

Self-similar collapse of a scalar field in higher dimensions

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Abstract. This paper constructs the continuously self-similar solution of the spherically symmetric gravitational collapse of a scalar field in n dimensions. The qualitative behaviour of these solutions is explained, and closed-form answers are provided where possible. Equivalence of scalar field couplings is used to show a way to generalize minimally coupled scalar field solutions to the model with general coupling.

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1. Introduction

Choptuik's discovery of critical phenomena in the gravitational collapse of a scalar field [1] sparked a surge of interest in gravitational collapse just at the threshold of black hole formation. Discovery of critical behaviour in several other matter models followed quickly [2–7]. While perhaps the presence of critical behaviour in gravitational collapse is not in itself surprising, some of its features are, in particular the conclusion that black holes of arbitrary small mass can be formed in the process. Moreover, the critical solution often displays additional peculiar symmetry—so-called self-similarity—and serves as an intermediate attractor for near-critical solutions.

The study of critical phenomena also throws new light on the cosmic censorship conjecture. The formation of a strong curvature singularity in critical collapse from regular initial data offers a new counterexample to the cosmic censorship conjecture.

Much work has been done, and general features of critical behaviour are now understood. However, there is a distinct and uncomfortable lack of analytical solutions. Due to the obvious difficulties in obtaining solutions of Einstein equations in closed form, most of the work seems to be done numerically. One of the few known closed-form solutions related to critical phenomena is Roberts' solution, originally constructed as a counterexample to the cosmic censorship conjecture [8], and later rediscovered in the context of critical gravitational collapse [9, 10]. It is a continuously self-similar solution of the spherically symmetric gravitational collapse of a minimally coupled massless scalar field in four-dimensional spacetime. For a review of the role self-similarity plays in general relativity see [11].

This paper searches for continuously self-similar, spherically symmetric scalar field solutions in n dimensions. They might be relevant in the context of superstring theory, which is often said to be the next 'theory of everything', as well as for understanding how critical behaviour depends on the dimensionality of the spacetime. Roberts' solution would be a particular case of the solutions discussed here. These solutions provide reasonably simple toy

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models of critical collapse, although they are not attractors [12]. Some qualitative properties of the self-similar critical collapse of a scalar field in higher dimensions have been discussed in [13]. Here we aim to find explicit closed-form solutions.

The second part of this paper deals with the extension of minimally coupled scalar field solutions to a wider class of couplings. It is shown how different couplings of the scalar field are equivalent, and several particular models are examined in detail. The procedure discussed here can be applied to any solution of Einstein-scalar field equations.

2. Reduced action and field equations

Evolution of the minimally coupled scalar field in n dimensions is described by the action

$$S = \frac{1}{16\pi} \int \sqrt{-g} d^n x [R - 2(\nabla\phi)^2] \quad (1)$$

plus surface terms. Field equations are obtained by varying this action with respect to field variables $g_{\mu\nu}$ and ϕ . However, if one is only interested in spherically symmetric solutions (as we are), it is much simpler to work with reduced action and field equations, where this symmetry of the spacetime is factored out.

Spherically symmetric spacetime is described by the metric

$$ds^2 = d\gamma^2 + r^2 d\omega^2, \quad (2)$$

where $d\gamma^2$ is the metric on a 2-manifold and $d\omega^2$ is the metric of a $(n-2)$ -dimensional sphere. Essentially, the spherical symmetry reduces the number of dimensions to two, with spacetime fully described by the 2-metric $d\gamma^2$ and 2-scalar r . It can be shown that the reduced action describing field dynamics in spherical symmetry is

$$S_{\text{sph}} \propto \int r^{n-2} \sqrt{-\gamma} d^2 x [R[\gamma] + (n-2)(n-3)r^{-2}((\nabla r)^2 + 1) - 2(\nabla\phi)^2], \quad (3)$$

where curvature and differential operators are calculated using the two-dimensional metric γ_{AB} ; capital Latin indices run through $\{1, 2\}$ and a stroke $|$ denotes the covariant derivative with respect to the 2-metric γ_{AB} . By varying the reduced action with respect to field variables γ_{AB} , r and ϕ , we obtain Einstein-scalar field equations in spherical symmetry. After some algebraic manipulation they can be written as

$$R_{AB} - (n-2)r^{-1}r|_{AB} = 2\phi_{,A}\phi_{,B}, \quad (4a)$$

$$(n-3)[(\nabla r)^2 - 1] + r \square r = 0, \quad (4b)$$

$$\square\phi + (n-2)r^{-1}\gamma^{AB}\phi_{,A}r_{,B} = 0. \quad (4c)$$

As usual with the scalar field, the $\square\phi$ equation is redundant.

3. n -dimensional generalization of Roberts' solution

We are interested in generalization of Roberts' solution to n dimensions. To find it, we write the metric in double-null coordinates

$$d\gamma^2 = -2e^{-2\sigma(z)} du dv, \quad r = -u\rho(z), \quad \phi = \phi(z). \quad (5)$$

The dependence of metric coefficients and ϕ on $z = -v/u$ only reflects the fact that we are looking for a continuously self-similar solution, with z being a scale-invariant variable. We turn on the influx of the scalar field at the advanced time $v = 0$, so that the spacetime is Minkowskian to the past of this surface, and the initial conditions are specified by continuity.

Signs are chosen so that $z > 0$, $\rho > 0$ in the sector of interest ($u < 0$, $v > 0$). With this choice of metric, Einstein-scalar equations (4) become

$$(n-2)[\rho''z + 2\sigma'\rho - 2\sigma'\rho'z] = -2\rho z\phi'^2, \quad (6a)$$

$$2\rho(\sigma''z + \sigma') + (n-2)\rho''z = -2\rho z\phi'^2, \quad (6b)$$

$$(n-2)[\rho'' - 2\sigma'\rho'] = -2\rho\phi'^2, \quad (6c)$$

$$(n-3)[\rho'^2z - \rho'\rho + \frac{1}{2}e^{2\sigma}] + \rho''\rho z = 0, \quad (6d)$$

$$\phi''\rho z + (n-2)\phi'\rho'z - \frac{1}{2}(n-4)\phi'\rho = 0. \quad (6e)$$

A prime denotes the derivative with respect to z . Combining equations (6a) and (6c), we obtain that $\sigma = \text{constant}$. By appropriate rescaling of coordinates, we can put $\sigma = 0$. Then

$$(n-2)\rho'' = -2\rho\phi'^2, \quad (7a)$$

$$(n-3)[\rho'^2z - \rho'\rho + \frac{1}{2}] + \rho''\rho z = 0, \quad (7b)$$

$$\frac{\phi''}{\phi'} + (n-2)\frac{\rho'}{\rho} - \frac{1}{2}(n-4)z^{-1} = 0. \quad (7c)$$

For further derivation we will assume that $n > 3$, as the case $n = 3$ is trivial. Equation (7c) can be immediately integrated,

$$\phi'\rho^{n-2}z^{-(n-4)/2} = c_0. \quad (8)$$

Substituting this result back into equation (7a), we obtain the equation for ρ only

$$\rho''\rho^{2n-5} = -\frac{2c_0^2}{n-2}z^{n-4}. \quad (9)$$

It is easy to show that equation (9) is equivalent to equation (7b). No surprises here, since the system (4) was redundant. Combining both equations we obtain the first integral of motion

$$[\rho'^2z - \rho'\rho + \frac{1}{2}]\left(\frac{\rho^2}{z}\right)^{n-3} = \frac{2c_0^2}{(n-2)(n-3)}, \quad (10)$$

which contains only first derivatives of ρ , and for this reason is simpler to solve than either one of equations (7b) and (9). Equation (10) is a generalized homogeneous equation, and can be solved by substitution

$$x = \frac{1}{2}\ln z, \quad \rho = \sqrt{z}y(x), \quad \rho' = \frac{1}{2}z^{-1/2}(\dot{y} + y), \quad (11)$$

where a dot denotes the derivative with respect to the new variable x . With this substitution, equations (10) and (8) become

$$\dot{y}^2 = y^2 - 2 + c_1y^{-2(n-3)}, \quad (12)$$

$$\dot{\phi} = 2c_0y^{-(n-2)}, \quad (13)$$

where we have redefined the constant

$$c_1 = \frac{8c_0^2}{(n-2)(n-3)} > 0. \quad (14)$$

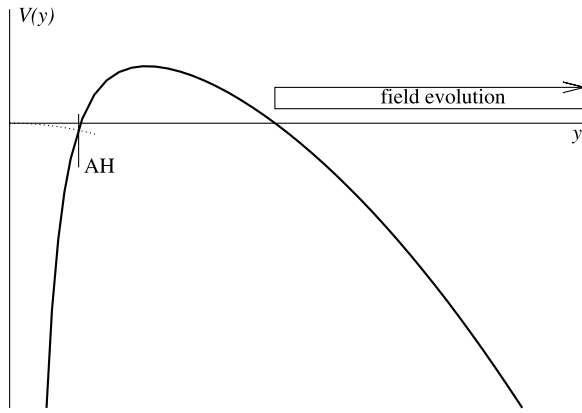


Figure 1. Subcritical field evolution.

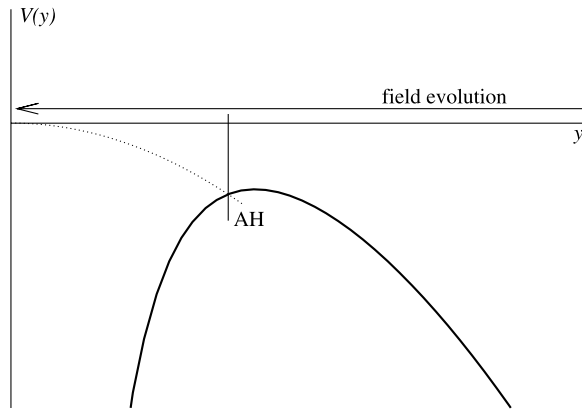


Figure 2. Supercritical field evolution.

The above equation (12) for y formally describes the motion of a particle with zero energy in the potential

$$V(y) = 2 - y^2 - c_1 y^{-2(n-3)}, \quad (15)$$

so we can describe the qualitative behaviour of y without actually solving equation (12).

Initial conditions are specified by continuous matching of the solution to Minkowskian spacetime on surface $v = 0$. Since on that surface $r \neq 0$, the value of $y = r/\sqrt{-uv}$ starts from infinity at $x = -\infty$, and rolls towards zero. What happens next depends on the shape of the potential. If there is region with $V(y) > 0$, as in figure 1, y will reach a turning point and will go back to infinity as $x = \infty$. If $V(y) < 0$ everywhere, as in figure 2, there is nothing to stop y from reaching zero, at which point a singularity is formed. Finally, if $V(y)$ has a second-order zero, as in figure 3, y will take forever to reach it.

Of course, variables separate, and equation (12) can be integrated

$$x = \pm \int \frac{dy}{\sqrt{y^2 - 2 + c_1 y^{-2(n-3)}}} + c_2. \quad (16)$$

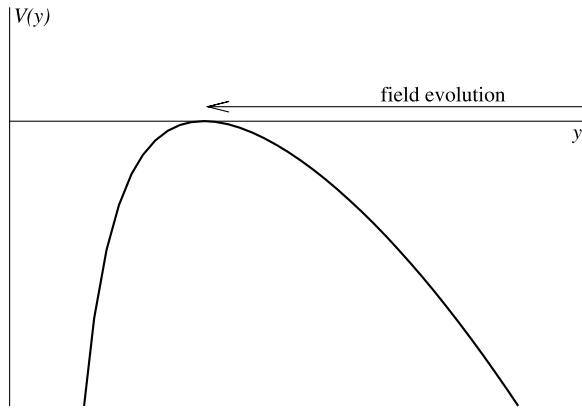


Figure 3. Critical field evolution.

The plus or minus sign in front of the integral depends on the sign of the derivative of y . Initial conditions imply that initially y comes from infinity towards zero, i.e. its derivative is negative, and so we must pick the branch of the solution which started out with a minus sign. Constant c_2 corresponds to a coordinate freedom in the choice of the origin of x , while constant c_1 is a real parameter of the solution.

Unfortunately, the integral cannot be evaluated in a closed form for arbitrary n . But if the integral is evaluated, and we can invert it to obtain y as a function of x , the solution for r is obtained by using definitions (11) and (5). The solution for ϕ is obtained by integrating relations (13) or (8).

4. Critical behaviour

The one-parameter family of self-similar scalar field solutions in n dimensions constructed above exhibits critical behaviour as the parameter c_1 is tuned, much as Roberts' family does in four dimensions. In this section we investigate black hole formation in the collapse.

In spherical symmetry, the existence and position of the apparent horizon are given by vanishing of $(\nabla r)^2 = 0$, which translates to $\rho' = 0$, or $\dot{y} + y = 0$ in our notation. Therefore, at the apparent horizon we have

$$\dot{y}^2 - y^2 = c_1 y^{-2(n-3)} - 2 = 0, \quad (17)$$

and the black hole is formed if the value of y reaches

$$y_{\text{AH}}^2 = \left(\frac{1}{2}c_1\right)^{1/(n-3)}. \quad (18)$$

As we have discussed above, depending on the value of c_1 , values of the field y either reach turning point and return to infinity, or go all the way to zero. The critical solution separates the two cases, and is characterized by the potential $V(y)$ having a second-order zero, i.e. $V(y_*) = V'(y_*) = 0$ at some point y_* . Differentiating expression (15) for potential $V(y)$, we see that it has a second-order zero at

$$y_*^2 = 2 \frac{n-3}{n-2}, \quad (19)$$

if and only if the value of the constant c_1 is

$$c_1^* = \frac{1}{n-3} \left[2 \frac{n-3}{n-2} \right]^{n-2}. \quad (20)$$

If the value of the parameter c_1 is less than critical, $c_1 < c_1^*$, the value of y turns around at the turning point, and never reaches the point of apparent horizon formation, which is located in the forbidden zone, as illustrated in figure 1. This case is subcritical evolution of the field. If $c_1 > c_1^*$, the value of y reaches a point where an apparent horizon is formed, and proceeds to go to zero, at which point there is a singularity inside the black hole. This supercritical evolution is illustrated in figure 2.

The mass of the black hole formed in the supercritical collapse is

$$M = \frac{1}{2}r_{\text{AH}} = -\frac{1}{2}u\sqrt{z_{\text{AH}}y_{\text{AH}}}. \quad (21)$$

It grows infinitely if we wait long enough, and will absorb all the field influx coming from past infinity. This happens because the solution is self-similar, and creates a problem for discussing mass scaling in the near-critical collapse. Cut and glue schemes [14] avoiding infinite black hole mass are a temporary means to lift this problem. However, the real answer to determining whether the critical solution is an intermediate attractor and calculating the mass-scaling exponent is to make a perturbative analysis of the critical solution. Similarity to Roberts' solution suggests that the results for four-dimensional spacetime [12] can be applied to higher dimensions as well.

5. Particular cases

In this section we consider several particular cases for which the general solution (16) is simplified. Particularly important is the $n = 4$ case, which is the already familiar Roberts' solution.

5.1. $n = 3$

As we have already mentioned, for $n = 3$ the only self-similar scalar field solution of the form (5) is trivial. To see this, note that equation (7b) implies that $\rho'' = 0$ if $n = 3$, and so $\rho = \alpha z + \beta$ and $r = \alpha v - \beta u$. From equation (7a) it then follows that $\phi = \text{constant}$. The spacetime is flat.

5.2. $n = 4$

Integration (16) can be carried out explicitly

$$\begin{aligned} x &= -\int \frac{dy}{\sqrt{y^2 - 2 + c_1 y^{-2}}} + c_2 \\ &= -\frac{1}{2} \ln |y^2 - 1 + \sqrt{y^4 - 2y^2 + c_1}| + c_2, \end{aligned} \quad (22)$$

and the result inverted

$$y^2 = \frac{1}{2}e^{-2(x-c_2)} + 1 + \frac{1}{2}(1 - c_1)e^{2(x-c_2)}, \quad (23)$$

to give the solution in the closed form

$$\rho = \sqrt{\frac{e^{2c_2}}{2} + z + \frac{1 - c_1}{2e^{2c_2}} z^2}. \quad (24)$$

By appropriately rescaling coordinates, we can put $e^{2c_2} = 2$. After redefining the parameter of the solution $p = (c_1 - 1)/4$, the solution takes on the following simple form:

$$\rho = \sqrt{1 + z - pz^2}, \quad r = \sqrt{u^2 - uv - pv^2}. \quad (25)$$

The scalar field ϕ is reconstructed from equation (8)

$$\phi' = c_0 \rho^{-2} = \frac{1}{2} \frac{\sqrt{1+4p}}{1+z-pz^2}, \quad (26)$$

to give

$$\begin{aligned} \phi &= \operatorname{arctanh} \frac{2pz-1}{\sqrt{1+4p}} + \text{constant} \\ &= \frac{1}{2} \ln \left[-\frac{2pz-1+\sqrt{1+4p}}{2pz-1-\sqrt{1+4p}} \right] + \text{constant}. \end{aligned} \quad (27)$$

The critical parameter value is $p^* = 0$, and for $p > 0$ the black hole is formed. The critical solution is

$$r = \sqrt{u^2 - uv}, \quad \phi = \frac{1}{2} \ln \left[1 - \frac{v}{u} \right]. \quad (28)$$

5.3. $n = 5, 6$

The integral (16) can be written in terms of elliptic functions, which becomes apparent with the change of variable $\bar{y} = y^{-2}$

$$x = \frac{1}{2} \int \frac{d\bar{y}}{\bar{y} \sqrt{1 - 2\bar{y} + c_1 \bar{y}^{n-2}}}, \quad (29)$$

and the solution $y(x)$ is given implicitly. However, integrals corresponding to critical solutions simplify, and can be taken in terms of elementary functions for $n = 5, 6$. The simplification happens because the potential factors, since it has second-order zero, therefore reducing the power of y in the radical by two. The results of integration for critical solutions are

$$\begin{aligned} n = 5: \quad x &= - \int \frac{y^2 dy}{(y^2 - \frac{4}{3}) \sqrt{y^2 + \frac{2}{3}}} \\ &= -\operatorname{arcsinh} \left(\sqrt{\frac{3}{2}} y \right) + \frac{1}{\sqrt{6}} \operatorname{arctanh} \left(\frac{\sqrt{3}y+1}{\sqrt{\frac{9}{2}y^2+3}} \right) \\ &\quad + \frac{1}{\sqrt{6}} \operatorname{arctanh} \left(\frac{\sqrt{3}y-1}{\sqrt{\frac{9}{2}y^2+3}} \right) \end{aligned} \quad (30)$$

$$\begin{aligned} n = 6: \quad x &= - \int \frac{y^3 dy}{(y^2 - \frac{3}{2}) \sqrt{y^4 + y^2 + \frac{3}{4}}} \\ &= -\frac{1}{2} \operatorname{arcsinh} \left(\sqrt{2} \left(y^2 + \frac{1}{2} \right) \right) + \frac{1}{2\sqrt{2}} \operatorname{arctanh} \left(\frac{1}{\sqrt{2}} \frac{\frac{4}{3}y^2+1}{\sqrt{y^4+y^2+\frac{3}{4}}} \right). \end{aligned} \quad (31)$$

The dependence $y(x)$ is still given implicitly. The critical value of the parameter c_1^* is $\frac{32}{27}$ for $n = 5$ and $\frac{27}{16}$ for $n = 6$.

5.4. Higher dimensions

For higher dimensions, the integral (16) cannot be taken in terms of elementary functions, so one has to be content with the solution in the integral-implicit form, or perform numerical calculations.

6. General scalar field coupling

In this section we discuss in detail how solutions of the minimally coupled scalar field model can be generalized to a much wider class of couplings. The fact that essentially all couplings of a free scalar field to its kinetic term and scalar curvature are equivalent has been used previously [15, 16] to study scalar field models with non-minimal coupling. In particular, this idea has been applied to extend the four-dimensional Roberts' solution to conformal coupling [17] and Brans–Dicke theory [18].

6.1. Equivalence of couplings

Suppose that the action describing evolution of the scalar field in n -dimensional spacetime is

$$S = \frac{1}{16\pi} \int \sqrt{-g} d^n x [F(\phi)R - G(\phi)(\nabla\phi)^2] \quad (32)$$

plus surface terms, where the couplings F and G are smooth functions of the field ϕ . Also suppose that the signs of couplings F and G correspond to the case of gravitational attraction. We will demonstrate that this action reduces to the minimally coupled one by redefinition of the fields $g_{\mu\nu}$ and ϕ . First, let us introduce a new metric $\hat{g}_{\mu\nu}$ that is related to the old one by the conformal transformation

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \hat{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}, \quad \sqrt{-\hat{g}} = \Omega^n \sqrt{-g}, \quad (33)$$

and denote quantities and operators calculated using $\hat{g}_{\mu\nu}$ by a hat. Scalar curvatures calculated using the old and new metrics are related

$$R = \Omega^2 \hat{R} + 2(n-1)\Omega \hat{\square} \Omega - n(n-1)(\hat{\nabla}\Omega)^2, \quad (34)$$

as are field gradients

$$(\nabla\phi)^2 = \Omega^2 (\hat{\nabla}\phi)^2. \quad (35)$$

Writing the action (32) in terms of the metric $\hat{g}_{\mu\nu}$, we obtain

$$S = \frac{1}{16\pi} \int \Omega^{-n} \sqrt{-\hat{g}} d^n x [F\{\Omega^2 \hat{R} + 2(n-1)\Omega \hat{\square} \Omega - n(n-1)(\hat{\nabla}\Omega)^2\} - G\Omega^2 (\hat{\nabla}\phi)^2]. \quad (36)$$

By choosing the conformal factor to be

$$\Omega^{n-2} = F, \quad (37)$$

the factor in front of the curvature \hat{R} can be set to one. Substitution of the definition of Ω into the above action, and integration by parts of $\hat{\square}$ operator yields

$$S = \frac{1}{16\pi} \int \sqrt{-\hat{g}} d^n x \left[\hat{R} - \left(\frac{G}{F} + \frac{n-1}{n-2} \frac{F'^2}{F^2} \right) (\hat{\nabla}\phi)^2 \right]. \quad (38)$$

The kinetic term in action (38) can be brought into a minimal form by the introduction of a new scalar field $\hat{\phi}$, related to the old one by

$$2(\hat{\nabla}\hat{\phi})^2 = \left(\frac{G}{F} + \frac{n-1}{n-2} \frac{F'^2}{F^2} \right) (\hat{\nabla}\phi)^2. \quad (39)$$

Thus, we have shown that with field redefinitions

$$\hat{\phi} = \frac{1}{\sqrt{2}} \int \left(\frac{G}{F} + \frac{n-1}{n-2} \frac{F'^2}{F^2} \right)^{1/2} d\phi, \quad \hat{g}_{\mu\nu} = F^{2/(n-2)} g_{\mu\nu}, \quad (40)$$

the generally coupled scalar field action (32) becomes minimally coupled

$$S = \frac{1}{16\pi} \int \sqrt{-\hat{g}} d^n x [\hat{R} - 2(\hat{\nabla}\hat{\phi})^2]. \quad (41)$$

This equivalence allows one to construct solutions of the model with general coupling (32) from the solutions of the minimally coupled scalar field by means of the inverse of relation (40), provided said inverse is well defined. However, there may be some restrictions on the range of ϕ so that field redefinitions give real $\hat{\phi}$ and positive-definite metric $\hat{g}_{\mu\nu}$. This means that not all the branches of the solution in general coupling may be covered by translating the minimally coupled solution. Technically speaking, the correspondence between solutions of minimally coupled theory and generally coupled theory is one-to-one where defined, but not onto.

However, one has to be careful making claims about global structure and critical behaviour of the generalized solutions based solely on the properties of the minimally coupled solution. The scalar field solutions encountered in critical collapse often lead to singular conformal transformations, which could, in principle, change the structure of spacetime.

6.2. Examples

To illustrate the above discussion, we consider two often used scalar field couplings as examples. They are non-minimal coupling and dilaton gravity.

6.2.1. Conformal coupling. Non-minimally coupled scalar field in n dimensions is described by the action

$$S = \frac{1}{16\pi} \int \sqrt{-g} d^n x [(1 - 2\xi\phi^2)R - 2(\nabla\phi)^2], \quad (42)$$

where ξ is the coupling parameter. Coupling factors are $F = 1 - 2\xi\phi^2$, $G = 2$ and so the field redefinition (40) looks like

$$\begin{aligned} \hat{\phi} &= \int \frac{\sqrt{1 - 2\xi\phi^2 + 2\xi_c^{-1}\xi^2\phi^2}}{1 - 2\xi\phi^2} d\phi \\ &= \frac{1}{\sqrt{2}} \sqrt{\xi^{-1} - \xi_c^{-1}} \arcsin \left[\sqrt{2} \sqrt{\xi - \xi_c^{-1}\xi^2} \phi \right] + \frac{1}{\sqrt{2\xi_c}} \operatorname{arcsinh} \left[\frac{\sqrt{2\xi_c^{-1}} \xi \phi}{\sqrt{1 - 2\xi\phi^2}} \right], \end{aligned} \quad (43)$$

where

$$\xi_c = \frac{1}{4} \frac{n-2}{n-1}. \quad (44)$$

Particularly interesting is the case of conformal coupling $\xi = \xi_c$ because field redefinition

$$\hat{\phi} = \frac{1}{\sqrt{2\xi_c}} \operatorname{arctanh}[\sqrt{2\xi_c}\phi] \quad (45)$$

can be inverted explicitly to give the recipe for obtaining conformally coupled solutions from minimally coupled ones. It is

$$\phi = \frac{1}{\sqrt{2\xi_c}} \tanh[\sqrt{2\xi_c}\hat{\phi}], \quad (46)$$

$$g_{\mu\nu} = \frac{\hat{g}_{\mu\nu}}{(1 - 2\xi_c\phi^2)^{2/(n-2)}} = \cosh^{4/(n-2)}[\sqrt{2\xi_c}\hat{\phi}] \hat{g}_{\mu\nu}. \quad (47)$$

In particular, the four-dimensional Roberts' solution becomes

$$\phi = \sqrt{3} \tanh \left[\frac{1}{\sqrt{3}} \operatorname{arctanh} \frac{2pz - 1}{\sqrt{1 + 4p}} \right], \quad (48)$$

$$ds^2 = \cosh^2 \left[\frac{1}{\sqrt{3}} \operatorname{arctanh} \frac{2pz - 1}{\sqrt{1 + 4p}} \right] \{-2du dv + (u^2 - uv - pv^2) d\omega^2\}, \quad (49)$$

in the conformally coupled model. This last expression was considered in [17].

6.2.2. Dilaton gravity. Another useful example is dilaton gravity described by the action

$$S = \frac{1}{16\pi} \int \sqrt{-g} d^n x e^{-2\phi} [R + 4(\nabla\phi)^2]. \quad (50)$$

Substituting coupling factors $F = e^{-2\phi}$, $G = -4e^{-2\phi}$ into relationship (40), one can see that the scalar field redefinition is a simple scaling

$$\hat{\phi} = \sqrt{\frac{2}{n-2}} \phi, \quad \phi = \sqrt{\frac{n-2}{2}} \hat{\phi}, \quad (51)$$

and metrics differ by exponential factor only

$$g_{\mu\nu} = \exp \left[\sqrt{\frac{2}{n-2}} 2\hat{\phi} \right] \hat{g}_{\mu\nu}. \quad (52)$$

In particular, the four-dimensional Roberts' solution becomes

$$\phi = \operatorname{arctanh} \frac{2pz - 1}{\sqrt{1 + 4p}}, \quad (53)$$

$$ds^2 = e^{2\phi} \{-2 du dv + (u^2 - uv - pv^2) d\omega^2\} \quad (54)$$

in dilaton gravity.

7. Conclusion

We have searched for and found continuously self-similar spherically symmetric solutions of minimally coupled scalar field collapse in n -dimensional spacetime. For spacetime dimensions higher than three they form a one-parameter family and display critical behaviour much like the Roberts solution. A qualitative picture of field evolution is easy to visualize in analogy with a particle travelling in a potential of upside-down U shape. Unfortunately, the solutions in dimensions higher than four can only be obtained in implicit form. Critical solutions are, in general, simpler than other members of the family due to the potential factoring. The strong similarity between Roberts' solution and its higher-dimensional generalizations allows one to conjecture that these higher-dimensional critical solutions are not attractors either. The absence of a non-trivial self-similar solution in three dimensions raises the question of whether a scalar field collapsing in three dimensions exhibits critical behaviour at all. Perhaps further numerical simulations will answer it.

We also use equivalence of scalar field couplings to generalize solutions of minimally coupled scalar field to a much wider class of couplings. For often-used cases of conformal coupling and dilaton gravity the results are remarkably simple. Some results of [19], applied for a single scalar field only, become trivial in view of this coupling equivalence.

However, the question of critical behaviour of these generalized solutions is complicated by the fact that the conformal factor Ω relating metrics (33) for minimally and generally coupled

solutions may be singular. In the simplest case of non-singular conformal transformation (i.e. when the coupling F is bounded and the lower bound is greater than zero) global properties of the minimally coupled solution are preserved, and all important features of near-critical collapse carry over on the generalized solution unchanged. If conformal transformation is suspected to be singular, a more careful study of global properties of the generalized solution (40) is necessary.

Acknowledgments

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