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GENERALIZATION OF PATTERNS: THE TENSION BETWEEN ALGEBRAIC THINKING AND ALGEBRAIC NOTATION

ABSTRACT. This study explores the attempts of a group of preservice elementary school teachers to generalize a repeating visual number pattern. We discuss students' emergent algebraic thinking and the variety of ways in which they generalize and symbolize their generalizations. Our results indicate that students' ability to express generality verbally was not accompanied by, and did not depend on, algebraic notation. However, participants often perceived their complete and accurate solutions that did not involve algebraic symbolism as inadequate.

1. INTRODUCTION

Patterns are the heart and soul of mathematics. However, unlike solving equations or manipulating integers, exploration of patterns does not always stand on its own as a curricular topic or activity. Some teachers see such an activity as recreational enrichment rather than curriculum core. We take the view that "algebra, and indeed all of mathematics is about generalizing patterns" (Lee, 1996, p. 103). Therefore we believe that it is essential in a study of mathematics to direct students' attention to patterns underlying a wide variety of mathematical topics.

In this article we describe the attempts of a group of preservice elementary school teachers to generalize a visual numeric pattern. We analyze the routes towards expressing generality followed or abandoned by these participants and the obstacles encountered on these routes. This way we extend the existing research on the development of algebraic thinking in general and on the pattern generalization approach to algebra in particular.

1.1. *Patterns*

There are several attempts to develop students' pattern finding strategies at different levels, from nursery school to secondary school (Ishida, 1997; Iwasaki and Yamaguchi, 1997; Orton and Orton, 1999; Radford, 2000). Research distinguishes between different kinds of patterns – number patterns, pictorial/geometric patterns, patterns in computational procedures,

Educational Studies in Mathematics **49:** 379–402, 2002.

linear and quadratic patterns, repeating patterns, etc. In what follows we comment on repeating patterns and on linear patterns.

Threlfall (1999) focused on one-dimensional repeating patterns in early primary years. Repeating patterns are patterns with a recognizable repeating cycle of elements, referred to as 'unit of repeat'. For example, ABCABCABC... can be seen as a repeating pattern with 3 attributes and a repeating cycle, or unit of repeat, of length 3; ABCabABCabABCab can be seen as a more complex repeating pattern, with 3 attributes and a cycle of length 5, in which not only the letter but also the case is varied. Varying some attributes of elements (such as size, color, orientation, etc.) while keeping other attributes constant adds complexity to a repeating pattern (Threlfall, 1999).

Among the reasons for working with repeating patterns, Threlfall (1999) acknowledges ideas of regularity and sequencing, and opportunities for teachers to draw students' attention to helpful aspects of the experience. Moreover, Threlfall advocates the use of repeating patterns as a vehicle for working with symbols, a conceptual stepping stone to algebra and a context for generalization. Young children can succeed in generating or continuing repeating patterns using a procedural or rhythmic approach. However, as a stepping stone towards generalization and algebra, it is essential to see particular patterns; that is, to perceive the unit of repeat in a repeating pattern. This goal may not be achieved if work with repeating patterns is undertaken only in early primary years, when students are not yet developmentally able to achieve the perception of a unit of repeat.

Stacey (1989) focused her exploration on linear patterns, presented pictorially as expanding ladders or trees. Participants were asked to determine the number of matches needed to make a ladder with 20 or 1000 rungs, or the number of lights in a Christmas tree of a given size. These patterns are labeled 'linear' because the n^{th} element can be expressed as $an+b$. Stacey found these problems were challenging for 8–13 year old students. The constant difference property was largely recognized and enabled students to find the n^{th} element of a pattern from the $(n-1)^{th}$ element. However, in an attempt to generalize, a significant number of students used an erroneous direct proportion method, that is, determining the n^{th} element as the n^{th} multiple of the difference. Stacey also reported inconsistencies in the methods chosen by students for 'near generalization' tasks (e.g., find the twentieth term), and 'far generalization' tasks (e.g., find the thousandth term). Similar results were reported by Zazkis and Liljedahl (2001, 2002) in their investigations of arithmetic sequences with preservice elementary school teachers. In these studies participants were provided with the first 4 or 5 elements in an arithmetic sequence and were asked to provide examples of large numbers in this sequence and to determine whether certain numbers belonged to the sequence if it continued infinitely. The direct proportion, or multiple of a difference approach, appropriate to sequences of multiples (e.g., 3,6,9,12 . . .) was also extended and applied to sequences of so called 'non-multiples' (e.g., 2,5,8,11 . . .).

Orton and Orton (1999) extended investigations of linear patterns (arithmetic sequences) to other sequences of numbers. They reported the tendency of students to use differences between the consecutive elements in a sequence as their preferred method. This method was successfully extended to quadratic patterns by taking the second differences, but led to a dead end in instances such as the Fibonacci sequence. Among the obstacles to successful generalization Orton and Orton mentioned students' arithmetical incompetence and fixation on a recursive approach. Although allowing students to generate the next element in a sequence based on a previous one, this approach prevented them from seeing the general structure of all the elements. A recursive approach was also mentioned by English and Warren (1998) as an approach that students preferred and often reverted to when more challenging patterns were presented to them.

1.2. *Generalization*

According to Dörfler (1991) generalization is both "an object and a means of thinking and communicating" (p. 63). Realizing the importance of generalization in mathematical activity, several researchers identify different kinds of generalization. Dörfler distinguishes between empirical generalization and theoretical generalization. Empirical generalization is based on recognizing common features or common qualities of objects. According to Dörfler it is considered 'problematic' in mathematics education in terms of determining qualities that are relevant for generalization. That is, empirical generalization is criticized for lacking a specific goal to decide what is essential, being limited without a possibility to generalize further and overreliance on particular examples. Theoretical generalization, in contrast, is both intentional and extensional. It starts with what Dörfler refers to as a "system of action", in which essential invariants are identified and substituted for by prototypes. Generalization is constructed through abstraction of the essential invariants. The abstracted qualities are relations among objects, rather than objects themselves.

Harel and Tall (1991) use the term generalization to mean "applying a given argument in a broader context" (p. 38). They distinguish between 3 different kinds of generalization: (1) *expansive*, where the applicability range of an existing schema is expanded, without reconstructing the schema; (2) *reconstructive*, where the existing schema is reconstructed in order to widen the applicability range; and (3) *disjunctive*, where a new schema is constructed when moving to a new context. It is noted that although a disjunctive generalization may appear as a successful generalization for an observer, it fails to be a cognitive generalization since it does not consider earlier examples as special cases of the general procedure. In fact, disjunctive generalization may be a burden for a weaker student, who constructs a separate procedure for a variety of cases, rather than creating a general case. Furthermore, expansive generalization is cognitively easier than reconstructive generalization, but may be insufficient in the long run.

1.3. *Generalization of patterns and algebra*

Attention to patterns is acknowledged in its importance as an introduction to algebra. Mason (1996) describes "expressing generality" as one of the roots of, and routes into, algebra. The use of patterns as a route to expressing generality has become popular over the past decade within school mathematics curricula in the UK (Orton and Orton, 1999). "Understanding patterns, relations and functions" is a continuous theme of the Algebra standard in the Principles and Standards for school mathematics (NCTM, 2000) at all grade levels.

English and Warren (1998) advocate a patterning approach to introducing a variable. They argue that, traditionally, variables are introduced as unknowns in equations, where they do not possess the varying nature. Furthermore, a patterning approach provides students with the opportunity to observe and verbalize their generalizations and to record them symbolically. They suggest that patterning activities need not end with the establishment of the concept of a variable, as they provide a useful and concrete base for work with symbols.

Attending to algebraic symbolization when exploring patterns in the context of an elementary algebra course for adults was one of the main foci of the teaching experiment reported by Lee (1996). According to Lee, the major problem for students was not in "seeing a pattern" but in perceiving an "algebraically useful pattern" (p. 95). Once students perceived a pattern in a certain way, it was hard for them to abandon their initial perception. A flexible view of patterns should be developed in order to help students find those patterns that may lead to algebraic symbolization (Lee, 1996; English and Warren, 1998).

2. Methodology

2.1. *The task*

The following array of numbers was presented to a group of 36 preservice elementary school teachers.

1	**2**	**3**	**4**	
	8	**7**	**6**	**5**
9	**10**	**11**	**12**	
	16	**15**	**14**	**13**
17	**18**	**19**	**20**	
...				

The participants were invited to explore patterns they identified in this array and keep a log/journal of their investigations. They had two weeks to complete the assignment and were advised to work on it for at least half an hour every other day. The following questions were intended to provide an initial guidance to their investigation:

- How can you continue this pattern[1] ?
- Suppose you continue it indefinitely. Are there numbers that you know 'for sure' where they will be placed? How do you decide?
- Can you predict where the number 50 will be? 150? And how about 86? 87? 187? 392? 7386? 546?
- In general, given any whole number, how can one predict where it will appear in this pattern? Explain the strategy that you propose.

The participants were asked to record carefully their processes, queries, conjectures and the results of testing them, their frustrations (if any), and celebrations. They were explicitly asked and expected to present the progress in their thinking, rather than just to provide a 'final solution'. Furthermore, it was suggested that they explain and justify every mathematical claim they made. The use of algebraic formalism was neither required nor assumed by the wording of the task.

After initial examination of the protocols, four participants were invited to a clinical interview. The interviews were intended to probe and clarify several claims that were provided but not justified in their written work.

[1] A more accurate way to present this question would have been "How can you extend this arrangement, preserving some regularity?" We are thankful to the Editor Anna Sierpinska for this comment and for helping us clarify the notion of 'pattern' in this article.

2.2. *Analysis of the task*

It is well known that no finite sequence of elements uniquely generates the next term (e.g. Mason, 2002). A finite array of 20 numbers can be extended in a variety of ways, each preserving some regularity. However, some extensions may be perceived as more 'natural' than others. For example, the next element in the sequence of 1,2,3,4,5,6... can be 727 if the sequence is defined by $a_n = (n-1)(n-2)(n-3)(n-4)(n-5)(n-6) + n$ or just 7 if the sequence is defined by $a_n = n$. We suggest that the latter extension is more 'natural' than the former. In our study the array was extended in the same way by all the participants. In what follows we attend only to this 'natural' extension.

The number array presented to the participants has a combination of pattern features mentioned in the literature. Its elements are numbers in a predetermined arrangement and the location of a number in this arrangement is an integral component of the array. Therefore, it can be perceived as both a numerical pattern and a visual pattern. Taken separately, the left, the right and middle columns (labeled A, E and C respectively) present arithmetic sequences, that in a context of patterns, are referred to as linear patterns. In addition, by observing every second row, one can detect a linear pattern in columns B and D as well.

Furthermore, we identify in this array features of a 'repeating pattern'. However, it is not a repeating pattern in some conventional way. There is a level of complexity added by making the unit of repeat implicit, or partially implicit. That is, what is explicitly repeating is the visual structure. Considering the visual structure and ignoring the numbers, we attend to the form that repeats every 2 lines or every 8 elements.

A	B	C	D	E
1	**2**	**3**	**4**	
	8	**7**	**6**	**5**
9	**10**	**11**	**12**	
	16	**15**	**14**	**13**
17	**18**	**19**	**20**	
...				

$\Longmapsto$

A	B	C	D	E
O	**O**	**O**	**O**	
	O	**O**	**O**	**O**
O	**O**	**O**	**O**	
	O	**O**	**O**	**O**
O	**O**	**O**	**O**	
...				

Although the numbers themselves do not repeat, applying the same transformation on each element produces an explicitly recognizable repeating cycle. This transformation replaces every element in the array by its remainder in division by 8.

1	**2**	**3**	**4**			**1**	**2**	**3**	**4**	
	8	**7**	**6**	**5**			**0**	**7**	**6**	**5**
9	**10**	**11**	**12**		$\Longmapsto$	**1**	**2**	**3**	**4**	
	16	**15**	**14**	**13**			**0**	**7**	**6**	**5**
17	**18**	**19**	**20**			**1**	**2**	**3**	**4**	
...						**...**				

Attending to this feature allows us to determine the location of any natural number in the array. For example, considering number 548 we note that 548 ÷ 8 = 68 Remainder 4. Therefore the column location of 548 is the same as of number 4 (column D). Furthermore, we conclude that 548 is found in the 69th set of 8 or in the137th row of the array.

The predominant pattern-related activity for students in school is extending number sequences and finding a 'general term', with the aim to express it algebraically. That is, given the position in a sequence, the goal is to determine the corresponding element. Defining such an element *t(n)* as a function of its position *n* expresses generality with standard algebraic symbolism. The task discussed in this article adds complexity on two counts. First, what constitutes a position is not predetermined. The planar, rather than sequential, presentation of numbers invites consideration of a location as an ordered pair of numbers, specifying either (column, row) or (place within set, set ordinal). Second, the task can be considered as an 'inverse task' as it reverses the usual roles of what is given and what is to be found. Unlike the usual goal of finding the element in any given place, the task is to locate a position for any element. In what follows we present one possible way to formalize the general description of the array.

In order to formalize the definition of a position of a number, we could first view the (infinite) array as consisting of sets of 8 numbers, where the place of each number in a set corresponds to the place of one of the first 8 numbers. For example, number 17 is positioned in place number 1, set 3; and number 13 is positioned in place number 5, set 2. We define *Position* as a function from the set of natural numbers to the set of ordered pairs, as follows:

Position: $N \rightarrow Z_8 \times N$, where N is a set of natural numbers, and

$$Z_8 = \{0,1,2,3,4,5,6,7\} \text{ is the set of remainders in division by 8}$$

Position(n) = (place within set(n), set ordinal(n),)
Place within set (n) = R(n, 8)

$$\text{Set ordinal (n)} = \begin{cases} 1 + \text{Q(n, 8)} & \text{if R(n, 8)} \neq 0 \\ \text{Q(n, 8)} & \text{if R(n, 8)} = 0 \end{cases}$$

where R(n,8) and Q(n,8) are the remainder and the quotient of n in division by 8 respectively.

We acknowledge that an interpretation of position as row and column may appear as a more natural pattern. This view can be formalized as follows:

$$\text{Column(n)} = \begin{cases} \text{A} & \text{if R(n, 8)} = 1 \\ \text{B} & \text{if R(n, 8)} = 2 \text{ or } 0 \\ \text{C} & \text{if R(n, 8)} = 3 \text{ or } 7 \\ \text{D} & \text{if R(n, 8)} = 4 \text{ or } 6 \\ \text{E} & \text{if R(n, 8)} = 5 \end{cases}$$

or recorded in a more concise form:

Column (n) = Map (R(n,8)),
where Map (1,2,3,4,5,6,7,0) = (A, B, C, D, E, D, C, B)

$$\text{Row(n)} = \begin{cases} 2 \times \text{Q(n, 8)} & \text{if R(n, 8)} = 0 \\ 2 \times \text{Q(n, 8)} + 1 & \text{if R(n, 8)} = 1, 2, 3, 4 \\ 2 \times \text{Q(n, 8)} + 2 & \text{if R(n, 8)} = 5, 6, 7 \end{cases}$$

This symbolization further clarifies the distinction made by Lee between 'seeing a pattern' and perceiving 'an algebraically useful pattern'. Flexibility in perceiving a pattern helps in choosing a way that leads more easily to a formal notation. We emphasize that this, or a similar, formalism was neither expected nor required from the participants. We present this in order to highlight a complexity that is involved in moving from a verbal description to formal symbols.

Having a predetermined solution in mind, initially we planned to investigate how participants take advantage of divisibility or division with remainder related ideas in their explorations of the array. However, facing the richness of participants' approaches we have extended our initial focus.

In our analysis of the 36 problem solving logs and 4 clinical interviews we address the following questions:

- What patterns were found and acknowledged in the given structure of numbers? What were the common tendencies or common obstacles?
- What patterns were generalized and how is it possible to describe the different kinds of generalization that took place?
- What means were used to express generality?

3. Results and analysis

The complexity of the number array and the data collection method resulted in a rich and diverse set of solution approaches. What follows is an analysis of results organized according to the themes that emerged in the participants' work. We describe students' solutions, their paths towards the solutions, and their perceptions of acceptability of the generated solutions. We discuss students' algebraic thinking, their use of algebraic symbolism, and the interplay between the two.

3.1. *What does it mean to 'solve' a pattern?*

Being able to continue a pattern can be taken as an understanding of a repeating pattern. Being able to describe a 'general' element can be seen as a solution of a linear pattern. In our case, we asked students to determine the location of every whole number. The following question directed students towards such generalization:

In general, given any whole number, how can one predict where it will appear in this pattern? Explain the strategy that you propose.

We provided students with no specific instructions as to the interpretation of 'where'. Only three students used a structure that was directly related to the calculation of quotient and remainder; they responded to the question 'where' with a pair of numbers, designating the ordinal set of 8 and the placement in such a set. As an example, the number 15 is found in the second set (of 8 numbers), seventh placement. The rest of the students, however, interpreted 'where' as a pair of numbers specifying the row and column location. In this case the number 15 is found in row 4 and column 3 (column C).

However, being able to determine row/column or set/position location of any number did not always satisfy the participants. During her first attempt at this problem, Myra determined that "multiples of 8 were at the end of the sets". She had shown that "50 divided by 8 equals 6.25, therefore 50 is two numbers into the seventh set". She further claimed that "with my current solution, I can place any number into the pattern". Myra didn't consider that this completed her solution. She claimed, "Although I have an answer to the problem, there has to be an easier solution". A search for an 'easier solution' was, at first, disappointing for Myra. She declared, "I have not been able to come up with any short-cuts for solving this problem. I'm sure that there is some sort of 'formula' to solve it quickly, but I have not found it".

After some persistence Myra noticed that "each set starts with 1+ multiple of 8" and suggested a solution by finding the first number in a set and counting up. The "first number in a set" was found by division by 8, "rounding down to a whole" where necessary, and adding 1. Her summary of the solution was presented as follows:

- *a divided by 8 = b (needs to be a whole number, if it is a decimal, the number should be rounded down, this number shows the amount of complete sets of 8)*
- *b* × *8 + 1 = the first number in the set*
- *count up to the number that has been selected (a)*

Myra found her second solution "much easier". We wondered what left Myra dissatisfied with her first solution and much happier with the second. There could be a combination of two things. The second solution attends to the first element in each unit of repeat, and as such it may appear as more accessible. In addition, Myra's second solution introduced algebraic symbolism, which was absent in her first solution. Ironically, it may have satisfied her search for a 'formula' and given her solution a perceived mathematical validity.

Myra exemplifies a common tendency among the participants. Search for a solution is a search for a single formula that will determine the location for any given number. Solutions by cases or solutions not involving algebraic formalism often left participants with a feeling of inadequacy.

3.2. *Spotting patterns*

As noted by Orton and Orton (1999) in their investigation of children's patterning abilities, the ability to continue a pattern comes well before the ability to describe the general term. With the specific number array in question, the ability to think of a way of extending the array does not easily translate into the ability to determine the place of any given element of the extension. Significant amount of pattern-spotting took place in participants' attempts to progress towards a generalized solution. The participants noticed and described patterns, however, at least initially they had no appreciation of what route it was beneficial to pursue. As Shirley pointed out, "I'm really unsure as to where to go with this. I see so many patterns, yet, I don't know how to use them."

Almost everyone started the exploration by recapitulating the visual structure of the array. It was referred to as 'right/left indented rows', 'snake-like', 'S-like' or 'zig-zagging'. A predominant observation attended to even and odd numbers. The fact that even numbers are in columns B and D, and the odd numbers are in rows A, C and E was easily spotted. (Regardless of the participants' choice of reference for labeling the columns, we use letters for consistency). The choice between B and D for evens and between A, C and E for odds was significantly more demanding.

The observation of differences between the consecutive numbers in the columns was another predominant focus of attention. The constant difference of 8 in columns A and E as well as the constant difference of 4 in column C was, for most, the first pattern described by the participants. Further, alternating differences of 2 and 6 were identified in columns B and D. Some participants generalized these disjunctive observations by noticing that 4+4=8 and also 2+6=8 and the constant difference of 8 persisted in every column when skipping every second row. However, attention to differences did not provide the impetus towards multiplicative generalization. Instead, it created a focus on recursive reasoning discussed in previous work on repeating patterns (Orton and Orton, 1999; Lee and Warren, 1998). “In row A I could count up by 8’s to find a number, however this could still take a long time for a large number”, summarized Shirley. “If it is odd, I know that I will only be able to find it in columns A, C or E. I could test if a number fits in columns A or E by subtracting 8 until the last number in my pattern was discovered. I know this is too tedious. I could test the middle number by subtracting 4 until I reached the desired number”, summarized Kate. These claims demonstrate the dominance of the recursive approach that prevents students from shifting their attention to the general structure. The exhibited dominance of additive thinking and lack of connection between additive and multiplicative structures is consistent with prior research findings (e.g., Zazkis and Campbell, 1996).

Patterns were also spotted in using specific points of reference. For example, the structure of multiples of 10 – that is, 10 in column B, 20 and 30 in column D, 40 and 50 in column B, 60 and 70 in column D, etc., – was often determined. For some students it served as a shortcut in counting up to a desired number, while for others it was “an interesting pattern that will not help”. Another interesting focus of attention was on multiples of 25, that Carol referred to as “main numbers”. Pam noticed that consecutive multiples of 25 create the following pattern in their columns ABCDEDCBABC .... Carol’s solutions employed a flexible use of multiples of 10 or 25, which was further developed as “flipping at 100” strategy, discussed in further detail in the next section.

However, spotting a pattern did not always lead to a solution. For example, Chris found that “column B has all the multiples of 8, but this doesn’t help to find all the numbers in this column”. She further noted that every number in column A can be written as $8 \times [\ ] + 1$. However, she wrote, this “only helps me to find out if the number is in column A or not, it doesn’t help me to place numbers in other columns”. Chris was not aware how a position of every number can be determined based on the information she had revealed.

Lee (1996) mentioned that students participating in her study had difficulty, not with spotting a pattern, but with recognizing an algebraically useful pattern. Chris determined a pattern that we considered 'algebraically useful', but failed to appreciate its usefulness. On the other hand, students focusing on reference points were able to develop complete solutions from patterns that we didn't appreciate at first as 'useful'. Therefore we suggest that 'usefulness' of an identified pattern is best considered as a spectrum, rather than as a dichotomy. Moreover, "usefulness" is not a feature of a pattern but a perception of the beholder.

3.3. *What numbers are 'familiar'?*

In the clinical interview students were presented with a slight variation of the original array. They were asked to consider the following:

	1	**2**	**3**	**4**
8	**7**	**6**	**5**	
	9	**10**	**11**	**12**
16	**15**	**14**	**13**	
	17	**18**	**19**	**20**
...				

It was a unanimous view that this variation was easier than the original. The reason for this perception is exemplified by Shirley's comments:

Shirley:	These are [Shirley added two more rows and pointed to the numbers in the left column] times 8 table
Interviewer:	How can it help you to find a location of a number, take 187 for example
Shirley:	What is there times 8? 180 – no, 182 – no, 184 – yes. So 184 is in this column on the left, then you continue down-right, 185,186,187, so it will be here [Shirley points to column D].

[. . .]

Interviewer:	Let's go back to the original. Look at the left column. What do you notice about the numbers?
Shirley:	They are 8 apart.
Interviewer:	Anything else?
Shirley:	They are all odd.
Interviewer:	Anything else?
Shirley:	Not really. There is no times 8 or times 9 pattern or something. Only the difference of 8 each time.

Shirley suggested that she could easily locate each number by considering the closest multiple of 8. However, she wasn't able to adapt this strategy for the original array. Prior research on students' understanding of arithmetic sequences outlined a similar perception: In sequences of multiples students were able to recognize both multiplicative (the essence of being a multiple) and additive (constant difference) invariants. In sequences of 'non-multiples' only the additive invariant was acknowledged (Zazkis and Liljedahl, 2001).

Orton and Orton (1999) mention arithmetic incompetence as one of the obstacles in pattern generalization. It could be the case that further competence with numbers and their possible arithmetic composition would help Shirley notice that there is not "only a difference of 8 each time", but also an embedded multiplicative relation.

3.4. *What's the unit of repeat?*

Perception of unit of repeat is critical in 'seeing' repeating patterns. (Threlfall, 1999). In this section we describe units of repeat identified by the participants and the attributes that might have influenced their perception.

As stated earlier, the visual structure of the array repeats every two rows or every eight elements. Consequently, the cycle of remainders in division of the elements by 8 generates the implicit repeating numerical pattern. However, only three of the participants attended to this unit of repeat as an organizing theme to their solution. Several alternative approaches were preferred.

Attending to the organizing structure of '4 numbers in a row' was one of the most popular methods of solution, suggested by 22 participants. Dan wrote, "We ought to be able to use multiples of 4, a multiple of 4 is the largest number in each row – any number that is not a multiple of 4 will appear in the same row with the next multiple of 4 greater than the number". Most solutions circled around the idea of groups of 4 and assigned the row and the column based on identification of the ordinal number of a row and its direction/indentation. (Further detail is provided in section 3.7, ‘Routes towards generalization’). It is the case that attention to 4 elements in a row and a familiar row/column organization prevented participants from noticing, what could be considered as a more general structure, a repeating unit of 8.

Another popular strategy, used by eight out of 36 students, was to consider 100 numbers as a unit. Though only one student made the mistake of claiming that adding 100 to a number will not change its position, that is, x is in the same column as 100+x. The remaining seven students identified that an increase of 100 will move the number into the ‘opposite’ column. (A and E, B and D were considered as pairs of ‘opposite columns’, and C was considered the opposite of itself). The perceived pattern was described by students as ‘flipping’ every 100 numbers. Therefore, in order to determine a column position of a large number (e.g., 7386), students first identified the position of the number determined by the last two digits (86 is in column D). From here some students simply ‘counted up’ by flipping every 100 between columns D and B. A more sophisticated approach along those lines suggested to ignore the thousands digit, explaining that adding 1000 will not change the position of the number. Surprisingly, no one looked at 200 as the unit of repeat, rather than 100 as the unit of ‘repeat and flip’.

The unit of repeat identified by Celia deserves some attention. Celia claimed that the pattern repeats every 40 numbers. Her initial strategy in placing large numbers was based on subtracting multiples of 40 until the result was smaller than 40 and on determining the position according to this result. She further improved her strategy by considering the quotient and the remainder in division by 40. For example,

392 ÷ 40 = 9 remainder 32,
32 is in column B, 8th row, therefore 392 is in column B, 98th row.

During the interview Celia explained that 98 was the result of 90+8, since 40 numbers take 10 rows, and the quotient of 9 indicated to her that “there were 9 sets of 40 (before the last set of 32) that occupied 90 rows”. Appreciating the thoughtfulness and originality of this solution, the interviewer wished to explore whether Celia could identify a different, smaller unit of repeat. Celia insisted on working with 40 as the ‘smallest cycle possible’. After considerable probing it became clear that Celia’s attention was focused on the pattern of last digits in column A: 1,9,7,5,3 alternating with blank spaces, in the first 10 lines. The number 41 is placed “back in the beginning of the pattern”, according to Celia. In the next ten lines, or 40 numbers, this pattern repeats. Like students in Lee’s (1996) study, Celia was reluctant to abandon her initial perception. With this last digit fixation any other approach seemed faulty.

3.5. *Algebraic symbols and algebraic generalization*

Marge started her investigation by assigning letters to numbers. She assigned a and b for the first and last numbers in the first row, c to the first number in the second row, d and e to the first and last numbers in the third row, and f to the first number in the fourth row. She further recorded the following equations, together with numerical substitution as justification:

b+4=c (4+4=8)
c+1=d (8+1=9)
e+4=f (12+4=16)

She continued the array for 32 rows and attempted to check what equations were applicable. This seemed to be a dead end. Marge further reflected on this attempt, by recording "I found myself so busy with the construction of the chart that I really didn't pay too much attention to the obvious patterns...". So Marge abandoned her equation building strategy and continued by attending to 'obvious patterns' in division by 4.

Mason (1996, p. 75) provided an example (medieval eggs problem) of students who rushed into building equations involving unknowns, but were unable to do anything with their equations. "This is an algorithm seeking question, not a simple algebra question" – noted Mason in explaining why symbol manipulation wasn't an appropriate strategy. We make a similar observation. Marge's symbols were not helpful as "this is a pattern seeking question, not a simple algebra question". Another example of symbolization detached from meaning could be seen in the work of Ann. In the beginning of her investigation Ann spotted the '+8 pattern' in columns A and E. She recorded that numbers in column A can be written as 1+8r, whereas numbers in column E can be written as 5+8r. (She didn't specify what r stands for). However, this observation did not provide a fruitful hint for Ann. She continued in alternative directions. The next day of her investigation she reported with excitement, "Eureka! Numbers in column A are 1 more than multiples of 8!" This finding allowed Ann first to determine whether or not a given number is in column A and further to be able to find a position of any number by considering a close number in column A. For example, in order to place 86, Ann considered 89, which is "1 more than a multiple of 8", and then counted down to 86 to determine its position in column D.

Ann's realization of the general structure of numbers in column A was significantly delayed compared to her ability to describe numbers in column A as 1+8r. The obvious expression of celebration in her "Eureka!" shows that the verbal generalization resulted from considering the numbers themselves, rather than the expression 1+8r generated earlier. The algebraic symbolism was the result of an attempt to fit a formula, but originally these symbols did not entail any meaning. The meaningful generalization, and therefore the ability to 'solve the pattern', was expressed verbally, and not connected to previously used symbols.

3.6. *Connectedness of different representations*

English and Warren (1998) suggest that understanding the notion of equivalence and recognizing equivalence in generalizations is important in working with patterns.

Remainder in division by 8 seems to be an obvious indicator of the 8 possible placement options. Furthermore, consideration of remainder in division by 4 (4 possible outcomes), together with evenness or oddness of a quotient also produces 8 possible placement options. As stated earlier, most students focused on division by 4 rather than 8. The equivalence between the two perspectives was not recognized, even by students aware of both options.

Andy noted early in her investigation that columns A, C, and E contain odd numbers, while columns B and D contain even numbers. Later, she reported a new observation: numbers in column D were 'doubles'. She exemplified this observation by showing that 4=2+2, 6=3+3, 12=6+6, etc. The equivalence of naming numbers as even and recognizing them as 'doubles' was not noted.

Dan summarized his solution in the following way:

if 8|x then x is in column B
if 8|(x+1) then x is in column C
if 8|(x+2) then x is in column D
if 8|(x+3) then x is in column E

if 4|x but 8 does not, then x is in column D
if 4|(x+1) but 8 does not, then x is in column C
if 4|(x+2) but 8 does not, then x is in column B
if 4|(x+3) but 8 does not, then x is in column A

It is not clear why Dan deserted the analysis presented in the first 4 lines of his solution, where he determined the column of a number based on how "far" it was from the multiple of 8. He could have continued the same line of reasoning, claiming that if 8|(x+4) , then x is in column D, etc. Given the unnecessary complexity introduced by considering divisibility by 4 and by 8 simultaneously, we believe that Dan was unaware that, for example, divisibility of x by 4 and not by 8 entails the remainder of 4 in division by 8, or, in Dan's preferred notation $8|(x+4)$.

Lena pointed out in her investigation that for odd-numbered rows the remainder in division by 8 indicated the column position, that is, remainders of 1, 2, 3, and 4 indicated the number's position in column A, B, C, and D respectively. In her attempt to place number 38 in the array, Lena was confused by the remainder of 6 in division by 8, as previously the value of the remainder equaled the number of the column. (We note that Lena labeled columns by numbers and not letters). She concluded that if the remainder in division by 8 was greater than 4, then the numbers were found in an even-numbered row. Locating a column presented a challenge. From the observation that 38 = 8 x 5 – 2, Lena concluded that remainder of –2 indicated column D, or second column from the right. She extended her "subtracting from a multiple" strategy, to conclude that remainders of 0, –1, –2, and –3 indicated columns B, C, D, and E respectively. From her decision to desert a consideration of remainders and focus on what she referred to as "negative remainders", it is clear that Lena did not see the equivalence between remainders 5, 6, 7 and the corresponding "negative remainders" of –3, –2 and –1 respectively. It may be the case that the "four in a row" structure of the array prevented both Lena and Dan from considering "big" remainders in division by 8. In both cases, a separate correct strategy has been developed to accommodate "big" remainders. This is an example of disjunctive generalization, discussed in the next section.

Ann, mentioned in section 3.5, didn't recognize the equivalence between algebraic expression (1+8r) and a verbal expression "one more than a multiple of 8". In this section we discussed examples of students failing to recognize the equivalence between different verbal representations (see the discussion of Andy's work above) and between equivalent computational strategies (see the discussion of Dan's and Lena's work above). This lack of awareness of equivalent expressions could be an obstacle to students' attempts to generalize. In the next section we discuss several pathways that students took in generalizing their solutions.

3.7. *Routes to generalization*

Among initial observations the constant difference between numbers in columns A, C and E was noted by the participants. This led them to a partial solution, considering, either implicitly or explicitly, remainders in division by 8 and 4. That is,

If the remainder of n in division by 8 is 1, then n is in column A;
if the remainder of n in division by 8 is 5, then n is in column E;
if the remainder of n in division by 4 is 3, then n is in column C.

An equivalent way to record these findings, with no explicit reference to remainders, was preferred by some participants:

If (n-1)/8 is a whole number, then n is in column A;
if (n-5)/8 is a whole number, then n is in column E,
if (n-3)/4 is a whole number, then n is in column C.

Taken together with the observation that odd numbers were placed in columns A, C and E, while even numbers were placed in columns B and D, the above generalization scheme provides a column position for odd numbers. A natural question then arises regarding the position of even numbers. There are several strategies that our participants employed. In fact, a minority opted to give up the investigation at this point, providing a solution for odd numbers and claiming the even numbers would be found in either column B or D. However, most students attempted to extend their investigation to include even numbers as well. These extensions are being considered here in the context of evolving levels of generalization identified by Harel and Tall (1991) (see section 1.2).

For a majority of students (22 out of 36) the visual perception of 4 numbers in a row served as a guiding principle to consideration of remainder in division by 4. However, this remainder did not provide a unique answer to the column location. Therefore, the case of even numbers invited reconsideration and attention to factors other than the remainder. Kelly noticed that numbers divisible by 4 were in column D when the quotient was odd, and in column B when the quotient was even. Since attention to the parity of quotients proved useful, she extended this consideration for even numbers that were not divisible by 4, that is, leaving a remainder of 2. Remainder of 2 and odd quotient identified the number's position in column B, while remainder 2 and even quotient identified number's position in column D. Kelly did not attempt to reexamine her solution and see the applicability of her new findings to the case of odd numbers. Therefore, we see her generalization as disjunctive – a new case deserved a new treatment, a new decision making scheme was constructed for even numbers. However, we also recognize an element of expansive generalization in utilizing the parity of quotients first in numbers divisible by 4 and then in those leaving a remainder of 2.

Rachel, from a similar starting point above, considered remainders in division by 8 for even numbers. She concluded that numbers in column B "when divided by 8 will have 0 left over or 2 left over", and numbers in column D will have "either 4 or 6 left over". This is an expansive generalization of previously considered cases for columns A and E. However, Rachel left column C as a separate disjunctive case, identified by remainder of 3 in division by 4. She made no attempt to reconsider this case and to accommodate it within her schema.

Laura, after giving separate consideration to even and odd numbers, extended her consideration of even and odd quotients in division by 4 to odd numbers as well. This is an example of expansive generalization: a solution that was developed for a specific case has been extended to accommodate other cases, that is, the applicability range of a schema has been extended.

Jane's final solution was very similar to Laura's, but her route was different. Having observed that all numbers in column C leave a remainder of 3 in division by 4, she tried to extend this strategy to other rows. However, other remainders did not provide a conclusive result. Numbers divisible by 4, as well as numbers having remainder of 2 in division by 4, were found both in column B and in column D. Numbers leaving the remainder of 1 in division by 4 were found in columns A and E. Since the attempt of expansive generalization failed, there was a need for reconstructive generalization. The scheme was reconstructed by attending to the parity of the whole number quotient. Numbers divisible by 4 were placed in column D when the quotient was odd and in column B when the quotient was even. Even quotient together with remainders 1, 2, and 3 indicated number placements in columns A, B, and C respectively. Odd quotients together with remainders 1, 2 and 3 indicated number placements in Columns E, D and C respectively. Note that remainder of 3 points to column C regardless of the parity of the quotient. Therefore the new reconstructed scheme includes the previously constructed scheme as a special case.

Reconstructive, as well as expansive, generalization were not a frequent phenomenon in this group of students, if we consider these as applied to the task as a whole. Once students found a solution by cases, even those who were not entirely happy with the solution had little motivation to look back and try to integrate different cases under one scheme. However, elements of both reconstructive and expansive generalization were present when considering separate components of the array. For example, extending a consideration of remainder in division by 8 from column A to column E can be seen as an element of expansive generalization.

The tendency to stay with disjunctive generalizations can be attributed to several factors. First, as Harel and Tall (1991) note, solution by cases (disjunctive generalization), puts less cognitive demands on a learner. Second, an equivalence between claims or computations may not be recognized (as seen in cases of Rachel above and Dan and Lena in section 3.6), and therefore, students do not 'see' how their separate cases fit together. Possibly, students' appreciation of the elegance and beauty of a complete generalized schema has been insufficiently developed to seek it as their goal. However, despite the low regard for disjunctive generalization (see section 1.3), we suggest that it can provide an essential starting point in approaching a new content and solving a new problem.

4. Synthesis and Concluding Remarks

School algebra instruction has been continuously criticized for "rushing from words to single letter symbols" (Mason, 1996, p. 75). As an alternative, several researchers have heralded pattern exploration as a preferred introduction to algebra. Typically, this has involved the search for "algebraically useful patterns" (Lee, 1996, p. 95), followed by a move towards algebraic notation in order to generalize the perceived pattern. The question regarding this two-step approach is, when does the algebraic thinking emerge and what could indicate its presence?

When the term *algebra* is used it encompasses two distinct concepts: algebraic thinking and algebraic symbolism. There is a lack of agreement among researchers as to the relationship between the two. Some view the algebraic symbols as a necessary component of algebraic thinking, while others consider them as an outcome or as a communication tool. Further, different perspectives are argued on the relationship between algebraic reasoning and generalization.

For Kieran (1989), "generalization is neither equivalent to algebraic thinking, nor does it even require algebra. For algebraic thinking to be different from generalization, [...] a necessary component is the use of algebraic symbolism to reason about and to express that generalization." (p. 165). She further suggests that "for meaningful characterization of algebraic thinking it is not sufficient to see the general in the particular, one must also be able to express it algebraically" (ibid.). On the other hand, Charbonneau (1996) considers symbolism as central to algebra, but "not the whole of algebra" (p. 35). He considers symbolism as a language that may condense the presentation of an argument and as a means to solve problems.

A more recent tendency among researchers is to separate algebraic symbolism from algebraic thinking. This separate consideration is fostered by two factors: (1) further acknowledgment of the possibility of mindless symbol manipulation and (2) a movement for 'early algebra', that is, focus on structure rather than on computation in elementary school. For Kaput and Blanton (2001), generalizing and formalizing patterns and constraints is one of the forms of the 'complex composite' of algebraic reasoning (p. 346). They see "generalization (which includes deliberate argumentation) and the progressively systematic expression of that generality [...] as underlying all the work we do [in algebra]" (ibid.). More specifically, by algebraic reasoning Kaput (1999) refers to students' activity of generalizing about data and mathematical relationships, establishing those generalizations through conjecture and argumentation and expressing them in increasingly formal ways.

Mason (1996) brings further itemization into algebraic thinking as an activity. He sees the roots of algebraic thinking in detecting sameness and difference, in making distinctions, in classifying and labeling, or simply in 'algorithm seeking'. The very formation of this algorithm in the mind of the student, in whatever form it is envisioned, is algebraic thinking. Algebraic symbolism, according to Mason, is the language that gives voice to this thinking, the language that expresses the generality. Dörfler (1991) suggests that theoretical generalization needs a certain symbolic description. However, he believes that symbolic description does not necessarily entail the use of letters. According to Dörfler these symbols can be verbal, iconic, geometric or algebraic in nature. This is consistent with Sfard (1995), who uses the term algebra "with respect to any kind of mathematical endeavor concerned with generalized computational processes, whatever the tools used to convey this generality" (p. 18).

We adopt the latter, more inclusive, views on algebraic thinking. The task set for the participants in our study does not lead to a 'smooth' algebraic notation, presented in one 'neat' formula that connects the element n to its location in the number array. An algebraic expression of the array requires either function definition by cases or a composition of functions. This was neither required nor expected from this group of participants. However, in exploring patterns participants engaged in detecting sameness and differences, in classifying and labeling, in seeking algorithms, in conjecturing and argumentation, in establishing numerical relationships among components or, more generally , in "generalizing about data and mathematical relationships" – activities identified as components of algebraic thinking by Mason (1996) and Kaput (1999). We used a categorization of Harel and Tall (1999) to describe and analyze the different kinds of generalization employed by participants. The task presented to participants in this study provided an opportunity for a variety of approaches in exploring the number array and generalizing its structure, as well as expressing this structure in increasingly formal ways.

Moreover, our participants were actively engaged in seeking a way to express their generalization. Their attempts to use algebraic notation, beyond simple labeling of elements and columns, often appeared unhelpful. The algebraic thinking emerged through alternate forms of communicating their findings; similar to Radford's (2000) conclusion that the "students were already thinking algebraically when they were dealing with the production of a written message, despite the fact that they were not using the standard algebraic symbolism" (p. 258). Furthermore, when our participants demonstrated both algebraic thinking and the ability to use algebraic notation, they lacked synchronization between the two. Therefore, neither the presence of algebraic notation should be taken as an indicator of algebraic thinking, nor the lack of algebraic notation should be judged as an inability to think algebraically.

There is a gap between students' ability to express generality verbally and their ability to employ algebraic notation comfortably. This gap, together with the rush "to single letter symbols [that] has marked school algebra instruction for over a hundred years" (Mason, 1996, p. 75), leaves students with a feeling of inadequacy at not meeting expectations. Several participants expressed an explicit concern that their solutions were incomplete because they lacked a 'formula'. This is consistent with Schoenfeld's (1988) observation that, for students, form of expression is what matters most and failing to use the proper form, regardless of the substance of what has been produced, is being 'unmathematical'. Rather than insisting on any particular symbolic notation, this gap should be accepted and used as a venue for students to practice their algebraic thinking. They should have the opportunity to engage in situations that promote such thinking without the constraints of formal symbolism. Problems that are rich in patterns, such as the one presented to our participants, offer students such opportunities. They are particularly useful for preservice elementary school teachers, for whom these problems serve not only as a rich mathematical activity, but also as a venue to gain appreciation of various ways of expressing generality.

A FINAL COMMENT

> What are the significant products of research in mathematics education? I propose two simple answers: 1. The most significant products are the transformations in the being of the researchers. 2. The second most significant products are stimuli to other researchers and teachers to test out conjectures for themselves in their own context. (Mason, 1998, p. 357)

As stated in section 2.2, we started this journey with a predetermined solution in mind. It was based on one particular view of the array of numbers, to which we initially referred as 'the pattern'. The participants in our study helped open our eyes to a variety of patterns that can be identified in the array and its extension, and also changed our perception regarding the 'usefulness' of certain ways of perceiving patterns. This is a researcher's transformation referred to by Mason (1998, p. 357) as "the most significant products" of research in mathematics education. Only time will be able to testify to Mason's "second most significant products".

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