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A REACTION TO BURN'S "WHAT ARE THE FUNDEMENTAL CONCEPTS OF GROUP THEORY?"

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In his abstract, Burn (this journal Vol. 31 issue No. 4) calls his paper a "... critical analysis of Dubinsky et al. (1994) ..." and we are profoundly puzzled over what appears to be an interesting discussion, but which certainly does not seem to be about our paper. Consider for example, the following two quotes, first from the first paragraph in Burn and then from the first page of Dubinsky et al. (op cit).

Burn: "It (Dubinsky et al.) is a report of a novel teaching procedure using the computer software ISETL in the teaching of group theory."

Dubinsky et al.: "In this paper we hope to open a discussion concerning the nature of knowledge about abstract algebra, in particular group theory, and how an individual may develop an understanding of various topics in this domain ... We include, at the end, a brief discussion of some pedagogical suggestions arising out of our observations, but a full consideration of instructional strategies and their effect on learning this subject must await future investigations yet to be conducted."

The paper is *not* about teaching abstract algebra, with ISETL or otherwise. ISETL itself is not mentioned at all in the body of the paper, nor are there any examples of computer activities, or indeed any specifics of the pedagogy. The teaching method used in the course under discussion was mentioned only incidentally (in a single paragraph), as part of the background for the research. In fact we make no claims whatever in this paper on the teaching of abstract algebra.

Rather, this paper presents research that attempts to contribute to knowledge of how students' understanding of certain group concepts (group, subgroup, coset, normality, quotient group) may develop. As such, it has a clearly stated research methodology and a theoretical framework within which it analyzes the data. Most of the discussion is devoted to excerpts from in-depth interviews of students and interpretations that try to relate students' responses to our theoretical analyses. And certainly, our agenda is not (as Burn suggests) to "criticise" students for their partial understanding any more than Piaget's research is a "criticism" of 3-year-old children for their inability to conserve.

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Our main purpose in this paper is to compare theory with data. It seems that a criticism of such a research effort ought to present conflicting data, or question our interpretations of the data we present, or criticize our theoretical perpsective. Having written such a research paper, we would certainly welcome criticism of our methodology, our theory, and our interpretations. We consider this paper to be a very preliminary study of a new and largely uncharted area. As such it has many gaps and imperfections, but it is a beginning nonetheless. We hope that efforts to understand students' learning of abstract algebra concepts will not become a debate over our common teaching experience, but will concentrate rather on refining and extending that research. We invite Burn to join in.

On the other hand, it is certainly true, as Burn points out, that the authors of this paper have been actively engaged in developing a course in Abstract Algebra. We have written a textbook and an expository article in the American Mathematical Monthly. The latter article has led to an exchange of letters in the Monthly in which we have participated, since we welcome this sort of discussion in an appropriate forum.

We believe that two sorts of discussion are essential to the development of innovative mathematics pedagogies and that it is important to distinguish between them. Some discussions should make use of our collective teaching experience and inspire public conversations such as the one in the Monthly. But other discussions should make specific research assertions where data is presented, intepretations are made and conclusions are drawn. We reiterate that Dubinsky et al. is a discussion of the second kind since it represents a systematic investigation of the nature of knowledge and how it develops in an individual. We hope, therefore, that others will analyze Dubinsky et al. in the spirit in which it was written. This study and its successors should form the foundation for research into particular pedagogical practices, but should not be confused with more general discussions of pedagogy derived from classroom lore.

Having said all that, we see value in responding to some of the specifics in Burn's article, not in defense of Dubinsky et al., but as a continuation of the discussions that have appeared in the American Mathematical Monthly on issues relevant to Abstract Algebra pedagogy.

When Burn suggests that some of the student difficulties we report could be due to our particular teaching methods, we understand his concern. However, in 1992-3 Zazkis followed students in a traditionally taught abstract algebra class and her results were very close to those reported in Dubinsky et al. Thinking the work showed little that was "new", Zazkis has not published this study, but perhaps such a discussion is needed to shed light on Burn's question. There does not yet exist, in fact, published research studies which specifically examine our own pedagogical approach for abstract algebra. We are hard at work on preparing such reports but the reader will understand they require a great deal of meticulous time consuming work. We can say that the first papers have been submitted and several more are on the way.

Burn questions whether our research actually covers the fundamental concepts of group theory by suggesting that the four properties (closed, associative, identity, inverse) be included as well as sets, functions, permutations and symmetry. We can hardly disagree; all of these concepts play major roles in our thinking about group theory, and in other papers (Breidenbach et al., 1991; Zazkis & Dubinsky, 1996) we have indicated our awareness of the importance of functions, permutations and symmetries. We probably should have inserted the word "some" in our title to make it "On Learning *Some* Fundemental Concepts of Group Theory". But other than the title, we are not sure what Burn is objecting to.

We would remind Burn that our research is about students' thinking, and not about our own mature view of mathematics. Our knowledge of students' thinking comes from the interview data where we probed students' use of certain concepts of group theory. But when Burn questions our treatment of student understanding of $\mathcal{Z}_3$ as a subgroup of $\mathcal{Z}_6$ (Burn's point 4), we think he is confusing *his own* thinking with that of *the students*. Of course, mathematically sophisticated thinkers may find it possible to consider $\mathcal{Z}_3$ as (isomorphic to) a subgroup of $\mathcal{Z}_6$. Students whose understanding was mature in this way showed that maturity by the way they explained their work. But our analysis of less mature conceptions was not based merely on students' contention that $\mathcal{Z}_3$ is a subgroup of $\mathcal{Z}_6$ but on the way they used the fact that the elements 0, 1, 2 of $\mathcal{Z}_3$ are also in $\mathcal{Z}_6$. We believe that our description of the various student conceptions of this matter sheds useful light on student understanding of group and subgroup.

We also contend that Burn is confusing student understanding with the understanding of mature mathematicians when he challenges (in his point 5) our view that quotient groups are difficult for students. He offers the fact that the two element group of even and odd integers is "familiar to many school children" and we all know that this is a quotient group of the integers. In fact, only someone who already possesses a good understanding of quotient groups can understand these examples as quotient groups. If G is a group and H is a quotient group, the fact that one understands H as a group does not in any way require, or even suggest, that one understands the quotient relation between G and H. We agree that the multiplication table of even and odd is easy and familiar. But this doesn't make the tremendous abstraction of quotient groups easy – no more so than, say, knowing elementary plane geometry makes understanding abstract vector spaces easy. We believe with Mason and Pimm (1984) that "seeing the general in the particular" is one of the most mysterious and difficult learning tasks students have to perform. Frankly we are surprised at this particular criticism since in the many talks and papers we have devoted to these matters, Burn is the first abstract algebra teacher who has questioned the difficulty of quotient groups for students. If others have such doubts, then it might be worth conducting a study to demonstrate this rather generally accepted assumption.

We have a similar reaction to Burn's claim that "... shuffling a pack of cards is a sufficient introduction to the notion of permutation to embark on a major study of group theory." For justification of this rather optimistic view, Burn refers to a treatise on group theory. We don't think this work tells us much about the requirements for student understanding as opposed to the requirements for logico/mathematical completeness. Asiala et al. (1996) have written an excellent description of student difficulties with permutations and symmetries and what might be done to help.

We are very concerned about a paper in a research journal which makes, without the slightest attempt at justification, statements like the following, "But even for the most abstract of *finite* groups, a visual concrete illustration of the quotient group is available which has been used to excellent effect on the outside cover of Fraleigh (1967) ... If the computer software for the course were to facilitate permutations of the rows and columns of a group table, the construction of quotient groups could become as concrete and participatory as the rest of the course." Burn is certainly welcome to this opinion, but it is our understanding that articles in a research journal such as this one should not contain unsupported claims. If Burn has such as view, then he, or someone else should conduct research on it and report the results. Our own study of visualization (Zazkis et al., 1996) addresses some of the issues raised in Dubinsky et al. and Burn's response. As that study shows, the effects of visualization may be considerably more complex than people generally think.

We agree with Burn that a historical view is useful to designing research and instruction with respect to group theory. History is certainly a part of our methodology, but we are influenced not only by the record of who proved what and when, but also with the mechanisms by which mathematical progress was made. Piaget & Garcia (1983) show that there is a much closer connection between historical and individual development at the level of cognitive mechanism (how a concept is constructed) than at the level of specific mathematical facts and relationships. Thus we find it difficult to accept without qualification Burn's assertion that "... attention to historical origins provides the strongest pointer available to an authentic genesis of this subject."

We are grateful to Burn for providing us with a thought provoking essay setting out his pedagogical views on learning group theory. But we have tried to clarify here the distinction between our two approaches to this important work and look forward to seeing a continuation of this exchange in appropriate forums. We also hope that Dubinsky et al. eventually serves to stimulate further systematic research into how the concepts of group theory may be learned.

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