Challenging basic assumptions: mathematical experiences for pre-service teachers

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This article is a contribution to an ongoing discussion on what mathematical experiences are essential in pre-service teachers' education. Examples of activities and investigations for pre-service teachers are presented. Most of the activities are targeted towards pre-service secondary school teachers, however, some are appropriate for elementary school teachers as well. The underlying theme in these activities is experiences with non-conventional mathematical objects, such as numerals and coordinate systems. A theoretical perspective addresses the issue of constructing richer schemas for mathematical concepts by introducing and overcoming disequilibrium. Potential benefits of the suggested mathematical experiences for personal and professional growth of teachers are discussed.

1. Prologue

How would you react to the following statements?

- 1(a) A graph of a function $y = x$ is a parabola.
- 1(b) A number is divisible by 5 if and only if the sum of its digits is divisible by 5.
- 1(c) $a + (b \cdot c) = (a + b) \cdot (a + c)$.

When presented to a class of pre-service secondary mathematics teachers, without an explicit hint given in the name of this paper, there seemed to be an agreement that all the three statements 'did not make sense' or simply were false. There was a tendency to consider the elements that did not fit as misprints or typographical errors and therefore to make some corrections, like taking x to the second power in (a), changing 5 to 3 in (b), and switching addition and multiplication symbols in (c).

Now consider another set of statements.

- 2(a) A graph of a function $y = kx$ is a straight line.
- 2(b) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.
- 2(c) The sum of the interior angles of a triangle is always 180 degrees.

At first glance the reaction to each is 'Yes, of course. (Every 9th grader has already learned that!)'. At the second glance we may recognize some hidden assumption in our immediate response. The statement 2(a) is true only if we take it for granted

that the function is represented with Cartesian coordinates. The statement 2(b) is true only if we take it for granted that the number is represented in the base ten number system. The statement 2(c) is true only if we take it for granted that a triangle lies on the Euclidean plane.

But WHAT IF NOT? What if we change these conventions? Following Brown and Walter [1], an attempt is made to apply the 'what if not' strategy to some of the fundamental assumptions of school mathematics. In the following article it is argued that such a change creates valuable engagement for mathematics teachers at all levels, enriching their mathematical experience as well as their pedagogical encountering.

2. Introductory remarks

It is widely agreed that a profound mathematical background is not only a benefit but a necessity for a mathematics teacher at a secondary school. However, it is debatable what constitutes such a background. Many institutions require a particular list of mathematics courses or a particular number of courses at a specified level as a pre-requisite for teaching certification in mathematics. While it is still debatable what courses should be taken, and how many of them, to provide a sufficient mathematical background for a teacher, many students are left wondering about how a course in differential equations or advanced numerical analysis helps them in becoming a (better) teacher.

What mathematics is essential for a teacher? Some researchers argue in favour of specific courses and specific strategies for implementing these courses. Abstract Algebra is one such, which teaches students to appreciate form and reaches an abstraction level far beyond all its pre-requisites [2, 3]. A recommendation of a special MAA committee for undergraduate education argues in favour of a mathematical modelling course for teachers [4]. In fact, by deciding which mathematics courses are core for mathematics teacher certification, different institutions and their academic curriculum designers express their opinion regarding what is seen as an essential background for a mathematics teacher. Courses in Non-Euclidean Geometry, History of Mathematics and Number Theory are among the frequent choices.

This article argues in favour of a specific set of investigations, tasks or engagements, that could constitute a separate course but could also be incorporated into any methods or problem-solving course. A common theme in these investigations is experience with non-conventional objects, such as numerals, operations, coordinate or axiom systems. 'Conventional' here means objects of school mathematics. It is believed that the activities suggested below are essential to any teacher education programme.

3. Theoretical considerations

Learning mathematics can be seen as constructing mathematical concepts in one's mind. This is the basic assumption of constructivist epistemology, prevalent in mathematics education research and adopted in this article.

According to Skemp, 'to understand something means to assimilate it into an appropriate schema' [5, p. 46]. A question to entertain with respect to mathematics teacher education is how can one understand better what has been already understood, that is, assimilated. Skemp's claim is extended by suggesting that to

understand something better means to assimilate it in a richer or more abstract schema. It is suggested that when mathematical concepts become in one's mind particular examples of more general mathematical concepts, a richer schema is constructed. This happens, for example, when familiar integer numbers become an example of a commutative additive group, or when a familiar square becomes a particular example of a parallelogram. What stipulates a construction of a richer schema? A central tenet of Piaget's theory is that an individual, dis-equilibrated by a perceived problem situation in a particular context, will attempt to re-equilibrate by assimilating the situation to existing schemas or, if necessary, reconstruct particular schemas enabling the individual to accommodate the situation. It is argued that activities presented in this paper dis-equilibrate learners and therefore create a need for schema reconstruction.

In what follows four examples of topics for activities are presented and their contribution to teacher education in general and to mental construction of richer schemas in particular, is discussed.

4. Example 1: Non-conventional numeration systems

Those who learned to count, add and multiply in the 20th century, and even the grandparents of their grandparents, tend to take our Hindu-Arabic numeration system for granted. However, a brief look at History reveals that even though there is evidence indicating that the idea of positional notation with zero was available to mathematicians of India as early as the third century AD, it was not until the 13th century AD that this system became widely accepted in Europe [6]. This means that what is obvious today for a second grader was beyond the reach of the great mathematicians of Alexandria and Athens. Furthermore, the common notation for decimal fractions came into common use only in the 16th century. The author believes that the Hindu-Arabic number system is one of the greatest inventions of humanity. To enhance its understanding and appreciation, consideration of other-than-ten base numeration systems is invited.

Simple counting in base five could be the first challenge. But if this is not ambitious enough, maybe counting backwards by threes is. Activities of performing arithmetic operations, mostly addition and subtraction, in different bases, as well as converting from other bases to base ten and vice versa, get some attention in mathematics courses for pre-service elementary school teachers. Here we take the notion of other than decimal representation a few steps further.

The idea of numbers represented in other-than-ten bases can be extended to fractional numbers. What does 12.34_{five} mean? In Zazkis and Whitkanack [7] fractions represented in different bases were introduced and called 'non-decimals'. Using expanded notation and noting the place value of different digits 12.34_{five} can be converted to base ten as follows: $12.34_{\text{five}} = 1 \times 5 + 2 \times 1 + 3 \times 1/5 + 4 \times 1/25 = 7.76_{\text{ten}}$.

Conversion in the opposite direction is an exciting and challenging activity for pre-service secondary teachers. They can be motivated by a question like how would you write $1/4$ in base 7 or $1/7$ in base 4 and left to figure out the solution. This task brings together a variety of mathematical topics, like geometric progressions, the division algorithm and Fermat's little theorem (see [7] for details). If the first steps in students' investigations are not very fruitful, a variety of activities can

precede and support the above investigation. For example, among the initial explorations with non-decimals students can be directed to look for patterns:

$$\frac{1}{2} = 0.\overline{1}_{\text{three}} \quad \frac{1}{3} = 0.\overline{1}_{\text{four}} \quad \frac{1}{4} = 0.\overline{1}_{\text{five}}$$

and in general, $1/(b-1) = 0.\overline{1}_{\text{base } b}$.

Another fascinating pattern occurs when expressing $\frac{1}{2}$ in different bases:

$$\frac{1}{2} = 0.1_{\text{two}} \quad \frac{1}{2} = 0.\overline{1}_{\text{three}}$$

$$\frac{1}{2} = 0.2_{\text{four}} \quad \frac{1}{2} = 0.\overline{2}_{\text{five}}$$

$$\frac{1}{2} = 0.3_{\text{six}} \quad \frac{1}{2} = 0.\overline{3}_{\text{seven}}$$

The following questions may follow:

Can you predict whether $\frac{1}{2}$ in base eight will be a finite or repeating non-decimal? And how about $\frac{1}{3}$? In what bases are the representations of one-half finite and in what bases are they repeating? Can you predict, before converting, whether $\frac{1}{12}$ is a finite or repeating decimal, when represented in base 10. How about non-decimal place value notation in base 6? Can you draw a general conclusion, that is, given a fraction p/r and a base b , how one can decide whether the representation is finite or repeating?

It may be the case that pre-service teachers recall why $\frac{7}{15}$ is a repeating decimal and $\frac{7}{16}$ is a finite one. However, for those that do not remember, it is a valuable task to figure out and then extend this finding to different bases.

Another investigation invites a search for divisibility patterns in different bases. Many pre-service mathematics teachers remember some popular divisibility rules, like those for divisibility by 2, 3, 4, 5, 9 and 10, but may have difficulty proving them or explaining why they ‘work’. Even when a proof for divisibility by 3, for example, is presented, some treat it as a magic trick. Figuring out and proving divisibility rules for different bases may help the understanding of not only how divisibility rules work but also why they do. This activity can be as open as one wishes it to be. Students can be left on their own with a task to find as many divisibility rules as they can for different bases and try to prove what they find. Alternatively, a guided progression, such as the following, can be the instructor’s choice.

List familiar divisibility rules. Which ones are true for base 6? Explain why. Which ones are true for base 5? Explain why. What is your prediction for base 7? Check for other bases. Are there other rules that are true for base 6? other bases? There is something here for everyone. Some students will focus only on the last digit(s) of a number, while others could discover a general rule for any number and any base.

And why is this important? The reader is invited to bear with other examples or to skip to the section that discusses the benefits of this approach. Here it is mentioned only that the importance of ‘this’ depends on your interpretation of what ‘this’ is. For those who wish to see mathematics as ‘applicable’ it is definitely irrelevant to decide whether a seven digit number represented in base six is divisible by five. However, the learning that takes place and the understanding of the number system, its patterns and structure, that can be acquired when dealing with the ‘irrelevant’ task above, is, the author believes, substantial.

Additional activities with different bases may invite students to re-examine their knowledge of introductory number theory. For example, all the following statements are true in base ten.

If a is divisible by c , and b is divisible by c , then $(a + b)$ is divisible by c .

Every number that is divisible by 12 is divisible by 3.

Numbers that have a last digit of 0 are even.

81, 100, 121, 144, 169, 196 and 225 are perfect squares.

Are they true in base six? base five? For what bases other-than-ten are the preceding statements true? Can you rephrase the statements to make them true?

We often identify an object with its representation [8]. Activities like the above help in separating the two by distinguishing properties of numbers as conceptual entities from properties of their specific representation. Programming seems to be a good way of constructing or reconstructing one's understanding of non-conventional number systems. Almost all the activities discussed above can be extended to programming tasks. Several examples are: write a program that 'counts' in a given base, that is, receives the base as an input and returns a sequence of the first 50 numerals represented in the given base; write a program that 'converts' numbers from base 10, that is, receives as an input a number x represented in base ten and a base b and returns representation of the number x in base b ; write a program that adds or multiplies in a different base. 'Teaching' a computer how to count, convert or add may present a challenge even for those who are fluent with these operations.

The author supports introduction of non-place-value number systems, such as Roman or Hebrew, to pre-service teachers, as well as asking them to search in the literature for alternative systems and compare them. While trying to add Hebrew or Roman numerals helps appreciate the place value invention, working with non-decimal place value systems helps understand the conventional one in a more profound fashion.

5. Example 2: Non-conventional operations

Statement 1(c) $a + (b \cdot c) = (a + b) \cdot (a + c)$ above is true if ' $\cdot$ ' and ' $+$ ' are taken as Boolean operations 'and' and 'or'. A concrete visual introduction to these operations can be presented as electric circuits with series or parallel connections, represented by ' $\cdot$ ' and ' $+$ ' respectively. House [9] suggested a variety of activities in which a correspondence between circuit diagrams and mathematical expression is to be established and equivalence of expressions is to be checked by a construction of truth tables. An example is presented in figure 1.

In fact, Boolean expressions that correspond to true statements with real numbers were much more easily verified and accepted by a class of pre-service secondary school teachers, than expressions that generate false statements with real numbers. For example, the distributive property in 1(c) was accepted with less ease than its counterpart $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$. Even when both columns of the truth table were filled correctly with identical values (see figure 1), students had a tendency to double-check and not to trust their results over their intuition.

If a similar activity is to be implemented with high school students the adoption of a different notation is recommended, that is, avoiding the use of conventional multiplication and addition symbols in order to avoid confusion. However, for pre-service teachers a benefit is seen in using 'confusing' symbolism.

a	b	c	$a + (b \cdot c)$	$(a + b) \cdot (a + c)$
1	1	1	1	1
1	1	0	1	1
1	0	1	1	1
1	0	0	1	1
0	1	1	1	1
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0

Figure 1. An example of assigning expressions to electric circuits and checking their equivalence.

This helps one reconsider automatic responses, de-automate the operation and pay attention to what it really means.

In considering arithmetic operations, it was noted by Hadar and Hadass [10] that conventional arithmetic operations to which students are exposed in school are either both commutative and associative (like addition and multiplication of real numbers) or neither commutative nor associative (like division and subtraction of real numbers). This combination was seen as rather unfortunate and a variety of alternative operations preserving one of these properties only was suggested. For example, the operation $*$ that assigns to any ordered pair its second member, $a * b = b$, is associative but not commutative. The operation $\bullet$ that assigns to any ordered pair of real numbers their geometric mean, $a \bullet b = \sqrt{ab}$, is commutative, but not associative. The authors encourage students to create their own examples of operations and check what properties hold for their inventions. Further research has shown that a substantial number of pre-service teachers perceive the properties of associativity and commutativity as logically dependent [11]. This finding was explained that despite intensive instruction in mathematics, these pre-service teachers were still influenced by initial prototype examples of basic arithmetic

operations. However, simple and short intervention proved fruitful to increase pre-service teachers' awareness of mutual independence of these properties. It has been suggested that exposing students to a variety of examples of non-conventional operations not only helps overcome their initial misconception regarding the associative and commutative properties, but also demonstrates the importance of counterexamples and encourages teachers to analyse the sources of their own misconceptions. The author suggests further, that the main importance of the exposure to a variety of operations is in generalizing the idea of a mathematical operation as a function on an ordered pair and therefore in reducing dependence on particular examples. This awareness is essential for a mathematics teacher.

6. Example 3: Non-conventional coordinate systems

The Cartesian coordinate system on a plane is defined by a choice of two perpendicular lines, of a positive direction on each line and of a unit which is equal on both lines. This coordinate system induces one-to-one correspondence between ordered pairs of real numbers and points on the plane. However, the Cartesian system of coordinates is only one particular example of affine coordinates, a coordinate system defined by any three non-collinear points (O , I , J). Lines OI and OJ establish two axes intersecting at the origin O , directed segments OI and OJ determine a positive direction and a unit on each of the axes [12]. Coordinates of a point on a plane are found by parallel projection, that is, drawing parallel lines to the axes through this point, and noting points of intersection of these lines with the axes (figure 2). Among the first tasks when introducing an affine coordinate

Figure 2. Assigning affine coordinates.

system can be a proof that every point on a plane has a unique representation as an ordered pair of numbers and vice versa.

After becoming familiar with plotting points and assigning coordinates, we reflect on graphs of polynomials of the first degree, or so called 'linear' equations. This may be a good moment to reconsider the knowledge of linear equations plotted with the Cartesian coordinates and decide what holds for any Affine coordinate system. What is defined by parameters m and b in $y = mx + b$? Do two lines have a unique point of intersection? Does a graph of $y = x$ form a 45 degrees angle with the positive direction of the X-axis? If not, how can this angle be determined? What predicts how steep the slope is?

Additional discussion can look at affine coordinates as a transformation of the Cartesian plane. Following Klein's Erlanger Program, we take a view that what defines geometry is invariance under a group of transformations. What are the invariants here? Do straight lines remain straight lines? Do triangles remain triangles? Squares? Circles? Conic sections? If yes—why? If not, what do they become? Investigation of these issues followed by a class discussion has the potential to be fruitful and enlightening.

Our only constraint on the choice of affine axes was that they must intersect, to define the origin of (0,0). However, a pair of parallel lines can also be seen as a system of coordinates [13]. A graph of a function $y = f(x)$ is attained by drawing lines connecting x on the X-axis with $f(x)$ on the Y-axis. Figure 3 sketches graphs of $y = x$, $y = 2x$, and $y = 3x$ in the system of parallel axes. A graph of $y = x$ is a set of parallel lines, each of the graphs for $y = 2x$ and $y = 3x$ is a burst of lines intersecting at one point, called the focus. Of course the diagram can show only several out of the infinity of lines in each graph of a linear function.

Observation of these diagrams invites several questions. Do all graphs of linear functions, where $m \neq 1$, have a focus? The answer to this question is positive and a proof is based on consideration of similar triangles and proportional segments.

High school students often struggle with a task of assigning equations to graphs in the Cartesian coordinate system, when specific coordinates are not marked on the axes (see figure 4). A similar challenge can be presented to pre-service mathematics teachers when working with a system of parallel axes (figure 5).

However, in order to face these challenges, there are a few questions that must be addressed. Does every point on a plane determine a focus of some linear function or linear equation? How can this equation be found? Is it unique? To be prepared for this investigation it is beneficial to engage first in a task of finding what parameters m and b in $y = mx + b$ determine. More specific questions to consider are: Where should one expect to find the focus of the graph if m (or b) is positive, negative, between 0 and 1, between 0 and -1? What happens to the focus when m (or b) increases or decreases? For what values of m and b is the focus to the left of both axes? to the right of both axes? between the axes? When observing the examples in figure 3 and noting that the focus is to the left of the axes, the most common conjecture is that for the negative values of m the focus will be to the right of the axes. However, further investigations refute this conjecture. For what values of m and b is the focus below, above or exactly on the 'zero' line, which is the line connecting 0 on one axis to the 0 on the other axis. Can the focus be on one of the axes? What can be said about the equation then? If three foci are collinear, what can be said about the corresponding equations?

(a) $y=x$

(b) $y=2x$

Figure 3. Examples of graphs in a system of parallel axes.

Figure 4. Exercise in Cartesian coordinates: what functions could these graphs represent?

Figure 5. Exercise in parallel axes coordinate system: what functions could graphs with these foci represent?

There are many other systems of coordinates that can be introduced. Friedlander and Dreyfus [14] list and name several: bidirectrix, bipolar, bifocal, polar, and focus-directrix coordinate systems. In each of these systems a graph of a function or equation is considered in terms of loci. In a bidirectrix system, for example, the graph of $y = x$ is the locus of points equidistant from two perpendicular lines, say p and q . In the same system the graph of $y = 2x$ is the locus of points whose distance from p is twice the distance from q . In bifocal and bipolar systems the loci are considered in terms of distances from two fixed points or angles from two fixed rays respectively. Among the systems mentioned above the author's favorite is the focus-directrix system. In this system a locus is defined in terms of distances from a fixed point and a fixed line. An investigation could start with a simple task of graphing $y = x$, $y = 2x$, $y = 3x$, $y = (\frac{1}{2})x$ and $y = (\frac{1}{3})x$ (figure 6).

An observation that some of these graphs look like ellipses, others look like hyperbolas and only one looks like a parabola invites a discussion of conic sections and their definitions in terms of loci. An 'AHA! effect' is a common reaction when noticing that what is defined in the focus-directrix coordinate system by $y = x$ is a set of points that are equidistant from a chosen line (directrix) and a chosen point (focus), which is, surprisingly or not, the definition of a parabola. How are different parabolas obtained?—can be a question to motivate further investigation.

(a) $y = x$

(b) $y = \frac{1}{2}x$

Figure 6. Examples of graphs in focus-directrix coordinate system.

Another interesting issue to be explored is the relationship between the distance from the focus to the directrix and the shape of the graph of the function.

The initial graphing of the functions in the focus-directrix coordinate system can be made on a special grid, as shown in figure 6, in which concentric circles intersect parallel lines. For a further investigation of possible variations and relationships, a geometry software with dynamic features can be helpful. Geometer's Sketchpad, Dynamic Geometry or Cabri are among software packages that have appropriate tools for this investigation. They allow the user to define a locus of points under given constraints and then to investigate a change in this locus as the constraints change.

Graphing functions is one of the central topics in school mathematics at any level. Activities with alternative coordinate systems shed a new light on the topic. They assist in constructing a richer understanding of conventional mathematics as well as creating bridges to other mathematical topics, like transformations, loci, and more.

7. Example 4: Un-conventional axiomatic geometry

Usually we do not take an axiomatic approach to school geometry. When teaching Euclidean geometry attempts are made to have students experience geometric relationships and theorems with specific examples and calculations, and based only on these experiences draw general conclusions and attempt to prove them. Proofs in geometry are known to present a challenge to students and their

teachers and this difficulty led at times to teachers questioning the whole idea of teaching the subject of Euclidean geometry or avoiding teaching proofs altogether [15]. One of the difficulties for students in learning to make a geometric proof is in getting accustomed to the idea of mathematical deduction, that is, in learning to distinguish what information is to be used, what is 'given', on what an argument can be based, and what seemingly obvious information should not be trusted. A similar challenge can be presented to pre-service teachers when starting their inquiry into geometry with the following definitions and axioms [12]:

Definition 1: If m and n are lines in a plane, then m and n are called parallel if and only if $m = n$ or $m \cap n = \emptyset$.

Definition 2: Three or more points are collinear if and only if there is a line containing all of the points. If there is no such line, the points are noncollinear.

Axiom 1

A plane is a set of points, and there are at least two lines in each plane.

A line is a set of points, and every line contains at least two points.

Axiom 2

For every two points in a plane, there is one and only one line containing them.

Axiom 3

If m is a line in plane E , and A is a point in E , there is exactly one line containing point A which is parallel to line m .

What is the minimal number of points on the plane to satisfy these axioms? It is easy to show that this number is four. A system of three points, say A , B , C , does not satisfy the third axiom because there exists no line parallel to a chosen line AB for example, containing a point C . A system of five points, say A , B , C , D and E , does not satisfy the third axiom because there exists more than one parallel line to any given line containing a given point. For example, given line AB and point C , both CD and CE are distinct lines containing C and parallel to AB .

This playful experiment may give an insight into the development of non-Euclidean geometries that arose when alternatives to the Euclidean fifth postulate, that is equivalent to the above Axiom 3, were suggested. There are two possible alternatives to the statement of existence of a unique parallel line to a given line through a given point: (1) existence of no parallel lines, that is, all lines intersect, and (2) existence of more than one parallel line. These alternatives are fundamental assumptions in Riemannian (elliptic) and Lobachevskian (hyperbolic) geometries respectively.

It is quite a challenge, even for individuals with a profound mathematical background, to ignore the diagram and to claim that in 4-point geometry AC and BD are parallel and in 5-point geometry AB is parallel to both CD and CE (figure 7). Euclidean intuition is an obstacle when considering the definition and claiming that AC and BD have no point in common, their intersection is an empty set and therefore they are parallel. Overcoming this obstacle is an important step in mathematics teacher education.

A variety of theorems can be proved based on these axioms. Several are listed below.

- (1) If m is a line in plane E , then there is a point on the plane which is not on line m .
- (2) In plane E , there are at least three noncollinear points.

Figure 7. Models for finite geometries.

- (3) Two distinct lines in a plane have at most one point in common.
- (4) If A is a point in a plane, then there is a line in the plane which does not contain A .
- (5) Every point in a plane lies on at least two lines.
- (6) In a plane there are at least three lines that do not all contain the same point.

Some proofs may not be as obvious as the statements suggest. Consider for example one possible proof for theorem (6). In this theorem we want to show the existence of three lines that do not all contain the same point. By theorem (2) there are at least three non-collinear points on a plane. Let us call them A , B , C . By theorem (5) there are at least two lines containing point A , let us call them m and n . By axiom 2 there is a line containing points B and C , let us call this line k . This line k does not contain point A , because we started with A , B and C as three noncollinear points. Therefore line k must be distinct from lines m and n . By axiom 2, m and n do not contain a common point other than A . Since k does not contain A , the three lines k , m and n are the lines we want.

More theorems are found in [12], and they present valuable exercises in deductive reasoning. Proving these theorems provides not only a mathematical challenge but also helps one realize one's dependence on conventional learned facts. The benefits of this experiment are in helping to restrict the arguments to what is given or proved previously, in not trusting the diagram or common sense, and in realizing how misleading a diagram can be. This exercise in pure deduction teaches how difficult a formal mathematical deduction can be for a novice learner of geometry.

8. Benefits

A frequently asked question is 'What is this good for?'. After all, we check a grocery bill using Hindu-Arabic numeration and build bridges using Euclidean geometry with Cartesian coordinates. In general, the author believes that any

intellectually motivating activity is good for human beings, and in particular a mathematical problem-solving activity is ‘good for’ pre-service teachers. However, the author argues in favour of ‘challenging basic assumptions’ topics (denoted CBA) for several additional reasons.

Modelling an activity-based approach

If we are to support an activity-based approach to school mathematics, it is necessary for teacher education programmes to provide pre-service teachers with rich and meaningful engagement with mathematical problem-solving activities. The CBA topics provide opportunities for modelling an activity-based approach with pre-service teachers. The activities are simple enough to start at any level of mathematical expertise, but on the other hand they can be developed as challenging problems and investigations for ‘mathematically sophisticated’ students.

Achieving a deeper understanding of the concepts—constructing richer schemas

CBA activities provide opportunities for pre-service teachers to reinforce their knowledge by making variations on parts of it. They force students to reconsider their automatic responses and pay attention to the conventional assumptions we make when considering a mathematical problem.

These activities also help to generalize knowledge and to see the ‘general in the particular’, that is, recognize a general feature in its particular appearance [16]. An example of such a generalization can be a conclusion that a number ending in zero or b , represented in base $2b$, is divisible by b . A familiar particular appearance of this claim is in the divisibility rule for 5.

Such a generalization can be interpreted through a theoretical lens of schema construction discussed earlier. Working with various bases a learner constructs a schema for ‘place value number representation’, in which the Hindu-Arabic number system is only one, though generic, example. Working with various operations a learner constructs a schema for ‘binary operations’ as functions of two variables, in which addition and subtraction are representative examples, and so on.

In addition, the above suggested activities establish (sometimes unexpected) connections among a variety of mathematical topics. Loci in investigation of graphs of linear equations [14] and Fermat’s little theorem in investigation of fractions represented in bases other than ten [7] are among these topics.

Experiencing mathematics as human endeavor

CBA activities help students see mathematics as a human creation and activity and to recognize human choices that created conventions in the domain of mathematical knowledge. At times the origin and the purpose of a convention are clear, such as the choice of ten as a base for the representation of numbers. However, a convention can also be a random choice, such as a positive direction of axes in Cartesian coordinates.

Gaining a perspective on students’ work

Experiencing CBA helps pre-service teachers to gain an appreciation of the work of students and students’ difficulties. Having teachers perform division successfully with base-5 blocks, but failing to do so, or at least facing some

difficulty in doing it without the blocks, is a wonderful demonstration of how difficult a transfer from a concrete situation to an abstract idea can be to a student and how important it is to build this bridge gradually and firmly. Teachers counting in base five and saying (or thinking) 42, 43, 44, 45—oh-no!, 50—oh-no—100 could be more sympathetic to a student counting twenty nine, twenty ten, twenty eleven. . . .

CBA activities can provide an insight into an individual's understanding that may not be apparent in a conventional setting. For example, pre-service elementary school teachers' understanding of fractional numbers in other-than-ten bases was investigated in detail in [17, 18] and [19]. The authors provided a detailed discussion on students' attempts to convert non-decimal fractions to decimals fractions and on how these attempts revealed students' understanding of numbers and number system. One of the frequent answers when asked to convert 12.34 from base five to base ten was 7.19. A closer look uncovered that this answer was obtained by considering the integer and the fractional parts of the number as two separate numbers, converting two numbers separately and then 'gluing' the two answers into a decimal by placing a point between them. A conjecture was made that this error strategy of separation of the fractional and integer parts of the number had its origin in the popular way of reading the decimal fractions as 'twelve-point-thirty-four', rather than 'twelve and thirty four hundredths'. In fact, a careless use of Dienes' blocks manipulatives (base ten blocks) may reinforce this misconception, because using these blocks the numbers 34 and .34 are traditionally represented with exactly the same physical objects. This of course, is not to recommend against the use of Dienes' Blocks, but to increase awareness of potential problems.

Putting on students shoes and performing standard investigations in an unconventional setting gives pre-service teachers a perspective on the investigational approach itself. Consider for example an investigation of the parameters m and b in graphing $y = mx + b$ with parallel axes. Encouraging pre-service teachers to reflect on their own ways to solve this problem increases their awareness of strategies for a fruitful investigation. This awareness could become beneficial for supporting students in their more standard investigations, such as investigation of the role of the parameters m and b in graphing $y = mx + b$ with Cartesian coordinates.

Recently the author worked with a class of pre-service secondary school teachers on graphing simple linear equations, like $y = x$, $y = 2x$, $y = (\frac{1}{2})x$ in a focus-directrix system. After the graphing was completed and the graphs presented on the overhead projector, the author asked a naive question: are these functions? There was strong disagreement among the students. About one half of the students claimed that the graphs failed the 'vertical line test' and therefore did not represent functions. This inappropriate generalization revealed that the 'test' was applied algorithmically, without understanding its meaning. Fortunately, after prolonged discussion, another opinion was accepted, claiming that the linear equations in question were indeed functions and non-standard graphical representation did not influence the nature of functional correspondence.

As experienced teachers of geometry we know that lines looking parallel on a diagram cannot be treated as such until this relationship is proved. However, didn't we all once fall in a pit claiming that lines AC and BD on figure 7 are *not* parallel? Trusting our eyes and ignoring the definition we could easily point to the

point of intersection. It is suggested that experience with a non-conventional system of axioms increases teachers' understanding of students' difficulties in making deductive arguments and in making an appropriate use of diagrams.

9. An issue of applicability and usefulness

While some students are happy with the intellectual challenge they are facing when engaging in CBA activities, others argue the usefulness, the utility and the applicability of the topics explored. 'Are these ever used?' is a troubling question. While it may indicate a curious mind seeking information, it usually has some negative connotations.

Hal Saunders in 'When are we ever gonna have to use this?' [20] suggests a variety of examples for high-school students, showing mathematics applied in different situations by different professions. However, it is surprising when this question is asked not by teenage students, not by student-carpenters or student-bankers, but by student-teachers of mathematics. The author's instantaneous reaction is to admit that she does not care about applicability, and that CBA topics will not help in balancing a cheque book or calculating taxes. The author believes that relevant mathematical activities are those that engage students in meaningful problem solving and help them to see and appreciate the internal beauty and power of mathematics, beyond its applicability to 'real life'. However, knowing the reluctance of student-teachers to accept such an answer, several others have proved useful.

A boxer spends hours every day jumping rope. Is he ever going to use this? Most likely, he will not jump rope in the ring. However, the purpose of this activity is to develop a skill, which is essential for boxing. A common gymnastics exercise is to touch the floor just in front of your toes with your finger tips without bending your knees. The purpose here is probably not to touch the floor, but to stretch a certain leg muscle. Similarly, when we perform division in different bases, or construct graphs in different coordinate systems, the goal is not in finding the quotient or visual representation, but in developing thinking, investigation and problem-solving skills, fundamental for the teaching of mathematics.

What is non-conventional in school mathematics could be a conventional approach elsewhere. Computers definitely utilize the binary and octal (base eight) number systems. Polar coordinates are used in physics and astronomy. Boolean algebra describes the behaviour of an electric circuit with given switches. From a historical perspective, there are examples of pure theoretical mathematical developments that have been utilized years later. The development of symbolic logic and its implementation in computing science is one such example. Therefore just because today no engineering or scientific tool uses bifocal coordinates, or base 7 number representation, this does not mean it could not happen tomorrow.

Aside from being applicable to a variety of situations, non-conventional representations provide a powerful problem-solving tool within mathematics. For example, binary number representation is the basic tool in suggesting a winning strategy in the NIM game [21].

Affine coordinates can be helpful in generating elegant proofs for geometric theorems. Consider, for example, a triangle and its three medians. A theorem claims that all three medians intersect at one point and the ratio of the segment connecting a triangle vertex to the point of intersection to the segment on the same

Figure 8. Three medians in a triangle.

median connecting the point of intersection to the opposite side is 2:1. This theorem is within the immediate repertoire of mathematics teachers, however, its proof is non-trivial and filled with algebraic manipulations, unless we set the given triangle on the affine coordinate system (figure 8).

It is easy to calculate the line equation of each median, because at least two points on each line are given. Looking at the point of intersection by solving systems of equations with two unknowns that stand for coordinates, we find that each two lines intersect at the point $(\frac{2}{3}, \frac{2}{3})$. Considering the difference in each coordinate at the points K, O, and A, we find that $AO:OK = 2:1$. Similar calculations reveal the same ratio for the segments on the two other medians. In order to appreciate the elegance and beauty of this proof, one could try to prove the theorem without the tool of affine coordinates.

In summary the author believes that one's conclusion of whether a tool is useful or not, probably depends on one's perception of usefulness. Can we change this perception? What would be necessary for pre-service teachers to see their own mathematical experiences as relevant to their teaching of mathematics? It is suggested that pedagogical reflection on mathematical activities is one necessary but definitely not sufficient condition.

10. Conclusion

These activities and many similar ones are utilized by a range of mathematicians and mathematics educators on different occasions. The author's call here is to bring them under a common umbrella, to recognize the similarity in emotional and intellectual dissonance they present and the possible benefits for teachers in facing this dissonance and finding equilibrium. The author believes that investigations like the above, that invite students to reconsider their basic mathematical assumptions and analyse their automated responses, represent more

than a curiosity or enrichment puzzle, they constitute an essential tool for the development of critical thinking in mathematics teacher education.

Learning a foreign language helps recognize structure, consistency, rules and exceptions in your mother tongue. Similarly, considering non-conventional structures helps in gaining a better understanding and appreciation of those that were chosen as 'conventional' and learned as a mother tongue. Working with non-conventional structures helps students in constructing richer and more abstract schemas, in which new knowledge will be assimilated.

It is commonly declared that very often pre-service teachers do not have a good understanding of the mathematics they will be teaching. Re-teach them school mathematics? It could be perceived as boring and even insulting. Teach them more higher mathematics? It could be perceived as irrelevant to their teaching career and the transition is not always made. Provide them with experiences and challenges that re-examine and enrich their understanding of mathematics and its pedagogy? This article has presented one possible way to accomplish a step in this direction.

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