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IRRATIONAL NUMBERS: THE GAP BETWEEN FORMAL AND INTUITIVE KNOWLEDGE

ABSTRACT. This report focuses on prospective secondary mathematics teachers' understanding of irrational numbers. Various dimensions of participants' knowledge regarding the relation between the two sets, rational and irrational, are examined. Three issues are addressed: richness and density of numbers, the fitting of rational and irrational numbers on the real number line, and operations amongst the elements of the two sets. The results indicate that there are inconsistencies between participants' intuitions and their formal and algorithmic knowledge. Explanations used by the vast majority of participants relied primarily on considering the infinite non-repeating decimal representations of irrationals, which provided a limited access to issues mentioned above.

KEY WORDS: prospective secondary teachers, irrational numbers, intuitive knowledge, dimensions of knowledge

1. Introduction

This report is part of an ongoing study on prospective secondary mathematics teachers' understanding of irrationality. The purpose of the study was to provide an account of participants' understandings and misunderstandings of irrational numbers, to interpret how the understanding of irrationality is acquired, and to explain how and why difficulties occur. Understanding of irrational numbers is essential for the extension and reconstruction of the concept of number from the system of rational numbers to the system of real numbers. Previously we focused our analysis on how irrational numbers can be (or cannot be) represented and how different representations influence participants' responses with respect to irrationality (Zazkis and Sirotic, 2004; Zazkis, 2005). In this article we consider participants' knowledge, intuitions and beliefs with respect to the relationship between the two number sets, rational and irrational. This report can be understood as a story of the ingenious ways in which participants strive to harmonize their intuitions with what they formally know to be true.

2. Background

Understanding irrationality of numbers is mentioned in literature in the research on proofs, limits and infinity (e.g. Tall, 2002). However, to the best

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of our knowledge, there are very few studies in the educational research literature that explicitly focus on the concept of irrational numbers. All these studies point to, at times surprising, deficiencies in understanding of irrational numbers by students and teachers alike. The main objective of the study by Fischbein, Jehiam, and Cohen (1995) was to survey the knowledge that high school students and preservice teachers possess with regard to irrational numbers. These researchers used a written response questionnaire administered to 62 students (grades 9 and 10) and 29 prospective teachers. Their study assumed, on historical and psychological grounds, that the concept of irrational numbers faced two major intuitive obstacles, one related to the incommensurability of irrational magnitudes and the other related to the nondenumerability of the set of real numbers. Contrary to expectations, the study found that these intuitive difficulties were not manifest in the participants' reactions. Instead, it was reported that subjects at all levels were not able to define correctly the concepts of rational, irrational, and real numbers. Many students could not even identify correctly various examples of numbers as being whole, rational, irrational, or real. The study concluded that the two intuitive obstacles mentioned above are not of primitive nature – they imply a certain intellectual maturity that the subjects of this study did not possess.

A study of Peled and Hershkovitz (1999), which involved 70 prospective teachers in their second or third year of college mathematics, focused on the difficulties that prevent student teachers from integrating various knowledge pieces of the concept into a flexible whole. Contrary to the Fischbein et al. study, these researchers found that student teachers knew the definitions and characteristics of irrational numbers but failed in tasks that required a flexible use of different representations. They identified the misconceptions related to the limit process as the main source of difficulty.

In their work on using history of mathematics to design pre-service and in-service teacher courses, Arcavi et al. (1987) report several findings that are of interest here as they relate to teachers' conceptions, and/or misconceptions regarding irrational numbers. This study was conducted on 84 in-service teachers who attended a summer teacher training program related to a national mathematics curriculum for junior high schools in Israel. One of the most striking discoveries of their study is that there is a widespread belief among teachers that irrationality relies upon decimal representations. Consistent with the Fischbein et al. study, Arcavi et al. found that many teachers had trouble recognizing numbers as being rational or irrational. Similar difficulty was reported by Tirosh et al. (1998) as a part of their study on prospective elementary teachers' conceptions of rational numbers.

Our research complements the above mentioned studies in an attempt to provide a more comprehensive account of understanding irrationality of numbers. Unlike Fischbein et al., we do not start with preconceived assumptions about intuitive obstacles. Rather, we present participants with tasks and, based on their responses, we draw conclusions about their knowledge. Our study brings to light a refined description of ideas and understandings held by teachers, and builds a foundation for an informed instructional intervention. Furthermore, our contribution to the existing body of research is in illuminating the inconsistencies between the different dimensions – formal, algorithmic and intuitive – of teachers' knowledge.

3. Dimensions of knowledge

For the purpose of our study, we adopt the conceptual framework suggested by Tirosh et al. (1998), in the study of teachers' understanding of rational numbers. The basic assumption of this framework is that learners' mathematical knowledge is embedded in a set of connections among algorithmic, intuitive and formal dimensions of knowledge.

We use the following explanations of dimensions of knowledge, as specified by Tirosh et al. (1998). The *algorithmic dimension* is procedural in nature – it consists of the knowledge of rules and prescriptions with regard to a certain mathematical domain and it involves a learner's capability to explain the successive steps involved in various standard operations. The *formal dimension* is represented by definitions of concepts and structures relevant to a specific content domain, as well as by theorems and their proofs; it involves a learner's capability to recall and implement definitions and theorems in a problem solving situation. The *intuitive dimension of knowledge* (also referred to as intuitive knowledge) is composed of a learner's intuitions, ideas and beliefs about mathematical entities, and it includes mental models used to represent number concepts and operations. It is characterized as the type of knowledge that we tend to accept directly and confidently – it is self-evident and psychologically resistant (Fischbein, 1987). To clarify further, we consider learners' beliefs to be the psychologically resistant components of the intuitive dimension of knowledge, while we consider intuitions to be the components that are self-evident and intrinsically necessary. That is, we draw the subtle distinction between intuitions and beliefs by their robustness in participants' views.

The three dimensions of knowledge are not discrete; they overlap considerably. Ideally, the three dimensions of knowledge should cooperate in any mathematical activity and their vitality depends upon the student's constructing consistent connections among algorithms, intuitions, and concepts.

According to Tirosh et al. (1998), many prospective teachers, especially those that did not have a major in mathematics, based their conceptions of numbers almost entirely on their experience with natural numbers. For instance, there was a widespread belief that "division always makes smaller". This belief is grounded in the partitive model of division applied to natural numbers, which is essentially the "sharing" metaphor, most often used in the earliest stages of learning about division. However, despite this belief, all the subjects who argued that "division always makes smaller", correctly computed at least some of the division problems that resulted in a quotient that was greater that the dividend.

It seems that people tend to adapt their formal knowledge and their algorithms to accommodate their beliefs, perhaps as a result of a natural tendency towards consistency. Inconsistencies then, might be the result of the counteraction of the deeply engrained procedures that emerge when the person is not watchful of his or her beliefs, but does things automatically instead. In this study we explore several inconsistencies in the participants' knowledge and discuss those in terms of different dimensions of knowledge.

4. METHODOLOGY

As part of a larger study on understanding irrationality, we investigated prospective secondary mathematics teachers' dimensions of knowledge regarding three distinct threads, as addressed by the following research questions:

1. What are the participants' ideas about the relative "sizes" of the two infinite sets, that is, about the relative abundance of irrational vs. rational numbers? What dimensions of knowledge are employed to support these ideas?
2. What are the participants' ideas about how the rational and irrational numbers fit together, in order, on the real number line? What dimensions of knowledge are employed to support these ideas?
3. What are the participants' ideas with respect to the effects of operations of addition and multiplication on the (ir)rationality of the results? What dimensions of knowledge are employed to support these ideas?
4. What is the interplay between the different dimensions of knowledge, in relation to 1, 2 and 3 above? Is there consistency or conflict?

4.1. *The tasks*

In this section we introduce the tasks that were presented to participants and explain our choices. First, we explored their formal and intuitive knowledge about the relative "sizes" of the two infinite sets. To examine this we designed the following item:

1. (a) *Which set do you think is "richer", rationals or irrationals (i.e. which do we have more of)?*
   (b) *Suppose you pick a number at random from [0,1] interval (on the real number line). What is the probability of getting a rational number?*

Mathematically, the correct answer to question 1(b) is 0. It was not expected that participants were familiar with this result or were able to come up with a formal proof.[1] It was also not expected that all participants would be familiar with the notion of denumerability of rationals and non-denumerability of irrationals in responding to 1(a). Our aim was to examine what intuitions participants had regarding the relative abundance or "richness" of rational versus irrational numbers. In the designed task we have chosen to use the word "richer", metaphorically, to avoid the use of words such as "size" or "larger", which are misleading in the discussion of infinite sets, and also to avoid the term "cardinality" that could have been unfamiliar to some participants.

Item 1(a) addresses the issue of relative abundance of the two sets directly. Item 1(b) addresses a related issue indirectly, presenting a hypothetical situation. That is to say, if one suggests that both sets are equally abundant, then the probability of picking an element that belongs to one of these sets should be 1/2. Similarly, if one claims that one of the sets is "richer", or "greater in size", then the probability of picking an element of this larger set should be relatively high. We were interested in examining the consistency of participants' responses to these two questions.

Second, we looked at participants' dimensions of knowledge about how the rational and irrational numbers fit together, related to the density of both sets. The following tasks were used:

2. (a) *It is always possible to find a rational number between any two irrational numbers. Determine True or False and explain your thinking.*
   (b) *It is always possible to find an irrational number between any two irrational numbers. Determine True or False and explain your thinking.*
   (c) *It is always possible to find an irrational number between any two rational numbers. Determine True or False and explain your thinking.*
   (d) *It is always possible to find a rational number between any two rational numbers. Determine True or False and explain your thinking.*

All these statements are true, which can be shown, formally, by constructing the numbers in question. In 2(a) the desired rational can be constructed by "truncating" the decimal expansion of the larger irrational at a certain point. Item 2(d) can be confirmed simply by considering the average of the two rational numbers. For 2(b) the averaging strategy needs adjustment; if the average of the two irrational numbers is rational, an irrational number satisfying the requirement is the average of one of the irrationals and the rational average of the two irrationals. In 2(c) the desired irrational between rational numbers can be found by, for example, adding a sufficiently small irrational number to the smaller rational number.

Thirdly, we investigated how participants respond to questions about the effects of operations between irrational numbers, using the following questions:

3. *(a) If you add two positive irrational numbers the result is always irrational. True or false? Explain your thinking.*
   (b) If you multiply two different irrational numbers the result is always irrational. True or false? Explain your thinking.

Both claims are false. However, the constraint of "positive" in 3(a) was chosen to avoid the trivial choice of counterexample using the additive inverse, where the sum is zero, and therefore rational. The constraint of "different" in 3(b) was chosen to avoid the trivial choice of squaring the (irrational) square root of some rational number.

Unlike the questions in Item 1, where some participants lacked formal knowledge related to different cardinalities and therefore had to rely mostly on their intuition, all the questions in Items 2 and 3 could have been addressed using the formal and algorithmic dimensions of knowledge. We investigated whether participants built upon these dimensions of knowledge when dealing with the presented tasks or relied mainly on their intuitive knowledge.

4.2. *Participants and data collection*

Participants in this study were 46 preservice secondary mathematics teachers (PSTs) in their final term of studies towards certification for teaching at the secondary school level. The mathematical background of the participants varied widely. Approximately one third of them (15 out of 46) were mathematics majors, while the rest held a major in science. The majority of science majors as well as some mathematics majors have not been exposed to formal set theory in their studies. Furthermore, some participants enrolled in teacher certification program immediately after completing their degrees, while others completed their degrees between 5 and 12 years before joining the program. As such, advanced mathematical knowledge acquired in their studies was not necessarily in their "active repertoire". However, we found that the number of mathematics courses taken by the participants and the date of their degree completion had minor influence on their responses and therefore this information is not included in our analysis.

The data consists of two sources: A written questionnaire and a clinical interview. The questionnaire, that included the 3 tasks discussed in Section 4.1, was administered to all the participants during a session of their secondary mathematics methods course. The time for completing the questionnaire was not limited.

Upon completion of the questionnaire, 16 volunteers from the group participated in a clinical interview. They represented various levels of performance on the written questionnaire. The interviews were semi-structured. Initially, participants were presented with questions that they addressed in the written questionnaire and asked to clarify further and to justify their responses. Based on the responses, the interviewer probed for "strength of belief" (Ginsburg, 1981, 1997) and presented an opportunity for participants to expand upon their responses or to change their initial responses. Establishing strength of belief helps understand whether a participant's response to a presented task is an arbitrary choice or whether it is based on persistent and robust knowledge, either formal or intuitive (Zazkis and Hazzan, 1998). In particular, we examined the participants' capability to produce adequate intuitive models for representing number concepts to accommodate the evidence that the existence of irrational numbers mandates. By "adequate" we refer to models that do not create inconsistencies with the other two knowledge dimensions. In the next section we present and analyze the results of our investigation.

5. Results and analysis

We examined the prospective secondary teachers' dimensions of knowledge regarding the relations between the two sets, rational and irrational, such as density, richness, and how numbers fit together, as well as the operations between members of these sets. In our analysis of the participants' intuitive dimension of knowledge of these notions, we detected various inconsistencies in relation to the other two dimensions of knowledge, algorithmic and formal. These inconsistencies are often revealing of misconceptions, cognitive or epistemological obstacles (Herscovics, 1989; Sierpinska, 1994), and other common difficulties; as

TABLE I
Quantification of results for Item 1(a) – what set is "richer", rationals or irrationals... ($n = 46$)

Response category	Number of participants	[%]
Irrationals	22	[47.8]
Neither	11	[23.9]
Rationals	10	[21.8]
Other (not specified or no answer)	3	[6.5]

TABLE II
Quantification of results for Item 1(b) – probability of picking a rational number from [0, 1] interval. ($n = 46$)

Response category	Number of participants	[%]
"Equal to 0"	2	[4.3]
"Close to 0"	9	[19.6]
"Close or equal to 50%"	10	[21.7]
"Close or equal to 100%"	8	[17.4]
"Undefined"	1	[2.2]
No answer	16	[43.8]

such, our goal here is to describe them and attempt to identify their sources.

5.1. *Richness and density*

First, we explore PSTs' ideas about the relative "sizes" of the two infinite sets. What justifications are used in addressing the questions about the abundance and density of rational versus irrational numbers? Items 1(a) and 1(b) pertain to the order of infinity of rational numbers versus irrational numbers (denumerable versus non-denumerable set). Tables I and II show the quantitative summary of written responses.

In what follows we present the arguments that participants provided with respect to the perceived abundance of irrational numbers on the written questionnaire and we exemplify these arguments with excerpts from the clinical interviews. Furthermore, we discuss whether the arguments provided in 1(a) are in accord with determinations in 1(b).

5.1.1. *"Irrationals are much richer than rationals"*

Although 22 out of 46 participants correctly identified the set of irrational numbers as "richer", only 3 of these participants presented arguments that referred to the set cardinality. An example of such an argument is in Olga's response:

- Irrationals are richer, because they are not countable while rationals are a countable set. Bijection exists between integers and rationals; bijection does not exist between integers and irrationals.

As mentioned earlier, not all the participants were exposed to formal set theory, and even those who were, may have studied it a long time ago. We were interested in participants' intuitive notions rather than in recall of information. In the excerpt below, we present Ted's response. We find it interesting because he arrives very close to the mathematically correct result without any reference to cardinality of infinite sets.

Interviewer: You say, there are way more irrational numbers, in comparison to rationals. Why do you say so?

Ted: You see, irrational are never-ending numbers, right, so like Pi, they know a million of digits and there is more and more, yea. And rationals, they may also have never ending lists, but there must be repeats, right? and with irrationals you have no repeats, right, I mean you do not get like 123, 123, 123 repeating over and over, or even no longer string repeating. So if we were just making numbers by putting digits, like decimal digits, at random, *what are the chances that the same random string will show over and over again? Very unlikely.* I would say even more unlikely than winning 6-49. But, you know, some people win, believe it or not. So numbers that you are gonna make are irrationals, right? So that's why I think there are more of those irrationals.

From Ted – who has never been formally exposed to the idea of "different" infinities – we learned that it is possible to know intuitively that there are many more irrationals without ever having seen Cantor's diagonalization proofs or knowing about the possibility of infinities of different order. Of course, this simple and sound reasoning is more likely to occur in those who see irrationals primarily as infinite non-repeating decimals. Ted, as many other participants, displayed an accentuated decimal disposition, which seems to have contributed to devising this type of reasoning in showing that the number of irrationals must be far greater than the number of rationals.

5.1.1.1. Arguments revealing misconceptions. At times a "helpful" intuition[2] that the probability of picking a rational number is very low, close to 0, was based on a misconception that there is a finite number of rationals. In the following three responses to Item 1(b) from the written questionnaire, the decision which set is richer is correct, but the reasoning is flawed.

- Probability of picking a rational is 0 because we have an infinite number of irrationals between 0 and 1, but we only have a finite number of rationals.
- There is a finite number of rationals but an infinite number of irrationals, so the probability of getting a rational is very, very low.
- Rational numbers are defined as a number with a ratio. It seems there would be a finite amount of rational numbers and an infinite number of irrational numbers. Probability of getting a rational number is very small – say 1%.

The misconception lurking from these three responses came as a surprise to us. It is obvious to everyone, even from a very early age, that there are infinitely many natural numbers. It is also well known, even to most high school students, that natural numbers are (or correspond to) a proper subset of rational numbers. In light of this, the thinking that there is a finite number of rationals seems absurd. It could be the case, though, that the reference to the "finite number of rationals" was implicitly limited by the students to the segment [0, 1], that has a finite number of natural numbers. But even if this were the case, one could consider rationals of the form 1/n (for a natural n) in the given interval to derive an infinite set. Another possible source for this surprising error could be linked to the view of "consecutive rationals" resulting from the conventional limiting of the number of digits in the decimal expansion, that is, considering only the sequence 0.1, 0.2, ... 0.9 or 0.01, 0.02, ... 0.99. Such a view is consistent with the claim made by some participants, that it is not always possible to find a rational number between two rationals, discussed below in considering responses to item 2(d).

However, we propose a hypothesis – that needs to be examined in further research – that this thinking may be the result of advanced mathematical training while the required background is missing. We see it as an individual's abandonment of common sense and ideas which appear intuitively obvious in order to accommodate the formal dimension of knowledge, especially if this knowledge is counterintuitive. This misconception, we believe, may have developed in these individuals as a consequence of exposure to cardinal infinities, in a situation where the underlying conceptions of rational, irrational and real number were underdeveloped to begin with. Moreover, it could hardly be said that this misconception was an isolated case. In our small group of participants, we came upon it three times, both in the written responses as well as in the interviews.

5.1.1.2. Arguments involving mapping. In this category we placed responses that involve a mapping using either the operation of addition or multiplication to transform every rational number into an irrational number. This is intended to show that the set of irrationals is richer. Here are some examples.

- Irrationals are richer. If we take each element of Q and add $\sqrt{2}$ to each, all of those numbers are irrational. Then we could take each element of Q and add π to it. Already we have twice the amount of irrationals as rationals. We could do this forever, so the set of irrationals is much richer.
- Irrationals are richer because all irrational $\times$ rational $=$ irrational (eg. $\sqrt{2} =$ irrat, $2\sqrt{2} =$irrat, $3\sqrt{2} =$irrat, ...).

In the interview with Dave, who is responding to how he knows that the set of irrational numbers is richer than the set of rational numbers, we see a more explicit example of this "addition" argument.

Dave: So what I did was, in order to, like I could, I could take um one irrational number and I could add all of the rational numbers to it one at a time so I could have like pi plus 1, pi plus 2, pi plus 4, and I would have some set of numbers that has the same cardinality as the rationals. Then I can take another irrational number, like root 5 and add again all the rational numbers to it, so right there we got twice as many irrationals as rational numbers, so I can continue to do that with all of the irrational numbers that I can possibly think of. As well, um, no that's pretty much it.

Although this way of thinking leads to a correct conclusion, it is inconsistent with mathematical convention, unless it is already known that the set of irrationals is nondenumerable. Instead, these arguments seem to imply that $\aleph_0 \times \aleph_0 \neq \aleph_0$ which is not, formally, the case. These responses reflect the application of finite experience to infinite sets, in particular that a part is smaller than the whole or that infinity plus infinity is twice as large as the original infinity.

5.1.2. *"The two sets are equally abundant"*

The most common response in Item 1(b) was that the probability of picking a rational number is 50%, which implies a belief that the two sets are equally abundant. About the same number of participants chose that neither of the two sets is richer in Item 1(a). From looking at individual questionnaires, we see the consistency of this belief across both items. In the following list, we present some common justifications for this response:

- I think there is an infinite number of rationals and an infinite number of irrationals. You can't have one infinity greater than other infinity. So both sets are equally rich!
- Since we have an infinite number of both, neither is "richer".
- Neither is richer. There are infinite number of rationals and irrationals.
- For every rational there will be an irrational to follow, so the probability of picking a rational is equal to 50%.
- Probability of picking a rational is 1 in 2. For every rational number there is an irrational.

The intuition that, following some kind of order on the number line, there should be one irrational number for each rational number, as if the numbers were nicely packed like that, was detected again in the responses to the items on how numbers fit together. A more detailed analysis of these intuitions is presented in the next section. However, it is interesting to note that we found the claim that the two sets have equal cardinality among some of the very best (based on their achievement in mathematics courses) participants in the group. For participants that have not been exposed to formal set theory, this appears to be the most natural intuition. Furthermore, it is likely that even those with basic familiarity with different infinities tend to fall back to their naive intuition after some time of not having much use for this knowledge. We found this to be the case with two of the participants that we interviewed.

5.1.3. *"Rationals are more abundant"*

The response that rationals are more abundant was quite consistent across both Item 1(a) and Item 1(b). In the following list, we present some common justifications for this choice:

- The rational set is richer. Because any integer divided by another integer repeats and is rational. Each integer can be divided by infinitely many other integers.
- Rationals are richer. Because I cannot remember many numbers similar to π.
- The chance of picking a rational number is pretty good, since some numbers that seem like irrational numbers can be written as rational numbers.
- Wouldn't all the numbers between 0 and 1 be rational? The probability of choosing a rational is 1.
- Aren't we choosing a fraction every time? Probability is 100%.
- Probability is 100% because I don't know of many numbers like Pi.

In the interview, Amy initially expressed the view that all numbers in the given interval should be rational, but soon rectified her position upon finding the evidence that it could not be so.

Interviewer: If you pick at random a number on the closed interval 0 to 1, what is the probability that you will pick a rational number?

Amy: Any number I get, I would say (pause) 1, I would say any number is rational...

Interviewer: Any number is rational, so you would say there are no irrational numbers on an interval between 0 and 1...

Amy: Mmm... (pause) It should be because I would say square root of 2 divided by 2 is between 0, no it's not, square root of 2 divided by, yes it is, it is, ...

Interviewer: So there are a few, you would say there are some irrational numbers, but there's way more rational numbers, is that your intuition?

Amy: 0 and 1, lots, lots of numbers, rational numbers and in between each of them we can find again and again and again more rational numbers. But square root of 2 divided by 2 is there too...

Interviewer: So would you change your answer now?

Amy: I would say yes. I have to, now I'm interested in, you know, about irrational and rational numbers.

Amy's initial confusion appears to be related to the density of rational numbers; if "again and again more rational numbers" can be found in between rational numbers, is there room left for irrational numbers? We find it interesting that she accessed the fact that there are some irrational numbers in the interval by estimating the value of the square root of 2 divided by 2, and not by considering the decimal representation.

5.1.4. *Intuitions on richness and density and consistency of knowledge*

As we can see, in Item 1(a) almost half of the participants (22 out of 46) identified irrationals as the "richer" of the two sets. Yet, in Item 1(b) we note a large drop in the corresponding response category. Only about a quarter of PSTs (11 out of 46) maintained that the probability of picking a rational number at random from the given interval [0, 1] was equal/close to zero or "very small". It is possible that this drop occurred as a result of insecurity that was caused by the conflict between intuitive and formal dimensions of knowledge. In addition, over 30% of participants provided no response to this item. Looking at the written data, we see that seven of the participants who said that irrationals are richer abstained from answering Item 1(b) altogether and two claimed that the chances of picking a rational number were close to 50%. We interpret that this pattern of responses is indicative of participants having acquired or recalled formal knowledge that irrationals are much more abundant than rational numbers, without having an understanding of why this may be. Therefore, for the most part, they did not bring this knowledge into the foreground in a notable way when the task (1b) relied on it implicitly. When probed more deeply as to how much more abundant, some participants could not respond. In addition, as exemplified earlier, several of the "correct" responses were found to be incidental, due to a misconception.

Although many participants (16 out of 46) abstained from answering Item 1(b), there was a much greater consistency between the two items with respect to the later two response categories, namely that the two sets are approximately equal in size, or that there are very few irrational numbers, if any, on the interval [0, 1]. It seems that beliefs of participants who fall into one of these categories are more resistant. That is to say, these participants who either expressed the view of equal cardinality, or thought that rational numbers far outnumber the irrationals, were much more likely to sustain the same belief across both items in comparison to those who expressed an opposing view.

5.2. *The fitting of numbers*

We looked at participants' ideas about how rational and irrational numbers fit together on the number line, in particular, how they reconcile the fact that between any two rationals, no matter how close they may be, there are infinitely many rational numbers, and yet it is still possible to fit irrational numbers amongst them.

Initially, in our analysis of the responses to the four questions of Item 2, we intended to focus on participants' formal/algorithmic dimension of knowledge, since the tools for a correct derivation were accessible to everyone. However, the answers were mostly intuitively based so our analysis focuses on participants' intuitive dimension of knowledge. Table III shows the quantification of responses (shaded fields signal correct responses).

Although each of the questions was correctly answered by the majority of participants, we see this majority as marginal for items 2(a) and 2(d). What we find very interesting here is that as many as one quarter of the participants expressed a belief that there are some closest irrational numbers such that no rational number could be found between them. In one of the interviews a participant referred to "consecutive irrationals" to describe his idea of the absence of gaps between irrational numbers. Even more interesting is the unexpectedly high frequency of belief that there exist some closest two rational numbers, such that no other rational number could be

TABLE III
Quantification of responses to Item 2 – how numbers fit together ($n = 46$)

Item	False	[%]	True	[%]	No answer	[%]
(a) Rational between two irrationals	12	[26.1]	24	[52.2]	10	[21.7]
(b) Irrational between two irrationals	5	[10.9]	32	[69.5]	9	[19.6]
(c) Irrational between two rationals	3	[6.5]	33	[71.6]	11	[23.9]
(d) Rational between two rationals	10	[21.7]	24	[52.2]	12	[26.1]

found in between. More than one fifth of the participants expressed this view, while over a quarter of them abstained from answering this question altogether. With this fact being so elementary, and the proof of it being easily within reach, especially for this group of participants given their educational background, we wanted to find an explanation for this. A possible source of error could be in the confusion between "countable" and "finite" which is presented in the excerpt from the interview with Erica later on in the article.

It should be noted that amongst those who answered question 2(d) correctly, there were very few (4 participants) who either used a general symbolic argument or verbalized that the "arithmetic mean of two rationals is also rational". Most of the explanations provided by the participants relied almost entirely on the decimal representation of numbers. This was prominent across all four items, as can be seen from the following collection of justifications. In what follows we present both those explanations that are mathematically valid and those that are not.

5.2.1. *Justifications consistent with the formal dimension of knowledge*

- Let a, b $\in$ Irrational and a $\neq$ b. There must exist (a + b)/2 which could be rounded to some nearby rational number so that this number would fall between a and b.
- If two rational numbers exist then there is certainly a midpoint between them, which would be found by adding the numbers and dividing by 2. This yields a rational number.
- You can find a rational number between any two irrationals by terminating the decimal expansion of the larger number such that you create a number bigger than one and smaller than the other.
- It is always possible to find an irrational number between any two rationals: just expand the decimal expansion so that it neither terminates or repeats *and* it is bigger than one and smaller than the other.

The unifying feature in these responses is that they not only claim the existence of an object, but also suggest a procedure for constructing such an object, be it a rational or irrational number. Though we aimed at exemplifying a range of responses, we note that variations on the last two examples appeared much more frequently than variations on the first two.

Further, as in responses to Item 1, we found instances of correct decisions based on faulty arguments. In examining the participants' justifications, we found three cases where an example was shown, and then on the basis of this example a claim was made that the statement is *always* true, that is, for *any* such pair of numbers. For example, one of the participants in response to whether it is always possible to find a rational number between any two irrational numbers, wrote "Take $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$; in between there is 1.6 =16/10, i.e. can be written in form m/n where n $\neq 0$; therefore this statement is True". The same individual also believed that it was always possible to find an irrational number between any two rational numbers, because "between 1 and 2 there are $\sqrt{2}$, $\sqrt{3}$, and between 2 and 3 there are $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$". This kind of nonmathematical argumentation, attempting to reach a generalized statement on the basis of a few cases, was found amongst three participants. It is possible that these individuals misinterpreted the questions, ignoring the requirements for "always" and "any".

5.2.2. *Justifications inconsistent with the formal dimension of knowledge*

- Between at least some rational numbers there are only irrational ones. This must be the case because there are many more irrational than rational numbers.
- There aren't as many rationals – irrationals fill in the gaps between rationals (justification for why 2(d) should be False).
- Irrational numbers are so dense, you can find two that do not have a rational in between.
- Spaces between irrational numbers can be infinitely small. There will be two irrational numbers that are closest to one another.
- Two non-patterned decimals can exist without a number that has a pattern existing between them. The two irrational numbers can be very close, but not the same.
- I believe numbers alternate: rational, irrational, rational, irrational, ... So there will be some closest rational numbers where only an irrational will be found. Similarly, between any two closest irrationals, you'd find a rational, not an irrational.

The idea of alternating rational and irrational numbers, as introduced in the last example above, was mentioned in four responses. However, "many more irrationals" – was the main reason provided to justify that it may be impossible to find a rational number either between two rationals or between two irrational numbers. The sources of misconceptions that these responses exemplify can be attributed to the participants' conceptions of the infinite and are discussed in detail in Section 5.2.3.

In the following passage, we consider an excerpt from the interview with Erica, which may help explain the unexpected discrepancy in questions 2(a) and 2(d) mentioned earlier. Erica was one of the participants who held the belief (identified and discussed in previous section) that there are finitely many rationals and infinitely many irrationals. We interpret this belief as a distorted remnant of her encounter with the theory of cardinal infinities; in particular, we see it as the confusion between denumerable set and finite set, and between non-denumerable set and infinite set. She changed her mind as the interview unfolded. This was likely due to the probing questions, giving her a chance to rethink the absurdity of maintaining that the set of rational numbers was finite in size. Still, in the end she is left puzzled, but certainly equipped to resolve the conflict.

Interviewer: How about this, is it always possible to find a rational number between any two irrational numbers?

Erica: No...

Interviewer: So, sometimes not?

Erica: Sometimes not...

Interviewer: Can you explain please...

Erica: *Back to my idea that there is an infinite number of irrational numbers, but a finite number of rational numbers, and if that holds true, then there mustn't, there can't possibly be a rational number that fits between every two irrationals ...*

Interviewer: And vice versa, is it always possible to find an irrational number between two rational numbers?

Erica: Yeah, because if there's an infinite number of them, then there must, (pause) *okay this logic is based on my idea of, of there being an infinite number of irrational numbers and a finite number of rational numbers ...*

Interviewer: And how about this, between any two rational numbers there's always another rational number, you say this is false, because?

Erica: Any two rational numbers there's always another..., well again that will be *based on my assumption that there's a finite number of rational numbers*, therefore, you have to at some point have gone as far as you can go, *but I'm starting to think that that may not be true (laugh), I'm starting to think that how could you actually have a finite number of rational numbers,* because even though rational number is a repeating or terminating decimal, you can still take it to an infinite number of decimal places, I mean you can still divide that space on that number line into an infinitely small subdivisions, so (laugh) .. my thinking on that at all. . .

Interviewer: That's okay. . .

Erica: Um, *I think that my whole idea of infinite and finite number of either one is something that I was either told, or somehow thought I was told and put into my brain and just took it as face value and didn't really actually think about what that meant, or what that would look like*.

Interviewer: Um hm, but you believe that there is an infinite number of natural numbers. . .

Erica: Yeah. . .

Interviewer: You do, right?

Erica: Um hm. . .

Interviewer: And you told me before, that rational numbers include natural numbers. . .

Erica: Yeah. . .

Interviewer: Already from that. . .

Erica: It doesn't, it's, *there must be an infinite number of both*, (pause) . . . going back to trying to like as I said, I need to kind of picture it, see it. . .

Interviewer: Maybe you meant in a given interval. . .

Erica: In a given interval, there's still an infinite amount of numbers that can be both rational and irrational.

Interviewer: And they never overlap?

Erica: Uh, they never overlap, no they don't overlap. . .

Interviewer: Yet we have infinitely many of both kinds?

Erica: Yeah (laugh)...

Interviewer: That is interesting. . .

Erica: (laugh) It is. . ., yeah I don't know how that's possible but that's what um, yeah. . .

Interviewer: Okay. So, so you would change it now again, this one too?

Erica: Uh, between any two rational numbers there's always another rational number, yeah, I don't, um I'm not sure, but I'm, *my thoughts right now are leading me to believe that my whole conception of rational numbers having a finite number and there being a finite number of rational numbers is not, is false*, that there's also an infinite number of rational numbers and if that's the case, then there's going to be a rational number in between two rational numbers. . . that's possible . . . I can't picture any of it, so it's very hard for me to. . .

Interviewer: You can't picture any. . .
Erica: For that number, I just, I keep seeing this string of numbers that just keep going into infinity that. . .
Interviewer: You mean you can't picture an irrational number, am I right in saying that?
Erica: Yeah, or even that there's an infinite number of rational numbers like.
Interviewer: Um hm. . .
Erica: I just, this too, it's not something to understand, I, I can't, I don't know, yeah.

According to common sense, "countable" means "what can be counted", and it implies that what is countable must necessarily be finite. Although the usage of "countable" in Cantor's theory of infinite sets is entirely different, meaning that the elements of the set can be put into one-to-one correspondence with the set of natural numbers, it is possible that students, expressing the view that the set of rational numbers is finite, adopted this more colloquial meaning of the word. Further research could examine whether this is indeed a "verbal obstacle" and, if so, what its extent is.

5.2.3. *The sources of misconceptions regarding the fitting of numbers*
We suggest that the reasons for many of these "unhelpful" intuitions regarding the fitting of numbers lie in the non-intuitive character of the infinite; for example, that rational numbers, given their dense order compared to the discrete order in the natural numbers, can be put in bijection with natural numbers. Or, that the rational numbers, in spite of being everywhere dense, are in fact very sparse in comparison to irrationals. All this is non-obvious, and often not convincing. Cantor's proofs are both simple yet very sophisticated, leaving many who have contemplated them still in doubts. Sometimes these doubts come not from what they show us, but from what new questions they open up for us, and leave unanswered. For example, one may be left wondering why the famous Cantor's diagonal argument cannot be applied to show that the rationals are nondenumerable too (the proof that another real number can always be found, different from all the ones in the supposedly complete list of all the real numbers presented as infinite decimal expansions).

Therefore, there are cognitive obstacles that may account for the difficulties preventing learners from concluding that there is a rational number between any two irrationals, and also that there is a rational number between any two rational numbers. Rational numbers are seen to be both very dense, and very sparse, and moving between these two conflicting ideas may cause inconsistencies to erupt. Furthermore, the formal knowledge that the irrationals by far outnumber the rationals, encourages the thinking that there must be some closest, neighbouring irrationals between which no rational number can be found. Excerpts from interviews with Kyra and Kathryn exemplify this.

(. . .responding to whether a rational number can always be found between two irrationals..)

Kyra: No, just because it's so, it's so dense, *the amount of irrational numbers is so dense*, I don't think, I don't think in every case you would find, because if you could find a rational number between any two irrational numbers, that would mean that the richness, that wouldn't hold, it would have to be equal richness, in order to find one, so to be consistent, I would have to say no. . .

Kathryn: Not always though, because I mean there are going to be, if you look at your irrational numbers, they're, in one case *there will be an irrational number that's right beside another one*, right. . . *So there can't be anything between those two*. . . Like if you think of um 1234, if you think of those as the only numbers that exist, then you can't put anything between 1 and 2, right. . . So in the same way, there has to be, like you have to go down far enough that there will be two irrational numbers that are right next to each other right.. . So between those two, you can't put anything else, but between any two that you pick sort of arbitrarily, then you should be able to.

What we see here, is the mind's desperate effort to accommodate new evidence brought about by the exposure to new formal knowledge, such as that of cardinal infinities. Sometimes consistent connections fail to be created, and some more basic conceptions fall apart; for example, the understanding that there is always a rational number between two rational numbers may be lost. This can be seen as a failure to integrate different items properly, and reorganize one's knowledge, to reach a better understanding of the subject following this exposure. As it is, there are inherent difficulties in contemplating the infinite. Beyond the naïve notion as "something that goes on and on", infinity is difficult to imagine. Compounding the problem is that the majority of participants based their thinking entirely on a single type of representation, namely the decimal.

One of the participants of the study, Dave, who initially reached the same conclusion as Kyra was later convinced by another student of the fallacy of such thinking. Still, he admitted feeling troubled by the contradiction this acceptance purports. We present this excerpt because it was the most successful resolution of this conflict, albeit still unsatisfying for Dave.

Interviewer: Is it always possible to find a rational number between any two irrational numbers?

Dave: Between any two irrational numbers, okay at first I thought no, so I put false, but I talked to Jody about it later and he came up with a really good example and I thought that convinced me. So what he said was, he says, so let's say I have two irrational numbers, and one of them is obviously, one on the top is bigger than the other one, so what I do is I take the larger one, or I take the two of them and line them up and pair up the places or match up the places, as soon as there's a place value that they differ the larger one I can just chop off the remaining and that gives me a rational number that then is smaller than the larger one, but bigger than the smaller one that I had...

Interviewer: Wow!

Dave: This is a convincing argument for me, so I'm going with that from now on (laugh).

Interviewer: Okay, but then how do you reconcile that with your previous discussion here, in that irrationals are so much richer, that there's way more irrational numbers and yet here you're just showing me how you can always insert a rational number between two irrationals...

Dave: Good question, I don't know, I had thought no way, I thought the irrationals are so dense, there are so many more of them that I probably could find two, that there wasn't a rational number in between, but then Jody said that to me and I thought, yeah that seems right, that seems like you can do it, so I don't know. I was torn. But he was convincing, it was a very convincing argument but I don't know.

Dave has a method – he knows how to do it, and that is convincing enough. At the same time, he is also convinced that irrationals by far outnumber the rationals (in fact, he showed us a "proof' for that too, using the mapping argument). He is torn, because he believes both are true, and yet they seem to contradict each other. However, most individuals could not sustain this contradiction, and thus consciously or subconsciously resolved the conflict in one of the two ways. They either decided that both infinities are equipotent, reflecting the intuition of infinity as absolute; that is, there cannot be many kinds of infinity, and if two sets are infinite then they have an equal number of elements, that is, "infinity" of them. Or, it was concluded that there are some closest irrationals between which no other number could be inserted.

TABLE IV
Quantification of responses to Item 3 – Operations ($n = 46$)

Item	False	[%]	True	[%]	No answer	[%]
(a) irrational + irrational = irrational (always?)	19	[41.3]	23	[50]	4	[8.7]
(b) irrational × irrational = irrational (always?)	16	[34.8]	21	[45.6]	9	[19.6]

5.3. *Effects of operations*

We investigated participants' knowledge regarding the effects of addition and multiplication on irrational numbers, using questions from Item 3. A quantitative summary of responses is presented in Table IV.

Looking at these results, what really stood out was that the majority of responses to both items in question 3 were incorrect. The majority of participants justified their decisions considering decimal representations of numbers.

5.3.1. *Justifications inconsistent with the formal dimension of knowledge*

- When you multiply two numbers each with an infinite number of digits together, the result will still be a number with an infinite number of digits.
- I use decimal representations. Because the decimal representations of irrational numbers cannot be terminated, the sum of such numbers will be a decimal that cannot be terminated.
- You cannot add $\sqrt{2} + \pi$, but you can add their decimal representations. The sum cannot be a terminating decimal.
- Two numbers that have an infinite number of non-repeating digits to the right of the decimal will have an infinite number of non-repeating digits in their sum.
- If we think of the product of two irrational numbers as an irrational number of irrational things, the question becomes "will these numbers somehow add up to rational?" I don't think so.
- The sum of two irrational numbers is irrational because $2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$.
- No pattern × no pattern = no pattern. You can't create a pattern through multiplication. You are just increasing the numbers, not changing the relationship.
- You cannot add two irrational numbers because they both continue forever so you would be adding infinitely.

Moreover, of those individuals that did answer question 3(b) correctly, many of them ignored the requirement that the two irrational numbers must be different and gave examples such as $\sqrt{2} \times \sqrt{2} = 2$. Further, hidden in these results are errors in fundamental logic, such as determining the truth value of a statement by considering one confirming example. For instance, one of the participants wrote $\sqrt{3} + \sqrt{2}$ as a justification that it is true that the sum of two irrational numbers is necessarily an irrational number.

We suggest that the low performance on these two items is due to participants' difficulty in conceiving an irrational number as an object (Dubinsky, 1991; Sfard, 1991). That is to say, the concept of irrational number has not been encapsulated; instead, an irrational number is viewed as stuck in the process of becoming by an endless summation of its decimals. An illustrative example of explicit process conception is demonstrated in the last comment in our list; this is an example of perceiving a number with infinite decimals as being constructed in time and not as an already made object.

Furthermore, it appears that one of the reasons for difficulties is the disposition towards closure of operations within a number set, in this case, the set of irrational numbers. The idea that the sum of two irrationals is irrational, and the product likewise, was expressed many times either implicitly of explicitly. The following three responses from the interviews exemplify this.

(In response to whether the sum of two irrationals is necessarily irrational.)

Steve: I still believe that that's the case. Um, because if you cannot, I think this is how I thought of this one, I was again thinking in fraction form, so if I had two numbers that cannot be in a fraction form, I don't see how I could all of a sudden put in a fraction form. Because when we add fractions, we always look at the denominator and add it, find the common denominator and all that, but here I don't think it was possible you can do that because you can't put them in, in the fraction form. So that the addition of them would always have to be irrational.

Claire: No, ... simply because if you have 2 irrational yes, a and b, irrational numbers, and you add them, addition is closed, addition is stable in that set, in irrational numbers, yeah the answer is still um, it's still irrational.

(In response to whether the product of two irrationals is necessarily irrational.)

Kathryn: You can't multiply two different numbers that cannot be expressed as fractions, and get fractions. I think you might be able to do it, but it's going to take until Christmas to find an example...

(Note: the interview took place in June)

The responses of Steve and Kathryn demonstrate another kind of leap in logic; namely, they confuse a rule and its converse. According to their reasoning, "a sum is rational if and only if the summands are rational" which is quite different from the true statement "if the summands are rational the sum is rational".

5.3.2. *Justifications consistent with the formal dimension of knowledge*

In contrast, the following responses utilize the formal and algorithmic knowledge of operations with irrational numbers.

- $(2 + \sqrt{3}) + (2 - \sqrt{3}) = 4$
- $\sqrt{3} \times 2\sqrt{3} = 2 \times 3 = 6$
- $\sqrt{5} \times \frac{1}{\sqrt{5}} = 1$
- There must be two irrational numbers whose digits will "cancel" when added to result in a rational number.
- Proof by counterexample. You can find two irrational numbers that create a repeating decimal expansion.

$$\begin{array}{r} 0.12122122212222\ldots \\ +0.21211211121111\ldots \\ \hline 0.33333333333333\ldots \end{array}$$

The first three justifications demonstrate a high level of concept development; that is to say, $(2 + \sqrt{2})$ or $2\sqrt{3}$ is conceived as an object and not only as an instruction for adding or multiplying two numbers. On the basis of both written questionnaire and clinical interviews, we found only six instances of participants exhibiting the thinking of irrational numbers as objects. We interpret the strikingly poor performance on item 3 as an indication that the notion of irrational number, such as $5 + \sqrt{2}$ for example, is commonly conceived operationally (as a process) rather than structurally (as an object) (Sfard, 1991). Interestingly, nobody offered a product of conjugates as a counterexample by which it can be easily demonstrated that a product of two different irrationals can indeed be rational. Despite the fact that all participants were competent in performing algorithms such as "rationalizing the denominator", 30 out of 46 PSTs either responded incorrectly or did not respond at all when asked whether the product of two irrational numbers could ever be rational (3b). Of those that responded nobody drew upon this familiar procedure. This reveals that algorithmic knowledge can become highly procedural and rote for the learner, to the extent where the very purpose of using such procedures may be completely lost. It indicates there is a problem in the integration of algorithmic, formal and intuitive dimensions of knowledge.

In the interview excerpt below, Claire expresses her views about the challenges and the importance of helping students understand irrational numbers, incidentally exposing a deficiency in her own thinking.

Interviewer: If you multiply two different, okay, so we want to have different irrational numbers, together, the result is always irrational. Is this true or false?

Claire: It's false, because you can take one number being 5 square root of 2 times 7 square root of 2, this is an irrational number, a product between, so I think, no I think. I'm, this is an irrational. So I think it should be set, these kind of questions should be, when you teach irrational, you have to show to the students, you know what, when you have..., I heard many, and I think they're wrong, saying that 7 square root of 2 are two numbers...

Interviewer: Hmm...

Claire: You understand? And the student is confused with those two numbers. No it's only one number, you have to see it like a symbol, 7 square root of 2 a number, don't see it like two, as long as you see it, but it should be somewhere in the definition or somewhere when you teach the lesson, it should be pointed out that this is a number, instead of saying two different numbers. Like, you know, 1a, 2a, 1 square root of 2, 2 root 2, yes, it's a slightly difference, and you don't have to, you have to be very careful how you, you say it in front of the students. Otherwise they will come and say, oh those two numbers I don't know...

It is interesting to note the inconsistency between Claire's responses to questions 3(a) shown earlier, and 3(b) . Although she clearly treats $7\sqrt{2}$ as an object in the case of multiplication, she does not do so in the case of addition, in $7 + \sqrt{2}$. We wonder whether it has to do with the writing of the number, that is $7\sqrt{2}$ is more likely to be thought of as "one thing" in comparison to $7 + \sqrt{2}$, which is seen as a process. In other words, the operation of multiplication is implicit in $7\sqrt{2}$ while the operation of addition is explicit in $7+\sqrt{2}$. It could be that as a result of this, she maintains that irrationals are closed under the operation of addition.

In summary, the majority of participants incorrectly argued that adding two positive irrational numbers will always produce an irrational number, and likewise, that multiplying two different irrational numbers must result in an irrational number (See Table IV). This general lack of competency in evaluating the adequacy of statements related to operations with irrationals could be attributed to three factors: (1) a great reliance on decimal representation (even when considering symbolic representations, such as roots, would be more appropriate and revealing), (2) understanding irrational numbers as processes rather than objects, and (3) disposition towards closure.

6. Conclusion

In this report, we centered our attention on the complex notion of intuition as manifested in the participants' responses regarding the relations between the two infinite sets (rationals and irrationals) that comprise the set of real numbers. Our findings indicate that underdeveloped intuitions are often related to weaknesses in formal knowledge and to the lack of algorithmic experience. Constructing consistent connections among algorithms, intuitions and concepts is essential for having a vital (as opposed to rote) knowledge of any mathematical domain, and therefore also for understanding irrationality. As anticipated, participants' responses to the presented tasks revealed a great deal about their understanding of numbers in general, and about their formal and intuitive knowledge of irrational numbers in particular. We described the inconsistencies between the participants' formal and intuitive dimensions of knowledge and their attempts to resolve these inconsistencies. We note that, at times, participants' difficulties related to their general ability to evaluate the truth value of mathematical statements, rather than to issues involving irrational numbers.

If so, one may ask, how important is the knowledge of irrational numbers to teaching at the secondary school? Before formulating our answer, we would like to share with the reader an incident we witnessed in observing a grade 9 mathematics lesson:

Student: Is π the only irrational number?
Teacher: No, remember, there is also $\sqrt{2}$.

Mathematically speaking, the teacher is right. To show that some object is not unique in its kind it is enough to point to another object of the same kind. However, we see in this teacher's response a missed pedagogical opportunity to open students' minds at least to a variety of examples of irrational numbers, if not to their "richness" or relative abundance. Indeed, $\sqrt{2}$ and π are generic examples for irrational numbers. In examining our data, we believe that about a fifth of the participants of our study were not aware of the existence of irrational numbers beyond π, e, and some commonly seen square roots, or at least did not communicate such awareness. There is a danger of communicating this view to students, either implicitly or explicitly. While we recognize that the discussion of infinities of different cardinality has no direct implementation in the secondary school curriculum, we feel that this is a useful knowledge for teachers, as lacking this background may introduce or reinforce students' misconceptions. We further maintain that the ideas of "fitting" and effects of operations are within or closely related to the secondary school curriculum.

While some participants' errors and misconceptions described in this research may be due to the inherent difficulty of the topic of irrational numbers, described in literature as epistemological or cognitive obstacles (Herscovics, 1989; Sierpinska, 1994), others may be a direct outcome of instructional choices. It could be the case – and a question for future research – to what degree our observations are linked to the educational choices and teaching practices, both at the secondary school and at the university level. Teacher education programs may be an appropriate place for reconsidering these particular choices and practices, as a part of an overarching attempt to strengthen teachers' overall mathematical competence.

The notion of number is one of the main notions underlying mathematics curriculum. The concept of an irrational number is inherently difficult; yet, understanding of irrational numbers is essential for the extension and reconstruction of the concept of number from the system of rational numbers to the system of real numbers. Therefore, careful didactical attention is essential for proper development of this concept.

We believe that teachers' knowledge is a prerequisite for developing understanding in their students. The results of our investigation help in understanding the challenges that the concept of irrational number presents to learners at all levels.

NOTES

1. The formal proof is based on the notion of "measure zero" and the fact that rational numbers are a "measure zero set".
2. We refer to intuition as "helpful" if it assists in reaching a correct decision

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