)T + C(\Omega)\int_0^T R'(s)\,\mathrm{d}s + \epsilon C(\Omega)\int_0^T (R')^2(s)\mathrm{d}s\\ &\le C(\Omega_0) + C(\Omega)T + C(\Omega)\sqrt{T} + \epsilon C(\Omega)\\ &\le C(\Omega_0) + (\sqrt{T}+\epsilon)C(\Omega)\\ &\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,,\end{aligned}$$

for $T\in(0,1]$. □

Remark: From the proof of Theorem 4.3 it is clear that for any $m\in\mathbb{N}$, a similar estimate as in (4.36) holds for $\|\Lambda_\epsilon^{m+1}(\epsilon p,\epsilon u,\theta)\|_{H^{s+1}}$. In particular, the estimate for $\|\Lambda_\epsilon^{m+1}(\epsilon p,\epsilon u,\theta)\|_{H^{s+1}}$ is closed in terms of $\|(u,p)\|_{H^s}$ and $\|(\epsilon p,\epsilon u,\theta)\|_{H^{s+1}}$ for all $m\in\mathbb{N}$.

4.4. A Priori Estimate of $\mathrm{curl}(\mathrm{e}^{-\theta}\mathrm{u})$**.** In addition to $(\epsilon p,\epsilon u,\theta)$, we have one part of u which varies on the time scale of order one as in [2]. By taking curl on both sides of the u-equation, we have the equation for $\mathrm{curl}(e^{-\theta}u)$ as

$$\begin{aligned}(\partial_t + u\cdot\nabla_x)\left(\mathrm{curl}\left(e^{-\theta}u\right)\right) &= \mathrm{curl}\left(e^{-\epsilon p}\left(\nabla_x\cdot\Sigma + \nabla_x\cdot\tilde{\Sigma}\right)\right) + \mathrm{curl}(e^{-\theta}u\,\partial_t\theta)\\ &\quad + [\mathrm{curl},\, u]\cdot\nabla_x(e^{-\theta}u) + \mathrm{curl}(e^{-\theta}u(u\cdot\nabla_x\theta))\,.\end{aligned} \tag{4.57}$$

Because there is no singular terms in the above equation, the Sobolev norms of $\mathrm{curl}(e^{-\theta}u)$ is done by directly differentiating the equations.

Lemma 4.7. *Let $s>5$. Let (p,u,θ) be a solution to the fluctuation equations (1.7). Then there exists an increasing function such that*

$$\sup_{[0,T]}\|\mathrm{curl}(e^{-\theta}u))\|^2_{H^{s-1}} + \alpha_0\int_0^T \|\nabla_x\mathrm{curl}(e^{-\theta}u)\|^2_{H^{s-1}}(\tau)\,\mathrm{d}\tau \le C(\Omega_0)e^{\sqrt{T}\,C(\Omega)}. \tag{4.58}$$

Proof. We rewrite (4.57) in terms of $\operatorname{curl}(e^{-\theta}u)$:

$$(\partial_t + u\cdot\nabla_x)\left(\operatorname{curl}\left(e^{-\theta}u\right)\right) = \nabla_x\cdot(e^{-\epsilon p}\mu(\theta)\nabla_x(\operatorname{curl}(e^{-\theta}u)) + \mathcal{R}_1\,, \tag{4.59}$$

where

$$\begin{aligned}
\mathcal{R}_1 &= \nabla_x(e^{-\epsilon p}\mu(\theta))\cdot\nabla_x(\operatorname{curl}(e^{-\theta}u)) + \nabla_x(e^{-\epsilon p}\mu(\theta))\times(\Delta_x u + \tfrac{1}{3}\nabla_x(\nabla_x\cdot u))\\
&\quad + \operatorname{curl}(e^{-\epsilon p}\mu'(\theta)\nabla_x\theta\cdot(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{3}(\nabla_x\cdot u)I))\\
&\quad + \nabla_x(e^{-\epsilon p}\tau_1)\times(\tfrac{2}{3}\nabla_x\Delta_x\theta) + \operatorname{curl}\nabla_x\cdot\left(\tau_2\left(\nabla_x\theta\otimes\nabla_x\theta - \tfrac{1}{3}(\nabla_x\theta\cdot\nabla_x\theta)I\right)\right)\\
&\quad + \operatorname{curl}\nabla_x\cdot\left(\tau_3\left(\nabla_x(\epsilon u)(\nabla_x u)^T - (\nabla_x(\epsilon u))^T\nabla_x u\right)\right)\\
&\quad + \operatorname{curl}(e^{-\theta}u\,\partial_t\theta) + [\operatorname{curl},\,u]\cdot\nabla_x(e^{-\theta}u) + \operatorname{curl}(e^{-\theta}u(u\cdot\nabla_x\theta))\,.
\end{aligned}\tag{4.60}$$

Notice that we do not have a fourth-order term from $\nabla_x\cdot\tilde{\Sigma}$ because the leading order term of $\operatorname{curl}(\nabla_x\cdot\tilde{\Sigma})$ vanishes. We do have a fourth-order term from $\operatorname{curl}(e^{-\theta}u\,\partial_t\theta)$ but it is pre-multiplied by ϵ. Thus the leading order term of $\operatorname{curl}(e^{-\theta}u\,\partial_t\theta)$ becomes third-order of $\Lambda_\epsilon(\epsilon u)$. Now apply ∂_x^σ on both sides with $1\le|\sigma|\le s-1$:

$$\begin{aligned}
&(\partial_t + u\cdot\nabla_x)\partial_x^\sigma\left(\operatorname{curl}\left(e^{-\theta}u\right)\right)\\
&= \nabla_x\cdot(e^{-\epsilon p}\mu(\theta)\nabla_x\partial_x^\alpha(\operatorname{curl}(e^{-\theta}u)) + \partial_x^\sigma\mathcal{R}_1 + \mathcal{R}_2\,,
\end{aligned}\tag{4.61}$$

where

$$\mathcal{R}_2 = [u,\ \partial_x^\sigma]\cdot\nabla_x(\operatorname{curl}(e^{-\theta}u)) + [\nabla_x\cdot(e^{-\epsilon p}\mu(\theta)),\ \partial_x^\sigma]\,\nabla_x(\operatorname{curl}(e^{-\theta}u))\,. \tag{4.62}$$

By standard energy estimates, multiplying both sides by $\partial_x^\sigma(\operatorname{curl}(e^{-\theta}u))$ and integrate over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\sup_{[0,T]}\|\operatorname{curl}(e^{-\theta}u))\|^2_{H^{s-1}} + \alpha_0\int_0^T\|\nabla_x\operatorname{curl}(e^{-\theta}u)\|^2_{H^{s-1}}(\tau)\,\mathrm{d}\tau\\
&\le c_0\|\operatorname{curl}(e^{-\theta}u)(0,\cdot)\|^2_{H^{s-1}} + c_0\left(\int_0^T(\|\partial_x^\sigma\mathcal{R}_1\|_{L^2} + \|\mathcal{R}_2\|_{L^2})(\tau)\,\mathrm{d}\tau\right)^2\\
&\le c_0\|\operatorname{curl}(e^{-\theta}u)(0,\cdot)\|^2_{H^{s-1}} + c_0\sqrt{T}\int_0^T(\|\partial_x^\sigma\mathcal{R}_1\|^2_{L^2} + \|\mathcal{R}_2\|^2_{L^2})(\tau)\,\mathrm{d}\tau\,.
\end{aligned}\tag{4.63}$$

Notice that the leading order terms in $\mathcal{R}_1$ are third-order terms in $(\Lambda_\epsilon(\epsilon u),\ \theta)$ and second-order terms in u. The leading order terms in $\mathcal{R}_2$ are terms of order $s+1$ in u and of order $s+1$ in $(\epsilon p,\theta)$. Therefore,

$$\int_0^T(\|\partial_x^\sigma\mathcal{R}_1\|^2_{L^2} + \|\mathcal{R}_2\|^2_{L^2})(\tau)\,\mathrm{d}\tau \le C(R)\,,$$

which combined with (4.63) gives (4.58). □

4.5. Some Additional Bounds. In this subsection we show some bounds of terms $\|\Lambda_\epsilon^{2(s-k)}(\epsilon\partial_t)^k(\epsilon p,\epsilon u,\theta)\|_{H^{s+1-k}}$ and $\|(\epsilon\partial_t)^k(u,p)\|_{L^2}$ for $1\le k\le s$. These bounds will be used in the next section. Recall the following basic calculus inequality: for $|k_1|+|k_2|+\cdot+|k_n| = s$ with $s > d+2 = 5$,

$$\|\nabla_x^{k_1}u_1\,\nabla_x^{k_2}u_2\,\cdots\nabla_x^{k_n}u_n\|_{L^2} \le \|u_1\|_{H^s}\|u_2\|_{H^s}\cdots\|u_n\|_{H^s}\,,\qquad \forall n\in\mathbb{N}\,. \tag{4.64}$$

Recall the definitions (4.34) of R, R':

$$R := \|(p,u)\|_{H^s} + \|\Lambda_\epsilon^{2s+1}(\epsilon p, \epsilon u, \theta)\|_{H^{s+1}}\,,$$
$$R' := \|\nabla_x(u,\theta)\|_{H^s} + \|\nabla_x \Lambda_\epsilon^{2s+1}(\epsilon u, \theta)\|_{H^{s+1}}\,.$$

First we show the bounds for $\|\Lambda_\epsilon^{2(s-k)}(\epsilon\partial_t)^k(\epsilon p, \epsilon u, \theta)\|_{H^{s+1-k}}$. These are immediate consequences of the bound for the slow motion in Theorem 4.3.

LEMMA 4.8. *Let (p,u,θ) be the solution to (1.7) and $\psi = (\epsilon p, \epsilon u, \theta)$. Then for all $1 \le k \le s$,*

$$\begin{aligned}
&\|\Lambda_\epsilon^{2(s-k)}(\epsilon\partial_t)^k\psi\|_{H^{s+1-k}} \le C(R)\,,\\
&\|\nabla_x\Lambda_\epsilon^{2(s-k)}(\epsilon\partial_t)^k\psi\|_{H^{s+1-k}} \le C(R)(1+R')\,,\\
&\sup_{[0,T]}\|\Lambda_\epsilon^{2(s-k)}(\epsilon\partial_t)^k\psi\|_{H^{s+1-k}} \le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,,\\
&\int_0^T \|\nabla_x\Lambda_\epsilon^{2(s-k)}(\epsilon\partial_t)^k\psi\|_{H^{s+1-k}}^2(\tau)\,\mathrm{d}\tau \le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,,
\end{aligned} \tag{4.65}$$

where Ω_0, Ω are defined in (4.37).

Proof. We simply use equation (1.7) to write $(\epsilon\partial_t)\psi$ in terms of combinations of the spatial derivatives of ψ and apply Theorem 4.3. Roughly speaking, each $\epsilon\partial_t$ corresponds to the spatial derivatives which are at most of order $(\epsilon\nabla_x)^\alpha\nabla_x$ with $|\alpha| = 2$. More specifically, we claim that $(\epsilon\partial_t)^k\psi$ is a summation of products of functions in the form of $f_j(\psi)$ or $\nabla_x^{k_i^{(j)}}(\epsilon\nabla_x)^{m_i^{(j)}}\psi$, that is,

$$(\epsilon\partial_t)^k\psi = \sum_j\left(f_j(\psi)\prod_i\nabla_x^{k_i^{(j)}}(\epsilon\nabla_x)^{m_i^{(j)}}\psi\right)\,, \tag{4.66}$$

where f_j is a smooth function in ψ for all j and each summand satisfies that

$$\sum_i |k_i^{(j)}| \le k\,, \qquad \sum_i |m_i^{(j)}| \le 2k\,. \tag{4.67}$$

This claim can be proved by induction. First, (4.66) and (4.67) hold for $k = 1$ by (1.7). Suppose (4.66) holds for k. Then for $1 \le k+1 \le s$, we have

$$\begin{aligned}
&(\epsilon\partial_t)\left(f_j(\psi)\prod_i\nabla_x^{k_i^{(j)}}(\epsilon\nabla_x)^{m_i^{(j)}}\psi\right)\\
&= \nabla_\psi f_j\cdot(\epsilon\partial_t)\psi + f_j(\psi)\sum_i\left(\left(\nabla_x^{k_i^{(j)}}(\epsilon\nabla_x)^{m_i^{(j)}}(\epsilon\partial_t)\psi\right)\cdot\prod_{l\ne i}\nabla_x^{k_l^{(j)}}(\epsilon\nabla_x)^{m_l^{(j)}}\psi\right)\,.
\end{aligned}$$

Using (1.7) to replace $(\epsilon\partial_t)\psi$ in the above equation shows that (4.66) and (4.67) hold for $k+1$. Thus (4.66) and (4.67) hold for all $1 \le k \le s$. This further implies that inequalities in (4.65) hold by Theorem 4.3. ∎

LEMMA 4.9. *Let (p,u,θ) be the solution to (1.7). Let $1 \le k \le s$. Then*

$$\begin{aligned}
&\|(\epsilon\partial_t)^k(p,u)\|_{H^{s-k}} \le C(R)\,,\\
&\|(\epsilon\partial_t)^k(p,u)\|_{H^{s+1-k}} + \|(\epsilon\partial_t)^{k-1}\partial_t\theta\|_{H^{s+1-k}} \le C(R)(1+R')\,.
\end{aligned} \tag{4.68}$$

Proof. The idea of the proof of (4.68) is similar as that of (4.65): we use equation (1.7) to write $(\epsilon\partial_t)(p,u)$ in terms of combinations of the spatial derivatives of (p,u) and ψ and use norms of (p,u) and ψ given by R in (4.34). By (4.64), we only need to check the leading order terms in $(\epsilon\partial_t)^k(u,p)$. We claim that these terms are of the following forms:

$$f(\psi)\nabla_x^\alpha u\,, \quad f(\psi)\nabla_x^\alpha p\,, \quad f(\psi)\nabla_x^\beta \psi\,, \quad f(\psi)(\epsilon\nabla_x)^\gamma \nabla_x^\beta \psi\,, \tag{4.69}$$

where f is a smooth function in ψ. The multi-indices α,β,γ satisfy $|\alpha| = k$, $|\beta| = k+1$, and $|\gamma| = 2k-1$. Again we prove this claim by induction. First, (4.69) holds for $k = 1$ by (1.7). Now suppose (4.69) holds for k. Then the leading-orders in $(\epsilon\partial_t)^k(u,p)$ are the leading-orders in the following terms:

$$f(\psi)\nabla_x^\alpha(\epsilon\partial_t)u\,, \quad f(\psi)\nabla_x^\alpha(\epsilon\partial_t)p\,, \quad f(\psi)\nabla_x^\beta(\epsilon\partial_t)\psi\,, \quad f(\psi)(\epsilon\nabla_x)^\gamma\nabla_x^\beta(\epsilon\partial_t)\psi\,.$$

Using (1.7) to replace $(\epsilon\partial_t)(u,p,\psi)$ gives that (4.69) holds for $k+1$. Thus (4.69) holds for all $1 \le k \le s$. Therefore,

$$\|(\epsilon\partial_t)^k(p,u)\|_{H^{s-k}} \le C(\|(p,u)\|_{H^s}, \|\Lambda_\epsilon^{2s-1}\psi\|_{H^{s+1}}) \le C(R)\,,$$

and

$$\begin{aligned} &\|(\epsilon\partial_t)^k(p,u)\|_{H^{s+1-k}} + \|(\epsilon\partial_t)^{k-1}\partial_t\theta\|_{H^{s+1-k}} \\ &\le C(\|(p,u)\|_{H^s}, \|\Lambda_\epsilon^{2s-1}\psi\|_{H^{s+1}}) \cdot (\|\nabla_x(p,u)\|_{H^s} + \|\Lambda_\epsilon^{2s-1}\nabla_x\psi\|_{H^{s+1}}) \\ &\le C(R)(1+R')\,, \end{aligned} \tag{4.70}$$

by the definitions of R, R' in (4.34). □

Given Lemma 4.8 and 4.9, we recall the following bound in [2] (Remark 6.21):

LEMMA 4.10 ([2]). *Let $F \in C^\infty(\mathbb{R}^1 \times \mathbb{R}^3 \times \mathbb{R}^1)$ such that $F(0) = 0$. Let $\psi = (\epsilon p, \epsilon u, \theta)$ be the solution to (1.7). Then*

$$\|F(\psi)\|_{L^\infty(0,T;H^s)} \le C(\Omega_0) + \sqrt{T}C(\Omega) \le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,. \tag{4.71}$$

We will use this bound in next section for estimates of the fast motion.

5. A priori estimate – fast motion. In this section we establish estimates for p and the acoustic part of u. The proof is presented slightly differently from [2] where the frequency space is divided into high and low regimes and estimates are obtained for these regimes respectively. Here using the various bounds obtained for the slow motion in Section 4, we can show Sobolev bounds of $(\nabla_x\cdot u,\ \nabla_x p)$ by studying estimates for $(\epsilon\partial_t)^k(p,u)$ with $1 \le k \le s$.

5.1. Linear Estimate for the Fast Motion. First we prove a linear estimate for the fast motion. Recall system (4.1) for (p, u, θ)

(5.1)
$$
\begin{aligned}
\tfrac{3}{5}(\partial_t + u_0\cdot\nabla_x)p + \frac{1}{\epsilon}\nabla_x\cdot\left(u - \tfrac{2}{5}\, h(\epsilon p_0)\kappa(\theta_0)\nabla_x\theta\right) &= \tfrac{2}{5}\epsilon h(\epsilon p_0)\Big(\big(\Sigma+\tilde\Sigma\big):\nabla_x u_0 + \nabla_x\cdot\tilde q\Big)\\
&\quad - \tfrac{2}{5}h(\epsilon p_0)\kappa(\theta_0)\nabla_x p_0\cdot\nabla_x\theta + f_1\,,\\
g(\theta_0)(\partial_t + u_0\cdot\nabla_x)u + \frac{1}{\epsilon}\nabla_x p &= h(\epsilon p_0)\left(\nabla_x\cdot\Sigma + \nabla_x\cdot\tilde\Sigma\right) + f_2\,,\\
\tfrac{3}{2}(\partial_t + u_0\cdot\nabla_x)\theta + \nabla_x\cdot u &= \epsilon^2 h(\epsilon p_0)\Big(\big(\Sigma+\tilde\Sigma\big):\nabla_x u_0 + \nabla_x\cdot\tilde q\Big)\\
&\quad + h(\epsilon p_0)\nabla_x\cdot(\kappa(\theta_0)\nabla_x\theta) + f_3\,,\\
(p,u,\theta)(x,0) &= (p^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})(x)\,,
\end{aligned}
$$

where

(5.2)
$$
\begin{aligned}
& q = \kappa(\theta_0)\nabla_x\theta\,,\quad h(\epsilon p_0) = e^{-\epsilon p_0}\,,\quad g(\theta_0) = e^{-\theta_0}\,,\\
& \Sigma = \mu(\theta_0)\left(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{3}(\nabla_x\cdot u)I\right)\,,\\
& \tilde\Sigma = \tau_1(\epsilon p_0,\theta_0)\left(\nabla_x^2\theta - \tfrac{1}{3}(\Delta_x\theta)I\right) + \tau_2(\epsilon p_0,\theta_0)\Big(\nabla_x\theta_0\otimes\nabla_x\hat\theta - \tfrac{1}{3}(\nabla_x\theta_0\cdot\nabla_x\theta)I\Big)\\
&\qquad + \epsilon^2\tau_3(\epsilon p_0,\theta_0)\left((\nabla_x u_0)(\nabla_x u)^T - (\nabla_x u_0)^T\nabla_x u\right)\,,\\
& \tilde q = \tau_4(\epsilon p_0,\theta_0)\left(\Delta_x u + \tfrac{1}{3}\nabla_x\nabla_x\cdot u\right) + \tau_6(\epsilon p_0,\theta_0)\nabla_x\theta_0\cdot\left(\nabla_x u - (\nabla_x u)^T\right)\,,\\
&\qquad + \tau_5(\epsilon p_0,\theta_0)\nabla_x\theta_0\cdot\left(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{3}(\nabla_x\cdot u)I\right)
\end{aligned}
$$

The estimate for (p, u) states

THEOREM 5.1. *Let (p, u, θ) be a solution to the linear system (5.1). Then there exists an increasing function $C(\cdot)$ such that*

$$
\begin{aligned}
&\sup_{[0,T]}\Big(\|\tilde p\|_{L^2}^2 + \|\tilde u\|_{H^1_\epsilon}^2\Big) + \alpha_0\int_0^T \|\nabla_x\tilde u\|_{H^1_\epsilon}^2(\tau)\,\mathrm{d}\tau\\
&\le C(r_0)e^{TC(r)}\Big(\|\tilde p^{in}\|_{L^2}^2 + \|\tilde u^{in}\|_{H^1_\epsilon}^2\Big) + C(r)\int_0^T \|(\epsilon f_2(\tau), f_3(\tau)\|_{H^1_\epsilon}^2\,\mathrm{d}\tau\\
&\quad + \left(\int_0^T \|(f_1(\tau), f_2(\tau)\|_{H^1_\epsilon}\mathrm{d}\tau\right)^2 + TC(r)\sup_{[0,T]}\|\nabla_x(\epsilon u,\tilde\theta)\|_{H^1_\epsilon}^2\\
&\quad + (\|F_0(\psi_0)\|_{L^\infty} + \|G_0(\psi_0)\|_{L^\infty})\int_0^T \|\Delta_x\tilde\theta\|_{H^1_\epsilon}^2(\tau)\mathrm{d}\tau\,,
\end{aligned}
\tag{5.3}
$$

where $(\tilde p, \tilde u, \tilde\theta)$ is defined in (5.5) and $F_0(\cdot), G_0(\cdot)$ are defined in (5.32) and (5.34) respectively.

Proof. Similarly as in the slow motion estimate, we combine leading order term in the dispersive term $\nabla_x\cdot\tilde q$ in the p-equation with the singular term. In that way, the p-equation is transformed into

$$
\begin{aligned}
&\tfrac{3}{5}(\partial_t + u_0\cdot\nabla_x)p + \frac{1}{\epsilon}\nabla_x\cdot\left(T_3 u - \tfrac{2}{5}\,h(\epsilon p_0)\kappa(\theta_0)\nabla_x\theta\right)\\
&= \epsilon\Psi_2 u + \epsilon\Psi_1 u + \epsilon\Psi_2\theta + \Psi_1\theta + f_1\,,
\end{aligned}
$$

where the symmetric positive operator T_3 is

$$
T_3 = I - \epsilon^2\nabla_x\cdot\left(\tfrac{8}{15}h(\epsilon p_0)\tau_4(\epsilon p_0,\theta_0)\nabla_x\right) \triangleq I - \epsilon^2\nabla_x\cdot(g_3(\epsilon p_0,\theta_0)\nabla_x)\,. \tag{5.4}
$$

Again we use Ψ_i to denote a homogeneous i^{th}-order differential operator. Here the coefficients of Ψ_1 and Ψ_2 depends on $\nabla_x^{\alpha_1}(h, \tau_1, \tau_2, \tau_3, \tau_4, \theta_0)$ with $|\alpha_1| \le 2$ and $\nabla_x^{\alpha_2}(u_0, p_0)$ with $|\alpha_2| \le 1$.

As suggested by the singular term in the above p-equation, we consider the system for

$$(\tilde{p}, \tilde{u}, \tilde{\theta}) = \left(p,\ u - T_3^{-1}\left(\tfrac{2}{5}\, h(\epsilon p_0)\kappa(\theta_0)\nabla_x \theta\right),\ \theta\right) . \tag{5.5}$$

The equation for $\tilde{p}$ is

$$\tfrac{3}{5}(\partial_t + u_0 \cdot \nabla_x)\tilde{p} + \frac{1}{\epsilon}\nabla_x \cdot (T_3 \tilde{u}) = \epsilon \Psi_2 u + \epsilon \Psi_1 u + \epsilon \Psi_2 \tilde{\theta} + \Psi_1 \tilde{\theta} + f_1 , \tag{5.6}$$

Next, we compute the equation for $\tilde{u}$. By (5.1) the equation for $\nabla_x \tilde{\theta}$ is

$$\begin{aligned} &\tfrac{3}{2}(\partial_t + u_0 \cdot \nabla_x)\nabla_x \tilde{\theta} + \nabla_x \nabla_x \cdot u \\ &= \epsilon^2 \Psi_3 u + \epsilon^2 \Psi_2 u + \epsilon^2 \Psi_1 u + \tfrac{4}{3}\epsilon^2 h(\epsilon p_0)\tau_4(\theta_0)\nabla_x \Delta_x \nabla_x \cdot u \\ &\quad + h(\epsilon p_0)\kappa(\theta_0)\nabla_x \Delta_x \tilde{\theta} + \epsilon^2 \Psi_3 \tilde{\theta} + \Psi_2 \tilde{\theta} + \nabla_x f_3 . \end{aligned} \tag{5.7}$$

Therefore, we have

$$\begin{aligned} \tfrac{3}{2}(\partial_t + u_0 \cdot \nabla_x)\nabla_x \tilde{\theta} + \nabla_x \nabla_x \cdot (T_2 u) = \Psi_3 \tilde{\theta} + \Psi_2 \tilde{\theta} + \Psi_1 \tilde{\theta} + \epsilon^2 \Psi_3 u \\ + \epsilon^2 \Psi_2 u + \epsilon^2 \Psi_1 u + \nabla_x f_3 . \end{aligned} \tag{5.8}$$

Thus,

$$\begin{aligned} &(\partial_t + u_0 \cdot \nabla_x)\left(\tfrac{2}{5} h(\epsilon p_0)\kappa(\theta_0)\nabla_x \tilde{\theta}\right) + \tfrac{4}{15} h(\epsilon p_0)\kappa(\theta_0)\nabla_x \nabla_x \cdot T_2 \tilde{u} \\ &= \Psi_3 \tilde{\theta} + \Psi_2 \tilde{\theta} + \Psi_1 \tilde{\theta} + \epsilon^2 \Psi_3 u + \epsilon^2 \Psi_2 u + \epsilon^2 \Psi_1 u + \tfrac{4}{15} h(\epsilon p_0)\kappa(\theta_0)\nabla_x f_3 . \end{aligned} \tag{5.9}$$

Now apply T_3^{-1} to (5.9) to obtain the equation for $T_3^{-1}(\frac{2}{5} h(\epsilon p_0)\kappa(\theta_0)\nabla_x \tilde{\theta})$.

$$\begin{aligned} &(\partial_t + u_0 \cdot \nabla_x) T_3^{-1}\left(\tfrac{2}{5} h(\epsilon p_0)\kappa(\theta_0)\nabla_x \tilde{\theta}\right) + T_3^{-1}\left(\tfrac{4}{15} h(\epsilon p_0)\kappa(\theta_0)\nabla_x \nabla_x \cdot T_2 \tilde{u}\right) \\ &= T_3^{-1}\Psi_3 \tilde{\theta} + T_3^{-1}\Psi_2 \tilde{\theta} + T_3^{-1}\Psi_1 \tilde{\theta} + \epsilon^2 T_3^{-1}\Psi_3 u + \epsilon^2 T_3^{-1}\Psi_2 u \\ &\quad + T_3^{-1}\left(\tfrac{4}{15} h(\epsilon p_0)\kappa(\theta_0)\nabla_x f_3\right) + [\partial_t,\ T_3^{-1}]\left(\tfrac{2}{5} h(\epsilon p_0)\kappa(\theta_0)\nabla_x \tilde{\theta}\right) \\ &\quad + \epsilon^2 T_3^{-1}\Psi_1 u + [u_0 \cdot \nabla_x,\ T_3^{-1}]\left(\tfrac{2}{5} h(\epsilon p_0)\kappa(\theta_0)\nabla_x \tilde{\theta}\right) . \end{aligned} \tag{5.10}$$

The last two commutator terms satisfy that

$$\begin{aligned} &[\partial_t,\ T_3^{-1}]\left(\tfrac{2}{5} h(\epsilon p_0)\kappa(\theta_0)\nabla_x \tilde{\theta}\right) + [u_0 \cdot \nabla_x,\ T_3^{-1}]\left(\tfrac{2}{5} h(\epsilon p_0)\kappa(\theta_0)\nabla_x \tilde{\theta}\right) \\ &= T_3^{-1}\tilde{\Psi}_0\left(\tfrac{2}{5} h(\epsilon p_0)\kappa(\theta_0)\nabla_x \tilde{\theta}\right) , \end{aligned}$$

where $\tilde{\Psi}_0$ is a zeroth order pseudo-differential operator which maps H^s to H^s for all $s \in \mathbb{R}$. The L^2-norm of this operator depends on $\|\partial_t(h, \tau_4)\|_{W^{1,\infty}}$ and $\|u_0\|_{W^{2,\infty}}$. Recall that we have the u-equation as

$$g(\theta_0)(\partial_t + u_0 \cdot \nabla_x)u + \frac{1}{\epsilon}\nabla_x \tilde{p} = h(\epsilon p_0)\left(\nabla_x \cdot \Sigma + \nabla_x \cdot \tilde{\Sigma}\right) + f_2 . \tag{5.11}$$

Subtracting $g(\theta_0)\times$ (5.10) from the u-equation, we obtain the equation for $\tilde{u}$

(5.12)

$$\begin{aligned}
&g(\theta_0)(\partial_t + u_0\cdot\nabla_x)\tilde{u} + \frac{1}{\epsilon}\nabla_x\tilde{p} - g(\theta_0)T_3^{-1}\left(\tfrac{4}{15}h(\epsilon p_0)\kappa(\theta_0)\nabla_x\nabla_x\cdot T_2\tilde{u}\right)\\
&= h(\epsilon p_0)\mu(\theta_0)(\Delta_x + \nabla_x\nabla_x\cdot)\tilde{u} + \epsilon^2\Psi_2 u + \Psi_1 u + \epsilon^2 g(\theta_0)T_3^{-1}(\Psi_3 u + \Psi_2 u + \Psi_1 u)\\
&\quad + g(\theta_0)T_3^{-1}(\Psi_3\tilde{\theta} + \Psi_2\tilde{\theta} + \Psi_1\tilde{\theta}) + \Psi_3\tilde{\theta} + \Psi_2(T_3^{-1}\Psi_1\tilde{\theta}) + \Psi_2\tilde{\theta} + \Psi_1\tilde{\theta}\\
&\quad + g(\theta_0)T_3^{-1}\tilde{\Psi}_0(\tfrac{2}{5}h(\epsilon p_0)\kappa(\theta_0)\nabla_x\tilde{\theta}) - g(\theta_0)T_3^{-1}\left(\tfrac{4}{15}h(\epsilon p_0)\kappa(\theta_0)\nabla_x f_3\right) + f_2\,,
\end{aligned}$$

The norms of Ψ_1, Ψ_2, Ψ_3 and $\tilde{\Psi}_0$ depend on $\|(h,\tau_4,\tau_5,\tau_6)\|_{W^{1,\infty}}$, $\|\partial_t(h,\tau_4)\|_{W^{1,\infty}}$, and $\|u_0\|_{W^{2,\infty}}$. We summarize the system for $(\tilde{p},\tilde{u})$ given by (5.6) and (5.12):

(5.13)

$$\begin{aligned}
&\tfrac{3}{5}(\partial_t + u_0\cdot\nabla_x)\tilde{p} + \frac{1}{\epsilon}\nabla_x\cdot(T_3\tilde{u}) = \epsilon\Psi_2 u + \epsilon\Psi_1 u + \epsilon\Psi_2\tilde{\theta} + \epsilon\Psi_1\tilde{\theta} + f_1\,,\\
&g(\theta_0)(\partial_t + u_0\cdot\nabla_x)\tilde{u} + \frac{1}{\epsilon}\nabla_x\tilde{p} - g(\theta_0)T_3^{-1}\left(\tfrac{4}{15}h(\epsilon p_0)\kappa(\theta_0)\nabla_x\nabla_x\cdot T_2\tilde{u}\right)\\
&= h(\epsilon p_0)\mu(\theta_0)(\Delta_x + \nabla_x\nabla_x\cdot)\tilde{u} + \epsilon^2\Psi_2 u + \Psi_1 u + \epsilon^2 g(\theta_0)T_3^{-1}(\Psi_3 u + \Psi_2 u + \Psi_1 u)\\
&\quad + g(\theta_0)T_3^{-1}(\Psi_3\tilde{\theta} + \Psi_2\tilde{\theta} + \Psi_1\tilde{\theta}) + \Psi_3\tilde{\theta} + \Psi_2(T_3^{-1}\Psi_1\tilde{\theta}) + \Psi_2\tilde{\theta} + \Psi_1\tilde{\theta}\\
&\quad + g(\theta_0)T_3^{-1}\tilde{\Psi}_0(\tfrac{2}{5}h(\epsilon p_0)\kappa(\theta_0)\nabla_x\tilde{\theta}) - g(\theta_0)T_3^{-1}\left(\tfrac{4}{15}h(\epsilon p_0)\kappa(\theta_0)\nabla_x f_3\right) + f_2\,.
\end{aligned}$$

We still keep the notation u in some terms on the right-hand side and recall that $\tilde{u}$ and u are related by (5.5).

Multiplying the $\tilde{p}$-equation in (5.13) by $\tilde{p}$ and integrating the resulting equation over $\mathbb{R}^3$ we have

$$\begin{aligned}
\tfrac{3}{10}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{p}\|_{L^2}^2 + \left\langle\tilde{p},\ \frac{1}{\epsilon}\nabla_x\cdot(T_3\tilde{u})\right\rangle &= -\langle\tilde{p},\ u_0\cdot\nabla_x\tilde{p}\rangle + \langle\tilde{p},\ \epsilon\Psi_2\tilde{u}\rangle + \langle\tilde{p},\ \epsilon\Psi_1\tilde{u}\rangle\\
&\quad + \left\langle\tilde{p},\ \epsilon\Psi_2\tilde{\theta}\right\rangle + \left\langle\tilde{p},\ \Psi_1\tilde{\theta}\right\rangle + \langle\tilde{p},\ f_1\rangle\ .
\end{aligned}\tag{5.14}$$

The estimates for the right-hand side of (5.14) are

$$\begin{aligned}
|\langle\tilde{p},\ u_0\cdot\nabla_x\tilde{p}\rangle| &\le C(r)\|\tilde{p}\|_{L^2}^2\,,\\
\left|\left\langle\tilde{p},\ \epsilon(\Psi_2+\Psi_1)(\tilde{u},\tilde{\theta})\right\rangle\right| &\le C(r)(\|\tilde{p}\|_{L^2}^2 + \|(\tilde{u},\tilde{\theta})\|_{H^1_\epsilon}^2) + \|\Delta_x(\epsilon\tilde{\theta})\|_{L^2}^2 + \tfrac{1}{16}\|\nabla_x\tilde{u}\|_{H^1_\epsilon}^2\,,\\
\left|\left\langle\tilde{p},\ \Psi_1\tilde{\theta}\right\rangle\right| &\le C(r)\left(\|\tilde{p}\|_{L^2}^2 + \|\nabla_x\tilde{\theta}\|_{L^2}^2\right)\ .
\end{aligned}$$

Therefore we have for the $\tilde{p}$-equation

$$\begin{aligned}
&\tfrac{3}{10}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{p}\|_{L^2}^2 + \left\langle\tilde{p},\ \frac{1}{\epsilon}\nabla_x\cdot(T_3\tilde{u})\right\rangle\\
&\le C(r)(\|\tilde{p}\|_{L^2}^2 + \|(\tilde{u},\tilde{\theta})\|_{H^1_\epsilon}^2) + \|\nabla_x\theta\|_{H^1_\epsilon}^2 + \tfrac{1}{16}\|\nabla_x\tilde{u}\|_{H^1_\epsilon}^2 + \|\tilde{p}\|_{L^2}\|f_1\|_{L^2}\,.
\end{aligned}\tag{5.15}$$

To estimate the H^1_ϵ-norm of $\tilde{u}$, multiply $T_3\tilde{u}$ to the $\tilde{u}$-equation in (5.13) and integrate over $\mathbb{R}^3$. We bound each term similarly as in those estimates for the slow motion. First,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle T_3(\sqrt{g}\,\tilde{u}),\ \sqrt{g}\,\tilde{u}\rangle &= 2\langle T_3(\sqrt{g}\,\tilde{u}),\ \sqrt{g}\,\partial_t\tilde{u}\rangle + R_1\\
&= 2\langle T_3\tilde{u},\ g\,\partial_t\tilde{u}\rangle + 2\langle[T_3,\ \sqrt{g}],\ \sqrt{g}\,\partial_t\tilde{u}\rangle + R_1\,,
\end{aligned}\tag{5.16}$$

where

$$R_1 = \langle (I - \epsilon^2 \nabla_x \cdot ((\partial_t g_3)\, \nabla_x))(\sqrt{g}\tilde{u}),\ \sqrt{g}\tilde{u}\rangle \\ + \langle (I - \epsilon^2 \nabla_x \cdot (g_3\, \nabla_x))((\partial_t \sqrt{g})\tilde{u}),\ \sqrt{g}\tilde{u}\rangle \\ + \langle (I - \epsilon^2 \nabla_x \cdot (g_3\, \nabla_x))(\sqrt{g}\tilde{u}),\ (\partial_t \sqrt{g})\tilde{u}\rangle\, .$$

Therefore,

$$|R_1| \leq C(r) \|\tilde{u}\|^2_{H^1_\epsilon}\, , \tag{5.17}$$

with $C(r)$ depends on $\|(\partial_t g_3, \partial_t g)\|_{W^{1,\infty}}$. We estimate $2\langle [T_3,\ \sqrt{g}],\ \sqrt{g}\, \partial_t \tilde{u}\rangle$ using the $\tilde{u}$-equation. First

$$[T_3,\ \sqrt{g}]\tilde{u} = \epsilon^2 \Psi_1 \tilde{u} + \epsilon^2 \Psi_0 \tilde{u}\, ,$$

with the coefficients of Ψ_1, Ψ_0 depending on $\|(g, g_3)\|_{W^{2,\infty}}$. We show here the estimates for the leading order terms in $2\langle [T_3,\ \sqrt{g}],\ \sqrt{g}\, \partial_t \tilde{u}\rangle$. The lower order terms obey similar bounds. First we have

$$\epsilon^2 \langle \Psi_1 \tilde{u},\ \tfrac{1}{\epsilon} g^{-1}(\theta_0) \nabla_x \tilde{p}\rangle = \epsilon \langle \Psi_1 \tilde{u},\ \nabla_x \tilde{p}\rangle\, .$$

Hence,

$$\left| \epsilon^2 \langle \Psi_1 \tilde{u},\ \tfrac{1}{\epsilon} \nabla_x \tilde{p}\rangle \right| \leq C(r) \|\nabla_x (\epsilon \tilde{p})\|^2_{L^2} + \tfrac{1}{16}\alpha_0 \|\nabla_x \tilde{u}\|^2_{L^2}\, , \tag{5.18}$$

where $C(r)$ depends on $\|(g, g_3, \tau_4)\|_{W^{2,\infty}}$. Next,

$$\begin{aligned} &\left| \epsilon^2 \langle \Psi_1 \tilde{u},\ g(\theta_0) T_3^{-1} \left(\tfrac{4}{15} h(\epsilon p_0) \kappa(\theta_0) \nabla_x \nabla_x \cdot T_2 \tilde{u} \right) \rangle \right| \\ &= \left| \langle \epsilon^2 \nabla_x \nabla_x \cdot (\tfrac{4}{15} h(\epsilon p_0) \kappa(\theta_0) T_3^{-1} \Psi_1 \tilde{u}),\ T_2 \tilde{u}\rangle \right| \\ &\leq C(r)\, \|\Psi_1 \tilde{u}\|_{H^1_\epsilon}\, \|\tilde{u}\|_{H^1_\epsilon} + \epsilon\, C(r)\, \|\Psi_1 \tilde{u}\|_{H^1_\epsilon}\, \|\nabla_x \tilde{u}\|_{H^1_\epsilon} \\ &\leq C(r)\, \|\tilde{u}\|^2_{H^1_\epsilon} + \tfrac{1}{16}\alpha_0 \|\nabla_x \tilde{u}\|^2_{H^1_\epsilon} + \epsilon\, C(r) \|\nabla_x \tilde{u}\|^2_{H^1_\epsilon}\, , \end{aligned} \tag{5.19}$$

and

$$\left| \epsilon^2 \langle \Psi_1 \tilde{u},\ \Psi_2 \tilde{u}\rangle \right| = \langle \Psi_1(\epsilon \tilde{u}),\ \Psi_2(\epsilon \tilde{u})\rangle \leq C(r) \|\tilde{u}\|^2_{H^1_\epsilon} + \tfrac{1}{16}\alpha_0 \|\nabla_x \tilde{u}\|^2_{H^1_\epsilon}\, . \tag{5.20}$$

Third,

$$\begin{aligned} \left| \epsilon^2 \langle \Psi_1 \tilde{u},\ \Psi_3 \tilde{\theta}\rangle \right| &= |\epsilon \langle \Psi_1(\epsilon \tilde{u}),\ \Psi_3 \tilde{\theta}\rangle| \\ &\leq C(r) \|(\epsilon \tilde{u}, \tilde{\theta})\|^2_{L^2} + \epsilon C(r) \|\nabla_x \tilde{u}\|^2_{H^1_\epsilon} + \epsilon C(r) \|\Delta_x \tilde{\theta}\|^2_{L^2} \\ &\leq C(r) \|(\epsilon \tilde{u}, \tilde{\theta})\|^2_{L^2} + \tfrac{1}{16}\alpha_0 \|\nabla_x \tilde{u}\|^2_{H^1_\epsilon} + \epsilon C(r) \|\Delta_x \tilde{\theta}\|^2_{L^2}\, , \end{aligned} \tag{5.21}$$

by choosing ϵ small enough. Combining (5.18), (5.19), (5.20), and (5.21) we have

$$\begin{aligned} |\langle [T_3,\ \sqrt{g}],\ \sqrt{g}\, \partial_t \tilde{u}]\rangle| \leq{}& C(r) \|(\tilde{p}, \tilde{u}, \tilde{\theta})\|^2_{H^1_\epsilon} + \tfrac{1}{4}\alpha_0 \|\nabla_x \tilde{u}\|^2_{H^1_\epsilon} + \epsilon C(r) \|\Delta_x \tilde{\theta}\|^2_{L^2}\, , \\ &+ \langle \nabla_x \cdot (\tfrac{2}{5} h(\epsilon p_0) \kappa(\theta_0) T_3^{-1} (\Psi_1(\epsilon \tilde{u}) + \Psi_0(\epsilon \tilde{u}))),\ \epsilon f_3\rangle \\ &+ \langle \Psi_1(\epsilon \tilde{u}) + \Psi_0(\epsilon \tilde{u}),\ \epsilon f_2\rangle\, . \end{aligned} \tag{5.22}$$

By (5.16),

$$\langle T_3 \tilde{u},\ g\, \partial_t \tilde{u}\rangle = \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \langle T_3(\sqrt{g}\tilde{u}),\ \sqrt{g}\tilde{u}\rangle - \langle T_3(\sqrt{g}\tilde{u}),\ \sqrt{g}\, \partial_t \tilde{u}\rangle - \tfrac{1}{2} R_1\, , \tag{5.23}$$

where $\langle T_3(\sqrt{g}\tilde{u}),\ \sqrt{g}\,\partial_t\tilde{u}\rangle$ satisfies (5.22) and R_1 satisfies (5.17).

The rest of the terms are estimated similarly. First by integration by parts the convection term has the bound

$$|\langle T_3\tilde{u},\ g(\theta_0)u_0\cdot\nabla_x\tilde{u}\rangle| \le C(r)\|\tilde{u}\|^2_{H^1_\epsilon}\,, \tag{5.24}$$

where $C(r)$ depends on $\|(g,g_3,u_0)\|_{W^{1,\infty}}$. The pressure term has the form

$$\langle T_3\tilde{u},\ \frac{1}{\epsilon}\nabla_x\tilde{p}\rangle = \frac{1}{\epsilon}\langle T_3\tilde{u},\ \nabla_x\tilde{p}\rangle\,. \tag{5.25}$$

The third term on the left-hand side of the $\tilde{u}$-equation in (5.13) gives

$$\begin{aligned} &-\langle T_3\tilde{u},\ g(\theta_0)T_3^{-1}\left(\tfrac{4}{15}h(\epsilon p_0)\kappa(\theta_0)\nabla_x\nabla_x\cdot T_2\tilde{u}\right)\rangle \\ &\ge 2\alpha_0\|\nabla_x\cdot\tilde{u}\|^2_{H^1_\epsilon} - \tfrac{1}{16}\alpha_0\|\nabla_x\tilde{u}\|^2_{H^1_\epsilon} - C(r)\|\tilde{u}\|^2_{L^2}\,. \end{aligned} \tag{5.26}$$

The dissipative term satisfies

$$-\langle T_3\tilde{u},\ h(\epsilon p_0)\mu(\theta_0)(\Delta_x+\nabla_x\nabla_x\cdot)\tilde{u}\rangle \ge 2\alpha_0\|\nabla_x\tilde{u}\|^2_{H^1_\epsilon} - C(r)\|\tilde{u}\|^2_{L^2}\,. \tag{5.27}$$

The rest of the terms related to u have the bounds

$$\begin{aligned} &|\langle T_3\tilde{u},\ \epsilon^2\Psi_2 u\rangle| \le C(r)\|(\epsilon u,\tilde{\theta})\|^2_{H^1_\epsilon} + C(r)\|\nabla_x(\epsilon u,\ \tilde{\theta})\|^2_{H^1_\epsilon}\,. \\ &|\langle T_3\tilde{u},\ \Psi_1 u\rangle| \le C(r)\,\|(\tilde{u},\tilde{\theta})\|^2_{H^1_\epsilon} + C(r)\|\nabla_x\tilde{\theta}\|^2_{H^1_\epsilon} + \tfrac{1}{16}\|\nabla_x\tilde{u}\|^2_{H^1_\epsilon}\,, \\ &|\langle T_3\tilde{u},\ \epsilon^2 g(\theta_0)T_3^{-1}(\Psi_3 u+\Psi_2 u+\Psi_1 u)\rangle| \le C(r)\,\|(\tilde{u},\tilde{\theta})\|^2_{H^1_\epsilon} + C(r)\,\|\nabla_x(\epsilon u,\ \tilde{\theta})\|^2_{H^1_\epsilon} \\ &\qquad + \tfrac{1}{16}\|\nabla_x\tilde{u}\|^2_{H^1_\epsilon}\,. \end{aligned} \tag{5.28}$$

Terms involving $\tilde{\theta}$ are bounded as

$$\begin{aligned} |\langle T_3\tilde{u},\ (\Psi_2\tilde{\theta}+\Psi_1\tilde{\theta})\rangle| &\le C(r)\|(\tilde{u},\tilde{\theta})\|^2_{H^1_\epsilon} + C(r)\|\nabla_x(\epsilon u,\ \tilde{\theta})\|^2_{H^1_\epsilon} \\ &\quad + \tfrac{1}{16}\alpha_0\|\nabla_x u\|^2_{H^1_\epsilon}\,, \\ |\langle T_3\tilde{u},\ g(\theta_0)T_3^{-1}(\Psi_2\tilde{\theta}+\Psi_1\tilde{\theta})\rangle| &\le C(r)\|(\tilde{u},\tilde{\theta})\|^2_{H^1_\epsilon} + C(r)\|\nabla_x(\epsilon u,\ \tilde{\theta})\|^2_{H^1_\epsilon} \\ &\quad + \tfrac{1}{16}\alpha_0\|\nabla_x u\|^2_{H^1_\epsilon}\,. \end{aligned} \tag{5.29}$$

We need to be more specific with the bounds for terms involving $\Psi_3\tilde{\theta}$. The reason will be clear when we do the estimates for high-order derivatives for (p,u). There are two terms that have third-order derivatives in $\tilde{\theta}$. The term $g(\theta_0)T_3^{-1}\Psi_3\tilde{\theta}$ comes from (5.10). It has the form

$$g(\theta_0)T_3^{-1}\Psi_3\tilde{\theta} = g(\theta_0)T_3^{-1}(h(\epsilon p_0)\kappa(\theta_0)\Delta_x\nabla_x\tilde{\theta})\,.$$

Therefore,

$$\begin{aligned} |\langle T_3\tilde{u},\ g(\theta_0)T_3^{-1}\Psi_3\tilde{\theta}\rangle| &= |\langle T_3\tilde{u},\ g(\theta_0)T_3^{-1}(h(\epsilon p_0)\kappa(\theta_0)\Delta_x\nabla_x\tilde{\theta})\rangle| \\ &= |\langle T_3^{-1}(g(\theta_0)T_3\tilde{u}),\ h(\epsilon p_0)\kappa(\theta_0)\Delta_x\nabla_x\tilde{\theta}\rangle| \\ &\le |\langle g(\theta_0)\tilde{u},\ h(\epsilon p_0)\kappa(\theta_0)\Delta_x\nabla_x\tilde{\theta}\rangle| \\ &\quad + |\langle T_3^{-1}[g(\theta_0),T_3]\tilde{u}),\ h(\epsilon p_0)\kappa(\theta_0)\Delta_x\nabla_x\tilde{\theta}\rangle|\,. \end{aligned} \tag{5.30}$$

The first term on the right-hand side of (5.30) has the bound

$$
\begin{aligned}
&|\langle g(\theta_0)\tilde{u},\ h(\epsilon p_0)\kappa(\theta_0)\Delta_x\nabla_x\tilde{\theta}\rangle| \\
&\le C(r)\|\tilde{u}\|_{L^2} + C(r)\|\nabla_x\tilde{\theta}\|_{L^2}^2 + \|F_0(\psi_0)\|_{L^\infty}\|\Delta_x\tilde{\theta}\|_{L^2}^2 + \tfrac{1}{16}\alpha_0\|\nabla_x\tilde{u}\|_{L^2}^2\,,
\end{aligned}
$$

where $F_0(\psi_0) = \frac{8}{\alpha_0}g(\theta_0)h(\epsilon p_0)\kappa(\theta_0)$. The second term on the right-hand side of (5.30) satisfies

$$
\begin{aligned}
|\langle T_3^{-1}[g(\theta_0), T_3]\tilde{u}),\ h(\epsilon p_0)\kappa(\theta_0)\Delta_x\nabla_x\tilde{\theta}\rangle| \le C(r)\|(\tilde{u},\tilde{\theta})\|_{H^1_\epsilon}^2 + C(r)\|\nabla_x(\epsilon u,\,\tilde{\theta})\|_{H^1_\epsilon}^2 \\
+ \tfrac{1}{16}\alpha\|\nabla_x\tilde{u}\|_{H^1_\epsilon}^2\,.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&|\langle T_3\tilde{u},\ g(\theta_0)T_3^{-1}\Psi_3\tilde{\theta}\rangle| \\
&\le C(r)\|(\tilde{u},\tilde{\theta})\|_{H^1_\epsilon}^2 + C(r)\|\nabla_x(\epsilon u,\,\tilde{\theta})\|_{H^1_\epsilon}^2 + \tfrac{1}{8}\alpha\|\nabla_x\tilde{u}\|_{H^1_\epsilon}^2 \\
&\quad + \|F_0(\psi_0)\|_{L^\infty}\|\Delta_x\tilde{\theta}\|_{L^2}^2\,,
\end{aligned}
\tag{5.31}
$$

with

$$
F_0(\psi_0) = \tfrac{8}{\alpha_0}g(\theta_0)h(\epsilon p_0)\kappa(\theta_0)
\tag{5.32}
$$

The other term that has the third-order derivative of $\tilde{\theta}$ is from the dispersive term in (5.11). It has the form

$$
\Psi_3\tilde{\theta} = h(\epsilon p_0)\tau_1(\epsilon p_0,\theta_0)\Delta\nabla_x\tilde{\theta}\,.
$$

Following similar estimates as we have done for (5.31), we have

$$
\begin{aligned}
|\langle T_3\tilde{u},\ \Psi_3\tilde{\theta}\rangle| \le C(r)\|(\tilde{u},\tilde{\theta})\|_{H^1_\epsilon}^2 + C(r)\|\nabla_x(\epsilon u,\,\tilde{\theta})\|_{H^1_\epsilon}^2 + \tfrac{1}{8}\alpha\|\nabla_x\tilde{u}\|_{H^1_\epsilon}^2 \\
+ \|G_0(\psi_0)\|_{L^\infty}\|\Delta_x\tilde{\theta}\|_{L^2}^2\,,
\end{aligned}
\tag{5.33}
$$

where

$$
G_0(\psi_0) = \tfrac{4}{\alpha_0}(1+g_3(\epsilon p_0,\theta_0))h(\epsilon p_0)\tau_1(\epsilon p_0,\theta_0)\,,
\tag{5.34}
$$

and g_3 is defined in (5.4).

Finally the forcing terms satisfy

$$
\begin{aligned}
&\langle T_3\tilde{u},\ g(\theta_0)T_3^{-1}\left(\tfrac{2}{5}h(\epsilon p_0)\kappa(\theta_0)\nabla_x f_3\right)\rangle + \langle T_3\tilde{u},\ f_2\rangle \\
&= -\langle \tfrac{2}{5}h(\epsilon p_0)\kappa(\theta_0)\nabla_x\cdot(T_3^{-1}(g(\theta_0)T_3\tilde{u})),\ f_3\rangle + \langle T_3\tilde{u},\ f_2\rangle\,.
\end{aligned}
\tag{5.35}
$$

Combining (5.23) through (5.35), the energy estimate for $\tilde{u}$ is

$$
\begin{aligned}
&\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \tilde{u},\ \sqrt{g(\theta_0)}\,T_3(\sqrt{g(\theta_0)}\tilde{u})\Big\rangle + \alpha_0\,\|\nabla_x\tilde{u}\|_{H^1_\epsilon}^2 + \frac{1}{\epsilon}\langle T_3\tilde{u},\ \nabla_x\tilde{p}\rangle \\
&\le C(r)\,\left(\|\tilde{p}\|_{L^2}^2 + \|(\tilde{u},\tilde{\theta})\|_{H^1_\epsilon}^2\right) + C(r)\,\|\nabla_x(\epsilon u,\,\tilde{\theta})\|_{H^1_\epsilon}^2 \\
&\quad + \left(\|F_0(\psi_0)\|_{L^\infty} + \|G_0(\psi_0)\|_{L^\infty}\right)\|\Delta_x\tilde{\theta}\|_{H^1_\epsilon}^2 + \langle \Psi_1(\epsilon\tilde{u}) + \Psi_0(\epsilon\tilde{u}),\ \epsilon f_2\rangle \\
&\quad + \langle T_3\tilde{u},\ f_2\rangle + \langle \tfrac{2}{5}h(\epsilon p_0)\kappa(\theta_0)\nabla_x\cdot(T_3^{-1}(g(\theta_0)T_3\tilde{u})),\ f_3\rangle \\
&\quad + \langle \nabla_x\cdot(\tfrac{2}{5}h(\epsilon p_0)\kappa(\theta_0)T_3^{-1}(\Psi_1(\epsilon\tilde{u}) + \Psi_0(\epsilon\tilde{u})),\ \epsilon f_3\rangle\,.
\end{aligned}
\tag{5.36}
$$

By (5.15) and (5.36) the energy estimate for $(\tilde{p}, \tilde{u})$ is

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{3}{5}\|\tilde{p}\|_{L^2}^2 + \left\langle \tilde{u},\ \sqrt{g(\theta_0)}\,T_3(\sqrt{g(\theta_0)}\tilde{u})\right\rangle\right) + \tfrac{3}{2}\alpha_0\,\|\nabla_x\tilde{u}\|_{H^1_\epsilon}^2 \\
&\le C(r)\,\left(\|\tilde{p}\|_{L^2}^2 + \|(\tilde{u},\tilde{\theta})\|_{H^1_\epsilon}^2\right) + \left(\|F_0(\psi_0)\|_{L^\infty} + \|G_0(\psi_0)\|_{L^\infty}\right)\|\Delta_x\tilde{\theta}\|_{H^1_\epsilon}^2 \\
&\quad + C(r)\,\|\nabla_x(\epsilon u,\,\tilde{\theta})\|_{H^1_\epsilon}^2 + \langle \Psi_1(\epsilon\tilde{u}) + \Psi_0(\epsilon\tilde{u}),\ \epsilon f_2\rangle + \langle T_3\tilde{u},\ f_2\rangle + \langle \tilde{p},\ f_1\rangle \\
&\quad + \langle \nabla_x\cdot(\tfrac{2}{5}h(\epsilon p_0)\kappa(\theta_0)T_3^{-1}(\Psi_1(\epsilon\tilde{u}) + \Psi_0(\epsilon\tilde{u})),\ \epsilon f_3\rangle \qquad (5.37)\\
&\quad + \langle \tfrac{2}{5}h(\epsilon p_0)\kappa(\theta_0)\nabla_x\cdot(T_3^{-1}(g(\theta_0)T_3\tilde{u})),\ f_3\rangle \\
&\le C(r)\,\left(\|\tilde{p}\|_{L^2}^2 + \|(\tilde{u},\tilde{\theta})\|_{H^1_\epsilon}^2\right) + C(r)\,\|\nabla_x(\epsilon u,\theta)\|_{H^1_\epsilon}^2 \\
&\quad + \left(\|F_0(\psi_0)\|_{L^\infty} + \|G_0(\psi_0)\|_{L^\infty}\right)\|\Delta_x\tilde{\theta}\|_{H^1_\epsilon}^2 + \|\epsilon f_2\|_{L^2}^2 \\
&\quad + \|\tilde{u}\|_{H^1_\epsilon}\|f_2\|_{H^1_\epsilon} + \|\tilde{p}\|_{L^2}\|f_1\|_{L^2} + C(r)\|f_3\|_{H^1_\epsilon}^2 + \tfrac{1}{2}\alpha_0\|\nabla_x\tilde{u}\|_{H^1_\epsilon}^2\,.
\end{aligned}
$$

By Gronwall's inequality and the equivalence of norms

$$
\left\langle \tilde{u},\ \sqrt{g(\theta_0)}T_3(\sqrt{g(\theta_0)}\tilde{u})\right\rangle \sim \|\tilde{u}\|_{H^1_\epsilon}^2\,,
$$

we have

$$
\begin{aligned}
&\sup_{[0,T]}\left(\|\tilde{p}\|_{L^2}^2 + \|\tilde{u}\|_{H^1_\epsilon}^2\right) + \alpha_0\int_0^T \|\nabla_x\tilde{u}\|_{H^1_\epsilon}^2(\tau)\,\mathrm{d}\tau \\
\le\ & C(r_0)e^{TC(r)}\left(\|\tilde{p}^{in}\|_{L^2}^2 + \|\tilde{u}^{in}\|_{H^1_\epsilon}^2\right) + C(r)\int_0^T \|(\epsilon f_2(\tau), f_3(\tau)\|_{H^1_\epsilon}^2\,\mathrm{d}\tau \\
&+ \left(\int_0^T \|(f_1(\tau), f_2(\tau)\|_{H^1_\epsilon}\mathrm{d}\tau\right)^2 + TC(r)\,\sup_{[0,T]}\|\nabla_x(\epsilon u,\tilde{\theta})\|_{H^1_\epsilon}^2 \\
&+ \left(\|F_0(\psi_0)\|_{L^\infty} + \|G_0(\psi_0)\|_{L^\infty}\right)\int_0^T \|\Delta_x\tilde{\theta}\|_{H^1_\epsilon}^2(\tau)\mathrm{d}\tau\,.
\end{aligned}
$$

□

5.2. A Priori Estimate for the Fast Motion. In this subsection we show a priori estimates for the fast motion (u, p). First we apply Theorem 5.1 to obtain bounds for $(\epsilon\partial_t)^\gamma(p,\ u)$, which then lead us to the bounds for the pressure and the acoustic part of the velocity field. The estimates for $(\epsilon\partial_t)^\gamma(p,\ u)$ with $1 \le \gamma \le s$ are stated in the following theorem:

THEOREM 5.2. *Let (p, u, θ) be a solution to (5.1). For $s > 5$, define*

$$
(p_\gamma, u_\gamma, \theta_\gamma) := (\epsilon\partial_t)^\gamma(p, u, \theta)\,, \tag{5.38}
$$

where $1 \le \gamma \le s$. Then there exists an increasing function $C(\cdot)$ such that

$$
\sup_{[0,T]}\left(\|p_\gamma\|_{L^2}^2 + \|u_\gamma\|_{H^1_\epsilon}^2\right) + \alpha_0\int_0^T \|\nabla_x(p_\gamma, u_\gamma)\|_{L^2}^2\,\mathrm{d}\tau \le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,. \tag{5.39}
$$

Before proving Theorem 5.2, we show how to use it to obtain the bounds for $\nabla_x\cdot u$ and $\nabla_x p$. To this end, we directly solve from equation (5.1) to get

$$
\begin{aligned}
\nabla_x\cdot u &= -\tfrac{3}{5}(\epsilon\partial_t)p - \tfrac{3}{5}(\epsilon u)\cdot\nabla_x p + \tfrac{2}{5}e^{-\epsilon p}\nabla_x\cdot(\kappa(\theta)\nabla_x\theta) + \tfrac{2}{5}\epsilon\, e^{-\epsilon p}\,\Sigma:\nabla_x(\epsilon u) \\
&\quad + \tfrac{2}{5}\epsilon\, e^{-\epsilon p}\,\tilde{\Sigma}:\nabla_x(\epsilon u) + \tfrac{2}{5}\epsilon^2\, e^{-\epsilon p}\,\nabla_x\cdot\tilde{q}\,, \qquad (5.40)\\
\nabla_x p &= -e^{-\theta}(\epsilon\partial_t)u - e^{-\theta}(\epsilon u)\cdot\nabla_x u + \epsilon e^{-\epsilon p}\left(\nabla_x\cdot\Sigma + \nabla_x\cdot\tilde{\Sigma}\right)\,.
\end{aligned}
$$

First we use induction to show that for every $0 \le k \le s$,

$$\sup_{[0,T]} \left(\|(\epsilon\partial_t)^k \nabla_x^\alpha p\|_{L^2}^2 + \|(\epsilon\partial_t)^k \nabla_x^{\alpha'} \nabla_x \cdot u\|_{L^2}^2 \right) \le C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}\,, \tag{5.41}$$

$$\int_0^T \|\nabla_x (\epsilon\partial_t)^k \nabla_x^{\alpha'} \nabla_x \cdot u\|_{L^2}^2(\tau)\,\mathrm{d}\tau \le C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}\,. \tag{5.42}$$

$$\int_0^T \|\nabla_x (\epsilon\partial_t)^k \nabla_x^{\alpha} p\|_{L^2}^2(\tau)\,\mathrm{d}\tau \le C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}\,. \tag{5.43}$$

where $|\alpha| = s-k$, and $|\alpha'| = s-k-1$. Notice that Theorem 5.2 shows (5.41) holds for $k = s$. Now suppose (5.41) holds for $k+1$ and we show that it also holds for k. Apply $(\epsilon\partial_t)^k \nabla_x^\beta$ with $|\beta| = s-k-1$ to (5.40). The resulting equations are

$$\begin{aligned}(\epsilon\partial_t)^k \nabla_x^\beta \nabla_x \cdot u = &-\tfrac{3}{5}(\epsilon\partial_t)^{k+1}\nabla_x^\beta p - \tfrac{3}{5}(\epsilon\partial_t)^k \nabla_x^\beta \left((\epsilon u)\cdot\nabla_x p\right) \\ &+ \tfrac{2}{5}(\epsilon\partial_t)^k \nabla_x^\beta \left(e^{-\epsilon p}\nabla_x\cdot(\kappa(\theta)\nabla_x\theta)\right) + \tfrac{2}{5}(\epsilon\partial_t)^k \nabla_x^\beta \left(\epsilon\, e^{-\epsilon p}\,\Sigma : \nabla_x(\epsilon u)\right) \\ &+ \tfrac{2}{5}(\epsilon\partial_t)^k \nabla_x^\beta \left(\epsilon\, e^{-\epsilon p}\,\tilde{\Sigma} : \nabla_x(\epsilon u)\right) + \tfrac{2}{5}(\epsilon\partial_t)^k \nabla_x^\beta \left(\epsilon^2\, e^{-\epsilon p}\,\nabla_x\cdot\tilde{q}\right)\,,\end{aligned} \tag{5.44}$$

$$\begin{aligned}(\epsilon\partial_t)^k \nabla_x^\beta \nabla_x p = &-(\epsilon\partial_t)^k \nabla_x^\beta \left(e^{-\theta}(\epsilon\partial_t)u\right) - (\epsilon\partial_t)^k \nabla_x^\beta \left(e^{-\theta}(\epsilon u)\cdot\nabla_x u\right) \\ &+ (\epsilon\partial_t)^k \nabla_x^\beta \left(\epsilon e^{-\epsilon p}\left(\nabla_x\cdot\Sigma + \nabla_x\cdot\tilde{\Sigma}\right)\right)\,.\end{aligned} \tag{5.45}$$

We estimate each term on the right-hand side (RHS) of (5.44). The estimates for p is similar. First, by the induction assumption, the first term on the RHS of (5.44) satisfies

$$\sup_{[0,T]} \|(\epsilon\partial_t)^{k+1}\nabla_x^\beta \nabla_x p\|_{L^2} \le C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}\,. \tag{5.46}$$

By Lemma 4.9, the second term on the RHS of (5.44) satisfies

$$\sup_{[0,T]} \|(\epsilon\partial_t)^k \nabla_x^\beta \left((\epsilon u)\cdot\nabla_x p\right)\|_{L^2} \le \epsilon C(\Omega)\,. \tag{5.47}$$

The rest of the terms on the RHS of (5.44) are all in terms of $\psi = (\epsilon p, \epsilon u, \theta)$ and their leading orders are at most $(\epsilon\partial_t)^k(\epsilon\nabla_x)\nabla_x^\beta\nabla_x^2$ with $|\beta| = s-k-1$. Therefore by Lemma 4.8, the L^2-norm of these terms are bounded by $C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}$. Together with (5.46) and (5.47), we have the desired bound for u in (5.41). The bounds (5.41) for p and (5.42), (5.43) are all shown in a similar way.

Combining (5.41), (5.42), and (5.43) we have the following theorem for p and the acoustic part of u (given Theorem 5.2):

THEOREM 5.3. *Let (p,u,θ) be a solution to the fluctuation equations (1.7). Then*

$$\begin{aligned}&\sup_{[0,T]} \left(\|p\|_{H^s}^2 + \|\nabla_x\cdot u\|_{H^{s-1}}^2\right) + \int_0^T (\|\nabla_x\nabla_x\cdot u\|_{H^{s-1}}^2 + \|\nabla_x p\|_{H^s}^2)(\tau)\,\mathrm{d}\tau \\ &\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,.\end{aligned}$$

Now we proceed to prove Theorem 5.2.

Proof. [Proof of Theorem 5.2] Apply the operator $(\epsilon\partial_t)^\gamma$ to the system (5.1). Recall the notation $(p_\gamma, u_\gamma, \theta_\gamma) = ((\epsilon\partial_t)^\gamma p, (\epsilon\partial_t)^\gamma u, (\epsilon\partial_t)^\gamma \theta)$. Then $(p_\gamma, u_\gamma, \theta_\gamma)$ satisfies

(5.48)

$$
\begin{aligned}
\tfrac{3}{5}(\partial_t + u\cdot\nabla_x)p_\gamma + \tfrac{1}{\epsilon}\nabla_x\cdot\left(u_\gamma - \tfrac{2}{5}h(\epsilon p)\kappa(\theta)\,\nabla_x\theta_\gamma\right) &= \tfrac{2}{5}\epsilon h(\epsilon p)\left(\left(\Sigma_\gamma + \tilde{\Sigma}_\gamma\right):\nabla_x u + \nabla_x\cdot\tilde{q}_\gamma\right)\\
&\quad + \tfrac{2}{5}h(\epsilon p)\kappa(\theta)\nabla_x p\cdot\nabla_x\theta_\gamma + g_1\,,\\
g(\theta)(\partial_t + u\cdot\nabla_x)\,u_\gamma + \tfrac{1}{\epsilon}\nabla_x p_\gamma &= h(\epsilon p)\left(\nabla_x\cdot\Sigma_\gamma + \nabla_x\cdot\tilde{\Sigma}_\gamma\right) + g_2\,,\\
\tfrac{3}{2}(\partial_t\theta_\gamma + u\cdot\nabla_x\theta_\gamma) + \nabla_x\cdot u_\gamma &= \epsilon^2 h(\epsilon p)\left(\left(\Sigma_\gamma + \tilde{\Sigma}_\gamma\right):\nabla_x u + \nabla_x\cdot\tilde{q}_\gamma\right)\\
&\quad + h(\epsilon p)\nabla_x\cdot(\kappa(\theta)\nabla_x\theta_\gamma) + g_3\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma_\gamma &= \mu(\theta)\left(\nabla_x u_\gamma + (\nabla_x u_\gamma)^T - \tfrac{2}{3}(\nabla_x\cdot u_\gamma)I\right)\,,\\
\tilde{\Sigma}_\gamma &= \tau_1(\epsilon p,\theta)\left(\nabla_x^2\theta_\gamma - \tfrac{1}{3}(\Delta_x\theta_\gamma)I\right) + \tau_2(\epsilon p,\theta)\left(\nabla_x\theta\otimes\nabla_x\theta_\gamma - \tfrac{1}{3}\nabla_x\theta\cdot\nabla_x\theta_\gamma I\right)\,,\\
&\quad + \tau_3(\epsilon p,\theta)\left((\nabla_x(\epsilon u))(\nabla_x(\epsilon u_\gamma))^T - (\nabla_x(\epsilon u))^T\nabla_x(\epsilon u_\gamma)\right)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{q}_\gamma &= \tau_4(\epsilon p,\theta)\left(\Delta_x u_\gamma + \tfrac{1}{3}\nabla_x\nabla_x\cdot u_\gamma\right) + \tau_6(\epsilon p,\theta)\left(\nabla_x u_\gamma - (\nabla_x u_\gamma)^T\right)\cdot\nabla_x\theta\\
&\quad + \tau_5(\epsilon p,\theta)\nabla_x\theta\cdot\left(\nabla_x u_\gamma + (\nabla_x u_\gamma)^T - \tfrac{2}{3}(\nabla_x\cdot u_\gamma)I\right)\,,\,.
\end{aligned}
$$

The commutator terms are

(5.49)

$$
\begin{aligned}
g_1 &= \left[\tfrac{3}{5}u,\,(\epsilon\partial_t)^\gamma\right]\cdot\nabla_x p - \frac{1}{\epsilon}\nabla_x\cdot\left[\tfrac{2}{5}h(\epsilon p)\kappa(\theta),\,(\epsilon\partial_t)^\gamma\right]\nabla_x\theta\\
&\quad + \epsilon\left[\Psi_2 + \Psi_1,\,(\epsilon\partial_t)^\gamma\right]u + \epsilon\left[\Psi_2 + \Psi_1,\,(\epsilon\partial_t)^\gamma\right]\theta\\
&\quad - \epsilon\left[\tfrac{2}{5}h(\epsilon p)\tau_4(\theta),\,(\epsilon\partial_t)^\gamma\right]\Delta_x\nabla_x\cdot u\,,\\
g_2 &= [g(\theta),\,(\epsilon\partial_t)^\gamma]\partial_t u + [g(\theta)u,\,(\epsilon\partial_t)^\gamma]\cdot\nabla_x u + [\Psi_2 + \Psi_1,\,(\epsilon\partial_t)^\gamma]u\\
&\quad - [h(\epsilon p)\tau_1(\theta),\,(\epsilon\partial_t)^\gamma]\Delta_x\nabla_x\theta + [\Psi_2 + \Psi_1, (\epsilon\partial_t)^\gamma]\theta\,,\\
g_3 &= [u,\,(\epsilon\partial_t)^\gamma]\nabla_x\theta + [\Psi_2 + \Psi_1,\,(\epsilon\partial_t)^\gamma]\,\theta\\
&\quad - \epsilon^2[h(\epsilon p)\tau_4(\theta),\,(\epsilon\partial_t)^\gamma]\,\Delta_x\nabla_x\cdot u + \epsilon^2[\Psi_2 + \Psi_1,\,(\epsilon\partial_t)^\gamma]\,u\,.
\end{aligned}
$$

Here Ψ_2 and Ψ_1 are homogeneous differential operators of order 2 and 1 respectively. Terms involving Ψ_2 in g_1, g_2 come from the dissipative and dispersive terms and their structures are

(5.50)
$$
[\delta(\psi)\partial_{x_k}\psi,\,(\epsilon\partial_t)^\gamma]\,\partial_{x_ix_j}\psi \quad\text{or}\quad [\delta(\psi),\,(\epsilon\partial_t)^\gamma]\,\partial_{x_ix_j}u\,,
$$

where $\psi = (\epsilon p, \epsilon u, \theta)$ and δ is a smooth function in ψ. Typical terms in (5.50) are

$$
C_{k_1,k_2,k_3}\left((\epsilon\partial_t)^{k_1}\delta(\psi)\right)\left((\epsilon\partial_t)^{k_2}\partial_{x_k}\psi\right)\left((\epsilon\partial_t)^{k_3}\partial_{x_ix_j}\psi\right)\,,
$$

and

$$
C_{k_4,k_5}\left((\epsilon\partial_t)^{k_4}\delta(\psi)\right)\left((\epsilon\partial_t)^{k_5}\partial_{x_ix_j}u\right)\,,
$$

where C_{k_1,k_2,k_3} and C_{k_4,k_5} are generic constants with

$$
\begin{gathered}
k_1 + k_2 + k_3 = |\gamma| = k\,,\quad k_1 + k_2 \geq 1\,,\\
k_4 + k_5 = k\,,\quad k_4 \geq 1\,.
\end{gathered}
$$

To estimate the H^1_ϵ-bounds of terms in (5.50), we only need to consider the leading-orders. These terms are

$$\delta(\psi)\left((\epsilon\partial_t)^\gamma\partial_{x_k}\psi\right)\,\partial_{x_ix_j}\psi\,,\quad \delta(\psi)\left((\epsilon\partial_t)\partial_{x_k}\psi\right)\,\left((\epsilon\partial_t)^{\gamma-1}\partial_{x_ix_j}\psi\right)\,,$$

and

$$\delta(\psi)\left((\epsilon\partial_t)^{\gamma-1}\,\partial_{x_ix_j}u\right)\,.$$

By Lemma 4.8 and 4.9,

$$\begin{aligned}\|(\epsilon\partial_t)^\gamma\partial_{x_k}\psi\|_{H^1_\epsilon}+\|(\epsilon\partial_t)^{\gamma-1}\partial_{x_ix_j}\psi\|_{H^1_\epsilon}&\le C(R)\,,\\ \|(\epsilon\partial_t)^\gamma\partial_{x_ix_j}u\|_{H^1_\epsilon}&\le C(R)(1+R')\,.\end{aligned}$$

Therefore,

$$\left\|\left[\delta(\psi)\partial_{x_k}\psi,\,(\epsilon\partial_t)^\gamma\right]\partial_{x_ix_j}\psi\right\|_{H^1_\epsilon}\le C(R)(1+R')\,. \tag{5.51}$$

By the linear estimate (5.3), we need the bounds of $\|(g_1,g_2,g_3)\|_{H^1_\epsilon}$. Again by (4.64) we only need to check the leading-orders in each term of g_1,g_2,g_3. The leading-order in the first term of g_1 has the form

$$\tfrac{3}{5}\left((\epsilon\partial_t)^\gamma u\right)\cdot\nabla_x p+\tfrac{3}{5}\left((\epsilon\partial_t)u\right)\cdot\left((\epsilon\partial_t)^{\gamma-1}\nabla_x p\right)\,.$$

By Lemma 4.8 and 4.9,

$$\|\tfrac{3}{5}\left((\epsilon\partial_t)^\gamma u\right)\cdot\nabla_x p+\tfrac{3}{5}\left((\epsilon\partial_t)u\right)\cdot\left((\epsilon\partial_t)^{\gamma-1}\nabla_x p\right)\|_{H^1_\epsilon}\le C(R)(1+R')\,.$$

Therefore,

$$\|\left[\tfrac{3}{5}u,\,(\epsilon\partial_t)^\gamma\right]\cdot\nabla_x p\|_{H^1_\epsilon}\le C(R)(1+R')\,. \tag{5.52}$$

The leading-order in the second term of g_1 is

$$-\tfrac{2}{5}\,\left(\partial_t\left(h(\epsilon p)\kappa(\theta)\right)\right)\left((\epsilon\partial_t)^{\gamma-1}\Delta\theta\right)-\tfrac{2}{5}\,\left((\epsilon\partial_t)^{\gamma-1}\partial_t\nabla_x(h(\epsilon p)\kappa(\theta))\right)\cdot\nabla_x\theta\,.$$

By Lemma 4.8 and 4.9,

$$\begin{aligned}\|-\tfrac{2}{5}\,\left(\partial_t\left(h(\epsilon p)\kappa(\theta)\right)\right)(\epsilon\partial_t)^{\gamma-1}\Delta\theta\|_{H^1_\epsilon}&\le C(R)\,,\\ \|-\tfrac{2}{5}\,\left((\epsilon\partial_t)^{\gamma-1}\partial_t\nabla_x(h(\epsilon p)\kappa(\theta))\right)\cdot\nabla_x\theta\|_{H^1_\epsilon}&\le C(R)(1+R')\,.\end{aligned}$$

Therefore,

$$\|-\tfrac{1}{\epsilon}\nabla_x\cdot\left[\tfrac{2}{5}h(\epsilon p)\kappa(\theta),\,(\epsilon\partial_t)^\gamma\right]\nabla_x\theta\|_{H^1_\epsilon}\le C(R)(1+R')\,. \tag{5.53}$$

The terms involving Ψ_2 are bounded as in (5.51). The leading-order in the last term in g_1 is

$$-\epsilon\tfrac{2}{5}\left((\epsilon\partial_t)^\gamma\left(h(\epsilon p)\tau_4(\theta)\right)\right)\left(\Delta_x\nabla_x\cdot u\right)-\epsilon\tfrac{2}{5}\left(\partial_t\left(h(\epsilon p)\tau_4(\theta)\right)\right)\left((\epsilon\partial_t)^{\gamma-1}\left(\Delta_x\nabla_x\cdot u\right)\right)\,,$$

where by Lemma 4.9,

$$\|-\epsilon\tfrac{2}{5}(\epsilon\partial_t)^\gamma\left(h(\epsilon p)\tau_4(\theta)\right)\left(\Delta_x\nabla_x\cdot u\right)\|_{H^1_\epsilon}\le C(R)\,,$$

and

$$\| -\epsilon \tfrac{2}{5}(\partial_t\,(h(\epsilon p)\tau_4(\theta)))\,(\epsilon\partial_t)^{\gamma-1}\,(\Delta_x\nabla_x\cdot u)\,\|_{H^1_\epsilon} \le C(R)(1+R')\,.$$

This gives

$$\| -\epsilon\left[\tfrac{2}{5}h(\epsilon p)\tau_4(\theta),\,(\epsilon\partial_t)^\gamma\right]\Delta_x\nabla_x\cdot u\|_{H^1_\epsilon} \le C(R)(1+R')\,. \tag{5.54}$$

Terms involving Ψ_1 in g_1 are commutators from Σ, $\tilde{\Sigma}$, and $\tilde{q}$. They have the form

$$[\delta(\psi)\partial_{x_k}u,\,(\epsilon\partial_t)^\gamma]\,\partial_{x_i}\psi\,,\quad \text{or}\quad [\delta(\psi)\partial_{x_k}p,\,(\epsilon\partial_t)^\gamma]\,\partial_{x_i}\psi\,,$$

or

$$\left[\delta(\psi)\partial_{x_k}\psi\,\partial_{x_j}\psi,\,(\epsilon\partial_t)^\gamma\right]\partial_{x_i}\psi\,.$$

Thus the leading-orders in Ψ_1-terms are

$$\left(\delta(\psi)\partial_{x_k}\psi\partial_{x_j}\psi\right)\,((\epsilon\partial_t)^\gamma\partial_{x_i}\psi)\quad\text{and}\quad(\delta(\psi)\partial_{x_i}\psi)\,((\epsilon\partial_t)^\gamma\partial_{x_k}(u,p))\,.$$

By Lemma 4.8 and 4.9,

$$\begin{aligned}\|\left(\delta(\psi)\partial_{x_k}\psi\partial_{x_j}\psi\right)\,((\epsilon\partial_t)^\gamma\partial_{x_i}\psi)\,\|_{H^1_\epsilon} &\le C(R)\,,\\ \|\,(\delta(\psi)\partial_{x_i}\psi)\,((\epsilon\partial_t)^\gamma\partial_{x_k}(u,p))\,\|_{H^1_\epsilon} &\le C(R)(1+R')\,.\end{aligned}$$

Therefore,

$$\|[\Psi_1,\,(\epsilon\partial_t)^\gamma]\psi\|_{H^1_\epsilon} \le C(R)(1+R')\,. \tag{5.55}$$

Combining (5.52), (5.53), (5.51), (5.55), and (5.54), we obtain that

$$\|g_1\|_{H^1_\epsilon} \le C(R)(1+R')\,. \tag{5.56}$$

The bound of $\|g_3\|_{H^1_\epsilon}$ can be obtained by considering the equation for $\epsilon p-\theta$:

$$\tfrac{3}{5}\epsilon g_1 - \tfrac{2}{3}g_3 = [u,\ (\epsilon\partial_t)^\gamma]\cdot\nabla_x(\epsilon p-\theta)\,. \tag{5.57}$$

The leading-orders in this term are

$$((\epsilon\partial_t)^\gamma u)\cdot\nabla_x(\epsilon p-\theta)\quad\text{and}\quad((\epsilon\partial_t)u)\cdot\left((\epsilon\partial_t)^{\gamma-1}\nabla_x(\epsilon p-\theta)\right)\,.$$

By Lemma 4.8 and 4.9,

$$\|\,((\epsilon\partial_t)^\gamma u)\cdot\nabla_x(\epsilon p-\theta)\|_{H^1_\epsilon} + \|((\epsilon\partial_t)u)\cdot\left((\epsilon\partial_t)^{\gamma-1}\nabla_x(\epsilon p-\theta)\right)\|_{H^1_\epsilon} \le C(R)\,.$$

By (5.57) and (5.56),

$$\|g_3\|_{H^1_\epsilon} \le C(R)(1+\epsilon R')\,. \tag{5.58}$$

Now we check the leading order terms in g_2. The leading-orders in the first term of g_2 are

$$\left((\epsilon\partial_t)^{\gamma-1}\partial_t g(\theta)\right)(\partial_t(\epsilon u))\quad\text{and}\quad(\partial_t g(\theta))\,(\epsilon\partial_t)^{\gamma-1}\partial_t(\epsilon u)\,.$$

By Lemma 4.8 and 4.9,

$$\| \big((\epsilon\partial_t)^{\gamma-1}\partial_t(g(\theta))\big)\,(\partial_t(\epsilon u))\|_{H^1_\epsilon} \leq C(R)\,,$$
$$\|(\partial_t g(\theta))\;((\epsilon\partial_t)^{\gamma} u)\,\|_{H^1_\epsilon} \leq C(R)\,.$$

Thus,

$$\|[g(\theta),\,(\epsilon\partial_t)^{\gamma}]\partial_t u\|_{H^1_\epsilon} \leq C(R)\,. \tag{5.59}$$

Next, the second term of g_2 has a similar structure as the first term in g_1. Therefore,

$$\|[g(\theta)u,\,(\epsilon\partial_t)^{\gamma}]\cdot\nabla_x u\|_{H^1_\epsilon} \leq C(R)(1+R')\,. \tag{5.60}$$

The Ψ_1 terms in g_2 are from Σ and $\tilde{\Sigma}$. They have similar structures as the Ψ_1 terms in g_1. Thus,

$$\|[\Psi_1,\,(\epsilon\partial_t)^{\gamma}]\theta\|_{H^1_\epsilon} + \|[\Psi_1,\,(\epsilon\partial_t)^{\gamma}]u\|_{H^1_\epsilon} \leq C(R)(1+R')\,. \tag{5.61}$$

The Ψ_2 terms in g_2 are bounded by (5.51). The leading-orders in the second last term in g_2 are

$$-\left((\epsilon\partial_t)^{\gamma}(h(\epsilon p)\tau_1(\theta))\right)\,(\Delta_x\nabla_x\theta) \quad\text{and}\quad -\left((\epsilon\partial_t)(h(\epsilon p)\tau_1(\theta))\right)\,(\epsilon\partial_t)^{\gamma-1}(\Delta_x\nabla_x\theta)\,.$$

By Lemma 4.8,

$$\|-\left((\epsilon\partial_t)^{\gamma}(h(\epsilon p)\tau_1(\theta))\right)\,(\Delta_x\nabla_x\theta)\|_{H^1_\epsilon} \leq C(R)\,,$$
$$\|-\left((\epsilon\partial_t)(h(\epsilon p)\tau_1(\theta))\right)\,(\epsilon\partial_t)^{\gamma-1}(\Delta_x\nabla_x\theta)\|_{H^1_\epsilon} \leq C(R)(1+R')\,.$$

Thus,

$$\|-[h(\epsilon p)\tau_1(\theta),\,(\epsilon\partial_t)^{\gamma}]\Delta_x\nabla_x\theta\|_{H^1_\epsilon} \leq C(R)(1+R')\,. \tag{5.62}$$

Combining (5.59), (5.60), (5.61), (5.51), and (5.62), we have

$$\|g_2\|_{H^1_\epsilon} \leq C(R)(1+R')\,. \tag{5.63}$$

By the linear estimate (5.3), we have

$$\begin{aligned}
&\sup_{[0,T]}\left(\|p_\gamma\|^2_{L^2} + \|\tilde{u}_\gamma\|^2_{H^1_\epsilon}\right) + \alpha_0\int_0^T \|\nabla_x\tilde{u}_\gamma\|^2_{H^1_\epsilon}(\tau)\,\mathrm{d}\tau \\
&\leq C(r_0)e^{TC(r)}\left(\|p^{in}_\gamma\|^2_{L^2} + \|\tilde{u}^{in}_\gamma\|^2_{H^1_\epsilon}\right) + C(r)\int_0^T \|(\epsilon g_2(\tau), g_3(\tau)\|^2_{H^1_\epsilon}\,\mathrm{d}\tau \\
&\quad + \left(\int_0^T \|(g_1(\tau), g_2(\tau)\|_{H^1_\epsilon}\mathrm{d}\tau\right)^2 + TC(r)\sup_{[0,T]}\|\nabla_x(\epsilon u_\gamma, \theta_\gamma)\|^2_{H^1_\epsilon} \\
&\quad + (\|F_0(\psi)\|_{L^\infty} + \|G_0(\psi)\|_{L^\infty})\int_0^T \|\Delta_x\theta_\gamma\|^2_{H^1_\epsilon}(\tau)\mathrm{d}\tau\,,
\end{aligned}$$

where

$$\psi = (\epsilon p, \epsilon u, \theta)\,, \qquad \tilde{u}_\gamma = u_\gamma - T_3^{-1}\left(\tfrac{2}{5}\,h(\epsilon p)\kappa(\theta)\nabla_x\theta_\gamma\right).$$

By the estimates for (g_1, g_2, g_3) and $(F_0(\psi), G_0(\psi))$, we then have

$$\begin{aligned}
&\sup_{[0,T]} \left(\|p_\gamma\|_{L^2}^2 + \|u_\gamma\|_{H^1_\epsilon}^2 \right) + \alpha_0 \int_0^T \|\nabla_x u_\gamma\|_{H^1_\epsilon}^2(\tau)\, \mathrm{d}\tau \\
\le\ & C(\Omega_0) e^{TC(r)} \left(\|p_\gamma^{in}\|_{L^2}^2 + \|\tilde{u}_\gamma^{in}\|_{H^1_\epsilon}^2 \right) + (T + \epsilon^2) C(\Omega) \\
&+ TC(r) \sup_{[0,T]} \|\nabla_x(\epsilon u_\gamma, \theta_\gamma)\|_{H^1_\epsilon}^2 + C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(r)} \int_0^T \|\Delta_x \theta_\gamma\|_{H^1_\epsilon}^2(\tau) \mathrm{d}\tau \\
&+ \sup_{[0,T]} \|T_3^{-1}(\tfrac{2}{5}\, h(\epsilon p)\kappa(\theta)\nabla_x\theta_\gamma)\|_{H^1_\epsilon}^2 + \alpha_0 \int_0^T \|\nabla_x T_3^{-1}(\tfrac{2}{5}\, h(\epsilon p)\kappa(\theta)\nabla_x\theta_\gamma)\|_{H^1_\epsilon}^2(\tau)\, \mathrm{d}\tau\,.
\end{aligned}$$

Estimates of terms involving $(\epsilon u_\gamma, \theta_\gamma)$ in the last four terms in the above inequality are as follows: by Lemma 4.8 we have

$$TC(r) \sup_{[0,T]} \|\nabla_x(\epsilon u_\gamma,\ \theta_\gamma)\|_{H^1_\epsilon}^2 \le TC(\Omega)\,,$$

$$\int_0^T \|\Delta_x \theta_\gamma\|_{H^1_\epsilon}^2(\tau)\, \mathrm{d}\tau + \sup_{[0,T]} \|T_3^{-1}(\tfrac{2}{5}\, h(\epsilon p)\kappa(\theta)\nabla_x\theta_\gamma)\|_{H^1_\epsilon} \le C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}\,,$$

and

$$\int_0^T \|\nabla_x T_3^{-1}(\tfrac{2}{5}\, h(\epsilon p)\kappa(\theta)\nabla_x\theta_\gamma)\|_{H^1_\epsilon}^2(\tau)\, \mathrm{d}\tau \quad \le C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}\,.$$

Therefore,

$$\sup_{[0,T]} \left(\|p_\gamma\|_{L^2}^2 + \|u_\gamma\|_{H^1_\epsilon}^2 \right) + \alpha_0 \int_0^T \|\nabla_x u_\gamma\|_{L^2}^2\, \mathrm{d}\tau \le C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}\,. \tag{5.64}$$

To conclude, we need the bound for $\int_0^T \|\nabla_x p_\gamma\|_{L^2}^2\, \mathrm{d}\tau$. To this end, we multiply the u_γ-equation in (5.48) by $\epsilon\nabla_x p_\gamma$ and integrate over $\mathbb{R}^3 \times [0, T]$. The resulting equation is

$$\begin{aligned}
\int_0^T \|\nabla_x p_\gamma\|_{L^2}^2(\tau)\, \mathrm{d}\tau = &- \int_0^T \int_{\mathbb{R}^3} g(\theta)\, \partial_t((\epsilon\partial_t)^\gamma(\epsilon u)) \cdot \nabla_x p_\gamma(x,\tau)\, \mathrm{d}x\, \mathrm{d}\tau \\
&+ \int_0^T \int_{\mathbb{R}^3} \Gamma \cdot (\nabla_x p_\gamma)(x,\tau)\, \mathrm{d}x\, \mathrm{d}\tau\,,
\end{aligned} \tag{5.65}$$

where

$$\Gamma = -g(\theta)(\epsilon u)\cdot\nabla_x\, u_\gamma + \epsilon h(\epsilon p) \left(\nabla_x \cdot \Sigma_\gamma + \nabla_x \cdot \tilde{\Sigma}_\gamma \right) + \epsilon g_2\,. \tag{5.66}$$

Note that by (5.64), the first term in Γ satisfies

$$\| - g(\theta)(\epsilon u)\cdot\nabla_x\, u_\gamma\|_{L^2(\mathbb{R}^3\times[0,T])} \le \epsilon C(\Omega) \cdot C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)} \le C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}\,.$$

By Lemma 4.8, the second term in Γ satisfies

$$\|\epsilon h(\epsilon p) \left(\nabla_x \cdot \Sigma_\gamma + \nabla_x \cdot \tilde{\Sigma}_\gamma \right) \|_{L^2(\mathbb{R}^3\times[0,T])} \le C(\Omega_0) e^{(\sqrt{T}+\epsilon)C(\Omega)}\,.$$

By the estimate for g_2 in (5.63), the last term in Γ satisfies

$$\|\epsilon g_2\|_{L^2(\mathbb{R}^3\times[0,T])} \le \epsilon \left(\int_0^T (C(R)(1+R'))^2(\tau)\,\mathrm{d}\tau \right)^{1/2} \le \epsilon C(\Omega)\,,$$

by the definitions of R, R', Ω in (4.34) and (4.37). Combining these bounds we have

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^3} \Gamma \cdot (\nabla_x p_\gamma)(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \right| &\le \|\Gamma\|_{L^2(\mathbb{R}^3\times[0,T])}\ \|\nabla_x p_\gamma\|_{L^2(\mathbb{R}^3\times[0,T])} \\ &\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)} + \tfrac{1}{4}\|\nabla_x p_\gamma\|^2_{L^2(\mathbb{R}^3\times[0,T])}\,. \end{aligned} \tag{5.67}$$

To estimate the first term on the right-hand side of (5.65), we integrate by parts in both t and x. Then

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} g(\theta)\,\partial_t((\epsilon\partial_t)^\gamma(\epsilon u)) \cdot \nabla_x p_\gamma(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ &= \int_{\mathbb{R}^3} (g(\theta)u_\gamma)\cdot((\epsilon\partial_t)^\gamma(\nabla_x(\epsilon p)))(x,T)\,\mathrm{d}x - \int_{\mathbb{R}^3} (g(\theta)u_\gamma)\cdot((\epsilon\partial_t)^\gamma(\nabla_x(\epsilon p)))(x,0)\,\mathrm{d}x \\ &\quad + \int_0^T \int_{\mathbb{R}^3} (\epsilon\partial_t g(\theta))(\nabla_x\cdot u_\gamma)\,p_\gamma\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^T \int_{\mathbb{R}^3} ((\epsilon\partial_t\nabla_x g(\theta))\cdot u_\gamma)\,p_\gamma\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + \int_0^T \int_{\mathbb{R}^3} (g(\theta)(\nabla_x\cdot u_\gamma))\,(\epsilon\partial_t p_\gamma)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^T \int_{\mathbb{R}^3} ((\nabla_x g(\theta))\cdot u_\gamma)\,(\epsilon\partial_t p_\gamma)\,\mathrm{d}x\,\mathrm{d}\tau\,. \end{aligned} \tag{5.68}$$

Estimates of each term are as follows. First, for each $t \in [0,T]$, by Lemma 4.8, Lemma 4.10, and (5.64),

$$\begin{aligned} \|((\epsilon\partial_t)^\gamma(\nabla_x(\epsilon p)))(\cdot,t)\|_{L^2(\mathbb{R}^3)} &\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,, \\ \|u_\gamma(\cdot,t)\|_{L^2(\mathbb{R}^3)} &\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,, \\ \|g(\theta)(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} &\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,. \end{aligned}$$

Therefore for $t = 0$ and $t = T$,

$$\left| \int_{\mathbb{R}^3} (g(\theta)u_\gamma)\cdot((\epsilon\partial_t)^\gamma(\nabla_x(\epsilon p)))(x,t)\,\mathrm{d}x \right| \le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,. \tag{5.69}$$

Similarly, by the θ-equation, Lemma 4.8 and (5.64),

$$\begin{aligned} \|p_\gamma\|_{L^2(\mathbb{R}^3\times[0,T])} &\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,, \\ \|\nabla_x\cdot u_\gamma\|_{L^2(\mathbb{R}^3\times[0,T])} &\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,, \\ \|(\epsilon\partial_t)\nabla_x g(\theta)\|_{L^\infty(\mathbb{R}^3\times[0,T])} &\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^3} (\epsilon\partial_t g(\theta))(\nabla_x\cdot u_\gamma)\,p_\gamma\,\mathrm{d}x\,\mathrm{d}\tau \right| &\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,, \\ \left| \int_0^T \int_{\mathbb{R}^3} ((\epsilon\partial_t\nabla_x g(\theta))\cdot u_\gamma)\,p_\gamma\,\mathrm{d}x\,\mathrm{d}\tau \right| &\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,. \end{aligned} \tag{5.70}$$

To estimate the last two terms in (5.68) we need to use the p_γ-equation in (5.48), which is

$$
\begin{aligned}
\epsilon\partial_t p_\gamma = &-\tfrac{5}{3}(\epsilon u)\cdot\nabla_x p_\gamma - \tfrac{5}{3}\nabla_x\cdot\big(u_\gamma - \tfrac{2}{5}h(\epsilon p)\kappa(\theta)\,\nabla_x\theta_\gamma\big) + \epsilon g_1 \\
&+ \tfrac{2}{3}\epsilon^2 h(\epsilon p)\left(\left(\Sigma_\gamma + \tilde{\Sigma}_\gamma\right):\nabla_x u + \nabla_x\cdot\tilde{q}_\gamma\right) + \tfrac{2}{3}h(\epsilon p)\kappa(\theta)\nabla_x(\epsilon p)\cdot\nabla_x\theta_\gamma\,.
\end{aligned}
\tag{5.71}
$$

Using (5.71) we show bounds for the second last term in (5.68). First, by Lemma 4.8 and (5.64), the contribution from the first term in $\epsilon\partial_t p_\gamma$ is

$$
\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^3}(g(\theta)(\nabla_x\cdot u_\gamma))\,(\tfrac{5}{3}(\epsilon u)\cdot\nabla_x p_\gamma)\,\mathrm{d}x\,\mathrm{d}\tau\right| \\
&\le \tfrac{5}{3}\|g(\theta)(\epsilon u)\|_{L^\infty(\mathbb{R}^3\times[0,T])}\|\nabla_x\cdot u_\gamma\|_{L^2(\mathbb{R}^3\times[0,T])}\|\nabla_x p_\gamma\|_{L^2(\mathbb{R}^3\times[0,T])} \\
&\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\|\nabla_x p_\gamma\|_{L^2(\mathbb{R}^3\times[0,T])} \\
&\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)} + \tfrac{1}{4}\|\nabla_x p_\gamma\|^2_{L^2(\mathbb{R}^3\times[0,T])}\,.
\end{aligned}
$$

The contribution from the term $\frac{5}{3}\nabla_x\cdot u_\gamma$ in $\epsilon\partial_t p_\gamma$ is

$$
\begin{aligned}
\left|\int_0^T\int_{\mathbb{R}^3}(g(\theta)(\nabla_x\cdot u_\gamma))\,(\tfrac{5}{3}\nabla_x\cdot u_\gamma)\,\mathrm{d}x\,\mathrm{d}\tau\right| &\le \tfrac{5}{3}\|g(\theta)\|_{L^\infty(\mathbb{R}^3\times[0,T])}\|\nabla_x\cdot u_\gamma\|^2_{L^2(\mathbb{R}^3\times[0,T])} \\
&\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,.
\end{aligned}
$$

By (5.56), the contribution from the term ϵg_1 in $\epsilon\partial_t p_\gamma$ is

$$
\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^3}(g(\theta)(\nabla_x\cdot u_\gamma))\,(\epsilon g_1)\,\mathrm{d}x\,\mathrm{d}\tau\right| \\
&\le \|g(\theta)\|_{L^\infty(\mathbb{R}^3\times[0,T])}\|\nabla_x\cdot u_\gamma\|_{L^2(\mathbb{R}^3\times[0,T])}\|\epsilon g_1\|_{L^2(\mathbb{R}^3\times[0,T])} \\
&\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}(\epsilon C(\Omega)) \\
&\le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,.
\end{aligned}
$$

The rest of the terms in $\epsilon\partial_t p_\gamma$ are all for the slow motion $(\epsilon\partial_t)^\gamma\psi = (\epsilon p, \epsilon u, \theta)$. The highest order for these terms are $(\epsilon\nabla_x)\nabla_x^2$. Therefore by Lemma 4.8, their $L^2(\mathbb{R}^3\times[0,T])$-norms are bounded by $C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}$. Hence, their contribution to the second last term of (5.68) is also bounded by $C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}$. Altogether we have

$$
\left|\int_0^T\int_{\mathbb{R}^3}(g(\theta)(\nabla_x\cdot u_\gamma))\,(\epsilon\partial_t p_\gamma)\,\mathrm{d}x\,\mathrm{d}\tau\right| \le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)} + \tfrac{1}{4}\|\nabla_x p_\gamma\|^2_{L^2(\mathbb{R}^3\times[0,T])}\,.
$$

The last term in (5.68) is bounded in a similar way. Therefore, we have at the end

$$
\int_0^T\|\nabla_x p_\gamma\|^2_{L^2}(\tau)\,\mathrm{d}\tau \le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)} + \tfrac{1}{2}\|\nabla_x p_\gamma\|^2_{L^2(\mathbb{R}^3\times[0,T])}\,,
$$

which gives

$$
\int_0^T\|\nabla_x p_\gamma\|^2_{L^2}(\tau)\,\mathrm{d}\tau \le C(\Omega_0)e^{(\sqrt{T}+\epsilon)C(\Omega)}\,.
$$

We thereby finish the proof of Theorem 5.2. □

Conclusion. Combining Theorem 4.3, Lemma 4.7, and Theorem 5.3, we complete the proof of Theorem 2.4.

6. Decay of the Local Energy. Given the uniform bound for (p, u, θ), we show in this section the local strong convergence of the fast wave (p^ϵ, u^ϵ). The theorem states as

THEOREM 6.1. *Let $T > 0$, and $s > 6$. Assume that the functions $(p^\epsilon, u^\epsilon, \theta^\epsilon)$ satisfy the system (1.7), and*

$$\sup_{[0,T]} \left(\|(p^\epsilon, u^\epsilon)\|_{H^s} + \|\theta^\epsilon\|_{H^{s+1}}\right) < \infty \,. \tag{6.1}$$

Assume further that θ^ϵ converges strongly in $C^0([0,T]; H^\sigma_{loc}(\mathbb{R}^3)$ for some $\sigma > 5/2$, to a limit θ satisfying for all $(t, x) \in [0,T] \times \mathbb{R}^3$

$$|\theta(t,x) - \underline{\theta}| \le C_0 |x|^{-1-\gamma}, \qquad |\nabla_x \theta(t,x)| \le C_0 |x|^{-2-\gamma} \,, \tag{6.2}$$

for some given positive constants C_0, γ, and $\underline{\theta}$. Then

$$\begin{aligned} p^\epsilon \to 0 \qquad & \textit{in} \quad L^2([0,T]; H^{s'}_{loc}(\mathbb{R}^3)) \,, \\ \nabla_x \cdot \left(u^\epsilon - \tfrac{2}{5} e^{-\epsilon p^\epsilon} \kappa(\theta^\epsilon) \nabla_x \theta^\epsilon\right) \to 0 \qquad & \textit{in} \quad L^2([0,T]; H^{s'-1}_{loc}(\mathbb{R}^3)) \end{aligned} \tag{6.3}$$

for all $1 \le s' < s$.

The proof of Theorem 6.1 is based on the following theorem via dispersive estimates by Métivier and Schochet [20] which is reformulated in [2].

THEOREM 6.2. *Let $T > 0$, and let w^ϵ be a bounded sequence in $C^0([0,T]; H^2(\mathbb{R}^3))$ such that*

$$\epsilon^2 \partial_t \left(a^\epsilon \partial_t w^\epsilon\right) - \nabla_x \cdot \left(b^\epsilon \nabla_x w^\epsilon\right) = c^\epsilon \,,$$

where c^ϵ converges to 0 strongly in $L^2(0,T; L^2(\mathbb{R}^3))$. Assume further that, for some $\sigma > 5/2$, the coefficients (a^ϵ, b^ϵ) are uniformly bounded in $C^0([0,T]; H^\sigma(\mathbb{R}^3))$ and converges in $C^0([0,T]; H^\sigma_{loc}(\mathbb{R}^3))$ to a limit (a, b) satisfying the decay estimate

$$\begin{aligned} |a(t,x) - \underline{a}| \le c_0 |x|^{-1-\gamma} \,, \qquad & |\nabla_x a(t,x)| \le c_0 |x|^{-2-\gamma} \,, \\ |b(t,x) - \underline{b}| \le c_0 |x|^{-1-\gamma} \,, \qquad & |\nabla_x b(t,x)| \le c_0 |x|^{-2-\gamma} \,, \end{aligned}$$

for some given positive constants $\underline{a}, \underline{b}, c_0$ and γ. Then the sequence w^ϵ converges to 0 in $L^2(0,T; L^2_{loc}(\mathbb{R}^3))$.

Proof. [Proof of Theorem 6.1] First we show that $p^\epsilon \to 0$ in $L^2([0,T]; H^{s'}_{loc}(\mathbb{R}^3))$. By the boundedness assumption (6.1) and interpolation arguments, it suffices to verify that $p^\epsilon \to 0$ in $L^2([0,T]; L^2_{loc}(\mathbb{R}^3))$. The equation for p^ϵ is

$$\begin{aligned} & \tfrac{3}{5}(\partial_t + u^\epsilon \cdot \nabla_x) p^\epsilon + \frac{1}{\epsilon} \nabla_x \cdot \left(u^\epsilon - \tfrac{2}{5} e^{-\epsilon p^\epsilon} \kappa(\Theta_\epsilon) \nabla_x \theta^\epsilon\right) \\ & = \tfrac{2}{5} \epsilon e^{-\epsilon p^\epsilon} \left(\left(\Sigma + \tilde{\Sigma}\right) : \nabla_x u^\epsilon + \nabla_x \cdot \tilde{q}\right) \,. \end{aligned}$$

Apply the operator $\epsilon^2 \partial_t$ to the p^ϵ-equation. We have

$$\begin{aligned} & \epsilon^2 \partial_t \left(\tfrac{3}{5}(\partial_t + u^\epsilon \cdot \nabla_x) p^\epsilon\right) + \epsilon \partial_t \nabla_x \cdot \left(u^\epsilon - \tfrac{2}{5} e^{-\epsilon p^\epsilon} \kappa(\Theta_\epsilon) \nabla_x \theta^\epsilon\right) \\ & = \tfrac{2}{5} \epsilon^3 \partial_t \left(e^{-\epsilon p^\epsilon} \left(\left(\Sigma + \tilde{\Sigma}\right) : \nabla_x u^\epsilon + \nabla_x \cdot \tilde{q}\right)\right) \,. \end{aligned} \tag{6.4}$$

By the u^ϵ-equation and θ^ϵ-equation in system (1.7), the above wave equation (6.4) has the form

$$\epsilon^2 \partial_t \left(\tfrac{3}{5}\partial_t p^\epsilon\right) + \nabla_x \cdot \left(e^{\theta^\epsilon} \nabla_x p^\epsilon\right) = \epsilon F(p^\epsilon, u^\epsilon, \theta^\epsilon)\,, \tag{6.5}$$

where F is a smooth function in its variables with $F(0) = 0$. By the boundedness assumption (6.1) on $(p^\epsilon, u^\epsilon, \theta^\epsilon)$, and the convergence assumption for θ^ϵ, we have

$$\epsilon F(p^\epsilon, u^\epsilon, \theta^\epsilon) \to 0 \quad \text{strongly in} \quad L^2(0, T; L^2(\mathbb{R}^3))\,,$$

and the coefficient e^{θ^ϵ} in $\nabla_x \cdot \left(e^{\theta^\epsilon} \nabla_x p^\epsilon\right)$ satisfies the requirement of b^ϵ in Theorem 6.2. Therefore, by Theorem 6.2, we have

$$p^\epsilon \to 0 \quad \text{strongly in} \quad L^2([0, T]; L^2_{loc}(\mathbb{R}^3))\,,$$

which also implies

$$p^\epsilon \to 0 \quad \text{strongly in} \quad L^2([0, T]; H^{s'}_{loc}(\mathbb{R}^3))\,,$$

for all $1 \le s' < s$ by interpolation.

The convergence of $\nabla_x \cdot \left(u^\epsilon - \frac{2}{5} e^{-\epsilon p^\epsilon} \kappa(\theta^\epsilon)\nabla_x \theta^\epsilon\right)$ is shown in the same way. Let

$$w^\epsilon \triangleq \nabla_x \cdot \left(u^\epsilon - \tfrac{2}{5} e^{-\epsilon p^\epsilon} \kappa(\theta^\epsilon)\nabla_x \theta^\epsilon\right)\,.$$

Then the wave equation for w^ϵ is

$$\epsilon^2 \partial_{tt} w^\epsilon + \nabla_x \cdot \left(e^{\theta^\epsilon} \nabla_x w^\epsilon\right) = \epsilon \tilde{F}(p^\epsilon, u^\epsilon, \theta^\epsilon)\,,$$

where $\tilde{F}$ is a smooth function in its variables with $\tilde{F}(0) = 0$. Therefore, the same argument guarantees that

$$\nabla_x \cdot \left(u^\epsilon - \tfrac{2}{5} e^{-\epsilon p^\epsilon} \kappa(\theta^\epsilon)\nabla_x \theta^\epsilon\right) \to 0 \quad \text{strongly in} \quad L^2([0, T]; H^{s'-1}_{loc}(\mathbb{R}^3))$$

for all $1 \le s' < s$. ☐

7. Passing to the limit. Combining the uniform bound given by Theorem 2.4 and the local strong convergence of the fast motion (6.3), we can prove the following convergence result.

THEOREM 7.1. *Using the same assumptions and notations in Theorem 6.1, the family* $\{(u^\epsilon, \theta^\epsilon) | \epsilon \in (0, 1]\}$ *converges weakly in* $L^\infty([0, T]; H^s(\mathbb{R}^3))$ *and strongly in* $L^2([0, T]; H^{s'}_{loc}(\mathbb{R}^3))$ *for all* $s' < s$ *to a limit* (u, θ) *satisfying the ghost effect system:*

$$\begin{aligned}
&\nabla_x \cdot \left(\tfrac{5}{2} u - \kappa(\theta)\nabla_x \theta\right) = 0\,,\\
&e^{-\theta}\left(\partial_t u + u \cdot \nabla_x u\right) + \nabla_x P^* = \nabla_x \cdot \Sigma + \nabla_x \cdot \tilde{\Sigma}\,,\\
&\tfrac{3}{2}(\partial_t + u \cdot \nabla_x)\theta + \nabla_x \cdot u = -\nabla_x \cdot q\,,\\
&(u, \theta)(x, 0) = (v^{\mathrm{in}}, \theta^{\mathrm{in}})(x)\,,
\end{aligned} \tag{7.1}$$

for some P^* *satisfying* $\nabla_x P^* \in C([0, T]; H^{s-2}(\mathbb{R}^3))$*, and* ϑ, Σ, $\tilde{\Sigma}$ *be defined as*

$$\begin{aligned}
&\Sigma = \mu(\theta)\left(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{3}(\nabla_x \cdot u) I\right)\,, \qquad q = -\kappa(\theta)\nabla_x \theta\,,\\
&\tilde{\Sigma} = \bar{\tau}_1(\theta)\left(\nabla_x^2 \theta - \tfrac{1}{3}\Delta_x \Theta I\right) + \bar{\tau}_2(\theta)\left(\nabla_x \theta \otimes \nabla_x \theta - \tfrac{1}{3}|\nabla_x \theta|^2 I\right)\,.
\end{aligned}$$

where $\bar\tau_1, \bar\tau_2$ are the transport coefficients such that $\bar\tau_1(\theta) = \tau_1(0,\theta)$ and $\bar\tau_2(\theta) = \tau_2(0,\theta)$ where τ_1, τ_2 are defined in (1.9). The initial data $(v^{\rm in}, \theta^{\rm in})$ satisfies that

$$\Pi(e^{-\theta^{\rm in}} v^{\rm in}) = u^{\rm in}\,, \qquad \nabla_x\cdot(v^{\rm in} - \tfrac{2}{5}\kappa(\theta^{\rm in})\nabla_x\theta^{\rm in}) = 0\,,$$

where Π is the projection onto the divergence free of $e^{-\theta^{\rm in}} v^{\rm in}$ and $u^{\rm in}, \theta^{\rm in}$ are defined in (2.4).

Proof. [Sketch of proof] First the uniform bound of the norm defined by (4.33) gives that

$$\sup_{[0,T]} (\|(p^\epsilon, u^\epsilon)\|_{H^s} + \|\theta^\epsilon\|_{H^{s+1}}) < \infty\,.$$

for $s \geq 6$. The equation for θ^ϵ then implies that $\partial_t\theta^\epsilon$ is uniformly bounded in $C([0,T]; H^{s-3}(\mathbb{R}^3)$. Extracting a subsequence, we can assume that for all $s' < s$,

$$\theta^\epsilon \to \theta \quad \text{strongly in} \qquad C^0([0,T]; H^{s'+1}_{loc}(\mathbb{R}^3))\,,$$

where the limit θ belongs to $C([0,T]; H^{s'+1}_{loc}(\mathbb{R}^3)) \cap L^\infty(0,T; H^{s+1}(\mathbb{R}^3))$. Meanwhile, one can check that the decay conditions (6.2) are satisfied by θ by checking the equations for $|x|^{1+\gamma}(\theta - \underline{\theta})$ and $\nabla_x\left(|x|^{1+\gamma}(\theta-\underline{\theta})\right)$. Therefore, by Theorem 6.1, we have the local strong convergence of p^ϵ and $\nabla_x\cdot\left(u^\epsilon - \frac{2}{5}e^{-\epsilon p^\epsilon}\kappa(\theta^\epsilon)\nabla_x\theta^\epsilon\right)$ as in (6.3).

By the uniform boundedness of (p^ϵ, u^ϵ) in $C([0,T]; H^s(\mathbb{R}^3))$, we can assume that

$$(p^\epsilon, u^\epsilon) \to (p, u) \quad \textit{weak}* \ \textit{in} \quad L^\infty(0,T; H^s(\mathbb{R}^3))\,, \tag{7.2}$$

By the limits

$$\begin{aligned} \theta^\epsilon &\to \theta && \text{in} \quad C^0([0,T]; H^{s'+1}_{loc}(\mathbb{R}^3))\,,\\ p^\epsilon &\to 0 && \text{in} \quad L^2([0,T]; H^{s'}_{loc}(\mathbb{R}^3))\,,\\ \nabla_x\cdot\left(u^\epsilon - \tfrac{2}{5}e^{-\epsilon p^\epsilon}\kappa(\theta^\epsilon)\nabla_x\theta^\epsilon\right) &\to 0 && \text{in} \quad L^2([0,T]; H^{s'-1}_{loc}(\mathbb{R}^3))\,, \end{aligned} \tag{7.3}$$

we obtain

$$\nabla_x\cdot u^\epsilon \to \nabla_x\cdot u \qquad \text{in} \quad L^2([0,T]; H^{s'-1}_{loc}(\mathbb{R}^3))\,, \tag{7.4}$$

Thus by (7.3) the limit (u,θ) satisfies the first equation in (7.1). By the equation for $\mathrm{curl}\,(\mathrm{e}^{-\theta^\epsilon}\mathrm{u}^\epsilon)$, there exists a subsequence of $\mathrm{curl}\,(e^{-\theta^\epsilon}u^\epsilon)$ such that for all $s' < s$,

$$\mathrm{curl}\,(e^{-\theta^\epsilon}u^\epsilon) \to \mathrm{curl}\,(e^{-\theta^\epsilon}u) \quad \text{strongly in} \quad C([0,T]; H^{s'-1}_{loc}(\mathbb{R}^3))\,, \tag{7.5}$$

Combining (7.4) and (7.5), we have up to a subsequence,

$$u^\epsilon \to u \quad \text{strongly in} \quad L^2([0,T]; H^{s'}_{loc}(\mathbb{R}^3))\,,$$

for all $s' < s$. The third equations in (7.1) is shown by direct applications of the uniform bounds of (p^ϵ, u^ϵ) in $L^\infty(0,T; H^s(\mathbb{R}^3))$, local strong convergence of (p^ϵ, u^ϵ) in $L^2([0,T]; H^{s'}_{loc}(\mathbb{R}^3)$ and strong convergence of θ^ϵ in $C^0([0,T]; H^{s'+1}_{loc}(\mathbb{R}^3))$. For the second equation in (7.1), first we have the limiting projected equation using various convergence of $(p^\epsilon, u^\epsilon, \theta^\epsilon)$

$$\Pi\left(\partial_t(e^{-\theta}u) + \nabla_x\cdot(e^{-\theta}u\otimes u)\right) = \Pi\left(\nabla_x\cdot\Sigma + \nabla_x\cdot\tilde\Sigma\right)\,, \tag{7.6}$$

where Π is the projection operator onto the divergence free part of $e^{-\theta}u$. Moreover, by the first and third equations in (7.1), we have $\partial_t u \in H^{s-2}(\mathbb{R}^3)$ for each $t \in [0,T]$. Therefore there exists $P*$ such that the second equation in (7.1) holds. Furthermore, by Theorem 3.2, the solution to the ghost effect system (7.1) is unique. Therefore, the above convergence in fact holds for the full sequence of $(p^\epsilon, u^\epsilon, \theta^\epsilon)$, which thereby completes the proof. □

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