Lecture 8 - Tidal forces

What's Important:
- tidal forces
- Roche limit

Text: Carroll and Ostlie, Sec. 18.2

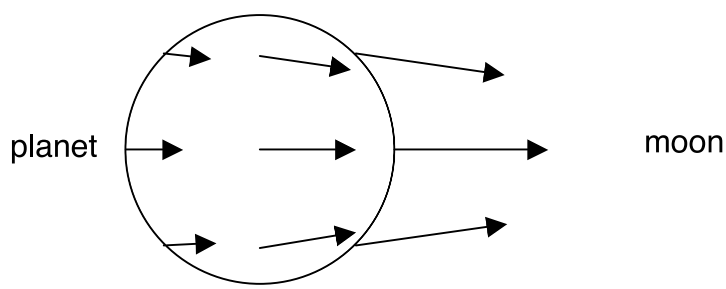
Tidal forces

Those of us living in coastal cities are aware of the effect of the gravitational pull of the Moon in creating ocean tides, although we may puzzle about why there are two high tides every day, rather than one. Tidal forces arise because of the distance dependence of the gravitational force, so that the gravitational attraction to an object like the Moon is stronger on one side of the Earth than the other. In this lecture, we find an approximate expression for tidal forces appropriate to planetary systems, although our interest is less in the motion of water than in the ability of tidal forces to tear apart satellites like the Moon.

According to Newton's law of gravity, the force between two masses is inversely proportional to the square of the distance between them. Using "planet" and "moon' to indicate the relative masses of the objects, we write

$$F = GM_{\text{planet}}m_{\text{moon}} / r^2.$$

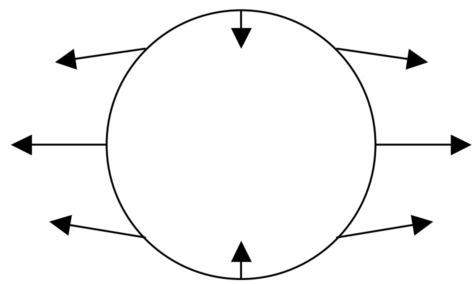
This means that the gravitational force of a moon on a planet has the schematic appearance



The force vectors on the far side of the planet are shorter than on the near side to the moon. Further, the vectors point towards the planet-moon axis.

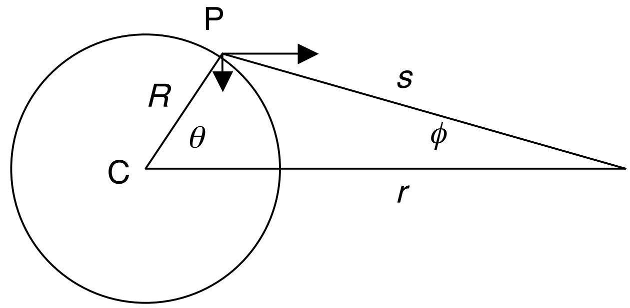
The moon has been deliberately placed close to the planet to emphasize the direction of the force vectors. This picture would intuitively lead to the "one tide" prediction.

However, what's important for tides is the gravitational force compared to the force at the center of the planet, as the entire planet experiences the force, not just its surface. Subtracting the force at the center from the above diagram leaves us with a force difference:



Once we see the pattern of the relative force, the cause of the "two tide" behaviour is obvious.

Let's calculate the force algebraically, following Carroll and Ostlie. We consider two points on the planet, labelled C and P, as in the diagram:



Resolving into components along the *x* and *y* axes:

$$F_{C,x} = GM_p m_m / r^2 \qquad F_{C,y} = 0$$

$$F_{P,x} = (GM_p m_m / s^2) \cos\phi \qquad F_{P,y} = -(GM_p m_m / s^2) \sin\phi$$

The difference in the force is

$$\Delta \mathbf{F} = \mathbf{F}_P - \mathbf{F}_C,$$

or

$$\Delta F_x = GM_p m_m (\cos\phi / s^2 - 1/r^2) \qquad \Delta F_y = -(GM_p m_m / s^2) \sin\phi \qquad (8.1)$$

These expressions are exact, but inconvenient in part because they are written in terms of ϕ and we would prefer θ of the planet. Further, they can be simplified because ϕ is small (although θ is not).

x-component:

From trig, $s^2 = (r - R\cos\theta)^2 + (R\sin\theta)^2 = r^2 + R^2 (\cos^2\theta + \sin^2\theta) - 2Rr\cos\theta$,

--> $s^2 = r^2 + R^2 - 2Rr\cos\theta$. (cosine rule)

Here, $r >> R$, so

$$s^2 \cong r^2 [1 - 2(R/r) \cos\theta]$$

and

$$\frac{1}{s^2} = \frac{1}{r^2}\frac{1}{[1-2(R/r)\cos\theta]} \cong \frac{1}{r^2}[1+2(R/r)\cos\theta].$$

Thus

$$\frac{1}{s^2} \cong \frac{1}{r^2} + \frac{2R\cos\theta}{r^3} \qquad (8.2)$$

Substituting Eq. (8.2) into (8.1) yields (when ϕ is small and $\cos\phi \cong 1$)

$$\Delta F_x \cong GM_p m_m\,(2R\cos\theta/r^3),$$

or

$$\Delta F_x \cong 2(GM_p m_m R/r^3)\cos\theta\,. \qquad (8.3a)$$

The second term involves $\sin\phi / s^2$. To the same order of approximation,

$1/s^2 \cong 1/r^2$ from (8.2)

and

$\sin\phi \cong (R\sin\theta) / r$,

so

$$\Delta F_y = -GM_p m_m \frac{1}{r^2}\frac{R\sin\theta}{r}$$

or

$$\Delta F_y \cong -\,(GM_p m_m R/r^3)\sin\theta\,. \qquad (8.3b)$$

We see that Eq. (8.3) has the expected behaviour as a function of θ:

ΔF_x elongates the planet at the equators, with the appropriate change in sign a $\theta = \pm\pi/2$.

ΔF_y compresses the planet at the poles $\theta = \pm\pi/2$ and changes sign at $\theta = 0$ and π.

Tides and synchrony

One can see from the motion of the ocean, that tidal forces can be dissipative, causing a loss of kinetic energy over time. In the case of the Earth-Moon system, tidal effects have already slowed the rotation of the Moon so that it co-rotates with the position of the Earth (the lighter rocks of the lunar surface are thicker on the far side of the Moon, the near side has a thinner layer of light rock). Tides are causing the rotational speed of the Earth to decrease at a rate of 0.0016 seconds per century.

The slowing of the Earth has a direct effect on the motion of the Moon. As the angular rotation ω of the Earth decreases, so too must its rotational angular momentum L. By conservation of angular momentum, the *orbital* angular momentum of the Moon must increase correspondingly. The Moon accomplishes this by moving further from the Earth, thus increasing its moment of inertia with respect to the Earth. Thus, the Moon is receding from the Earth at 3-4 cm per year.

Note: whether a satellite spirals inward or outward to compensate for changing angular momentum depends upon its orbital radius relative to the "synchronous" orbit (stationary with respect to an equatorial location). Satellites inside the synchronous orbit will spiral inward and crash (depends on whether the satellite leads or trails a planetary bulge; see Carroll and Ostlie).

Roche limit

Eq. (8.3) demonstrates that the tidal force scales like r^{-3}: the smaller the orbit the larger the force. Is it physically possible for the tidal force to become so strong that it tears a satellite apart? This would happen if the gravitational force were less than the tidal force. We use Gm_m^2/R_m^2 as a crude measure of the gravitational binding force of the moon, and set (R_m on rhs, not R_p, because tidal force is at the moon)

$$Gm_m^2/R_m^2 < 2GM_p m_m R_m / r^3$$

or

$$m_m/R_m^3 < 2M_p / r^3.$$

Reworking this a little to make the terms look like densities gives

$$m_m/R_m^3 < 2\,(M_p/R_p^3) \bullet (R_p^3/r^3). \tag{8.4}$$

Now, the mean density of a spherical object is

$$\overline{\rho} = \frac{m}{\frac{4\pi}{3}R^3}$$

so Eq. (31.4) can be written

$$\overline{\rho}_m < 2\overline{\rho}_p\left(\frac{R_p}{r}\right)^3$$

or

$$r < 2^{1/3}(\overline{\rho}_p/\overline{\rho}_m)^{1/3}R_p.$$

The factor of $2^{1/3} = 1.26$ is a little crude; Edouarde Roche considered this problem around 1850, finding that the correct prefactor is 2.456:

$$r < 2.456(\overline{\rho}_p/\overline{\rho}_m)^{1/3}R_p \qquad \text{Roche limit} \tag{8.5}$$

Example Suppose that a planet and its satellite have the same mean density. Then,

[*Roche limit*] = 2.456 R_p.

It is interesting to note that many (but not all) of the rings of the Jovian planets lie within the this equal-density planetary Roche limit, perhaps indicating the origin of some of the ring material.