

Chaotic scattering and capture of strings by a black hole

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Abstract. We consider scattering and capture of circular cosmic strings by a Schwarzschild black hole. Although it is *a priori* a very simple axially symmetric two-body problem, it shows all the features of chaotic scattering. In particular, it contains a fractal set of unstable periodic solutions; a so-called strange repeller. We study the different types of trajectories and obtain the fractal dimension of the basin boundary separating the space of initial conditions according to the different asymptotic outcomes. We also consider the fractal dimension as a function of energy, and discuss the transition from order to chaos.

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Chaos in general relativity and cosmology is by now a well established subject. Some studies, pioneered by the work of Hawking [1] and Page [2], have been concerned with the behaviour of solutions of the Einstein equations themselves; the most famous example now being the mixmaster universe (see, for instance, [3], and references given therein). In other studies, the subject of interest has been the behaviour of test-particle trajectories in black-hole spacetimes (see, for instance, [4, 5]). In the latter case, since point-particle motion is completely integrable in the generic Kerr–Newman background [6], it was necessary to consider quite complicated multi-black-hole spacetimes (typically of the Majumdar–Papapetrou type [7, 8]) to obtain chaotic point-particle dynamics. Taking into account that these systems represent (at least) three-body problems, and comparing with Newtonian dynamics, it was certainly no surprise that chaotic dynamics appeared. (It should be mentioned, however, that in the two-black-hole case, chaos is really a relativistic effect [4].) The main interest in these systems therefore was also to use and further develop coordinate-invariant descriptions and measures of chaotic behaviour [9, 10], suitable for general relativity where space and time are not absolute.

In the present paper, we study a conceptually simpler and more symmetric system in general relativity that nevertheless, and maybe somewhat surprisingly, leads to chaotic behaviour. We consider the two-body problem of a circular test string in the background of a Schwarzschild black hole. The circular string is taken to be coaxial with the black hole and is allowed to oscillate in its plane and to propagate in the direction perpendicular to its plane, as illustrated in figure 1. The combined system is therefore axially symmetric.

The physical picture we have in mind is that of a cosmic string near an astrophysical black hole. For a grand unified theory (GUT) string with string tension of the order of $G\mu \approx 10^{-6}$ [11], the test-string approximation should be valid even for a string initially up to (say) 10^4 times longer than the black-hole horizon radius. Moreover, for a GUT string we can use the

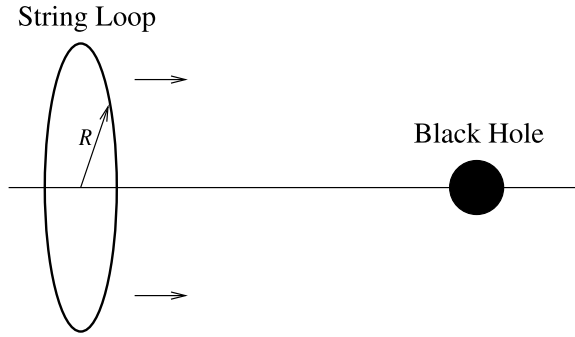


Figure 1. Schematic illustration of a string loop approaching a black hole.

leading-order thickness approximation, where the cosmic string is described by the Nambu–Goto action [11]

$$S = -\mu \int d\tau d\sigma \sqrt{-\det[G_{ab}]}, \quad (1)$$

where $G_{ab} = g_{\mu\nu} X_{,a}^{\mu} X_{,b}^{\nu}$ is the induced metric on the string worldsheet. In this case, the string equations of motion and constraints (in conformal gauge) take the standard form

$$\ddot{X}^{\mu} - X''^{\mu} + \Gamma_{\rho\sigma}^{\mu} (\dot{X}^{\rho} \dot{X}^{\sigma} - X'^{\rho} X'^{\sigma}) = 0, \quad (2)$$

$$g_{\mu\nu} \dot{X}^{\mu} X'^{\nu} = g_{\mu\nu} (\dot{X}^{\mu} \dot{X}^{\nu} + X'^{\mu} X'^{\nu}) = 0, \quad (3)$$

where a dot and a prime denote derivatives with respect to the string worldsheet coordinates τ and σ , respectively. Using Schwarzschild coordinates for the background metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (4)$$

and parametrizing the circular string by ansatz

$$t = t(\tau), \quad r = r(\tau), \quad \theta = \theta(\tau), \quad \phi = \sigma, \quad (5)$$

one finds that the string equations of motion (2) and constraints (3) lead to the following system of ordinary differential equations:

$$\dot{t} = E \left(1 - \frac{2M}{r}\right)^{-1}, \quad (6)$$

$$\ddot{r} = (r - 3M)\dot{\theta}^2 - (r - M)\sin^2 \theta, \quad (7)$$

$$\ddot{\theta} = -\frac{2}{r} \dot{r} \dot{\theta} - \sin \theta \cos \theta, \quad (8)$$

supplemented by the constraint

$$\dot{r}^2 + (r^2 - 2Mr)(\dot{\theta}^2 + \sin^2 \theta) = E^2. \quad (9)$$

The integration constant E , appearing here, will play the role of an external control parameter ('order parameter') for the system. It is equal to the total conserved energy of the string divided by $2\pi\mu$. In the absence of first integrals other than (9), we are thus dealing with a three-dimensional phase space. Notice also that the worldsheet time τ is not equal to the proper time. The worldsheet time τ is, however, finite for a string falling into the black hole.

Equations (6)–(9) are equivalent to the following Hamiltonian system:

$$H = \frac{1}{2} \left(1 - \frac{2M}{r} \right) P_r^2 + \frac{1}{2r^2} P_\theta^2 - \frac{E^2}{2} \left(1 - \frac{2M}{r} \right)^{-1} + \frac{1}{2} r^2 \sin^2 \theta, \quad (10)$$

with the constraint $H = 0$. This is very similar to the Hamiltonian describing zero angular momentum photons in the Schwarzschild background [12], except for the last term in the potential which is due to the string tension. This is precisely the term that leads to non-integrability and, as we shall see, chaos.

The Hamiltonian system (10) has previously been considered and solved in the equatorial plane ($\theta = \pi/2$) [13], and some trajectories in the general case were computed numerically in [14]. In this paper, we make a more complete analysis of the dynamics associated with the Hamiltonian (10). In particular, we shall demonstrate the presence of chaos in this simple system.

Due to the non-integrable nature of the dynamical system (6)–(9), the analysis of the string evolution was done numerically. The string trajectories for various initial conditions were obtained by integrating equations (7) and (8) using the fifth-order embedded Runge–Kutta method with adaptive step size control [15]. The constraint (9) was used to check independently for numerical precision.

It is easy to see that there are three possible asymptotic outcomes of the string dynamics. The string can either fly by the black hole and escape to $(r, \theta) = (\infty, \pi)$, or the string can backscatter and escape to $(r, \theta) = (\infty, 0)$, or the string can be captured by the black hole $r \leq 2M$. Some examples of string trajectories illustrating these outcomes are shown in figure 2, where we plot the string radius $R = r \sin \theta$ (vertically) as a function of $Z = r \cos \theta$ (horizontally, the string comes in from the left). In the three examples shown, the string is initially collapsed at a position a few horizon radii outside of the black hole. It then expands and propagates to the right—towards the black hole. In figure 2(a), the string passes the black hole but then returns and is captured, $r \leq 2M$. In figure 2(b), the string passes the black hole

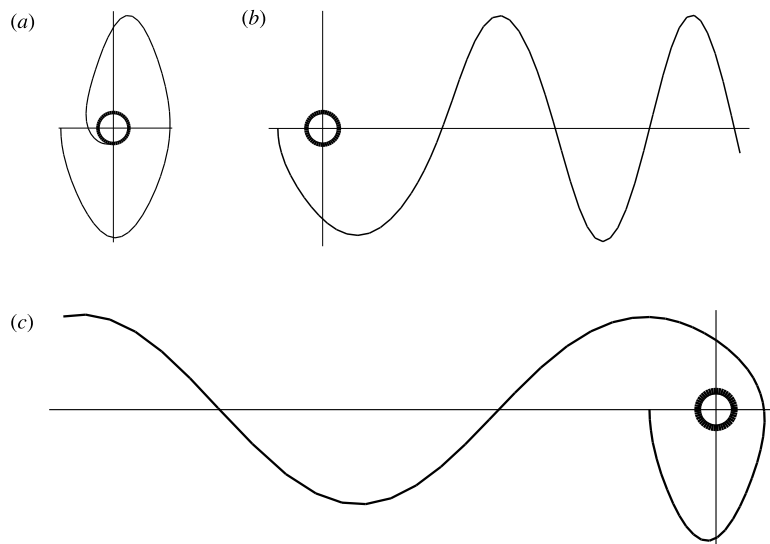


Figure 2. Possible asymptotic outcomes of the string evolution: (a) capture, (b) escape and (c) escape with backscatter. Axes are $(r \cos \theta, r \sin \theta)$, and the thick circle represents the event horizon of the black hole.

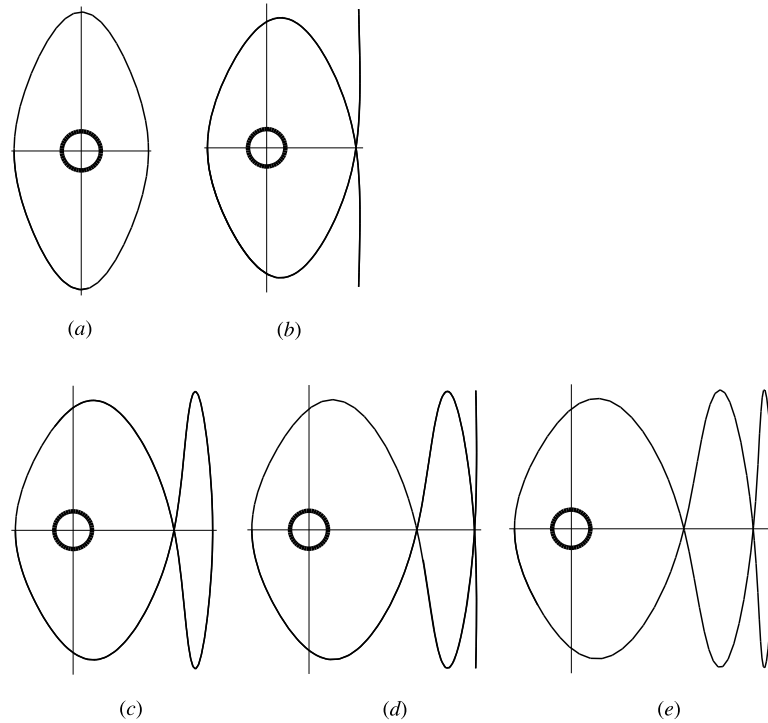


Figure 3. Unstable periodic orbits.

and escapes to the right, $(r, \theta) = (\infty, \pi)$. In figure 2(c), the string passes the black hole, then returns and finally escapes to the left, $(r, \theta) = (\infty, 0)$.

Besides the solutions with these three asymptotic outcomes, there is an infinite set of unstable periodic orbits, which separate the solutions with different asymptotic outcomes in the phase space of all solutions. Typical examples of periodic orbits are shown in figure 3. Besides the ones shown, there are also periodic orbits when the string starts a little further away from the black hole and oscillates a number of times before reaching it. There are no stable periodic orbits in our system.

To gain a better understanding of the string dynamics, we now consider a two-dimensional slice of the four-dimensional space of initial conditions in more detail. It is most convenient to fix the constant ‘energy’ E and then to impose one more relation between initial values of $(r, \theta, \dot{r}, \dot{\theta})$ at $\tau = 0$. Following the procedure of the basin-boundary method for chaotic scattering [9], we then colour this two-dimensional slice of initial conditions according to the three different asymptotic outcomes mentioned above. Due to numerical reasons, the determination of the two asymptotic outcomes corresponding to escape is done not at infinity but at some large, but finite, value of r . While in general the string can pass this cut-off radius in one direction but change its mind later, for sufficiently large r , this procedure will only lead to a wrong colour for very few points, as follows from the asymptotic behaviour of the potential in the Hamiltonian (10) (this should be contrasted with the case of the mixmaster universe [16], where all the trajectories eventually bounce back).

In figure 4 we show examples of this colouring procedure applied to the phase space of string solutions. Figures 4(a) and (b) show two different two-dimensional slices of the space of initial conditions. Figures 4(c) and (d) show magnifications of the regions of figures 4(b)

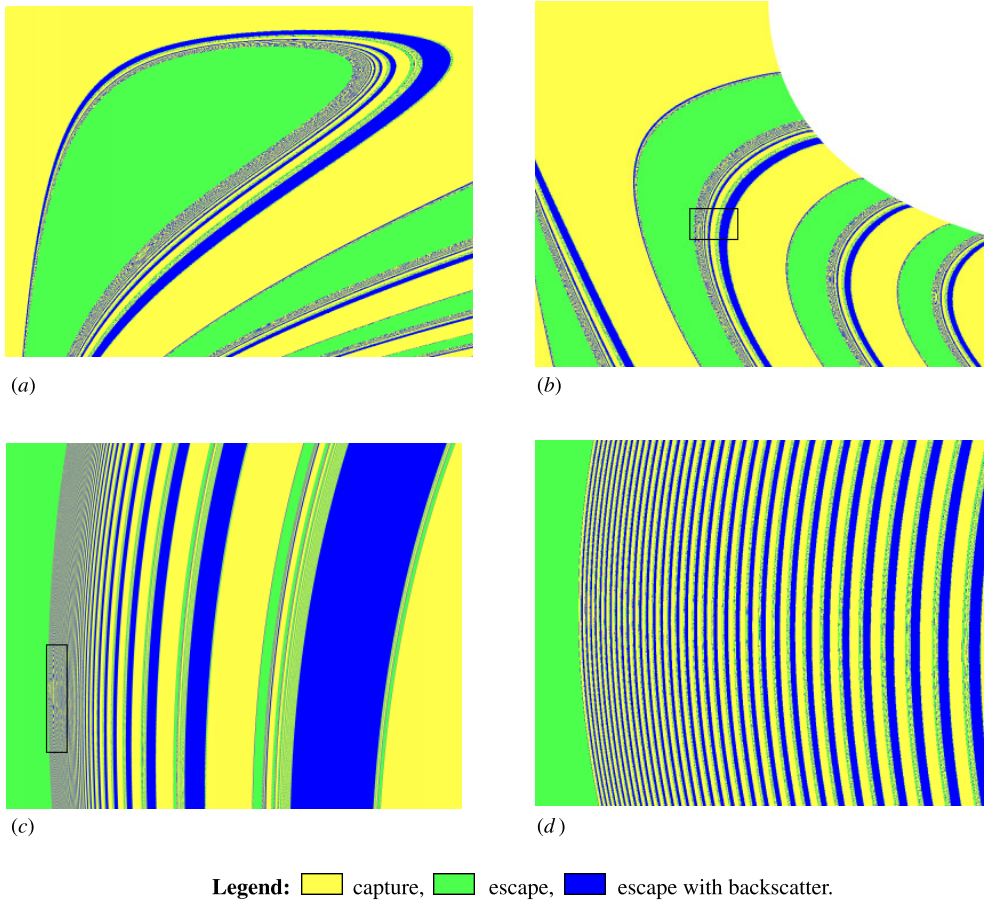


Figure 4. Basin boundaries, plotted for $E = 14.0M$: (a) slice $\theta = 0$, with $r \in [2.0M, 29.2M]$ on the horizontal axis and $-\dot{r} \in [0, E]$ on the vertical axis; (b) slice $\frac{d}{dt}(r \cos \theta) = 0$, with $r \in [2.0M, 27.5M]$ on the horizontal axis and $\theta \in [0, \pi/2]$ on the vertical axis; (c) and (d) show fractal detail of figures (b) and (c), respectively.

and (c) indicated in the previous figures. The boundaries between different colours in these figures correspond to the unstable periodic orbits. The magnifications of these so-called basin boundaries reveal a fractal structure (see figures 4(b)–(d)). This provides the coordinate-invariant indication that the dynamics is in fact chaotic [9].

To obtain a quantitative measure we can also determine the fractal dimension. Consider for instance figure 4(b), which corresponds to the two-dimensional slice in the space of initial conditions, given by

$$E = 14.0M, \quad \frac{d}{d\tau}(r \cos \theta) = 0 \quad \text{at} \quad \tau = 0. \tag{11}$$

The box-counting dimension is defined by [9]

$$D = \lim_{\epsilon \rightarrow 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}, \tag{12}$$

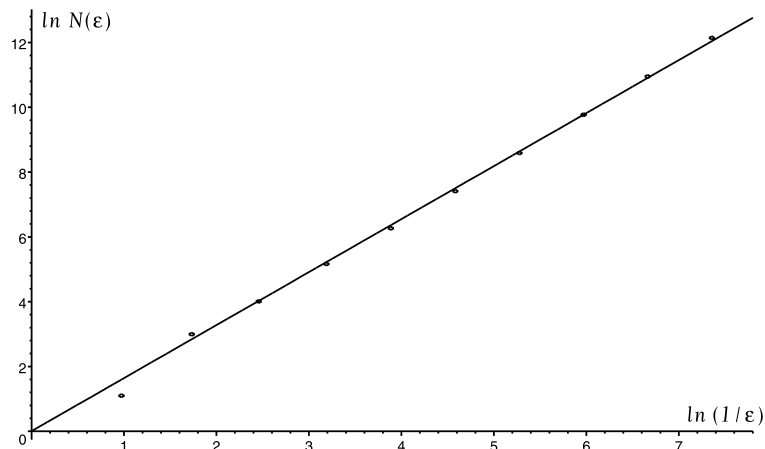


Figure 5. Number of fractal points as a function of resolution, calculated for the slice in figure 4(b).

where $N(\epsilon)$ is the number of squares of side length ϵ needed to cover the basin boundary. A square should be counted only if it contains at least two different colours, otherwise it is not part of the basin boundary. The result of this counting is shown as a plot of $\ln N(\epsilon)$ versus $\ln(1/\epsilon)$ in figure 5. The straight line is a least-squares fit to the data points and shows that the dependence is indeed linear over a very wide range of resolution. The fractal dimension of figure 4(b), calculated from the highest-resolution grid of 4000×3200 points, is

$$D = 1.65 \pm 0.03. \quad (13)$$

The non-integer value shows in a coordinate-invariant way [17] that the basin boundary is indeed a fractal; a so-called strange repeller. The error in the result is due to the finite size of the numerical grid and the slow convergence rate of definition (12), and can be estimated by examining D calculated at lower resolutions.

It is interesting to also examine the fractal dimension as a function of ‘energy’ E . We considered the slice (11) in the energy range $E \in [0, 10^3 M]$ and computed the fractal dimension (12) of the basin boundary (corresponding to figure 4(b) and its analogue for different energies) as a function of energy E . The result is shown in figure 6, where we plot the fractal dimension D as a function of $\ln(E/M)$. For low energies, $E \lesssim 4.37M$, the dynamics is completely regular in the sense that there is only one asymptotic outcome—namely capture. For slightly larger energies, with values of E between $4.37M$ and $5.67M$, escape becomes possible but the different asymptotic outcomes are connected by ‘regular transitions’, corresponding to one-dimensional curves in the space of initial conditions (and thus corresponding to fractal dimension 1). However, at energy $E \simeq 5.67M$, the picture changes dramatically from regular to highly chaotic dynamics. In a very narrow energy range, the fractal dimension changes abruptly from 1 to approximately 1.6, whereafter it increases slightly for higher energies. It must be stressed that the numerical values given above for energies separating the different ‘phases’ are slice dependent. The phenomenon explained and demonstrated by figure 6 is, however, generic.

In conclusion, we have shown in a coordinate-invariant way that the axially symmetric system of a circular test string in the Schwarzschild black-hole background is chaotic. Chaos sets in abruptly at a certain ‘critical’ value of the external control parameter, which is related to

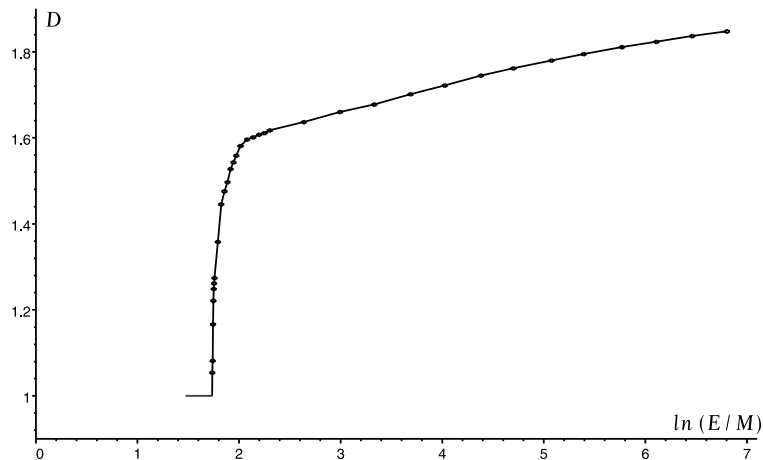


Figure 6. Fractal dimension as a function of energy calculated for the slice in figure 4(b). The radial size of the box is scaled with energy as $r \in [2.0M, (2.0 + (25.5/14.0) E/M)M]$ to keep the choice of region consistent.

the conserved energy of the string. Below this critical energy, the dynamics is regular. Above the critical energy, the dynamics is highly irregular and chaotic.

In all fairness, it should be mentioned that we have neglected some physical effects, including the backreaction and gravitational radiation of the cosmic string. However, the system considered here represents the simplest and most symmetric example of string dynamics in black-hole spacetimes and therefore suggests quite generally that string dynamics in black-hole spacetimes is chaotic (notice, however, that the problem of *stationary* strings in black-hole spacetimes is completely integrable [18]). It would thus be interesting to consider other dynamical string configurations as well, and we notice that there has recently been some interest in the scattering and capture of open strings by a black hole; see [19, 20] and references given therein.

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